Received: 10 September 2021 | Accepted: 4 January 2022
DOI: 10.1002/mma.8127

RESEARCH ARTICLE

WILEY

Second-order differential equations for the power converters dynamical performance analysis

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Communicated by: B. Harrach

Funding information
PRIN2017, Grant/Award Number: 2017AYM8XW

The paper is devoted to the study of a second-order differential equation of type Sturm-Liouville with coefficients sign changing, and the existence of one positive solution is obtained by requiring a specific growth of the nonlinearity. In particular, this study is useful in the analysis of the dynamical performance of a class of power converter.

KEYWORDS
boundary value problems, critical point theory, nonlinear differential problems, ordinary differential equations, power converters analysis

MSC CLASSIFICATION
34B10

1 | INTRODUCTION

This paper considers the existence of one positive solution for the following boundary value problem with a Sturm-Liouville second-order differential equation and Dirichlet conditions

$$\begin{cases} -u'' + \gamma(t)u' + \sigma(t)u = \lambda f(t,u) & \text{in }]a,b[, \\ u(a) = u(b) = 0. \end{cases} \tag{D_λ}$$

where $\lambda \in \mathbb{R}^+$, $f \in L^1([a,b]\times\mathbb{R})$ is a function that satisfies the Carathéodory hypothesis and $\gamma, \sigma \in L^\infty([a,b])$ such that $\operatorname*{ess\,inf}_{[a,b]} \sigma > -\left(\frac{\pi}{b-a}\right)^2$.

This type of problems describes, for instance, physical and chemical events, as well as the Boyd equation about eddies in the atmosphere,[1] Laplace tidal wave equation,[2] and Meissner equation which arises in a model of a one-dimensional crystal.[3] From a mathematical point of view, such problems are often studied in particular cases when the associated energy functional is well known, that is, for instance, if $\sigma \geq 0$ and $\gamma = 0$, while in the previous models the coefficients may be even non-positive. In this paper, we present our results considering that the functions γ and σ can be non-zero or even sign changing, in order to offer a more effective link between pure and applied mathematics. Therefore, the existence of one positive solution for the problem (D_λ) is investigated. In particular, a mathematical approach for tuning a proportional voltage control of a DC-DC buck converter has been performed. In detail, for a given time dynamic response of the system and by imposing the voltage error null before and after the voltage transient, maximum and minimum values of the proportional constant of the P controller are obtained. The mathematical results have been compared with simulations of the DC-DC converter by using the Matlab/Simulink platform by obtaining a good matching. In our paper, we apply a recent critical point result to equations of Sturm-Liouville type in a complete form, that is, also with the presence of the term $\gamma(t)u$. We recall that, in this case, the variational formulation of the corresponding problem is not natural. Indeed, we use an appropriate functional I_λ which is more general than the usual energy functional, and it can be applied to the sign changing case (see Proposition 2.3). Moreover, our results extend previous results obtained for the Sturm-Liouville equation in the incomplete form and with nonnegative coefficients (see, for instance, Bonanno and D'Aguı[4]). Our main assumption, that is (jj) of Theorem 3.1, describes a growth of the nonlinearity which is more than quadratic in an interval $[c, d]$ (see its simpler form (A) in Theorem 3.3), and as a conclusion, we obtain a suitable interval of parameters for which problem admits at least one non-zero solution which is a local minimizer of the Euler-Lagrange functional corresponding to the problem (D_λ).

We emphasize that to achieve our goal, we do not assume any growth condition at infinity on the nonlinearities. Finally, we also want to point out that, to the best of our knowledge, it is the first time that an application of the nonlinear problem is given in order to study the stability of a power electric devise with the aim to have a control low of a Buck converter. The paper is organized as follow. In Section 2, we prove some basic properties among them, the equivalence between the usual norm in $W_0^{1,2}([a,b])$ (see Proposition 2.2) and a useful norm with the problem, and moreover, we mention a non-zero local minimum for functionals of class C^1 parameter depending (Theorem 2.1), which is our main tool. In Section 3, we present our result on the existence of one classical solutions for the problem (D_λ) and its consequence in particular cases. Finally, Section 4. is devoted to the application of our results to the analysis of the dynamical performance of a class of power converter.

2 | BASIC PROPERTIES

Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be an L^1–Carathéodory function, that is a function such that

(*i*) $t \to f(t, x)$ is measurable for all $x \in \mathbb{R}$;
(*ii*) $x \to f(t, x)$ is continuous for almost every $t \in [a, b]$;
(*iii*) for all $\rho > 0$ the function $\sup_{|x| \le \rho} |f(t, x)|$ belongs to $L^1([a, b])$.

Moreover, let $\gamma, \sigma : [a, b] \to \mathbb{R}$ be two functions belonging to $L^\infty([a, b])$. We assume that

$$\operatorname*{ess\,inf}_{[a,b]} \sigma > -\left(\frac{\pi}{b-a}\right)^2. \tag{2.1}$$

Finally, λ is a positive real number. Consider the following problem

$$\begin{cases} -u'' + \gamma(t)u' + \sigma(t)u = \lambda f(t, u) \text{ in } [a, b], \\ u(a) = u(b) = 0. \end{cases} \tag{D_λ}$$

Denote by X the Sobolev space $W_0^{1,2}([a,b])$ with the usual norm $\|u\| = \|u'\|_2$, where $\|\cdot\|_2$ is the norm of the Lebesgue space $L^2([a, b]$, that is,

$$\|u\| = \left(\int_a^b |u'(t)|^2 dt\right)^{\frac{1}{2}}.$$

We point out the following Poincaré inequalities.

Proposition 2.1. *For all $u \in X$, one has*

(*j*) $\max_{t\in[a,b]} |u(t)| \le \frac{(b-a)^{\frac{1}{2}}}{2} \|u\|$
(*jj*) $\|u\|_2 \le \frac{b-a}{\pi} \|u\|$.

Proof. (j) Since, in particular, $u' \in L^1([a,b])$ (and hence u is absolutely continuous), one has $u(t) = \int_a^t u'(\tau)d\tau$ and $-u(t) = \int_t^b u'(\tau)d\tau$. Therefore, taking also the Hölder inequality into account, one has $|u(t)| \leq \frac{1}{2}\int_a^b |u'(\tau)|d\tau \leq \frac{(b-a)^{\frac{1}{2}}}{2}\left(\int_a^b |u'(\tau)|^2 d\tau\right)^{1/2}$ and the inequality is proved.

(jj) Denoting by λ_1 the first eigenvalue of the problem $-u'' = \lambda u$, $u(a) = u(b) = 0$, from a classical result (see, for instance, Boccardo and Croce[5, Theorem 8.4, page 69]), one has

$$\lambda_1 = \min_{u \in X, u \neq 0} \frac{\|u'\|_2^2}{\|u\|_2^2}. \tag{2.2}$$

Since $\lambda_1 = \left(\frac{\pi}{b-a}\right)^2$, the conclusion follows. □

Remark 2.1. The constant $\frac{(b-a)^{1/2}}{2}$ is the best constant among the constants c for which the equality $\max_{t \in [a,b]} |u(t)| \leq c\|u\|$ is true for all $u \in X$, as the function

$$\varphi(t) = \begin{cases} \frac{t-a}{b-a} & \text{if } t \in [a, \frac{a+b}{2}] \\ \frac{b-t}{b-a} & \text{if } t \in [\frac{a+b}{2}, b], \end{cases}$$

shows. The same remark also for (jj) holds as (2.2) shows for the eigenfunction $\rho(t) = \sin(\frac{\pi}{b-a}t)$ for all $t \in [a,b]$.

Now, put

$$\|u\|_X = \left(\int_a^b e^{-\Gamma(t)}|u'(t)|^2 dt + \int_a^b e^{-\Gamma(t)}\sigma(t)|u(t)|^2 dt\right)^{\frac{1}{2}},$$

where $\Gamma(t) = \int_a^t \gamma(\xi)d\xi$ for all $t \in [a,b]$.

We have the following result.

