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Research Paper

Classifying simple superalgebras with automorphism and pseudoautomorphism ☆

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Let F be an algebraically closed field of characteristic zero. We give a complete classification of finite dimensional simple superalgebras over F endowed with a graded automorphism or a pseudoautomorphism.

1. Introduction

Let A be an associative algebra over a fixed field F of characteristic zero. The algebra A is called simple if $A^2 \neq 0$ and it has no non-trivial ideals. The most famous example of a simple algebra is given by $M_n(F)$, the algebra of $n \times n$ matrices over F. Actually,

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under the hypothesis of an algebraically closed field, this is the only example of a finite dimensional simple algebra, as stated in the well-known Wedderburn-Artin Theorem.

Theorem 1. *If A is a finite dimensional algebra over an algebraically closed field F, then A is simple if and only if $A \cong M_n(F)$, for some $n \geq 1$.*

The main goal of this paper is to generalize the Wedderburn-Artin Theorem in the setting of algebras with additional structures. More precisely, we assume that the algebra A, defined over an algebraically closed field, has a structure of superalgebra. Moreover it is endowed with particular graded linear maps.

We recall that a linear map φ defined on $A = A_0 \oplus A_1$ is graded if it preserves the grading, i.e., $\varphi(A_j) \subseteq A_j$, for any $j = 0, 1$. Graded involutions, superinvolutions and pseudoinvolutions are examples of graded maps. For superalgebras with such graded maps a classification theorem of *simple algebras* is known (see [1,2,5,6]).

In this paper we are interested in automorphism-like graded maps.

Recently the authors started the study of superalgebras with superautomorphisms, i.e., superalgebras endowed with graded linear maps φ of order ≤ 2 such that $\varphi(ab) = (-1)^{|a||b|}\varphi(a)\varphi(b)$. Here $|a|, |b|$ denote the homogeneous degree of a, b. In [4] a complete classification of the finite dimensional simple superalgebras with superautomorphism was given.

Now assume that the superalgebra A is endowed with a graded automorphism of order at most 2 or with a pseudoautomorphism. We recall that φ is a pseudoautomorphism on A if $\varphi^2(a) = (-1)^{|a|}a$ and $\varphi(ab) = (-1)^{|a||b|}\varphi(a)\varphi(b)$.

In 2022 the first author showed that superautomorphisms and pseudoautomorphisms represent the connection link between graded involutions, superinvolutions and pseudoinvolutions ([3]).

In this paper we obtain a classification theorem in the setting of simple superalgebras with graded automorphism or pseudoautomorphism.

2. Superalgebras and automorphisms

Throughout this paper F will denote an algebraically closed field of characteristic zero and $A = A_0 \oplus A_1$ an associative superalgebra, that is an F-algebra graded by $\mathbb{Z}_2$, the cyclic group of order 2.

The subspaces A_0 and A_1 satisfy the properties $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$; their elements are called homogeneous of degree zero (or even elements) and of degree one (or odd elements), respectively.

It is well-known that there is a one-to-one correspondence between $\mathbb{Z}_2$-gradings and automorphisms of order ≤ 2. If $A = A_0 \oplus A_1$ is a superalgebra then A can be endowed with an automorphism $\psi \colon A \to A$ such that $\psi(a_0 + a_1) = a_0 - a_1$, for all $a_0 \in A_0$, $a_1 \in A_1$. Conversely, let A be endowed with an automorphism ψ of order ≤ 2 and let

$A = A_0^\psi + A_1^\psi$, where $A_0^\psi = \{a \in A \mid \psi(a) = a\}$ and $A_1^\psi = \{a \in A \mid \psi(a) = -a\}$. Then $A = A_0 \oplus A_1$ is a superalgebra with grading $A_0 = A_0^\psi$ and $A_1 = A_1^\psi$.

We say that an ideal I of a superalgebra $A = A_0 \oplus A_1$ is graded if

$$I = (I \cap A_0) \oplus (I \cap A_1).$$

In case A does not contain non-trivial graded ideals we say that A is a simple superalgebra.

