JORDAN ALGEBRAS OF A DEGENERATE BILINEAR FORM: SPECHT PROPERTY AND THEIR IDENTITIES

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Abstract. Let K be a field and let $J_{n,k}$ be the Jordan algebra of a degenerate symmetric bilinear form b of rank $n-k$ over K. Then one can consider the decomposition $J_{n,k} = B_{n-k} \oplus D_k$, where B_{n-k} represents the corresponding Jordan algebra, denoted as $B_{n-k} = K \oplus V$. In this algebra, the restriction of b on the $(n-k)$-dimensional subspace V is non-degenerate, while D_k accounts for the degenerate part of $J_{n,k}$. This paper aims to provide necessary and sufficient conditions to check if a given multilinear polynomial is an identity for $J_{n,k}$. As a consequence of this result and under certain hypothesis on the base field, we exhibit a finite basis for the T-ideal of polynomial identities of $J_{n,k}$. Over a field of characteristic zero, we also prove that the ideal of identities of $J_{n,k}$ satisfies the Specht property. Moreover, similar results are obtained for weak identities, trace identities and graded identities with a suitable $\mathbb{Z}_2$-grading as well. In all of these cases, we employ methods and results from Invariant Theory. Finally, as a consequence from the trace case, we provide a counterexample to the embedding problem given in [8] in case of infinite dimensional Jordan algebras with trace.

Keywords: Jordan algebra, polynomial identities, weak identities, graded identities, trace identities, embedding problem.

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1. Introduction

Let A be an algebra over a field K, then A is said to be an algebra with polynomial identities (or simply PI-algebra) if it satisfies a non-trivial polynomial identity. The development of the theory of PI-algebras started in 1950 with [2], a celebrated paper by Amitsur and Levitzki about standard polynomials on matrix algebras, that posed the following crucial question: given any PI-algebra A, are we able to describe the polynomial identities satisfied by A? In other terms, if $T(A)$ is the T-ideal of identities of A, how to compute a basis for $T(A)$?

Around the 80's, Kemer developed a sophisticated theory which gave a complete description of the verbally prime (also called T-prime) algebras in terms of their T-ideals (that is the ideals of their polynomial identities). He showed that the structure of the T-ideals in the free associative algebra resembles in many aspects that of the ideals in the usual polynomial algebra in several variables. Kemer's work is of fundamental importance in the theory of PI-algebras, see for more details [18], and the monograph [19].

In spite of the extensive research in this area, little is known about the concrete form of the identities satisfied by such algebras. In the setting of associative algebras, we can cite the monograph [9] and its references for an exhaustive discussion about algebras with polynomial identities. Kemer's results led to the solution of the long-standing Specht problem. It asks whether there exists a finite basis for the ideal of identities of an algebra in a given variety (associative, Lie, Jordan, etc.) Equivalently, the Specht problem asks whether the ideal $T(A)$ of identities of an algebra A and all T-ideals containing $T(A)$ are finitely generated as T-ideals. Kemer proved that the variety of the associative algebras, over a field of characteristic 0, satisfies the Specht property (see [19]). Later on Iltyakov [14] transferred Kemer's theorem to large classes of Lie algebras which include the finite dimensional ones, and Vajs and Zelmanov [35] obtained an analogous theorem for finitely generated Jordan algebras (under an additional restriction). Iltyakov [13] obtained the Specht property for finitely generated alternative algebras. Finally in [12], Iltyakov showed that the variety of unitary algebras generated by B_n satisfies the Specht property. We recall that the Specht property does not hold in general. The counterexamples in the case of associative algebras turned out to be much more

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intricate; such counterexamples were provided simultaneously and independently by Grishin, Belov and Shchigolev [11, 17, 31].

Now identities for Jordan algebras even less is known. For instance, Isaev in [15] obtained a basis for the T-ideal of identities of the Jordan algebra of a non-degenerate symmetric bilinear form, denoted by B_n, where $n = 2, 3, \dots, \infty$, over a finite field. Vasilovsky [36] obtained an analogous result in case of an infinite field of characteristic different from 2, 3, 5 and 7. There is an intrinsic fact related to B_∞. According to a theorem of Sverchkov [33], over a field of characteristic 0, the variety generated by B_∞ is special, meaning that each algebra in it is special. There are not many examples with this property, thus such algebra plays a major role in the variety of Special Jordan algebras. In addition to these, we also mention that in [24] the second author of this paper exhibited a basis for the T-ideal of identities of the Jordan algebra of a degenerate symmetric bilinear form of rank $n-1$ when the base field has characteristic zero. Furthermore, in [23], together with Koshlukov, a basis for the T-ideal of identities for the Jordan algebra of upper triangular matrices of order 2 was described. This last result was obtained when the field is infinite and of characteristic different from 2 and 3. Recently, Gonçalves and Salomão in [10] obtained an analogous result over a finite field. Until now, these are the only Jordan algebras in which a basis of their T-ideals of identities is known.

In the light of the above discussion, an interesting problem is to investigate other types of polynomial identities such as weak, trace, and graded ones. The trace identities for $M_n(K)$, with $\operatorname{char} K = 0$, were independently described by Procesi in [26] and by Razmyslov in [29]. More precisely, the authors proved that all trace identities of $M_n(K)$ follow from the Cayley-Hamilton polynomial of degree n. It is important to mention that both works have great importance also for ring theory. In [37] Vasilovsky exhibited a finite basis for the polynomial identities with trace of B_n when $n \geq 2$ and the field is infinite of characteristic $\neq 2$. On the other hand, also graded polynomial identities play an important role in the study of PI-algebras, in fact such identities are easier to describe in many important cases and they are related to the ordinary ones. For instance, two algebras having the same graded identities also have the same ordinary identities. For Jordan algebras, very little is as yet known about the concrete descriptions, here we can mention [10, 22, 23, 38].

Another interesting matter is the so-called embedding problem. More precisely, how to answer the question: given a ring (or an algebra) that satisfies polynomial identity, what can one say about the structure of its subrings (subalgebras)?

The matrix rings are "good" ones: they are quite well understood and their importance in Ring theory is enormous without any doubt, thus describing conditions for embedding a ring into the $n \times n$ matrices is a matter of great importance. Of course, an immediate necessary condition for embedding a ring S into the $n \times n$ matrices is that S must satisfy all polynomial identities of the $n \times n$ matrices. This condition turned out not sufficient, in fact in the 70's, Amitsur [3] and Small [32] gave independently examples of rings satisfying all the identities of $n \times n$ matrices over a field but not embeddable into matrices of order n over any commutative domain.

An exhaustive answer to the embedding problem in general is not known yet. For instance, Procesi in [28] and Berele in [4] studied such a problem in the setting of associative matrix algebras with trace and associative matrix algebras with involution-trace, respectively. Moreover, in [8] the authors extend the previous results, studying the embedding problem for the Jordan algebra B_n with trace.