Proposition 2.2. *Assume (2.1). Then $\|\cdot\|_X$ is a norm on X, and it is equivalent to the usual norm. In particular, one has*

$$m\|u\| \leq \|u\|_X \leq M\|u\| \tag{2.3}$$

for all $u \in X$, where m, M, with $M \geq m > 0$, are given by

$$m = \begin{cases} \left(\min\limits_{[a,b]} e^{-\Gamma}\right)^{1/2} & \textit{if } \operatorname*{ess\,inf}\limits_{[a,b]} \sigma \geq 0 \\ \left[\min\limits_{[a,b]} e^{-\Gamma}\left(1 + \operatorname*{ess\,inf}\limits_{[a,b]}\sigma\left(\frac{b-a}{\pi}\right)^2\right)\right]^{1/2} & \textit{if } \operatorname*{ess\,inf}\limits_{[a,b]} \sigma < 0, \end{cases}$$

and

$$M = \begin{cases} \left[\max\limits_{[a,b]} e^{-\Gamma}\left(1 + \operatorname*{ess\,sup}\limits_{[a,b]}\sigma\left(\frac{b-a}{\pi}\right)^2\right)\right]^{1/2} & \textit{if } \operatorname*{ess\,sup}\limits_{[a,b]} \sigma \geq 0 \\ \left(\max\limits_{[a,b]} e^{-\Gamma}\right)^{1/2} & \textit{if } \operatorname*{ess\,sup}\limits_{[a,b]} \sigma < 0. \end{cases}$$

Proof. First, we prove (2.3). To this end, one has

$$\|u\|_X^2 = \int_a^b e^{-\Gamma(t)}|u'(t)|^2 dt + \int_a^b e^{-\Gamma(t)}\sigma(t)|u(t)|^2 dt \geq \int_a^b e^{-\Gamma(t)}|u'(t)|^2 dt + \operatorname*{ess\,inf}_{[a,b]}\sigma \int_a^b e^{-\Gamma(t)}|u(t)|^2 dt$$
$$\geq \min_{[a,b]} e^{-\Gamma}\left(\int_a^b |u'(t)|^2 dt + \operatorname*{ess\,inf}_{[a,b]}\sigma \int_a^b |u(t)|^2 dt\right).$$

So, if $\operatorname{ess\,inf}_{[a,b]}\sigma \geq 0$ one has $\|u\|_X^2 \geq \min_{[a,b]} e^{-\Gamma}\|u'\|_2^2$. While, if $\operatorname{ess\,inf}_{[a,b]}\sigma < 0$, taking also into account (2.1) and (jj) of Proposition 2.1, one has $\|u\|_X^2 \geq \min_{[a,b]} e^{-\Gamma}\left(\|u'\|_2^2 + \operatorname{ess\,inf}_{[a,b]}\sigma\|u\|_2^2\right) \geq \min_{[a,b]} e^{-\Gamma}\left(\|u'\|_2^2 + \operatorname{ess\,inf}_{[a,b]}\sigma\left(\frac{b-a}{\pi}\right)^2\|u'\|_2^2\right) = \min_{[a,b]} e^{-\Gamma}\left(1 + \operatorname{ess\,inf}_{[a,b]}\sigma\left(\frac{b-a}{\pi}\right)^2\right)\|u'\|_2^2$,with $\min_{[a,b]} e^{-\Gamma}\left(1 + \operatorname{ess\,inf}_{[a,b]}\sigma\left(\frac{b-a}{\pi}\right)^2\right) > 0$. On the other hand, one has $\|u\|_X^2 \leq \max_{[a,b]} e^{-\Gamma}\left(\|u'\|_2^2 + \operatorname{ess\,sup}_{[a,b]}\sigma\|u\|_2^2\right)$. So, if $\operatorname{ess\,sup}_{[a,b]}\sigma \geq 0$, taking again (jj) of Proposition 2.1 into account, one has$\|u\|_X^2 \leq \max_{[a,b]} e^{-\Gamma}\left(\|u'\|_2^2 + \operatorname{ess\,sup}_{[a,b]}\sigma\left(\frac{b-a}{\pi}\right)^2\|u'\|_2^2\right)$, while if $\operatorname{ess\,sup}_{[a,b]}\sigma < 0$, it follows $\|u\|_X^2 \leq \max_{[a,b]} e^{-\Gamma}\|u'\|_2^2$. Hence (2.3) is proved.

Now, it is easy to verify that $\|\cdot\|_X$ is a norm, by using (2.3) and by standard computations. □

Remark 2.2. Clearly, the dot product

$$\langle u; v\rangle = \int_a^b e^{-\Gamma(t)}u'(t)v'(t)dt + \int_a^b e^{-\Gamma(t)}\sigma(t)u(t)v(t)dt$$

induces the norm $\|\cdot\|_X$.

Remark 2.3. From (j) of Proposition 2.1 and Proposition 2.2, we obtain that

$$\max_{[a,b]} |u(x)| \leq \frac{(b-a)^{1/2}}{2m}\|u(x)\|_X \ \forall u \in X, \tag{2.4}$$

where m is given in Proposition 2.2.

Remark 2.4. We explicitly observe that the inequality (2.3) remains true by substituting the interval $[a, b]$ with a nonempty bounded open set $A \subseteq \mathbb{R}$. In particular, one has

$$m_A^2\int_A |u'(t)|^2dt \leq \int_A e^{-\Gamma(t)}|u'(t)|^2dt + \int_A e^{-\Gamma(t)}\sigma(t)|u(t)|^2dt$$

for all $u \in W_0^{1,2}(A)$, where $m_A^2 = \min\left\{\inf_A e^{-\Gamma}, \inf_A e^{-\Gamma}\left(1 + \operatorname{ess\,inf}_A\sigma\left(\frac{|A|}{\pi}\right)^2\right)\right\}$. We will use the present remark in the proof of Lemma 2.1.

Now we recall the definition of a generalized solution for (D_λ). We say that $u : [a, b] \to \mathbb{R}$ is a generalized solution of (D_λ) if $u \in C^1([a, b])$, $u' \in AC([a, b])$, which is the set of all absolutely continuous function, $u(a) = u(b) = 0$ and $-u''(t) + \gamma(t)u'(t) + \sigma(t)u(t) = \lambda f(t, u(t))$ for a.e. $t \in [a, b]$. Clearly, if $f \in C([a, b] \times \mathbb{R})$, $\gamma, \sigma \in C([a, b]$, any generalized solution u is a classical solution, that is $u \in C^2([a, b])$, $u(a) = u(b) = 0$ and $-u''(t) + \gamma(t)u'(t) + \sigma(t)u(t) = \lambda f(t, u(t))$ for all $t \in [a, b]$.

Now, put $F(t, x) = \int_0^x f(t, \xi)d\xi$ for all $(t, x) \in [a, b] \times \mathbb{R}$. Clearly, one has (i') $t \to F(t, x)$ is measurable for all $x \in \mathbb{R}$; (ii') $x \to F(t, x)$ belongs to $C^1(\mathbb{R})$ for a.e. $t \in [a, b]$; (iii') $|F(t, x)| \leq \left(\sup_{|\xi|\leq|x|} |f(t, \xi)|\right)|x|$ for a.e. $t \in [a, b]$, for all $x \in \mathbb{R}$. Moreover, put

$$\Psi(u) = \int_a^b e^{-\Gamma(t)}F(t, u(t))dt.$$

for all $u \in X$. From (iii'), taking also into account that $X \subseteq C([a, b])$, $\Psi : X \to \mathbb{R}$ is well defined since

$$e^{-\Gamma(t)}F(t, u(t)) \leq \max_{[a,b]} e^{-\Gamma}\left(\sup_{|\xi|\leq\max_{[a,b]}|u|} |f(t, \xi)|\right)\max_{[a,b]} |u| \leq CK\left(\sup_{|\xi|\leq K} |f(t, \xi)|\right) \in L^1([a, b]).$$

Standard computations show that Ψ is Gâteaux differentiable, and one has

$$\Psi'(u)(v) = \int_a^b e^{-\Gamma(t)}f(t, u(t))v(t)dt,$$

for all $u, v \in X$. Further, it is easy to verify that $\Psi' : X \to X^*$ is a continuous operator, that is Ψ is a C^1-function.

Now, put $\Phi(u) = \frac{1}{2}\|u\|_X^2$ for all $u \in X$. Taking into account of Remark 2.2, the Gâteaux derivative of Φ is

$$\Phi'(u)(v) = \int_a^b e^{-\Gamma(t)} u'(t)v'(t)dt + \int_a^b e^{-\Gamma(t)}\sigma(t)u(t)v(t)dt,$$

for all $u, v \in X$. Moreover, Φ is a C^1-function.

Finally, put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

for all $u \in X$. Clearly, I_λ is a C^1-function and, in particular, one has

$$I'_\lambda(u)(v) = \int_a^b e^{-\Gamma(t)} u'(t)v'(t)dt + \int_a^b e^{-\Gamma(t)}\sigma(t)u(t)v(t)dt - \lambda \int_a^b e^{-\Gamma(t)} f(t, u(t))v(t)dt,$$

for all $u, v \in X$.

We have the following result.

Proposition 2.3. *u is a generalized solution of $(D_\lambda) \iff u$ is a critical point of I_λ.*

Proof. Assume that u is a generalized solution of (D_λ). In particular, one has $-u''(t) + \gamma(t)u'(t) + \sigma(t)u(t) = \lambda f(t, u(t))$ for a.e. $t \in [a, b]$. Fix $v \in X$ and multiplying by $e^{-\Gamma}v$, which belongs again to X, integrating and then integrating by parts the first term, it follows that

$$\int_a^b -u''(t)e^{-\Gamma(t)}v(t)dt + \int_a^b \gamma(t)u'(t)e^{-\Gamma(t)}v(t)dt + \int_a^b \sigma(t)u(t)e^{-\Gamma(t)}v(t)dt = \lambda \int_a^b f(t, u(t))e^{-\Gamma(t)}v(t)dt$$

$$\int_a^b u'(t)e^{-\Gamma(t)}v'(t)dt + \int_a^b \sigma(t)u(t)e^{-\Gamma(t)}v(t)dt = \lambda \int_a^b f(t, u(t))e^{-\Gamma(t)}v(t)dt,$$

that is, u is a critical point of I_λ. Now, assume that u is critical point of I_λ. Fix $w \in X$. Clearly, $v = e^\Gamma w \in X$. So, $I'_\lambda(u)(v) = 0$, that is

$$\int_a^b e^{-\Gamma(t)}u'(t)v'(t)dt + \int_a^b e^{-\Gamma(t)}\sigma(t)u(t)v(t)dt = \lambda \int_a^b e^{-\Gamma(t)} f(t, u(t))v(t)dt.$$