The structure of simple superalgebras is given in the following result (see [7]).

Theorem 2. *Let A be a finite dimensional simple superalgebra over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following:*

- $Q(n) = M_n(F \oplus cF)$, $c^2 = 1$, *where* $Q(n)_0 = M_n(F)$ *and* $Q(n)_1 = cM_n(F)$.
- $M_{k,h}(F)$, *the algebra of* $n \times n$ *matrices,* $n = k + h$, $k \geq h \geq 0$, *with the following* $\mathbb{Z}_2$*-grading*

$$M_{k,h}(F) = \left\{ \begin{pmatrix} K & 0 \\ 0 & H \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & R \\ S & 0 \end{pmatrix} \right\},$$

where K, R, S, H *are* $k \times k$, $k \times h$, $h \times k$, $h \times h$ *matrices, respectively.*

Now we want to restate the previous theorem in the language of algebras with an automorphism. Clearly we say that an algebra is simple, as an algebra with automorphism ψ, if it has no non-trivial ψ-ideals, i.e., ideals invariant under the action of ψ. Moreover, (A_1, ψ_1) and (A_2, ψ_2) are isomorphic (as algebras with automorphism) if there exists an isomorphism of algebras $\tau\colon A_1 \to A_2$ such that $\tau(\psi_1(a)) = \psi_2(\tau(a))$, for any $a \in A_1$.

Theorem 3. *Let A be an algebra over an algebraically closed field F of characteristic zero endowed with an automorphism of order ≤ 2. If A is simple, as an algebra with automorphism, then it is isomorphic to one of the following:*

- $M_n(F) \oplus M_n(F)$ *with the exchange automorphism* $(a, b) \mapsto (b, a)$.
- $M_{k,h}(F) = \left\{ \begin{pmatrix} K & R \\ S & H \end{pmatrix} \right\}$, $k \geq h \geq 0$ *and* $\psi\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix} \right) = \begin{pmatrix} K & -R \\ -S & H \end{pmatrix}$.

The following remark will be useful in the next section.

Remark 4. Let $M_n(F)$ be endowed with two automorphisms ψ_1 and ψ_2 defined for any $L \in M_n(F)$ as

$$\psi_1(L) = PLP^{-1} \qquad \text{and} \qquad \psi_2(L) = QLQ^{-1},$$

where P and Q are invertible matrices in $M_n(F)$. If $(M_n(F), \psi_1)$ and $(M_n(F), \psi_2)$ are isomorphic, then we can choose the matrices P and Q to be similar.

Proof. By definition, since $(M_n(F), \psi_1)$ and $(M_n(F), \psi_2)$ are isomorphic, there exists an isomorphism $\tau\colon M_n(F) \to M_n(F)$ defined by $\tau(L) = CLC^{-1}$, for some $n \times n$ invertible matrix C, such that $\tau(\psi_1(L)) = \psi_2(\tau(L))$, for any $L \in M_n(F)$. It follows that $\tau\psi_1\tau^{-1}(L) = \psi_2(L)$. Hence

$$\tau\psi_1(C^{-1}LC) = \tau(PC^{-1}LCP^{-1}) = (CPC^{-1})L(CPC^{-1})^{-1} = QLQ^{-1}.$$

We get that $(Q^{-1}CPC^{-1})L = L(Q^{-1}CPC^{-1})$. So $Q^{-1}CPC^{-1}$ commutes with any matrix $L \in M_n(F)$ and, so, $Q^{-1}CPC^{-1} = \alpha I$ is a scalar matrix, $\alpha \in F$. Hence P is similar to αQ and, since Q and αQ determine the same automorphism, we get the desired conclusion. □

3. Simple φ-superalgebras

In this section we define on the superalgebra $A = A_0 \oplus A_1$ some particular graded linear maps. We recall that a linear map $\varphi\colon A \to A$ is graded if it preserves the grading of A, that is $\varphi(A_j) \subseteq A_j$, for $j = 0, 1$.