This paper deals with the Jordan algebra $J_{n,k}$ which arises from a degenerate symmetric bilinear form b with rank $n-k$ over a field K, where n and k are positive integers such that $n > k$. If we denote by V_n an n-dimensional vector space equipped with a symmetric bilinear form b, then it is well known that we can choose a canonical basis for V such that $V_n = V_{n-k} \oplus D_k$, where the restriction of b on the $(n-k)$-dimensional subspace V_{n-k} is non-degenerate and D_k is its degenerate part of dimension k. Under this notation, it is easy to verify that $J_{n,k} = K \oplus V_n$ with the multiplication $(\alpha + u)(\beta + v) = (\alpha\beta + b(u, v)) + (\alpha v + \beta u)$ is a Jordan algebra. This algebra is called *the Jordan algebra of the bilinear form b*. It can be interesting to consider the decomposition $J_{n,k} = B_{n-k} \oplus D_k$, where B_{n-k} is the corresponding Jordan algebra $B_{n-k} = K \oplus V_{n-k'}$. Notice that one should formally write $J_{n,k}(b)$ for $J_{n,k}$ since this algebra obviously depends on the form b. In fact if b and b' are two symmetric bilinear forms on V_n, the algebras $J_{n,k}(b)$ and $J_{n,k}(b')$ are isomorphic if and only if the forms b and b' have the same rank, in other words, they are equivalent. When the base field K is algebraically closed, there exists, up to isomorphism, only one algebra $J_{n,k}$ that depends on its rank k. However, over an arbitrary field, $J_{n,k}$ should interpreted as the class of Jordan algebras, which are not necessarily isomorphic, having bilinear form of the same rank $n-k$. Moreover, it is clear that

$T(J_{n,k}) \subseteq T(B_{n-k})$, since B_{n-k} is a subalgebra of $J_{n,k}$. Analogously, it turns out that, under some technical conditions, B_n is a graded and trace subalgebra of $J_{n,k}$, therefore, it is possible and interesting to compute a basis for the T-ideal of identities of such an algebra in each one of the aforementioned cases.

The paper is organized as follows. In Section 2, we provided a basis for the T-ideal of ordinary polynomial identities of $J_{n,k}$, assuming that a basis for the T-ideal of B_{n-k} is known. Section 3 is devoted to the study of the $\mathbb{Z}_2$-graded polynomial identities of $J_{n,k}$ endowed with a suitable $\mathbb{Z}_2$-grading called the scalar grading. In order to obtain the description of the identities of the latter case, we exhibit a basis for the T-ideal of the weak identities for the pair $(J_{n,k}, V_n)$. Recall that V_n was defined in the previous paragraph. In Section 4, a basis for the T-ideal of trace identities of $J_{n,k}$ will be computed. In particular, we prove that the algebra $J_{n,k}$ and B_{n-k} are PI-equivalent, as ordinary, trace and graded algebra when n is infinite, but the same conclusion does not hold for n finite. Moreover, we provide a counterexample to the embedding problem in the sense of [8, Theorem 4.7] in the case of infinite dimensional Jordan algebras with trace. Finally, in Section 5 we list some conjectures that may lead to possible generalizations of the results presented here.

2. Ordinary polynomial identities

From now on, unless specified otherwise, K will denote a field of characteristic different from two. All algebras will be unitary and considered over K. Let A be an associative algebra, then one can always construct a Jordan algebra considering $A^{(+)}$ as the vector space A equipped with the Jordan product $a \circ b = (ab+ba)/2$, for all a and b in A. Conversely, given a Jordan algebra J, the existence of an associative algebra A such that $J \subseteq A^{(+)}$ is not guaranteed. Jordan algebras sharing this property are called special and the associative algebra A is called associative envelope for J. Otherwise, they are called exceptional.

Unless otherwise stated, we shall assume that X is an infinite countable set, and we denote by $K(X)$ the free nonassociative algebra freely generated by X over K. The elements of $K(X)$ are called polynomials and a polynomial f is called polynomial identity for A (written as $f \equiv 0$), if $f(a_1, \ldots, a_m) = 0$ for all a_1, $\ldots$, $a_m \in A$. Let S be a subset of $K(X)$, the set of all algebras satisfying the polynomials in S is called the variety generated by S and we denote it by $\mathcal{V}(S)$. We say that $A \in \mathcal{V}(S)$, if S is contained in the set of all identities for A, denoted by $T(A)$. Moreover, we say that a variety $\mathcal{V}$ is generated by the algebra A if $\mathcal{V}$ is the variety generated by $T(A)$, and we denote $\mathcal{V}$ by $\mathcal{V}(A)$. We also denote by $K\langle X\rangle$ and $J(X)$ the free associative and Jordan algebra freely generated by X over K, respectively. Moreover, it is possible to construct $SJ(X)$, the free special Jordan algebra, as the subalgebra of $K\langle X\rangle^{(+)}$ generated by the set X by means of the definition of the Jordan product as previously stated.

Now, let J be a Jordan algebra, the set $T(J) = \{f \in J(X) \mid f \equiv 0\}$ is an ideal of $J(X)$ that is closed under the endomorphisms of $J(X)$; such ideals are called T-ideals. A set of identities $\{g_1, g_2, \ldots\}$ is a basis of the T-ideal I if $\{g_1, g_2, \ldots\}$ generates I as a T-ideal and in this case we write $I = \langle g_1, g_2, \ldots\rangle_T$. It is well-known (see for instance [9, Theorem 1.3.7]) that, in case of characteristic zero, every T-ideal is generated by the multilinear polynomials it contains. Over an infinite field of positive characteristic, one has to take into account the multihomogeneous polynomials instead of the multilinear ones. Recall that a multilinear polynomial is a polynomial of the vector subspace

$$P_n = \mathrm{span}_K\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\},$$

where S_n is the symmetric group and $x_{\sigma(1)} \cdots x_{\sigma(n)}$ stands for a monomial with all possible brackets arrangements. It turns out that if $f \in P_n$, then in order to establish whether $f \in T(J)$, it suffices to evaluate f on the elements of a basis of J.

A polynomial identity f is a consequence of the identity g (or follows from g) if $f \in \langle g\rangle_T$. Similarly, we say that f and g are equivalent if each one is a consequence of the other, i.e. $\langle f\rangle_T = \langle g\rangle_T$. More generally, if A and B are two algebras we say that A and B are PI-equivalent if $T(A) = T(B)$.

An important role in the theory is played by the so-called Capelli-type polynomials defined as follows.