Therefore, taking into account that $e^{-\Gamma(t)}v'(t) = (e^{-\Gamma(t)}v(t))' + \gamma(t)e^{-\Gamma(t)}v(t)$, one has

$$\int_a^b u'(t)(e^{-\Gamma(t)}v(t))'dt + \int_a^b u'(t)\gamma(t)e^{-\Gamma(t)}v(t)dt + \int_a^b e^{-\Gamma(t)}\sigma(t)u(t)v(t)dt = \lambda \int_a^b e^{-\Gamma(t)} f(t, u(t))v(t)dt,$$

that is

$$\int_a^b u'(t)w'(t)dt + \int_a^b \gamma(t)u'(t)w(t)dt + \int_a^b \sigma(t)u(t)w(t)dt = \lambda \int_a^b f(t, u(t))w(t)dt.$$

Hence, u' admits the weak derivative which is $\gamma u' + \sigma u - \lambda f(\cdot, u(\cdot))$, which is a L^1-function and so, by standard arguments, the conclusion is achieved. □

Our main tool is a non-zero local minimum theorem obtained in Bonanno[6] as a consequence of the local minimum theorem established in Bonanno.[7] We recall it below. To this end, let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions, put $I = \Phi - \Psi$ and fix $r > 0$. We say that I satisfies the *Palais-Smale condition cut off upper at r* (in short $(PS)^{[r]}$-condition) if any sequence $\{u_n\}$ such that $I(u_n)$ is bounded, $I'(u_n) \to 0$ and $\Phi(u_n) < r$ for all $n \in \mathbb{N}$ has a convergent subsequence. Moreover, put

$$\underline{\varphi}(r) = \frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r}; \qquad \overline{\varphi}(r) = \sup_{u \in \Phi^{-1}(]0, r[)} \frac{\Psi(u)}{\Phi(u)}.$$

Now, we recall the non-zero local minimum theorem (see Bonanno[6, Theorem 2.3]).

Theorem 2.1. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there is a r > 0 such that*

$$\underline{\varphi}(r) < \overline{\varphi}(r) \tag{2.5}$$

and for each $\lambda \in \Lambda_r = \left]\frac{1}{\overline{\varphi}(r)}, \frac{1}{\underline{\varphi}(r)}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition.

Then, for each $\lambda \in \Lambda_r$ there is $u_\lambda \in \Phi^{-1}(]0, r[)$ (hence, $u_\lambda \neq 0$) such that $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]0, r[)$ and $I'_\lambda(u_\lambda) = 0$.

Now, we point out the following results which allows to obtain nonnegative or positive solutions to problem (D_λ). Put

$$f^+(t,x) = \begin{cases} f(t,0), & \text{if } t \in [a,b],\ x < 0, \\ f(t,x), & \text{if } t \in [a,b] x \geq 0, \end{cases}$$

and consider the problem

$$\begin{cases} -u'' + \gamma(t)u' + \sigma(t)u = \lambda f^+(t,u) \text{ in } [a,b], \\ u(a) = u(b) = 0. \end{cases} \tag{D_λ^+}$$

The following result is useful to obtain nonnegative solutions.

Lemma 2.1. *Assume that*

$$f(t,0) \geq 0 \ for \ a.e. \ t \in [a,b].$$

Then, any generalized solution of the problem (D_λ^+) is nonnegative, and it is also a generalized solution of (D_λ).

Proof. Let $\bar{u}$ be a generalized solution of problem (D_λ^+). So, taking into account Proposition 2.3, one has $\int_a^b e^{-\Gamma(t)}\bar{u}'(t)v'(t)dt + \int_a^b e^{-\Gamma(t)}\sigma(t)\bar{u}(t)v(t)dt = \lambda\int_a^b e^{-\Gamma(t)} f^+(t,\bar{u}(t))v(t)dt$ for all $v \in X$. Now, put $\bar{u}^- = \min\{\bar{u}, 0\}$. Clearly, $\bar{u}^- \in X$ (see, for instance, Gilbarg and Trudinger[8], Lemma 7.6), for which we can choose $v = \bar{u}^-$ in the previous equality. We have $\int_a^b e^{-\Gamma(t)}\bar{u}'(t)(\bar{u}^-)'(t)dt + \int_a^b e^{-\Gamma(t)}\sigma(t)\bar{u}(t)\bar{u}^-(t)dt = \lambda\int_a^b e^{-\Gamma(t)} f^+(t,\bar{u}(t))\bar{u}^-(t)dt$. Now, put $A = \{t \in [a,b] : \bar{u}(t) < 0\}$ and arguing by contradiction, assume that A is non empty. Taking into account that $\operatorname{ess\,inf}_A \delta \geq \operatorname{ess\,inf}_{[a,b]} \delta > -\left(\frac{\pi}{b-a}\right)^2 \geq -\left(\frac{\pi}{|A|}\right)^2$, from Remark 2.4 one has

$$0 \leq (m_A)^2 \int_A [(\bar{u}^-)'(t)]^2 dt \leq \int_A e^{-\Gamma(t)}[(\bar{u}^-)'(t)]^2 dt + \int_A e^{-\Gamma(t)}\sigma(t)[\bar{u}^-(t)]^2 dt =$$

$$= \int_A e^{-\Gamma(t)}\bar{u}'(t)(\bar{u}^-)'(t)dt + \int_A e^{-\Gamma(t)}\sigma(t)\bar{u}(t)\bar{u}^-(t)dt =$$

$$= \int_a^b e^{-\Gamma(t)}\bar{u}'(t)(\bar{u}^-)'(t)dt + \int_a^b e^{-\Gamma(t)}\sigma(t)\bar{u}(t)\bar{u}^-(t)dt = \lambda\int_a^b e^{-\Gamma(t)} f^+(t,\bar{u}(t))\bar{u}^-(t)dt =$$

$$= \lambda\int_A e^{-\Gamma(t)} f^+(t,\bar{u}(t))\bar{u}^-(t)dt = \lambda\int_A e^{-\Gamma(t)} f(t,0)\bar{u}^-(t)dt \leq 0, \text{ that is}$$

$$(m_A)^2 \int_A [(\bar{u}^-)'(t)]^2 dt = 0.$$

So, $\bar{u}(t) = 0$ for all $t \in A$, and this is absurd. Hence, $A = \emptyset$ for which $\bar{u}(t) \geq 0$ for all $t \in [a,b]$.

Finally, it is simple to verify that $\bar{u}$ is a generalized solution of (D_λ). Indeed, the definition of f^+ and taking again Proposition 2.3 into account, one has $\int_a^b e^{-\Gamma(t)}\bar{u}'(t)v'(t)dt + \int_a^b e^{-\Gamma(t)}\sigma(t)\bar{u}(t)v(t)dt = \lambda\int_a^b e^{-\Gamma(t)} f^+(t,\bar{u}(t))v(t)dt = \lambda\int_a^b e^{-\Gamma(t)} f(t,\bar{u}(t))v(t)dt$ for all $v \in X$. □

The next result allows us to obtain positive solutions. It is based on the strong maximum principle.

Lemma 2.2. *Assume that*

$$f(t,x) \geq 0 \; for \; a.e. \; t \in [a,b], \; for \; all \; x \geq 0.$$

Then, any non-zero generalized solution of the problem (D_λ^+) is positive, and it is also a generalized solution of (D_λ).

Proof. Let $\bar{u}$ a non-zero generalized solution of the problem (D_λ^+). Owing to Lemma 2.1, it is a nonnegative solution of the problem (D_λ). Therefore, taking the Proposition 2.3 into account, one has

$$\int_a^b e^{-\Gamma(t)}\bar{u}'(t)v'(t)dt + \int_a^b e^{-\Gamma(t)}\sigma(t)\bar{u}(t)v(t)dt = \lambda \int_a^b e^{-\Gamma(t)} f(t,\bar{u}(t))v(t)dt \geq 0$$

for all $v \in X$ such that $v(t) \geq 0$ for every $t \in [a,b]$. Therefore, taking into account that $\bar{u} \in C^1([a,b])$ and $\bar{u}(t) \geq 0$ for every $t \in [a,b]$, the strong maximum principle (see, for instance, Pucci and Serrin[9], Theorem 11.1) ensures $\bar{u} > 0$ for every $t \in]a,b[$, that is the conclusion. □

3 | EXISTENCE THEOREMS OF AT LEAST ONE NON-ZERO SOLUTION

In this section, we present our main results. Put

$$K = \frac{1}{2}\frac{m^2}{M^2}\frac{\min\limits_{t\in[a,b]} e^{-\Gamma(t)}}{\max\limits_{t\in[a,b]} e^{-\Gamma(t)}} \tag{3.1}$$

where m, M are given in Proposition 2.2 and Γ is the primitive of γ as defined in the previous section. We observe that $0 < K \leq \frac{1}{2}$. Our main result is the following.

Theorem 3.1. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a L^1-Carathédory function and put $F(t,x) = \int_0^x f(t,\xi)d\xi$ for all $(t,x) \in [a,b] \times \mathbb{R}$. Assume that there are two positive constants c, d, with $d < c$, such that*

(j) $F(t,x) \geq 0$ for a.e. $t \in [a, a+\frac{b-a}{4}] \cup [b-\frac{b-a}{4}, b]$ and for all $x \in [0,d]$

(jj) $\frac{\int_a^b \max_{x\in[-c,c]} F(t,x)dt}{c^2} < K\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2}$.