We say that φ is a graded automorphism if, for any $a, b \in A$,

$$\varphi^2(a) = a \qquad \text{and} \qquad \varphi(ab) = \varphi(a)\varphi(b).$$

Instead, φ is a pseudoautomorphism if, for any homogeneous elements $a, b \in A_0 \cup A_1$,

$$\varphi^2(a) = (-1)^{|a|}a \qquad \text{and} \qquad \varphi(ab) = (-1)^{|a||b|}\varphi(a)\varphi(b).$$

In what follows we shall denote by φ a graded automorphism or a pseudoautomorphism on A and we shall say that A is a φ-superalgebra. In this setting a φ-ideal of A is a graded ideal such that $\varphi(I) = I$. In case the only φ-ideals of A are those trivial we say that A is simple, as a φ-superalgebra, or φ-simple.

The main goal of this section is to classify the simple φ-superalgebras.

The following result goes in this direction.

Lemma 5. *Let A be a finite dimensional simple φ-superalgebra. Then A is either*

- *simple as a superalgebra or*
- *$A = B \oplus B^\varphi$, for some simple superalgebra B, where $B^\varphi = \varphi(B)$.*

Proof. Suppose that A is φ-simple but not simple as a superalgebra. Then there exists a proper non-zero graded ideal B of A. Notice that both $B + B^\varphi$ and $B \cap B^\varphi$ are graded ideals of A stable under the action of φ. Since A is φ-simple we get that $A = B + B^\varphi$ and $B \cap B^\varphi = \{0\}$. Hence $A = B \oplus B^\varphi$.

We are left to show that B is simple as a superalgebra. Assume, by absurd, that there exists a proper non-zero graded ideal I of B. Then $I \oplus I^\varphi$ would be a proper non-zero graded ideal of A stable under φ, and this is a contradiction. □

Now given a superalgebra $B = B_0 \oplus B_1$, consider $B \oplus B$ as a superalgebra with grading induced by the grading on B. It is not difficult to see that the map

$$\begin{aligned} \mathrm{ex}\colon B \oplus B &\to B \oplus B \\ (a,b) &\mapsto (b,a) \end{aligned}$$

is a graded automorphism. In order to have an analogous map in the case of pseudoautomorphisms, let B^s denote the superalgebra with the same graded vector space structure as B but with distinct product $\circ$ given, for any $a, b \in B_0 \cup B_1$, by

$$a \circ b = (-1)^{|a||b|} ab.$$

If $B \oplus B^s$ is the superalgebra with grading induced by the grading on B, we can define the following pseudoautomorphism:

$$\begin{aligned} \mathrm{pex}\colon B \oplus B^s &\to B \oplus B^s \\ (a,b) &\mapsto ((-1)^{|(a,b)|} b, a), \end{aligned}$$

for any homogeneous element $(a, b) \in B \oplus B^s$.

Definition 6. Two φ-superalgebras (A, φ_1) and (B, φ_2) are said to be isomorphic (as φ-superalgebras) if there exists an isomorphism of superalgebras $g\colon A \to B$ such that $g(\varphi_1(a)) = \varphi_2(g(a))$, for any $a \in A$.

Now we can prove the following result.

Theorem 7. *Let $A = B \oplus B^\varphi$ be a finite dimensional simple φ-superalgebra over an algebraically closed field F of characteristic zero. If φ is a graded automorphism then A is isomorphic to either*

- $(M_{k,h}(F) \oplus M_{k,h}(F), \mathrm{ex})$ *or*
- $(Q(n) \oplus Q(n), \mathrm{ex})$.

If φ is a pseudoautomorphism then A is isomorphic to either

- $(M_{k,h}(F) \oplus M_{k,h}(F)^s, \mathrm{pex})$ *or*
- $(Q(n) \oplus Q(n)^s, \mathrm{pex})$.

Proof. Assume first that φ is a graded automorphism. The linear map

$$\begin{aligned}\tau\colon\ (B\oplus B^{\varphi},\varphi) &\to (B\oplus B,\mathrm{ex})\\ a+\varphi(b)\ &\mapsto (a,b)\end{aligned}$$

is an isomorphism of superalgebras with graded automorphism.