Definition 2.1. *A polynomial $f \in J(X)$ is a polynomial of Capelli-type of order m if f is multilinear and alternating in m variables. Moreover, we say that the algebra J satisfies the Capelli-type identities of order m, and we write $Cap_m \equiv 0$, if J satisfies all polynomials of Capelli-type of order m.*

It is clear that if $\dim_K J = k$, then $Cap_t \equiv 0$ on J for all $t \geq k+1$. Furthermore, Capelli-type polynomials represent an important tool in order to establish an equivalence among two T-ideals, as highlighted in the following lemma that we will use very often.

Lemma 2.2 ([6], [12]). *Let K be a field of characteristic zero. Moreover, let I and Q be T-ideals of $J(X)$ and let $I^{(m)} = I \cap J(x_1, \ldots, x_m)$ and $Q^{(m)} = Q \cap J(x_1, \ldots, x_m)$, for all $m \geq 1$. Then I and Q are equal modulo Cap_{k+1} if and only if $I^{(k)} = Q^{(k)}$.*

In what follows we are interested in the polynomial identities of a special Jordan algebra called the Jordan algebra of a symmetric bilinear form. Let V be a vector space, $\dim_K V = n \geq 2$, equipped with a symmetric bilinear form b and let $A = K \oplus V$. It is well known that A, with the multiplication $(\alpha + u) \circ (\beta + v) = (\alpha\beta + b(u,v)) + (\alpha v + \beta u)$, for all $\alpha, \beta \in K$ and for all $v, u \in V$, is a Jordan algebra with respect to the multiplication $\circ$. Thus, if b is non-degenerate, we denote it by B_n if $\dim_K V = n$ and by B_∞ if $\dim_K V = \infty$.

Assuming that V is an n-dimensional vector space, if b is degenerate with rank $n-k$, for some integer k satisfying $0 < k < n$, then A, with respect to the multiplication $\circ$, will be denoted by $J_{n,k}$. Here n and k can be infinite. We can also write $J_{n,k} = B_{n-k} \oplus D_k$, where $B_{n-k} = K \oplus V'$, V' is the subspace of V spanned by the non-degenerate vectors with respect to b and D_k is the degenerate part of V. Thus, it is clear that B_{n-k} is a subalgebra of $J_{n,k}$, for all $k \geq 0$. It should be noted that both B_m and $J_{n,k}$ are special algebras, and their associative envelopes are the Clifford algebras. Additionally, B_m is simple whereas $J_{n,k}$ is not. It is evident that these algebras depend on the form b.

A basis of $T(B_m)$ and $T(B_\infty)$ were found in [36], where the following theorems were proved.

Theorem 2.3. [36, Theorem 0.2] *The identities*

$$([x,y]^2, z, t) \equiv 0 \tag{1}$$

$$\sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)}, (x_{\sigma(2)}, x, x_{\sigma(3)}), x) \equiv 0 \tag{2}$$

form a basis for polynomial identities of the $var(B_\infty)$ of Jordan algebras over an infinite field of characteristic different from 2,3,5, 7.

Theorem 2.4. [36, Theorem 0.2] *The identities (1), (2),*

$$\sum_{\sigma \in S_{m+1}} (-1)^\sigma (x_{\sigma(1)}, y_1, x_{\sigma(2)}, \ldots, y_m, x_{\sigma(m+1)}) \equiv 0 \tag{3}$$

$$\sum_{\sigma \in S_{m+1}} (-1)^\sigma (x_{\sigma(1)}, y_1, x_{\sigma(2)}, \ldots, y_{m-1}, x_{\sigma(m)})(y_m, x_{\sigma(m+1)}, y_{m+1}) \equiv 0 \quad , \tag{4}$$

form a basis for polynomial identities of the $var B_m$, $m < \infty$, of Jordan algebras over an infinite field of characteristic different from 2,3,5, 7.

Here and in what follows $[x,y] = xy - yx$ is the commutator among the variables x and y, and $(x,y,z) = (xy)z - x(yz)$ stands for the associator among the variables x, y and z. One can also define by induction the associator among more than three elements of the algebra by left-normalizing the brackets. Thus $(x_1, x_2, x_3, x_4, x_5) = ((x_1, x_2, x_3), x_4, x_5)$ and so on.

It is well known that $SJ(X) \cong J(X)$ if and only if $|X| \leq 2$; see for example [16, p. 47] or [39, Theorem 3, p. 59]. Therefore we are identifying $[x,y]^2$ with the corresponding Jordan polynomial written by means of the multiplication in the associative envelope of $J(x,y) = SJ(x,y)$. The linearization and the translation by using Jordan multiplication of $[x,y]^2$ were computed. In fact, if we set $T(x_1,x_2,x_3,x_4) = (x_1x_2, x_3, x_4) - x_1(x_2,x_3,x_4) - x_2(x_1,x_3,x_4)$ in $J(X)$, then it is well know that

$$T(x_1,x_2,x_3,x_4) = \frac{1}{4}([x_1,x_3] \circ [x_4,x_2] + [x_1,x_4] \circ [x_3,x_2]).$$

Moreover, if K is an infinite field of characteristic different from 2 and 3, then by [23, Section 4], the right-hand side is readily seen to be, up to a scalar, the multilinearization of the polynomial $[x,y]^2$, therefore we can substitute the identity (1) by

$$(T(x_1,x_2,x_3,x_4), z, t), \tag{5}$$

which is multilinear, see also [36].

In the next subsection, a basis of $T(J_{\infty,k})$ and $T(J_{n,k})$, for all $k \geq 1$, will be computed. To this end, first remark that we may assume K algebraically closed eventually by considering $\bar{K}$, the algebraic closure of K, and observing that polynomials (1) – (4) have coefficients in K (see also [9, Section 1.4]). Therefore, if $J_{n,k} = K \oplus V$, then we can fix an orthonormal basis of V with respect to the bilinear form b, namely $\mathcal{B} = \mathcal{N} \cup \mathcal{D}$ where $\mathcal{N} = \{e_1, e_2, \ldots, e_{n-k}\}$ spans the non-degenerate subspace of V and $\mathcal{D} = \{d_1, d_2, \ldots, d_k\}$ spans the degenerate part. Hence,

$$e_i \circ e_j = \delta_{i,j}, \quad d_r \circ e_l = d_r \circ d_t = 0,$$

for all $1 \leq i, j, l \leq n-k$ and for all $1 \leq r, t \leq k$, where $\delta_{i,j}$ is the Kronacker's delta. The above notation also implies that B_{n-k} is spanned by $\{1\} \cup \mathcal{N}$ and D_k is spanned by $\mathcal{D}$. Notice that the previous arguments can also be easily generalized for the case $J_{\infty,k}$. In order to simplify the notation, in the Jordan product we shall replace $\circ$ with $\cdot$ or we shall omit it.