Then, for each $\lambda \in \Lambda_{c,d}$, where

$$\Lambda_{c,d} = \left] \frac{2m^2}{(b-a)\max\limits_{t\in[a,b]} e^{-\Gamma(t)}}\frac{1}{K}\frac{d^2}{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}; \frac{2m^2}{(b-a)\max\limits_{t\in[a,b]} e^{-\Gamma(t)}}\frac{c^2}{\int_a^b \max\limits_{x\in[-c,c]} F(t,x)dt}\right[,$$

the problem (D_λ) admits at least one non-zero generalized solution $\bar{u}$ such that $\|\bar{u}\|_\infty < c$, $\|\bar{u}\|_X < \frac{2m}{(b-a)^{1/2}}c$ and

$$\int_a^b e^{-\Gamma(t)}\left[\frac{1}{2}|\bar{u}'(t)|^2 + \frac{1}{2}\sigma(t)|\bar{u}(t)|^2 - \lambda F(t,\bar{u}(t))\right]dt \leq \int_a^b e^{-\Gamma(t)}\left[\frac{1}{2}|u'(t)|^2 + \frac{1}{2}\sigma(t)|u(t)|^2 - \lambda F(t,u(t))\right]dt$$

for all $u \in W_0^{1,2}([a,b])$ such that $\|u\|_X < \frac{2m}{(b-a)^{1/2}}c$.

Proof. Our aim is to apply Theorem 2.1. To this end, take $(X, \|\cdot\|_X)$, $\Phi, \Psi : X \to \mathbb{R}$ as defined in Section 2. As also seen there, Φ and Ψ satisfy the assumptions of regularity requested in Theorem 2.1. So we are going to verify (2.5). Put $r = \frac{2m^2}{b-a}c^2$. Taking Remark 2.3 into account, for each $u \in X$ such that $\frac{1}{2}\|u\|_X^2 < r$, one has $|u(t)| \leq \frac{(b-a)^{1/2}}{2m}\|u\|_X \leq \frac{(b-a)^{1/2}}{2m}\sqrt{2r} = \left(\frac{(b-a)}{2m^2}r\right)^{1/2} = c$ for all $t \in [a,b]$. Therefore, it follows that

$$\Psi(u) = \int_a^b e^{-\Gamma(t)}F(t,u(t))dt \leq \max_{t\in[a,b]} e^{-\Gamma(t)}\int_a^b F(t,u(t))dt \leq \max_{t\in[a,b]} e^{-\Gamma(t)}\int_a^b \max_{\xi\in[-c,c]} F(t,\xi)dt$$

for each $u \in X$ such that $\frac{1}{2}\|u\|_X^2 < r$.

Hence, one has $\underline{\varphi}(r) = \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} \leq \frac{\max_{t\in[a,b]} e^{-\Gamma(t)} \int_a^b \max_{\xi\in[-c,c]} F(t,\xi)dt}{\frac{2m^2}{b-a}c^2} =$

$= \frac{(b-a)}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_a^b \max_{\xi\in[-c,c]} F(t,\xi)dt}{c^2}$, that is,

$$\underline{\varphi}(r) \leq \frac{(b-a)}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_a^b \max_{\xi\in[-c,c]} F(t,\xi)dt}{c^2}. \tag{3.2}$$

Now, put

$$\tilde{u}(t) = \begin{cases} 4d\frac{t-a}{b-a} & \text{if } t \in [a, a+\frac{b-a}{4}[\\ d & \text{if } t \in \left[a+\frac{b-a}{4}, b-\frac{b-a}{4}\right] \\ 4d\frac{b-t}{b-a} & \text{if } t \in]b-\frac{b-a}{4}, b] \end{cases}$$

Clearly, $\tilde{u} \in X$ and one has $\|\tilde{u}\|^2 = \int_a^b |\tilde{u}'(t)|^2 dt = \int_a^{a+\frac{b-a}{4}} \left(\frac{4d}{b-a}\right)^2 dt + \int_{b-\frac{b-a}{4}}^b \left(\frac{4d}{b-a}\right)^2 dt = \frac{8d^2}{b-a}$. So, taking Proposition 2.2 into account, one has $\Phi(\tilde{u}) \leq \frac{1}{2}M^2\|\tilde{u}\|^2 = \frac{4M^2}{b-a}$, that is,

$$\Phi(\tilde{u}) \leq \frac{4M^2}{b-a}d^2.$$

Moreover, taking (j) into account, one has
$\Psi(\tilde{u}) = \int_a^b e^{-\Gamma(t)}F(t,\tilde{u}(t))dt \geq \min_{t\in[a,b]} e^{-\Gamma(t)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt$, that is,

$$\Psi(\tilde{u}) \geq \min_{t\in[a,b]} e^{-\Gamma(t)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt.$$

Therefore, one has

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \frac{(b-a)}{4M^2} \min_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2}.$$

Now, we verify that $\tilde{u} \in \Phi^{-1}(]0,r[)$. First, we observe that from $d<c$, owing to (jj), we obtain $\sqrt{2}\frac{M}{m}d < c$. Indeed, arguing by contradiction, we assume that $d < c \leq \sqrt{2}\frac{M}{m}d$. It follows that

$$\frac{\int_a^b \max_{\xi\in[-c,c]}F(t,\xi)dt}{c^2} \geq \frac{\int_a^b \max_{\xi\in[-c,c]}F(t,\xi)dt}{2\left(\frac{M}{m}\right)^2 d^2} \geq \frac{1}{2}\left(\frac{m}{M}\right)^2 \frac{\int_a^b F(t,d)dt}{d^2} \geq$$

$$\geq \frac{1}{2}\left(\frac{m}{M}\right)^2 \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2} \geq \frac{1}{2}\left(\frac{m}{M}\right)^2 \frac{\min_{t\in[a,b]}e^{-\Gamma(t)}}{\max_{t\in[a,b]}e^{-\Gamma(t)}} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2} = K\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2}$$

and this contradicts (jj). Next, from $\sqrt{2}\frac{M}{m}d < c$, we have $2\frac{M^2}{m^2}d^2 < c^2$, $\frac{4}{b-a}M^2d^2 < \frac{2}{b-a}m^2c^2$, that is, $\frac{4}{b-a}M^2d^2 < r$. Hence, $\Phi(\tilde{u}) \leq \frac{4M^2}{b-a}d^2 < r$, and our claim is proved.

Finally, taking into account that $\tilde{u} \in \Phi^{-1}(]0,r[)$, from $\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \frac{(b-a)}{4M^2} \min_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2}$, we obtain

$$\overline{\varphi}(r) \geq \frac{(b-a)}{4M^2} \min_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2} \tag{3.3}$$

Hence, from our assumption (jj), (3.2) and (3.3), we obtain

$$\underline{\varphi}(r) < \overline{\varphi}(r),$$

that is, (2.5) is verified. Moreover, again from (3.2) and (3.3), one has

$$\Lambda_{c,d} \subseteq \Lambda_r.$$

Now, we prove that $I_\lambda = \Phi - \lambda\Psi$, $\lambda > 0$, satisfies the $(PS)^{[r]}$-condition. Let $\{u_n\}$ be a sequence such that $I(u_n)$ is bounded, $I'(u_n) \to 0$ and $\Phi(u_n) < r$ for all $n \in \mathbb{N}$. From $\Phi(u_n) < r$, that is, $\|u_n\|_X < \sqrt{2r}$ for all $n \in \mathbb{N}$, we obtain that $\{u_n\}$ is bounded in X. Therefore, since X is reflexive, $\{u_n\}$ admits a subsequence which is weakly convergent to $u \in X$. Moreover, taking into account that the embedding of X into $C([a,b])$ is compact, there is a subsequence which is strongly convergent to u in $C([a,b])$. Summing up and renaming the subsequence again with $\{u_n\}$, we have

$$u_n \rightharpoonup u \text{ in } X \qquad \text{and} \qquad u_n \to u \text{ in } C([a,b]).$$

Clearly, since Φ' is a linear operator and $u_n \rightharpoonup u$, one has

$$\langle \Phi'(u), u_n - u\rangle \to 0. \tag{3.4}$$

Moreover, we prove that one has

$$\langle \Phi'(u_n); u_n - u\rangle \to 0. \tag{3.5}$$

Indeed, we have $\langle \Phi'(u_n); u_n - u\rangle = \langle I'_\lambda(u_n); u_n - u\rangle + \lambda \int_a^b e^{-\Gamma(t)} f(t, u_n(t))(u_n - u)dt$. So, since one has $\int_a^b e^{-\Gamma(t)} f(t, u_n)(u_n - u)dt \le \int_a^b C(u_n - u)dt$, being $f(t, u_n(t)) \le \max_{|\xi|\le k} f(t,\xi)$ since $\|u_n\|_\infty \le \|u_n - u\|_\infty + \|u\|_\infty \le k$, and $\langle I'_\lambda(u_n); u_n - u\rangle \le \|I'_\lambda(u_n)\|_{X^*}\|u_n - u\|_X \le L\|I'_\lambda(u_n)\|_{X^*}$
the condition (3.5) is proved.
Hence, from (3.4) and (3.5), one has

$$\langle \Phi'(u_n) - \Phi'(u); u_n - u\rangle \to 0. \tag{3.6}$$

So, taking into account that $\langle \Phi'(u_n), u\rangle \le \|u_n\|_X\|u\|_X$ and $\langle \Phi'(u), u_n\rangle \le \|u\|_X\|u_n\|_X$, one has $(\|u_n\|_X - \|u\|_X)^2 = \|u_n\|_X^2 + \|u\|_X^2 - \|u_n\|_X\|u\|_X - \|u\|_X\|u_n\|_X \le \|u_n\|_X^2 + \|u\|_X^2 - \langle \Phi'(u_n), u\rangle - \langle \Phi'(u), u_n\rangle = \langle \Phi'(u_n), u_n\rangle + \langle \Phi'(u), u\rangle - \langle \Phi'(u_n), u\rangle - \langle \Phi'(u), u_n\rangle = \langle \Phi'(u_n) - \Phi'(u); u_n - u\rangle$.
Therefore, from (3.6), it follows that

$$\lim_{n\to+\infty} \|u_n\|_X = \|u\|_X.$$

Hence, since X is uniformly convex, Brezis[10, Proposition III.30] ensures that

$$\lim_{n\to+\infty} \|u_n - u\|_X = 0,$$

that is, our claim is proved.