Analogously, in case φ is a pseudoautomorphism, the graded linear map

$$\begin{aligned}\tau\colon\ (B\oplus B^{\varphi},\varphi) &\to (B\oplus B^{s},\mathrm{pex})\\ a+\varphi(b)\ &\mapsto (a,b)\end{aligned}$$

is an isomorphism of superalgebras with pseudoautomorphism.

Now the result follows by using Theorem 2. □

In the following proposition we deal with the superalgebra $Q(n) = M_n(F) \oplus cM_n(F)$, $c^2 = 1$. We shall denote by $\mathrm{Aut}(M_n(F))$ the set of all automorphisms of order ≤ 2 on the matrix algebra $M_n(F)$. Let $i \in F$ be an element such that $i^2 = -1$.

Proposition 8. *We have that*

- *graded automorphisms on $Q(n)$ are of the form $\varphi(a + cb) = f(a) \pm cf(b)$,*
- *pseudoautomorphisms on $Q(n)$ are of the form $\varphi(a + cb) = f(a) \pm icf(b)$,*

where $a, b \in M_n(F)$ and $f \in Aut(M_n(F))$.

Proof. Let φ be a graded automorphism or a pseudoautomorphism on $Q(n)$. So, for $a, b \in M_n(F)$, one can write

$$\varphi(a + cb) = f(a) + cg(b),$$

where f, g are linear maps on $M_n(F)$, $f = \varphi|_{M_n(F)}$ and $g\colon M_n(F) \to M_n(F)$ is such that $g(b) = d$ if $\varphi(cb) = cd$. Clearly f is an automorphism of order ≤ 2 on $M_n(F)$.

Now assume that φ is a graded automorphism and let us prove that $g(1) = \pm 1$, where 1 denotes the identity $n \times n$ matrix. We have that:

$$\begin{aligned}cg(1)f(b) &= \varphi(c1)\varphi(b) = \varphi(c1b) = \varphi(cb) = \varphi(cb1)\\ &= \varphi(bc1) = \varphi(b)\varphi(c1) = f(b)cg(1) = cf(b)g(1).\end{aligned}$$

It follows that $g(1)$ commutes with $f(b)$, for any $b \in M_n(F)$. Since f is in particular surjective, $g(1)$ commutes with any element of $M_n(F)$ and, so, it is a scalar matrix. Moreover, we have that $g(1)^2 = 1$. In fact,

$$1 = f(1) = \varphi(1) = \varphi(c1 \cdot c1) = \varphi(c1)\varphi(c1) = g(1)g(1) = g(1)^2.$$

Now the proof is complete since, for any $b \in M_n(F)$, we have that

$$f(b) = f(b1) = \varphi(cbc1) = \varphi(cb)\varphi(c1) = g(b)g(1) = \pm g(b).$$

If φ is a pseudoautomorphism, one proves $g(1) = \pm i1$ and, so, $f(b) = \pm i g(b)$. □

We are left to determine the graded automorphisms and the pseudoautomorphisms on the matrix superalgebra $M_{k,h}(F)$.

Assume first $h = 0$, that is $M_{k,0}(F) \cong M_k(F)$ is endowed with the trivial grading. In this case a graded automorphism or a pseudoautomorphism on $M_k(F)$ is just an automorphism of order ≤ 2 and a description of such automorphisms is given in the second item of Theorem 3.

Now let us consider the case $h > 0$. We start with the following lemma.

Lemma 9. *Let $A = A_0 \oplus A_1$ be a simple φ-superalgebra with non-trivial grading, where φ is a graded automorphism or a pseudoautomorphism. Then either*

- *$(A_0, \varphi|_{A_0})$ is simple, as an algebra with automorphism of order ≤ 2, or*
- *$A_0 = C_1 \oplus C_2$, $A_1 = D_1 \oplus D_2$, where $(C_i, \varphi|_{C_i})$ are simple (as algebras with automorphism), D_j are irreducible A_0-bimodules such that $\varphi(D_j) = D_j$, $j = 1, 2$, and*
  - *$C_2D_1 = C_1D_2 = D_1D_1 = D_2D_2 = D_1C_1 = D_2C_2 = \{0\}$.*
  - *$D_1D_2 = C_1$, $D_2D_1 = C_2$, $C_lD_l = D_l$, $D_lC_j = D_l$, $l, j \in \{1, 2\}$, $l \neq j$.*

Proof. The result can be proved using the same approach of [6, Theorem 12]. □

Now we are in a position to prove the following theorem.