2.1. **Computing the T-ideal of identities.** In order to reach the goal, we first state the following proposition that relates the multilinear identities of $J_{n,k}$ to the ones of B_{n-k}.

Proposition 2.5. *Let $J_{n,k} = B_{n-k} \oplus D_k$ be the decomposition of $J_{n,k}$ into the direct sum of the non-degenerate part of the algebra plus the degenerate part where $k \geq 1$. A multilinear polynomial $f \in J(X)$ is a polynomial identity of $J_{n,k}$ if and only if $f \in T(B_{n-k})$ and*

$$f_{x_i} = f(n_1, \ldots, n_{i-1}, m_i, n_{i+1}, \ldots, n_{degf}) = 0, \tag{6}$$

for all $1 \leq i \leq \deg f$ and for all $n_j \in B_{n-k}$, $j \neq i$, $m_i \in D_k$.

Proof. If $f \in T(J_{n,k})$ then $f \in T(B_{n-k})$ since B_{n-k} is a subalgebra of $J_{n,k}$. Furthermore, condition (6) trivially holds and we are done.

Conversely, let us suppose that $f = f(x_1, \ldots, x_r) \in T(B_{n-k})$ is a multilinear polynomial satisfying (6). We claim that $f \in T(J_{n,k})$. Since f is multilinear, one can evaluate every variable x_i on elements of the basis $\mathcal{B} \cup \{1\}$. If we substitute each variable with an element of $\mathcal{N} \cup \{1\}$, then $f = 0$, since $f \in T(B_{n-k})$. Thus we may evaluate at least one variable on $\mathcal{D}$. Moreover, since D_k is an ideal of $J_{n,k}$ and $D_k^2 = 0$, we have that if we evaluate at least two distinct variables of f in $\mathcal{D}$, then such evaluation is automatically zero. Therefore, it suffices to consider only the evaluations in which exactly one variable x_i is mapped into $\mathcal{D}$, i.e. $x_i \mapsto m \in \mathcal{D}$ and $x_j \mapsto n_j \in \mathcal{N} \cup \{1\}$ for all $j \neq i$ and for all $1 \leq i \leq \deg f$. Since condition (6) holds, also in this case we get zero, hence $f \in T(J_{n,k})$ as claimed. □

We shall use the previous result in order to prove that polynomials (1), (2) and (4) are still identities of $J_{n,k}$, for all $k \geq 1$. First, we need the following technical lemma.

Lemma 2.6. *If $a_1, a_2, a_3 \in \mathcal{N} \cup \{1\}$, and $d \in \mathcal{D}$, then*

(1) $(a_1, d, a_2) = 0$*;*
(2) $d \cdot (a_1, a_2, a_3) = 0$*.*

Proof. If $a_i = 1$ for some i, then the result is trivial, thus let us suppose that $a_1, a_2, a_3 \in \mathcal{N}$. Since $da_i = 0$ for all $1 \leq i \leq 3$, then $(a_1, d, a_2) = (a_1 d)a_2 - a_1(da_2) = 0$ and the first item is proved. Moreover, if $(a_1, a_2, a_3) \in \operatorname{span}_K\{\mathcal{N}\}$ then also $d \cdot (a_1, a_2, a_3) = 0$. The latter claim shows the last item and we can conclude the proof. □

Lemma 2.7. *If K is an infinite field of characteristic different from 2 and 3, then the polynomial (1) is still an identity of $J_{n,k}$.*

Proof. Since $\operatorname{char} K \neq 2$ and 3, by the previous arguments, it suffices to prove that the polynomial (5) is an identity of $J_{n,k}$. To this end, let

$$f = ((x_1 x_2, x_3, x_4) - x_1(x_2, x_3, x_4) - x_2(x_1, x_3, x_4), x_5, x_6).$$

Moreover, by Theorem 2.3 we have that $f \in T(B_{n-k})$, thus in order to reach the goal we only need to check condition (6) of Proposition 2.5.

By Lemma 2.6, if either x_3 or x_5 takes value in D_k, then we get automatically zero, i.e. $f_3 = f_5 = 0$. Now let $d \in D_k$ and $a_i \in \mathcal{N}$, for all $1 \leq i \leq 6$. Let us analyse f_{x_1} :

$$f_{x_1} = ((da_2, a_3, a_4) - d(a_2, a_3, a_4) - a_2(d, a_3, a_4), a_5, a_6).$$

As $da_i = 0$ for all $a_i \in \mathcal{N}$, then each summand of the first element of this associator is equal to zero, i.e., $f_{x_1} = 0$. Since f is symmetric with respect to the variables x_1 and x_2, we get also $f_{x_2} = 0$.

Let now substitute x_4 by d. We get:

$$f_{x_4} = ((a_1a_2, a_3, d) - a_1(a_2, a_3, d) - a_2(a_1, a_3, d), a_5, a_6).$$

It can be easily noted that the first associator is equal to zero and $(a, b, d) \in \operatorname{span}\{d\}$, for all a, $b \in J_{n,k}$. Hence $f_{x_4} = 0$. Finally,

$$\begin{aligned} f_{x_6} &= ((a_1a_2, a_3, a_4) - a_1(a_2, a_3, a_4) - a_2(a_1, a_3, a_4), a_5, d) \\ &= (((a_1a_2, a_3, a_4) - a_1(a_2, a_3, a_4) - a_2(a_1, a_3, a_4))a_5)d. \end{aligned}$$

But $(a_1a_2, a_3, a_4) - a_1(a_2, a_3, a_4) - a_2(a_1, a_3, a_4)$ has degree 4 and all its variables evaluate in $\mathcal{N}$; then its evaluation is a scalar $\alpha \in K$. This implies that $f_{x_6} = \alpha a_5 d = 0$ and we are done. ☐

Lemma 2.8. *If K is an infinite field of characteristic different from 2, then the polynomial (2) is still an identity for $J_{n,k}$.*

Proof. Since $\operatorname{char} K \neq 2$, by multilinearizing (2), it is sufficient to deal with

$$g = \sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)}, (x_{\sigma(2)}, x, x_{\sigma(3)}), y) + \sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)}, (x_{\sigma(2)}, y, x_{\sigma(3)}), x).$$

By Theorem 2.3, $g \in T(B_{n-k})$, thus it suffices to prove condition (6).

As in the previous lemma, we can consider only evaluations in $\mathcal{N} \cup \mathcal{D}$. For all $d \in D_k$ and for all $a_i \in \mathcal{N}$, by Lemma 2.6 we have that

$$g_y = \sum_{\sigma \in S_3} (-1)^\sigma (a_{\sigma(1)}, (a_{\sigma(2)}, a, a_{\sigma(3)}), d) = \sum_{\sigma \in S_3} (-1)^\sigma (a_{\sigma(1)}(a_{\sigma(2)}, a, a_{\sigma(3)}))d.$$

If one expand the associator, as it was done in [24, Lemma 4], it turns out that $g_y = 0$. Furthermore, the polynomial g is symmetric with respect to the variables y and x thus $g_x = 0$.