Since all assumptions of Theorem 2.1 are verified, for each $\lambda \in \Lambda_r$ and, in particular, for each $\lambda \in \Lambda_{c,d}$, the functional I_λ admits a non-zero critical point $\bar{u} \in \Phi^{-1}(]0, r[)$ such that $I_\lambda(\bar{u}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]0, r[)$. Hence, from Proposition 2.3, taking Remark 2.3 also into account, $\bar{u}$ is a generalized solution of (D_λ) which satisfies the conclusion. □

Remark 3.1. Taking Theorem 2.1 into account, the assumptions (j) and (jj) of Theorem 3.1 can be expressed in the following way.

Assume that there is a positive constant c such that

(j') $F(t,x) \ge 0$ for a.e. $t \in [a, a + \frac{b-a}{4}] \cup [b - \frac{b-a}{4}, b]$ and for all $x \in [0, c]$

(jj') $\dfrac{\int_a^b \max_{x\in[-c,c]} F(t,x)dt}{c^2} < K \sup_{d\in]0,c[} \dfrac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2}.$

So that the interval for which the conclusion of Theorem 3.1 holds become

$$\Lambda_c = \left] \frac{2m^2}{(b-a)\max\limits_{t\in[a.b]} e^{-\Gamma(t)}} \frac{1}{K} \inf_{d\in]0,c[} \frac{d^2}{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}; \frac{2m^2}{(b-a)\max\limits_{t\in[a.b]} e^{-\Gamma(t)}} \frac{c^2}{\int_a^b \max\limits_{x\in[-c.c]} F(t,x)dt} \right[.$$

Remark 3.2. If in addition we assume that (iii^*) for all $\rho > 0$ the function $\sup\limits_{|x|\leq\rho} |f(t,x)|$ belongs to $L^\infty([a,b])$, then the assumption ((jj) of Theorem 3.1 can be expressed as follows:

(jj^*) $\dfrac{\operatorname{ess\,sup}\limits_{t\in[a,b]} \max\limits_{x\in[-c.c]} F(t,x)}{c^2} < K^* \dfrac{\operatorname{ess\,inf}\limits_{t\in[a,b]} F(t,d)}{d^2}$,

where

$$K^* = \frac{1}{2}\frac{m^2}{M^2} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} e^{-\Gamma(t)}dt}{\int_a^b e^{-\Gamma(t)}dt}.$$

In this case, the interval become

$$\Lambda^*{}_{c,d} = \left] \frac{2m^2}{(b-a)\|e^{-\Gamma}\|_1} \frac{1}{K} \frac{d^2}{\operatorname{ess\,inf}\limits_{t\in[a,b]} F(t,d)}; \frac{2m^2}{(b-a)\|e^{-\Gamma}\|_1} \frac{c^2}{\operatorname{ess\,sup}\limits_{t\in[a,b]} \max\limits_{x\in[-c.c]} F(t,x)} \right[.$$

Indeed, arguing as in the proof of Theorem 3.1, one has

$$\underline{\varphi}(r) \leq \frac{(b-a)}{2m^2} \|e^{-\Gamma}\|_1 \frac{\operatorname{ess\,sup}\limits_{t\in[a,b]} \left(\max\limits_{\xi\in[-c,c]} F(t,\xi) \right)}{c^2},$$

and

$$\overline{\varphi}(r) \geq \frac{(b-a)}{4M^2} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} e^{-\Gamma(t)}dt \frac{\operatorname{ess\,inf}\limits_{t\in[a,b]} F(t,d)}{d^2}.$$

Remark 3.3. We explicitly observe that when $\gamma \equiv \sigma \equiv 0$, one has $K = \frac{1}{2}$ (and $K^* = \frac{1}{4}$), and in this case, the formula (jj) (so also (jj^*)) is completely independent from the choice of interval $[a,b]$. On the contrary, the interval of parameters $\Lambda_{c.d}$ (the same for $\Lambda^*_{c.d}$), also in this particular case, depends on the length of $[a,b]$. We use this fact to obtain a result we will see later.

We also observe that also in general when one has the dependence of γ and σ, an appropriate choice of the interval $[a,b]$ does not allow us to verify directly the assumption (jj), being $K \leq \frac{1}{2}$.

The following result is a version of Theorem 3.1 in order to obtain positive solutions.

Theorem 3.2. *Let $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ be a L^1-Carathédory function such that $f(t,x) \geq 0$ for all $(t,x) \in [a,b]\times[0,+\infty[$ and put $F(t,x) = \int_0^x f(t,\xi)d\xi$ for all $(t,x) \in [a,b]\times\mathbb{R}$. Assume that there are two positive constants c, d, with $d < c$, such that*

($\tilde{j}j$) $\dfrac{\int_a^b F(t,c)dt}{c^2} < K \dfrac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2}$.

Then, for each $\lambda \in \Lambda_{c,d}$, where

$$\Lambda_{c,d} = \left] \frac{2m^2}{(b-a)\max\limits_{t\in[a.b]} e^{-\Gamma(t)}} \frac{1}{K} \frac{d^2}{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}; \frac{2m^2}{(b-a)\max\limits_{t\in[a.b]} e^{-\Gamma(t)}} \frac{c^2}{\int_a^b F(t,c)dt} \right[,$$

the problem (D_λ) *admits at least one positive generalized solution* $\bar{u}$ *such that* $\|\bar{u}\|_\infty < c$, $\|\bar{u}\|_X < \frac{2m}{(b-a)^{1/2}}c$ *and*

$$\int_a^b e^{-\Gamma(t)}\left[\frac{1}{2}|\bar{u}'(t)|^2+\frac{1}{2}\sigma(t)|\bar{u}(t)|^2-\lambda F(t,\bar{u}(t))\right]dt \leq \int_a^b e^{-\Gamma(t)}\left[\frac{1}{2}|u'(t)|^2+\frac{1}{2}\sigma(t)|u(t)|^2-\lambda F(t,u(t))\right]dt$$

for all $u \in W_0^{1,2}([a,b])$ *such that* $\|u\|_X < \frac{2m}{(b-a)^{1/2}}c$.

Proof. Let f^+ be the function as defined in Section 2. It satisfies all the assumptions of Theorem 3.1, for which the problem (D_λ^+) admits a positive generalized solution $\bar{u}$ that, owing to Lemma 2.2, is a positive generalized solution of the problem (D_λ). Hence, taking also into account that $F^+(t,\bar{u}(t)) = F(t,\bar{u}(t))$ for all $t \in [a,b]$, the conclusion is achieved. □

Remark 3.4. Clearly, Remarks 3.1,3.2, and 3.3 can also refer to Theorem 3.2.

The assumptions of Theorems 3.1 and 3.2 can have a simpler form when the nonlinear term is of separate variables type. As an example, here, we point out the following two cases.

Theorem 3.3. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $g(x) \geq 0$ *for all* $x \in [0,+\infty[$ *and* $\alpha \in L^1([a,b]$ *such that* $\alpha(t) \geq 0$ *for a.e.* $t \in [a,b]$*. Put* $G(x) = \int_0^x g(\xi)d\xi$ *for all* $x \in \mathbb{R}$ *and assume that there are two positive constants* c, d*, with* $d < c$*, such that*

$$\frac{G(c)}{c^2} < \tilde{K}\frac{G(d)}{d^2}, \tag{A}$$

where $\tilde{K} = \frac{1}{2}\frac{m^2}{M^2}\frac{\min_{t\in[a,b]} e^{-\Gamma(t)}}{\max_{t\in[a,b]} e^{-\Gamma(t)}}\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}}\alpha(t)dt}{\|\alpha\|_1}$

Then, for each $\lambda \in \tilde{\Lambda}_{c,d}$*, where*

$$\tilde{\Lambda}_{c,d} = \left]\frac{2m^2}{(b-a)\|\alpha\|_1 \max_{t\in[a,b]} e^{-\Gamma(t)}}\frac{1}{\tilde{K}}\frac{d^2}{G(d)};\frac{2m^2}{(b-a)\|\alpha\|_1 \max_{t\in[a,b]} e^{-\Gamma(t)}}\frac{c^2}{G(c)}\right[,$$

the problem

$$\begin{cases} -u''+\gamma(t)u'+\sigma(t)u = \lambda\alpha(t)g(u) \text{ in } [a,b], \\ u(a)=u(b)=0, \end{cases} \tag{$\tilde{D}_\lambda$}$$

admits at least one positive generalized solution $\bar{u}$ *such that* $\|\bar{u}\|_\infty < c$, $\|\bar{u}\|_X < \frac{2m}{(b-a)^{1/2}}c$ *and*

$$\int_a^b e^{-\Gamma(t)}\left[\frac{1}{2}|\bar{u}'(t)|^2+\frac{1}{2}\sigma(t)|\bar{u}(t)|^2-\lambda\alpha(t)G(\bar{u}(t))\right]dt \leq \int_a^b e^{-\Gamma(t)}\left[\frac{1}{2}|u'(t)|^2+\frac{1}{2}\sigma(t)|u(t)|^2-\lambda\alpha(t)G(u(t))\right]dt$$

for all $u \in W_0^{1,2}([a,b])$ *such that* $\|u\|_X < \frac{2m}{(b-a)^{1/2}}c$.