Theorem 10. *Let $M_{k,h}(F)$, $h > 0$, endowed with a graded automorphism or a pseudoautomorphism φ. Then it is isomorphic to one of the following:*

- *$(M_{k,k}(F), \varphi)$ with $\varphi\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} H & \alpha S \\ \alpha R & K \end{pmatrix}$, where*
  - *$\alpha = 1$ in case φ is a graded automorphism,*
  - *$\alpha = i$ in case φ is a pseudoautomorphism;*
- *$(M_{k,h}(F), \varphi)$ with $\varphi\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} PKP & \alpha PRQ \\ \alpha QSP & QHQ \end{pmatrix}$, where*
  - *$\alpha = \pm 1$ in case φ is a graded automorphism,*
  - *$\alpha = \pm i$ in case φ is a pseudoautomorphism.*

  Here $P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}$, $Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}$, the I_j's are identity matrices of order j and $k = k_1 + k_2$, $k_1 \geq k_2$, $h = h_1 + h_2$ and $h_1 \geq h_2$.

Proof. We shall prove the result when φ is a pseudoautomorphism. The case in which φ is a graded automorphism can be proved in a similar manner.

In order to simplify the notation write $A = M_{k,h}(F)$. According to Lemma 9, assume first that $A_0 = M_k(F) \oplus M_h(F)$ is simple as an algebra with automorphism. Hence, by Theorem 3, we must have $k = h$ and, up to isomorphism,

$$(A_0, \varphi = \varphi|_{A_0}) = (M_k(F) \oplus M_k(F), \varphi),$$

where $\varphi(a, b) = (\varphi(b), \varphi(a))$. Now let us consider the following elements:

$$a_{11} = \sum_{l=1}^{k} e_{ll}, \qquad a_{12} = \sum_{l=1}^{k} e_{lk+l}, \qquad a_{21} = \sum_{l=1}^{k} e_{k+ll}, \qquad a_{22} = \sum_{l=k+1}^{2k} e_{ll},$$

where the e_{lj}'s are elementary matrices. We get $\varphi(a_{11}) = a_{22}$, $\varphi(a_{22}) = a_{11}$ and

$$A_0 = M_k(F)a_{11} \oplus M_k(F)a_{22}, \quad A_1 = M_k(F)a_{12} \oplus M_k(F)a_{21}.$$

We have that $\varphi(a_{12}) = \varphi(a_{11}a_{12}a_{22}) = a_{22}\varphi(a_{12})a_{11}$, and, so, $\varphi(a_{12}) = ea_{21}$, for some $e \in M_k(F)$. Analogously, $\varphi(a_{21}) = e'a_{12}$, for some $e' \in M_k(F)$. Moreover, for any $b \in M_k(F)$, we have that

$$\begin{aligned} e\varphi(b)a_{21} &= ea_{21}\varphi(b)a_{11} = \varphi((a_{12})(ba_{22})) = \varphi(ba_{12}) \\ &= \varphi((ba_{11})a_{12}) = \varphi(b)a_{22}ea_{21} = \varphi(b)ea_{21}. \end{aligned}$$

It follows that $e \in Z(M_k(F)) \cong F$. Analogously, one gets that $e' \in Z(M_k(F)) \cong F$. Since φ is a pseudoautomorphism, we have that $ee' = -1$. In fact,

$$-a_{12} = \varphi^2(a_{12}) = \varphi(ea_{21}) = e\varphi(a_{21}) = ee'a_{12}.$$

So far we have proved that the pseudoautomorphism φ is of the kind

$$\varphi\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} H & \alpha S \\ \beta R & K \end{pmatrix},$$

where $\alpha\beta = -1$. Now we shall show that we can take $\alpha = \beta = i$. Let φ_i be the pseudoautomorphism on $M_{k,k}(F)$ defined by

$$\varphi_i\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} H & iS \\ iR & K \end{pmatrix}.$$