Moreover, since g is symmetric with respect to the variables x_1, x_2 and x_3, we shall compute f_{x_1} only. To this end, let us consider the evaluation φ such that $\varphi(x_1) = d \in \mathcal{D}$, $\varphi(x_i) = a_i$, $\varphi(x) = a$ and $\varphi(y) = b$, where $a_i, a, b \in \mathcal{N}$, $i = 1, 2$. By Lemma 2.6 and recalling that the equality $(a, b, c) = -(c, b, a)$ hold in every Jordan algebra, we have

$$\begin{aligned} g_{x_1} &= (d, (a_2, a, a_3), b) - (d, (a_3, a, a_2), b) + (d, (a_2, b, a_3), a) - (d, (a_3, b, a_2), a) \\ &= 2[(d, (a_2, a, a_3), b) + (d, (a_2, b, a_3), a)] \\ &= -2d[(a_2, a, a_3)b + (a_2, b, a_3)a] \\ &= -2d[(a_2a)(a_3b) - (aa_3)(a_2b) + (a_2b)(a_3a) - (a_3b)(a_2a)] = 0. \end{aligned}$$

Hence Proposition 2.5 applies the polynomial (2) is an identity of $J_{n,k}$. ☐

By putting together the previous lemmas, we easily obtain the following result.

Theorem 2.9. *Let K be an infinite field of characteristic different from 2, 3, 5 and 7, the Jordan algebras $J_{\infty,k}$ and B_∞ are PI-equivalent.*

Let us now focus our attention on the case n finite.

Lemma 2.10. *If $n < \infty$ then the polynomial (4), for $m = n - k$, is still an identity for $J_{n,k}$.*

Proof. The proof follows immediately from the second equality of Proposition 2.6, since the product of two associators where one of them is in D_k is zero. ☐

Things seem to be different in the case of polynomial (3), in fact it can be proved that it is not a polynomial identity for $J_{n,k}$. The following simple example shows a particular case in which one can find a non-zero evaluation.

Example 2.11. *Let $n = 2$ and $k = 1$. The polynomial* (3) *is equal to*

$$h = 2((x_3, y_1, x_1, y_2, x_2) + (x_2, y_1, x_3, y_2, x_1) + (x_1, y_1, x_2, y_2, x_3)).$$

We recall that the basis $\mathcal{B} \cup \{1\}$ of $J_{2,1}$ is $\{1, e_1, e_2, d\}$. If we consider the evaluation $\varphi(x_1) = \varphi(y_1) = e_1$, $\varphi(x_2) = \varphi(y_2) = e_2$ and $\varphi(x_3) = d$ we have

$$\varphi(h) = 2((d, e_1, e_1, e_2, e_2) + (e_1, e_1, e_2, e_2, d)) = 4d \neq 0.$$

This implies that h does not lie in $T(J_{2,1})$.

In order to provide a basis of $T(J_{n,k})$, as a T-ideal, we shall define a linear operator firstly introduced by Iltyakov in [12], that later on Vasilovsky used as a means to compute the T-ideal of identities of B_m (see [36, Section 1]). To this end, let $\mathcal{V}_0$ be the minimal subset of $K(X)$ such that $(a, b, c) \in \mathcal{V}_0$, for arbitrary a, b, $c \in \mathcal{V}_0' = \mathcal{V}_0 \cup X$. Set $\mathcal{V}_1 = \{uv \mid u, v \in \mathcal{V}_0\}$. It is well known that $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1$ is a linearly independent set of $K(X)$, see [12, Propostion 1].

Definition 2.12. *Let $K\mathcal{V} = span\{\mathcal{V}\}$. For all a, $b \in \mathcal{V}_0'$, we define the linear operator $L(a, b)$ on $K\mathcal{V}$ as follows:*

(a) *if $u = (u_1, u_2, u_3) \in \mathcal{V}_0$, then*

$$\begin{aligned} uL(a,b) &= (1/2)\{(u_3, a, u_1, b, u_2) + (u_3, b, u_1, a, u_2) + (u_2, a, b, u_3, u_1) \\ &+ (u_2, b, a, u_3, u_1) - (u_2, a, b, u_1, u_3) - (u_2, b, a, u_1, u_3)\}; \end{aligned}$$

(b) *$(uv)L(a, b) = (uL(a, b)) = (uL(a, b))v$, where $u, v \in V_0$;*

(c) *the operator $L(a, b)$ is extended to $K\mathcal{V}$ by linearity.*

Notice that the operator $L(a, b)$ is symmetric with respect to a and b.

For all $f \in K(X)$, we set f^+ as the image of the polynomial f under the natural homomorphism of $K(X)$ onto $SJ(X)$, i.e. f^+ is the associative polynomial which is obtained from f by means of substituting the Jordan product $x \circ y$. According to the above notation, it is easy to check that the equality

$$\{(x, y, z)L(a, b)\}^+ = (x, y, z)^+ \circ (a \circ b) \tag{7}$$

holds in $SJ(X)$.

In [36], there are a lot of interesting tools and informations about such operator. Among all, Vasilovsky proved that one can reduce the study of identities of the algebra B_m to that of the identities lying in a suitable quotient space. More precisely, the author exhibited such a space in the next lemma.

Lemma 2.13. [36, Lemma 1.6.] *The T-ideal of identities of B_m is generated by the identities lying in*

$$\begin{aligned} \overline{K\mathcal{V}} &= span\{\ (x,y,z)L(a_0,b_0)L(a_1,b_1)\cdots L(a_k,b_k);\ (x_1,y_1,z_1)(x_2,y_2,z_2)L(a_0,b_0)L(a_1,b_1)\cdots \\ &\quad \cdots L(a_k,b_k) \mid x,y,z,x_i,y_i,z_i,a_i,b_i \in X;\ i \geq 1, k \geq 0\}. \end{aligned}$$

The following identities are satisfied in the Jordan algebra B_∞ (see [36, p. 148-149]):

$$(x, y, z)L(a, b)L(c, d) \equiv (x, y, z)L(c, d)L(a, b); \tag{8}$$

$$(x, y, z, s, t) \equiv (z, s, t)L(x, y) - (x, s, t)L(y, z). \tag{9}$$

It is clear that these identities hold also on $J_{n,k}$, since $T(B_\infty) \subseteq T(J_{n,k})$.

In [36] the author characterizes the polynomial identities, modulo $T(B_\infty)$, using the invariants of the orthogonal group as described by De Concini and Procesi in [27]. Such a characterization was given in terms of double tableaux. In what follows, we shall introduce such a description for convenience of the reader.