Proof. It follows from Theorem 3.2. □

Remark 3.5. In the autonomous case, that is, when $\alpha \equiv 1$, and when $\gamma \equiv \sigma \equiv 0$ (see also Remark 3.3), the condition (A) and the interval of parameters assume the following simpler forms

$$\frac{G(c)}{c^2} < \frac{1}{4}\frac{G(d)}{d^2},$$

$$\tilde{\Lambda}_{c,d} = \left]\frac{8}{(b-a)^2}\frac{d^2}{G(d)};\frac{2}{(b-a)^2}\frac{c^2}{G(c)}\right[.$$

Theorem 3.4. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(x) \geq 0$ for all $x \in [0, +\infty[$ and $\alpha \in L^1([a, b]$ such that $\alpha(t) \geq 0$ for a.e. $t \in [a, b]$. Assume that*

$$\lim_{x\to 0^+} \frac{g(x)}{x} = +\infty. \tag{B}$$

Put

$$\tilde{\lambda} = \frac{2m^2}{(b-a)\|\alpha\|_1 \max\limits_{t\in[a.b]} e^{-\Gamma(t)}} \sup_{c>0} \frac{c^2}{G(c)},$$

where $G(x) = \int_0^x g(\xi)d\xi$ for all $x \in \mathbb{R}$.

Then, for each $\lambda \in]0, \tilde{\lambda}[$, the problem $(\tilde{D}_\lambda)$ admits at least one positive generalized solution $\bar{u}$.

Proof. Fix $\lambda \in]0, \tilde{\lambda}[$ and let $\tilde{c} > 0$ such that $\lambda < \frac{2m^2}{(b-a)\|\alpha\|_1 \max\limits_{t\in[a.b]} e^{-\Gamma(t)}} \frac{\tilde{c}^2}{G(\tilde{c})}$. From (B) one has $\lim_{x\to 0^+} \frac{2m^2}{(b-a)\|\alpha\|_1 \max\limits_{t\in[a.b]} e^{-\Gamma(t)}} \frac{1}{\tilde{K}} \frac{x^2}{G(x)} = 0 < \lambda$, so we can fix $\tilde{d} < \tilde{c}$ such that $\frac{2m^2}{(b-a)\|\alpha\|_1 \max\limits_{t\in[a.b]} e^{-\Gamma(t)}} \frac{1}{\tilde{K}} \frac{\tilde{d}^2}{G(\tilde{d})} < \lambda$. Hence, one has

$$\frac{2m^2}{(b-a)\|\alpha\|_1 \max\limits_{t\in[a.b]} e^{-\Gamma(t)}} \frac{1}{\tilde{K}} \frac{\tilde{d}^2}{G(\tilde{d})} < \lambda < \frac{2m^2}{(b-a)\|\alpha\|_1 \max\limits_{t\in[a.b]} e^{-\Gamma(t)}} \frac{\tilde{c}^2}{G(\tilde{c})},$$

for which Theorem 3.3 ensures the conclusion. □

Here, as a consequence of the theorems above, we point out a result where the problem is independent from the parameter λ.

Theorem 3.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(x) \geq 0$ for all $x \in [0, +\infty[$ and $\alpha \in L^1([a, b]$ such that $\alpha(t) \geq 0$ for a.e. $t \in [a, b]$. Put $G(x) = \int_0^x g(\xi)d\xi$ for all $x \in \mathbb{R}$ and assume that*

$$\frac{G(c)}{c^2} < \frac{1}{4}\frac{G(d)}{d^2}, \tag{C}$$

Then, for each $a \in \mathbb{R}$ there is $b \in \mathbb{R}$, with $b > a$, such that the problem

$$\begin{cases} -u'' = g(u) & \text{in } [a, b], \\ u(a) = u(b) = 0, \end{cases} \tag{D}$$

admits at least one positive classical solution $\bar{u}$.

Proof. Fix $a \in \mathbb{R}$ and put $b = a + \sqrt{\frac{4d^2}{G(d)} + \frac{c^2}{G(c)}}$. Since (C) holds, from Theorem 3.3 one has that for each $\lambda \in \frac{2}{(b-a)^2} \cdot \left] \frac{4d^2}{G(d)}, \frac{c^2}{G(c)} \right[$ the problem

$$\begin{cases} -u'' = \lambda g(u) & \text{in } [a, b], \\ u(a) = u(b) = 0, \end{cases}$$

admits at least one positive classical solution $\bar{u}$. Hence, taking into account that $\frac{2}{\frac{4d^2}{G(d)} + \frac{c^2}{G(c)}} \frac{\frac{4d^2}{G(d)} + \frac{c^2}{G(c)}}{2} = 1$, that is 1 is the middle point of the interval of parameters, the conclusion is achieved. □

Remark 3.6. As the proof shows, a point b which satisfies the conclusion of Theorem 3.5 is

$$b = a + \sqrt{\frac{4d^2}{G(d)} + \frac{c^2}{G(c)}}.$$

4 | APPLICATIONS TO A DC/DC BUCK CONVERTER

The Buck converter is a DC/DC switching mode power converter that steps-down the input voltage to a suitable level for powering a given load (for general references, see previous studies[11–16]). The basic Buck converter scheme, shown in Figure 1, is a second-order circuit consisting of two semiconductor devices (diode D and a transistor S) and two energy storage elements (capacitor C and an inductor L).

The Buck converter is a nonlinear system with a variable structure, because its circuit topology changes according to the state of the two semiconductor devices. More precisely, both the diode and the transistor can behave as an open or closed switch, hence, as shown in Figure 2, the converter may take three states, namely, powering, free-wheeling and idle, corresponding, respectively, to S on and D off, S off and D on, and S off and D off.

As shown in Figure 3, the powering state starts at $t = t_0$ when the transistor S is turned on and the diode D, which is reverse biased, turns off. The inductor current increases, and the inductor stores in the form of a magnetic field a part of the energy drawn from the input power source. The balance of the input energy is directly supplied to the load. At $t = t_0 + T_{on}$, the transistor S is turned off and the diode D, being forward biased, turns on. The converter takes the free-wheeling state, and the energy stored in the inductor is fed to the load; hence, the current in the inductor decreases. If the current in the inductor reaches zero at $t = t_0 + T_{on} + T_{off}$, the converter enters the idle mode until the end of the switching cycle at $t = t_0 + T_{on} + T_{off} = T_s$.

The Buck converter can operate in continuous conduction mode (CCM) or in discontinuous conduction mode (DCM). In the first case, the converter takes only the powering and free-wheeling states along a switching cycle and the inductor current never reaches zero. For efficiency reasons CCM is the most common operating mode. In CCM at steady state, the output voltage v_0 is related to the input voltage v_{in} and to the duty cycle δ by the following simple expression:

$$v_0 = v_{in}\delta \tag{4.1}$$

where $\delta = \frac{T_{on}}{T_s}$ $(0 < \delta < 1)$.

A constant structure approximation of the Buck converter operating in CCM can be obtained according to the time averaging approach, which gives the equivalent circuit shown in Figure 4.