It is easy to check that the φ-superalgebras $(M_{k,k}(F), \varphi_i)$ and $(M_{k,k}(F), \varphi)$ are isomorphic through the isomorphism

$$\begin{aligned} f\colon\ (M_{k,k}(F), \varphi_i) &\to \quad (M_{k,k}(F), \varphi) \\ \begin{pmatrix} K & R \\ S & H \end{pmatrix} &\mapsto \begin{pmatrix} K & -i\gamma R \\ \alpha^{-1}\gamma S & H \end{pmatrix}, \end{aligned}$$

where $\gamma \in F$ is such that $\gamma^2 = \alpha i$.

Now assume that $(M_{k,h}(F))_0$ is not simple as an algebra with automorphism. Clearly $(M_{k,h}(F))_0 = M_k(F) \oplus M_h(F)$ and $(M_{k,h}(F))_1 = M_{k\times h}(F) \oplus M_{h\times k}(F)$. By Lemma 9, we have that $M_k(F)$ and $M_h(F)$ are simple, as algebras with automorphism. Hence there exists $P \in M_k(F)$ with $P^2 = I_k$ such that

$$\varphi|_{M_k(F)}(K) = PKP,\ P \in M_k(F).$$

Analogously, there exists $Q \in M_h(F)$ with $Q^2 = I_h$ such that

$$\varphi|_{M_h(F)}(H) = QHQ,\ H \in M_h(F).$$

On the other hand, according to Lemma 9, we have that $M_{k\times h}(F)$ and $M_{h\times k}(F)$ are φ-invariant. Now, if we take a matrix unit e_{lj} with $l \in \{1, \dots, k\}$ and $j \in \{k+1, \dots, k+h\}$, we have that, for some $\alpha \in F$,

$$\varphi(e_{lj}) = \varphi(e_{ll}e_{lj}e_{jj}) = \varphi(e_{ll})\varphi(e_{lj})\varphi(e_{jj}) = P\left[e_{ll}P\varphi(e_{lj})Qe_{jj}\right]Q = \alpha Pe_{lj}Q.$$

Let $r \in \{1, \dots, k\}$ and $s \in \{k+1, \dots, k+h\}$. As before, we get that $\varphi(e_{rs}) = \beta Pe_{rs}Q$, for some $\beta \in F$. Next we prove that $\alpha = \beta$. In fact

$$\begin{aligned}\alpha Pe_{lj}Q = \varphi(e_{lj}) &= \varphi(e_{lr}e_{rs}e_{sj}) = \varphi(e_{lr})\varphi(e_{rs})\varphi(e_{sj})\\ &= (Pe_{lr}P)(\beta Pe_{rs}Q)(Qe_{sj}Q) = \beta Pe_{lj}Q.\end{aligned}$$

Moreover, $\alpha = \pm i$. In fact

$$-e_{lj} = \varphi^2(e_{lj}) = \varphi(\alpha Pe_{lj}Q) = \alpha^2 e_{lj}.$$

Now, with the same argument, we get that $\varphi(e_{jl}) = \alpha Qe_{jl}P$, for any $j \in \{k+1, \dots, k+h\}$ and $l \in \{1, \dots, k\}$. Hence

$$\varphi\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} PKP & \alpha PRQ \\ \alpha QSP & QHQ \end{pmatrix},$$

where $\alpha = \pm i$. In order to complete the proof we need just to show that the matrices P and Q can be chosen as in the statement. By Theorem 3 we know that $(M_k(F), \varphi|_{M_k(F)})$ is isomorphic to $(M_{k_1,k_2}(F), \psi)$ for some $k = k_1 + k_2$, $k_1 \geq k_2$. Hence, without loss of generality we may assume that P is similar to

$$P' = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix},$$

i.e., $P' = LPL^{-1}$, for some $k \times k$ invertible matrix L (see Remark 4). Analogously we get that $(M_h(F), \varphi|_{M_h(F)})$ is isomorphic to $(M_{h_1,h_2}(F), \psi)$ for some $h = h_1 + h_2$, $h_1 \geq h_2$. Hence, for some $h \times h$ invertible matrix M, we get $MQM^{-1} = Q'$, where

$$Q' = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}.$$

Then $(M_{k,h}(F), \varphi)$ is isomorphic to $(M_{k,h}(F), \sigma)$ where

$$\sigma\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} P'KP' & \alpha P'RQ' \\ \alpha Q'SP' & Q'HQ' \end{pmatrix}.$$