A double tableau is an array

$$T = \left(\begin{array}{cccc|cccc} p_{11} & p_{12} & \cdots & p_{1m_1} & q_{11} & q_{12} & \cdots & q_{1m_1} \\ p_{21} & p_{22} & \cdots & p_{2m_2} & q_{21} & q_{22} & \cdots & q_{2m_2} \\ \vdots & & & & \vdots & & & \\ p_{r1} & p_{r2} & \cdots & p_{rm_r} & q_{r1} & q_{r2} & \cdots & q_{rm_r} \end{array}\right) \tag{10}$$

where $m_1 \geq m_2 \geq \ldots \geq m_k$ and the p_{ij} and q_{ij} are positive integers. Moreover, T is called a double standard tableau if $p_{i1} < p_{i2} < \ldots < p_{im_i}$, $q_{i1} < q_{i2} < \ldots < q_{im_i}$, $p_{ij} \leq q_{ij}$ and $q_{ij} \leq p_{i+1,j}$. Hence, a double tableau is standard, if we get an ordinary standard tableau (in the sense of [9, Definition 2.2.5]) by inserting each row of q_{ij} just below its counterpart p_{ij}.

Set $\mu[T] = (m_1, m_2, \dots, m_r)$ and $h[T] = r$. We denote by $T^{(i)}$ the i-th row of the tableau T:

$$T^{(i)} = (p_{i1} \dots p_{im_i} \mid q_{i1} \dots q_{im_i}).$$

We shall call the double tableaux of the type (10) a 0-tableaux if $p_{11} = 0$ and all remaining entries of T are positive integers. For all $r > 0$, we shall associate to each double tableau-row $T = (p_1 \dots p_r \mid q_1 \dots q_r)$ a linear operator on $K\mathcal{V}$ given by

$$l[T] = \sum_{\sigma \in S_r} (-1)^\sigma L(x_{p_1}, x_{q_{\sigma(1)}}) \cdots L(x_{p_r}, x_{q_{\sigma(r)}}).$$

Notice that $L(x_p, x_q) = l(p \mid q)$. If $T_0 = (0\ p_2 \dots p_r \mid q_1 \dots q_r)$ is a double 0-tableau, then we set

$$F[T_0] = \frac{1}{2} \sum_{\sigma \in S_r} (-1)^\sigma (x_{q_{\sigma(1)}}, x_{p_1}, x_{q_{\sigma(2)}}) L(x_{p_3}, x_{q_{\sigma(3)}}) \cdots L(x_{p_r}, x_{q_{\sigma(r)}}). \tag{11}$$

If T is an arbitrary double 0-tableau with $\mu[T^{(1)}] \geq 2$, then

$$F[T] = F[T^{(1)}] l[T^{(2)}] \cdots l[T^{(h[T])}].$$

Finally, for $r \geq 3$ we set

$$F[(p_1 \dots p_r \mid q_1 \dots q_r)] = \frac{1}{4} \sum_{\sigma \in S_r} (-1)^\sigma (x_{p_1}, x_{q_{\sigma(1)}}, x_{p_2})(x_{q_{\sigma(3)}}, x_{p_3}, x_{q_{\sigma(2)}}) L(x_{p_4}, x_{q_{\sigma(4)}}) \cdots L(x_{p_r}, x_{q_{\sigma(r)}}). \tag{12}$$

and if $\mu[T^{(1)}] \geq 3$ we define $F[T]$ as before. For T with $\mu[T^{(1)}] = 2$ and $h[T] \leq 2$ we put

$$F[T] = F \left[\begin{array}{c} T^{(1)} \\ T^{(2)} \end{array} \right] l[T^{(3)}] \cdots l[T^{(h(T))}].$$

In [36, p. 163] it is proved that the polynomial (3) is equal to

$$2^{m-1} \sum_{\sigma \in S_{m+1}} (-1)^\sigma (x_{\sigma(1)}, y_1, x_{\sigma(2)}) L(x_{\sigma(3)}, y_2) \cdots L(x_{\sigma(m+1)}, y_m) = 2^m F[(0\ 1 \dots m \mid 1\ 2 \dots m+1)], \tag{13}$$

and (4) is

$$\begin{aligned} & 2^{m-2} \sum_{\sigma \in S_{m+1}} (-1)^\sigma (x_{\sigma(1)}, y_1, x_{\sigma(2)})(y_2, x_{\sigma(3)}, y_3) L(x_{\sigma(4)}, y_4) \cdots L(x_{\sigma(m+1)}, y_{m+1}) \\ = \ & 2^m F[(1\ 2 \dots m+1 \mid 1\ 2 \dots m+1)]. \end{aligned} \tag{14}$$

Moreover, the following proposition holds.

Proposition 2.14. [36, Proposition 2.3] *Every polynomial in $\overline{K\mathcal{V}}$ can be represented as a linear combination of polynomials $\{F[T]\}$, where the T's are double standard 0-tableaux.*

Since the previous arguments hold modulo $T(B_\infty)$, they are also valid modulo $T(J_{n,k})$. We are now in a position to prove that the polynomial (3) does not belong to $T(J_{n,k})$.

Lemma 2.15. *The polynomial in (3), for $m = n - k$, is not an identity for $J_{n,k}$.*

Proof. Write the polynomial (3) as

$$f = 2^{n-k-1} \sum_{\sigma \in S_{n-k+1}} (-1)^\sigma (x_{\sigma(n-k)}, y_{n-k}, x_{\sigma(n-k+1)}) L(x_{\sigma(n-k-1)}, y_{n-k-1}) \cdots L(x_{\sigma(1)}, y_1).$$

By Proposition 2.5, it is enough to prove that f_{x_i} is non-zero for some i.

First remark that if $a = d \in \mathcal{D}$ (as well as $b = d$) then, by definition of L, we have

$$\begin{aligned} (\bar{x}_1, \bar{y}_1, \bar{x}_2) L(d, \bar{b}) &= (1/2)\{(\bar{x}_2, d, \bar{x}_1, \bar{b}, \bar{y}_1) + (\bar{x}_2, \bar{b}, \bar{x}_1, d, \bar{y}_1) + (\bar{y}_1, d, \bar{b}, \bar{x}_2, \bar{x}_1) \\ &+ (\bar{y}_1, \bar{b}, d, \bar{x}_2, \bar{x}_1) - (\bar{y}_1, d, \bar{b}, \bar{x}_1, \bar{x}_2) - (\bar{y}_1, \bar{b}, d, \bar{x}_1, \bar{x}_2)\} \\ &= (1/2)\{(\bar{y}_1, \bar{b}, d, \bar{x}_2, \bar{x}_1) - (\bar{y}_1, \bar{b}, d, \bar{x}_1, \bar{x}_2)\} \\ &= 0. \end{aligned} \tag{15}$$