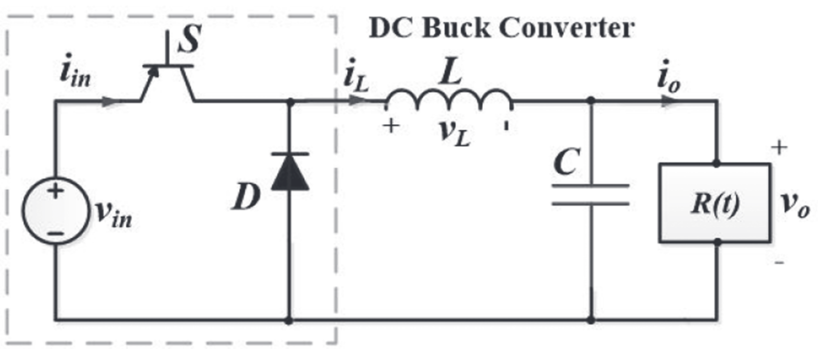


FIGURE 1 DC Buck converter

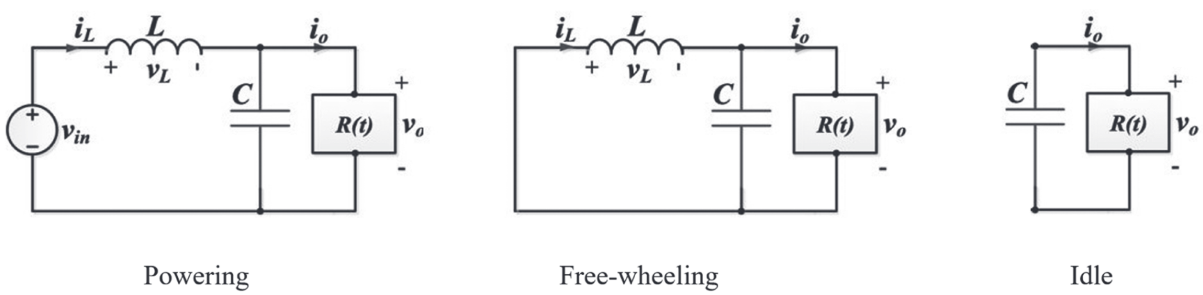


FIGURE 2 The three possible states of a Buck converter

FIGURE 3 Buck converter variables in a generic switching cycle

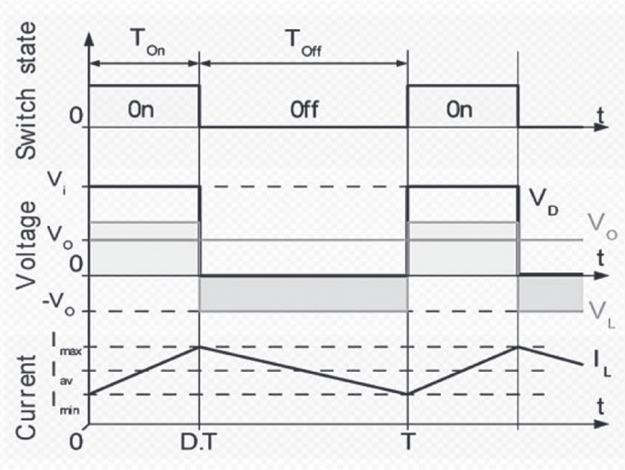


FIGURE 4 Time averaged model of the Buck converter

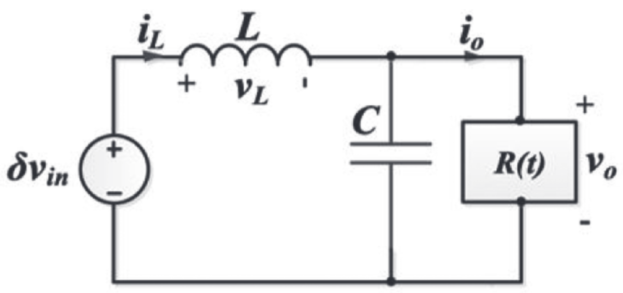
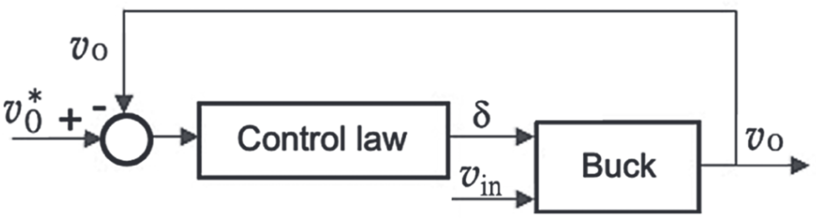


FIGURE 5 Closed loop control of the Buck converter output voltage v_0



A second-order differential equation can be written analytically to describe the circuit of Figure 4:

$$\delta(t, v_0) = LCv_0'' + \frac{v_0'}{R} + v_0 \tag{4.2}$$

with R, L, C constant coefficients.

In constant output voltage applications, the load voltage v_0 is kept at a given reference value v_0^* by acting on the duty cycle δ, in order to compensate the variation of the load $R(t)$ over the time. The duty cycle is regulated according to a closed loop control law processing the difference between the reference voltage v_0^* and the actual one v_0, as shown in the block scheme of Figure 5.

In the Laplace domain, the control law is described by a rational fractional function, called the transfer function, of the type:

$$T(s) = \frac{N(s)}{D(s)} = K \cdot \frac{a_m s^m + a_{m-1}s^{m-1} + \ldots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0} \quad \text{being } m \leq n, \tag{4.3}$$

which can be rewritten as

$$T(s) = K \cdot \frac{(1 + sT_1)(1 + sT_2) \ldots (1 + sT_m)}{(1 + s\tau_1)(1 + s\tau_2) \ldots (1 + s\tau_n)}. \tag{4.4}$$

The roots of the numerator are called the zeros of the transfer function.

$$z_1 = -\frac{1}{T_1}, \quad \dots \quad , z_m = -\frac{1}{T_m} \tag{4.5}$$

The roots of the denominator are called the poles of the transfer function.

$$p_1 = -\frac{1}{\tau_1}, \quad \dots \quad , p_n = -\frac{1}{\tau_n} \tag{4.6}$$

A control law with a transfer function with a pole in the origin ($s = 0$) ensures a null asymptotic error on a second-order system as the considered one, according to Equation 4.2. This means that, at least asymptotically, the error between the reference voltage and the actual one is in any case driven to zero. Once that the convergence of the system to the reference is ensured, the dynamic of the system can be managed through a simple proportional action, which means an additional term of the control transfer function featuring an output proportional to the error between the reference voltage and the actual one. In the time domain, a proportional control law is given by

$$\delta(t, v_0) = K_p f(v_0^* - v_0), \;\; e = (v_0^* - v_0). \tag{4.7}$$

Considering that the value of the duty cycle is positive and upper limited to 1, a nonlinear piecewise control function is obtained, which is shown in Figure 5 as a function of the normalized voltage error $e_v = \frac{(v_0^* - v_0)}{v_0^*}$. The slope of the first segment of the control function can be adjusted by acting on the gain k_p in order to achieve a reasonably low error after a given time.

The problem can be written in terms of the voltage error e:

$$\delta(t, e) = LC(v_0^* - e)'' + \frac{(v_0^* - e)'}{R} + (v_0^* - e), \tag{4.8}$$

$$\delta(t, e) = K_p(\beta\sqrt{e} - \eta(e)), \tag{4.9}$$

with β and η two positive constants. Figure 6 shows the proportional error of Equation (4.7), and the approximated error described in (4.9) by means the sqrt functions.

The term $R(t)$ which represents the electrical load can vary with the time t. By considering $e_0 = e(t_0)$, if at t_0 the value of R changes, the error e can be reported at the initial value for $t = \infty$, $e_\infty = e_0 = e(t_\infty)$, as shown in Figure 7. For a given value of K_p and v_0^*, the duty cycle will be changed in order to obtain $e_\infty = e_0 = e(t_\infty)$. Figure 7 shows the voltage error e due to R transient from 10Ω to 5Ω for different values of K_p. The error e_∞ decreases as K_p increases. Figure 8 shows the error response for different load transients R by keeping constant $K_p = 1$. The greater the variation of R, the greater the maximum error.

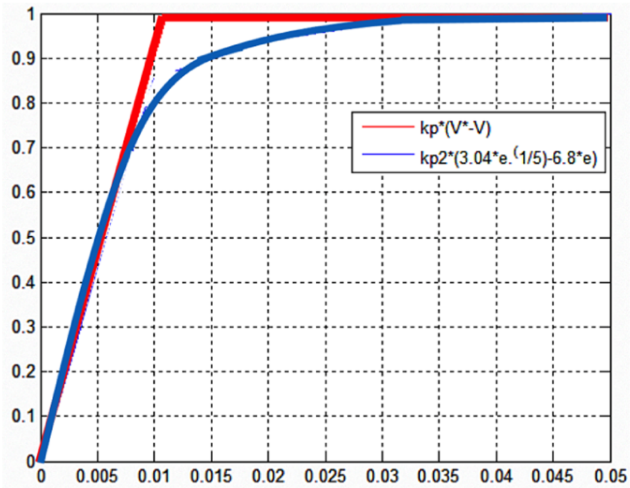


FIGURE 6 Proportional and approximated control functions [Colour figure can be viewed at wileyonlinelibrary.com]

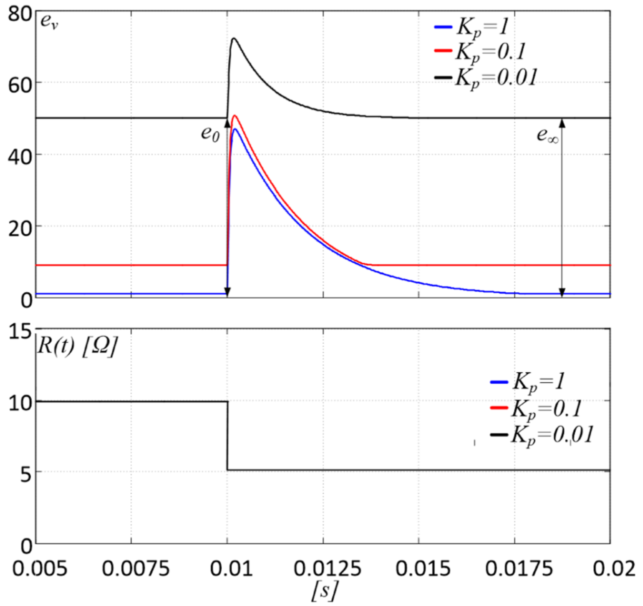


FIGURE 7 Voltage error versus load transient R for different values of K_p [Colour figure can be viewed at wileyonlinelibrary.com]