In fact the map $f\colon (M_{k,h}(F), \varphi) \longrightarrow (M_{k,h}(F), \sigma)$ defined by

$$f\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} LKL^{-1} & LRM^{-1} \\ MSL^{-1} & MHM^{-1} \end{pmatrix}$$

is an isomorphism of superalgebras with pseudoautomorphism. □

We summarize the results of this section in the following theorems, giving the classification of simple φ-superalgebras.

Theorem 11. *Let A be a finite dimensional simple superalgebra with graded automorphism over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following:*

(1) $M_{k,h}(F) \oplus M_{k,h}(F)$ *with the exchange graded automorphism* ex*;*
(2) $Q(n) \oplus Q(n)$ *with the exchange graded automorphism* ex*;*
(3) $(Q(n), \varphi)$*, where φ is the graded automorphism defined as*

$$\varphi(a + cb) = f(a) \pm cf(b),$$

for some automorphism f of order ≤ 2 on $M_n(F)$;

(4) $(M_{k,k}(F), \varphi)$ *with graded automorphism defined as*

$$\varphi\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} H & S \\ R & K \end{pmatrix};$$

(5) $M_{k,h}(F)$*, with graded automorphism φ defined as*

$$\begin{pmatrix} K & R \\ S & H \end{pmatrix}^{\varphi} = \begin{pmatrix} PKP & \alpha PRQ \\ \alpha QSP & QHQ \end{pmatrix},$$

where $\alpha = \pm 1$, $P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}$, $Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}$, $I_{k_1}, I_{k_2}, I_{h_1}, I_{h_2}$, are the identity matrices of orders k_1, k_2, h_1, h_2, respectively, $k = k_1 + k_2$, $h = h_1 + h_2$, $k_1 \geq k_2$ and $h_1 \geq h_2$.

Theorem 12. *Let A be a finite dimensional simple superalgebra with pseudoautomorphism over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following:*

(1) $(M_{k,h}(F) \oplus M_{k,h}(F)^s, \mathrm{pex})$ *with pseudoautomorphism* pex *defined on homogeneous elements as* $\mathrm{pex}(a,b) = ((-1)^{|(a,b)|}b, a)$*;*
(2) $Q(n) \oplus Q(n)^s$ *with the pseudoautomorphism* pex*;*
(3) $(Q(n), \varphi)$*, where* φ *is the pseudoautomorphism defined as*

$$\varphi(a + cb) = f(a) \pm icf(b),$$

for some automorphism f *of order* ≤ 2 *on* $M_n(F)$*;*
(4) $(M_{k,k}(F), \varphi)$ *with pseudoautomorphism defined as*

$$\varphi\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} H & iS \\ iR & K \end{pmatrix};$$

(5) $M_{k,h}(F)$*, with pseudoautomorphism* φ *defined as*

$$\begin{pmatrix} K & R \\ S & H \end{pmatrix}^{\varphi} = \begin{pmatrix} PKP & \alpha PRQ \\ \alpha QSP & QHQ \end{pmatrix},$$

where $\alpha = \pm i$*,* $P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}$*,* $Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}$*,* $I_{k_1}, I_{k_2}, I_{h_1}, I_{h_2}$*, are the identity matrices of orders* k_1, k_2, h_1, h_2*, respectively,* $k = k_1 + k_2$*,* $h = h_1 + h_2$*,* $k_1 \geq k_2$ *and* $h_1 \geq h_2$*.*

Data availability

No data was used for the research described in the article.

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