Here we denote by $\overline{x}_i$ the evaluation of the variable x_i in an element of the algebra. The latter one plus identity (8) and the first equality of Lemma 2.6, imply that $f_{y_i} = 0$, for all i. Now, since f is symmetric with respect to the variables x_i's, we compute f_{x_1} only, i.e., $\bar{x}_1 = d$. By the identity (9) we have

$$(x_1, y_1, x_2)L(a, b) \equiv (a, b, x_1, y_1, x_2) + (a, y_1, x_2)L(b, x_1) \pmod{T(J_{n,k})}$$

and so

$$(d, \bar{y}_1, \bar{x}_2)L(\bar{a}, \bar{b}) = (\bar{a}, \bar{b}, d, \bar{y}_1, \bar{x}_2).$$

Remark that if $\bar{y}_1 \neq \bar{x}_2$ then $(d, \bar{y}_1, \bar{x}_2) = 0$, and consequently $(d, \bar{y}_1, \bar{x}_2)L(\bar{a}, \bar{b}) = 0$. Moreover, if $\bar{a} \neq \bar{b}$ then $(d, \bar{y}_1, \bar{x}_2)L(\bar{a}, \bar{b}) = 0$. Hence, we may assume $\overline{x}_1 = d$, $\overline{y}_1 = \overline{x}_2$, and for each pair (a, b) inside of the operator L, $\overline{a} = \overline{b}$. Here $\overline{a}$ and $\overline{x}_2$ are elements in $\mathcal{N}$.

By taking into account (8) and (15), we have that the only non-zero monomials in f_{x_1} with the evaluation $\overline{x}_1 = d$ and $\overline{x}_{i+1} = \overline{y}_i = e_i$, for all $1 \leq i \leq n-k$, are

$$(d, e_{n-k}, e_{n-k})L(e_{n-k-1}, e_{n-k-1}) \cdots L(e_1, e_1)$$

and

$$(e_{n-k}, e_{n-k}, d)L(e_{n-k-1}, e_{n-k-1}) \cdots L(e_1, e_1).$$

Thus

$$f_{x_1} = 2^{n-k-1}\{(d, e_{n-k}, e_{n-k})L(e_{n-k-1}, e_{n-k-1}) \cdots L(e_1, e_1) - (e_{n-k}, e_{n-k}, d)L(e_{n-k-1}, e_{n-k-1}) \cdots L(e_1, e_1)\} = 2^{n-k}d \neq 0,$$

and we are done. $\square$

The connection between polynomial identities of B_m and double standard tableaux is based on [27, Section 5] as follows. Let V be the "generic" vector space with a basis consisting of the vectors $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$. Define an inner product by $x_i \circ x_j = \sum_k x_{ik}x_{j1}$. In [27], Procesi and De Concini associate the double tableau (10) to the polynomial f_T given by

$$f_T = f(T) = f_{T^{(1)}} f_{T^{(2)}} \cdots f_{T^{(r)}}$$

where

$$f_{T^{(i)}} = \sum_{\sigma \in S_{m_i}} (-1)^\sigma (x_{p_{i1}} \circ x_{q_{i\sigma(1)}}) \cdots (x_{p_{im_i}} \circ x_{q_{i\sigma(m_i)}}) \tag{16}$$

for $T^{(i)} = (p_{i1} \ldots p_{im_i} \mid q_{i1} \ldots q_{im_i})$. Moreover, the authors proved that the K-algebra A of invariants of the orthogonal group is generated by the products $x_i \circ x_j$, i.e., $A = K[x_i \circ x_j]$. More precisely, the following theorem holds.

Theorem 2.16. [27, Theorem 5.1] *The polynomials $\{f_T\}$, where T runs over all double standard tableaux of positive integers such that $m_1 \leq m$, form a basis of A over K.*

In order to simplify the exposition, we now give the definition of weak identity of B_n. Recall that we shall extensively study such kind of identities in the next section.

Definition 2.17. *An associative polynomial $f = f(x_1, \ldots, x_n)$ will be called a weak identity if, for all $v_1, \ldots, v_n \in V_m$, in an associative envelope algebra of $B_n = K \oplus V_m$, one has $f(v_1, \ldots, v_n) = 0$.*

Remark 2.18. *According to Theorem 2.16, if $\sum_i \alpha_i f_{T_i}$, where $\{T_i\}$'s are double standard tableaux with $\mu[T_i^{(1)}] \leq n$, is a weak identity of the algebra B_m, with $m < \infty$, then $\alpha_i = 0$, for all i.*

Lemma 2.19. [36, Lemma 1.10] *For all r satisfying $1 \leq r \leq m$, the following weak identity holds:*

$$\{(x, y, z)l[p_1 \ldots p_k \mid q_1 \cdots q_r]\}^+ = (x, y, z)^+ \circ f([p_1 \ldots p_r \mid q_1 \cdots q_r]).$$

From now until the end of the section, let char$K = 0$ and let I be the T-ideal generated by the polynomials (1), (2) and (4). By Lemmas 2.7, 2.8 and 2.10, we have $I \subseteq T(J_{n,k})$. Furthermore, it is clear that the identity (3) is Capelli-type of order $n - k + 1$. Thus, we have

$$I = T(B_{n-k}), \tag{17}$$

modulo the T-ideal generated by Cap_{n-k+1}. Then from Lemma 2.2 it follows that $I^{(n-k)} = T(B_{n-k})^{(n-k)}$.

Proposition 2.20. *Let K be a field of characteristic zero, we then have*

$$T(J_{n,k})^{(n-k)} = T(B_n)^{(n-k)}.$$

Proof. As B_{n-k} is a subalgebra of $J_{n,k}$, we have that $T(J_{n,k}) \subseteq T(B_{n-k})$. Moreover, from (17) we have

$$T(B_{n-k}) = I \subseteq T(J_{n,k}) \subseteq T(B_{n-k}),$$

modulo the T-ideal generated by Cap_{n-k+1}. Hence, $T(B_{n-k}) = T(J_{n,k})$, modulo the T-ideal generated by Cap_{n-k+1}. From Lemma 2.2, one gets $T(J_{n,k})^{(n-k)} = T(B_{n-k})^{(n-k)}$ and we are done. □

The previous proposition implies that in order to determine $T(J_{n,k})$, it is enough to verify which Capelli-type polynomial of order $n-k+1$ which is an identity of B_{n-k} lies also in $T(J_{n,k})$.