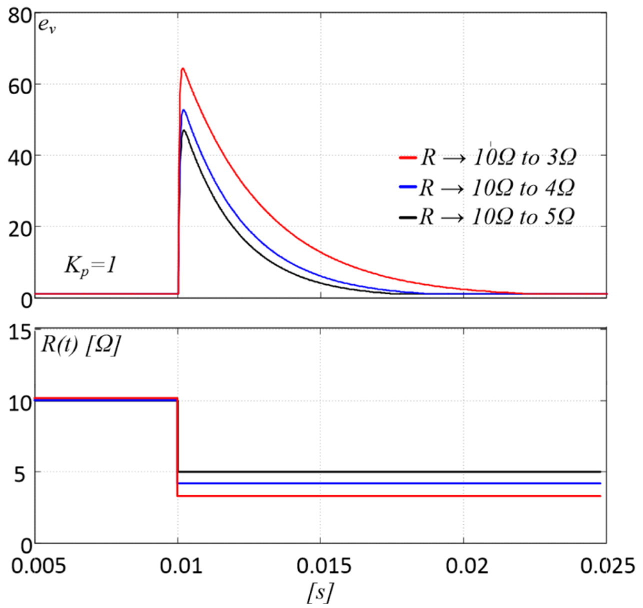


FIGURE 8 Voltage error versus different load transients R for $K_p = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

Consider the following problem

$$\begin{cases} -e'' - \frac{e'}{LCR(t)} = \frac{1}{LC} K_p \Delta\delta(t, v_0^* - e) & \text{in }]0.01, 0.015[, \\ e(0.01) = e(0.015) = 0. \end{cases} \tag{4.10}$$

where $L = C = 0.01$, $R(t) = \frac{t+10^{-2}}{10^{-3}}$ and $\Delta\delta(t, v_0^* - e) = \beta\sqrt{e} - \eta e$, with $\beta = 0.1$ and $\eta = 0.067$. Our aim is to apply Theorem 3.3, because problem (4.10) can be obtained from problem $(\tilde{D}_\lambda)$ by choosing

$$\sigma(t) = 0, \ \alpha(t) = 1, \ \lambda = K_p,$$

$$\gamma(t) = -\frac{1}{LCR(t)}, \ g(e) = \frac{1}{LC}\Delta\delta(t, v_0^* - e).$$

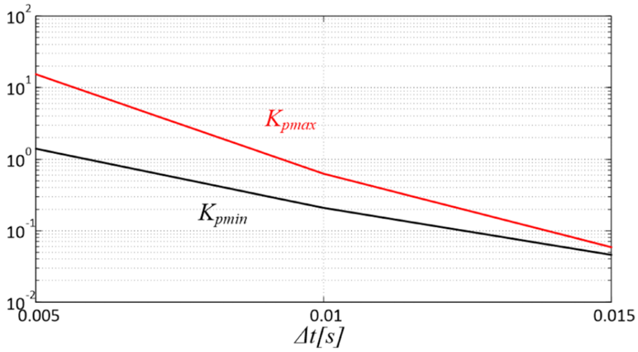


FIGURE 9 Trend of K_{pmax} and K_{pmax} versus $\triangle t$ [Colour figure can be viewed at wileyonlinelibrary.com]

Besides, the first hypothesis of Theorem 3.3 is that g has to be a continuous and positive function, so we put $e^* = \left(\frac{\beta}{2\eta}\right)^2 = 0.556917$, and we consider

$$\tilde{g}(e) = \begin{cases} g(e) & \text{if } 0 \le e \le e^*, \\ g(e^*) & \text{if } e > e^*. \end{cases}$$

Clearly, this function satisfies the required assumptions. Hence, by simple calculations, one has

$$\Gamma(t) = \int_{10^{-2}}^{t} \gamma(\xi)\, d\xi = -\ln\left(\frac{t+10^{-2}}{2\cdot 10^{-2}}\right)^{\frac{10^{-2}}{LC}},$$

$$G(x) = \begin{cases} \frac{1}{LC}\left(\frac{2}{3}\beta x^{\frac{3}{2}} - \frac{\eta}{2}x^2\right) & \text{if } 0 \le x \le e^*, \\ \frac{1}{LC}\left(\frac{\beta^2}{4\eta}x - \frac{\beta^4}{96\eta^3}\right) & \text{if } x > e^* \end{cases}$$

and

$$\tilde{K} = \frac{1}{4}\left(\frac{4}{5}\right)^{\frac{2\cdot 10^{-3}}{LC}}.$$

Choosing $c = e^*$ and, for instance, $d = 10^{-7}$, hypothesis (A) of Theorem 3.3 is satisfied; then, for each $K_p \in]1.414, 15.385[$, rounded down, the given problem admits at least one positive generalized solution $\bar{e}$ such that $0 < e < e^*$.

Now, we consider problem (4.10) in the interval $]10^{-2}, 2\cdot 10^{-2}[=]0.01, 0.02[$; hence, one has

$$\tilde{K} = \frac{1}{4}\left(\frac{2}{3}\right)^{\frac{2\cdot 10^{-3}}{LC}}.$$

Then, choosing $c = e^*$ and, for instance, $d = 10^{-9}$, for each $K_p \in]0.219, 0.621[$, rounded down, the considered problem admits at least one positive generalized solution $\bar{e}$ such that $0 < e < e^*$.

Finally, we consider problem (4.10) in the interval $]10^{-2}, 2.5\cdot 10^{-2}[=]0.01, 0.025[$; hence, one has

$$\tilde{K} = \frac{1}{4}\left(\frac{4}{7}\right)^{\frac{2\cdot 10^{-3}}{LC}}.$$

Then, choosing $c = e^*$ and, for instance, $d = 10^{-11}$, for each $K_p \in]0.046, 0.059[$, rounded down, the considered problem admits at least one positive generalized solution $\bar{e}$ such that $0 < e < e^*$. Let us consider $\triangle t$ the time range in which the problem has been considered. Hence, we have $\triangle t_1 = 0.015 - 0.01 = 0.005$ s, $\triangle t_2 = 0.02 - 0.01 = 0.01$ s, and $\triangle t_3 = 0.025 - 0.01 = 0.015$ s. The trend of K_{pmax} and K_{pmin} has been plotted as a function of $\triangle t$, as shown in Figure 9. Note as higher is the time range $\triangle t$ in which at the end the voltage error value is again zero, lower is the value of K_p. The waveform of the voltage error versus the $\triangle t$ has been evaluated in Figure 10, where for each time range $\triangle t_1 = 0.015 - 0.01 = 0.005$ s, $\triangle t_2 = 0.02 - 0.01 = 0.01$ s and $\triangle t_3 = 0.025 - 0.01 = 0.015$ s, estimated values of $K_{p1} = 2, 5, K_{p2} = 0.35$ and $K_{p3} = 0.05$ have been set. The simulation results are in accordance with mathematical results.

In conclusion, in this section, a mathematical approach for tuning a proportional voltage control of a DC-DC buck converter has been performed. In detail, for a given time dynamic response of the system and by imposing the voltage

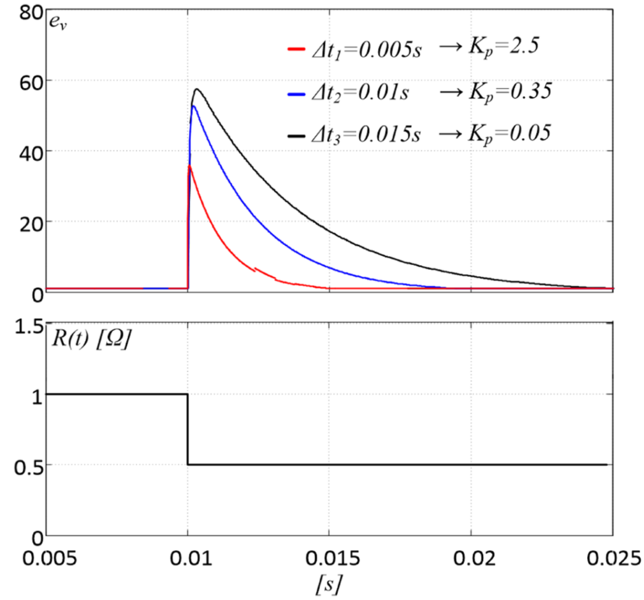


FIGURE 10 Voltage error during R transient for versus $\triangle t$ [Colour figure can be viewed at wileyonlinelibrary.com]

error null before and after the voltage transient, maximum and minimum values of proportional constant of the P controller are obtained. The mathematical results have been compared with simulations of the DC-DC converter by using the Matlab/Simulink platform by obtaining a good matching.

ACKNOWLEDGEMENTS

The first three authors have been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The paper is partially supported by PRIN 2017- Progetti di Ricerca di rilevante Interesse Nazionale, "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods" (2017AYM8XW). Open Access Funding provided by Universita degli Studi di Messina within the CRUI-CARE Agreement.

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How to cite this article: Amoroso E, Bonanno G, D'Aguí G, et al. Second-order differential equations for the power converters dynamical performance analysis. *Math Meth Appl Sci*. 2022;45(9):5573-5591. doi:10.1002/mma.8127