Lemma 2.21. *Let f be a Capelli-type polynomial of order $n-k+1$ such that $f \in T(B_{n-k})$. Then f modulo I is a linear combination of polynomials of the type $F[T]$, where T runs over the set of 0-tableaux containing the integers $1, 2, \dots, n-k+1$ in the first row.*

Proof. Since f is an identity of B_{n-k}, by Lemma 2.13 we can suppose that modulo I, f is a consequence of polynomial in (3) and so it can be written as a linear combination of polynomials of the form

$$(x, y, z)L(a_0, b_0)L(a_1, b_1)L(a_2, b_2)\cdots. \tag{18}$$

Suppose that f is alternating in x_1, x_2,..., x_{n-k+1}. Since the operator $L(a, b)$ is symmetric with respect to a and b, by taking into account the identities (8)-(9) and passing from Jordan algebras to Lie triple systems, we have that each polynomial in (18) can be written as

$$(x_1, y, x_2)L(x_3, y_3)L(x_4, y_4)\cdots L(x_{n-k+1}, y_{n-k+1})L(a_1, b_1)\cdots L(a_l, b_l) \text{ or} \tag{19}$$

$$(x_1, y, z)L(x_2, y_2)L(x_3, y_3)\cdots L(x_{n-k+1}, y_{n-k+1})L(a_1, b_1)\cdots L(a_l, b_l) \text{ or} \tag{20}$$

$$(x, y, z)L(x_1, y_1)L(x_2, y_2)L(x_3, y_3)\cdots L(x_{n-k+1}, y_{n-k+1})L(a_1, b_1)\cdots L(a_l, b_l). \tag{21}$$

First we consider any non-zero polynomial of f of the type (19) with non-zero coefficient $\beta_1 \in K$. Since every permutation can be written as a product of transpositions, an easy induction argument shows that for any permutation $\sigma \in S_m$,

$$(x_{\sigma(1)}, y, x_{\sigma(2)})L(x_{\sigma(3)}, y_3)L(x_{\sigma(4)}, y_4)\cdots L(x_{\sigma(n-k+1)}, y_{n-k+1})L(a_1, b_1)\cdots L(a_l, b_l)$$

appears in f with coefficient $(-1)^\sigma\beta_1$. It follows that there exists a double standard tableau T having $\{1, 2, \dots, n-k+1\}$ on its first row, such that $F[T]$ appears in the linear combination of f.

We consider now a non-zero polynomial of f of the form in (20) with non-zero coefficient $\beta_2 \in K$. As before, we have that for all $\sigma \in S_m$,

$$(x_{\sigma(1)}, y, z)L(x_{\sigma(2)}, y_2)L(x_{\sigma(3)}, y_3)\cdots L(x_{\sigma(n-k+1)}, y_{n-k+1})L(a_1, b_1)\cdots L(a_l, b_l)$$

appears in f with coefficient $(-1)^\sigma\beta_2$. Thus there exists a double standard 0-tableau T that corresponds to

$$(x_{\sigma(1)}, y, z)L(x_{\sigma(2)}, y_2)L(x_{\sigma(3)}, y_3)\cdots L(x_{\sigma(n-k+1)}, y_{n-k+1}).$$

In this case as well, we have successfully established that $\{1, 2, \dots, n+1\}$ lies in the first row of a double standard tableau T and we are done.

Finally, we consider in f a non-zero polynomial of the form (21) with non-zero coefficient $\beta_3 \in K$. This implies that for all $\sigma \in S_m$,

$$(x, y, z)L(x_{\sigma(1)}, y_1)L(x_{\sigma(2)}, y_2)L(x_{\sigma(3)}, y_3)\cdots L(x_{\sigma(n-k+1)}, y_{n-k+1})L(a_1, b_1)\cdots L(a_l, b_l)$$

appears in f with coefficient $(-1)^\sigma\beta_3$, and, again, the result follows. □

The next result characterizes the Capelli-type identities of B_{n-k}, which are not in $T(J_{n,k})$.

Proposition 2.22. *Let f be a multilinear Capelli-type polynomial of order $n+1$ which is an identity of B_n. If, modulo I, f is a linear combination of polynomials $F[T]$, where T's are double standard 0-tableaux, such that $\mu[T^{(1)}] = n-k+1$ and $\mu[T^{(2)}] < n-k+1$ (if it exists), then $f \notin T(J_{n,k})$.*

Proof. We have that f, modulo I, is written as linear combination of polynomials of the form

$$F[T] = F[T^{(1)}]l[T^{(2)}]\cdots l[T^{(h[T])}],$$

where T is an arbitrary double standard 0-tableau with $\mu[T^{(1)}] = n-k+1$.

If $h[T] = 1$, then the result follows from the previous lemma together with Lemma 2.15. Thus let us consider $h[T] > 1$ and write

$$T = \begin{pmatrix} T^{(1)} \\ T^{(2)} \\ \vdots \\ T^{(s)} \end{pmatrix}.$$

As f is a polynomial of Capelli-type of order $n-k+1$, by the previous lemma we get that the set $\{1, 2, \ldots, n-k+1\}$ lies in the same row. Moreover, since each 0-tableau T is standard, Proposition 2.14 implies that the first row is $(0\ r_1 \ldots r_n \mid 1 \ldots n(n+1))$. Hence we can write

$$f = \sum \alpha_i F[T_i],$$

where $\alpha_i \in K$ and for all i the first row of T_i is of the form $T' = (0\ r_1 \ldots r_{n-k} \mid 1 \ldots (n-k)(n-k+1))$, for some fixed integers $r_1, \ldots, r_{n-k}$. Therefore,

$$f = F[T']\left(\sum \alpha_i l[T_i']\right),$$

where

$$T_i' = \begin{pmatrix} T_i^{(2)} \\ \vdots \\ T_i^{(s-1)} \\ T_i^{(s)} \end{pmatrix}.$$

Suppose that f is an identity for $J_{n,k}$. By Lemma 2.19, together with the fact that $F[T']$ is not identity for $J_{n,k}$, we have that $\sum \alpha_i f(T_i)$ is a weak identity for B_n with $\mu[T_i^{(2)}] \leq n-k$. Thus by Remark 2.18 we get $\alpha_i = 0$, for all i and so $f = 0$, which is a contradiction since f is non-trivial. □

Corollary 2.23. *Every multilinear Capelli-type identity of order $n-k+1$ of B_{n-k}, modulo I, which lies in $T(J_{n,k})$ is a consequence of*

$$h_{n-k} = \sum_{\sigma \in S_{n-k+2}} (-1)^\sigma (x_{\sigma(1)}, y_1, x_{\sigma(2)}, \ldots, y_n, x_{\sigma(n-k+1)}, y_{n-k+1}, x_{\sigma(n-k+2)}). \tag{22}$$

Proof. Due to Proposition 2.22, we have that every Capelli-type polynomial of order $n+1$ which is an identity for $J_{n,k}$ can be expressed as a linear combination, modulo I, of polynomials of the form $F[T]$, where T's are double standard 0-tableaux with either $\mu[T^{(1)}] > n+1$ or $\mu[T^{(1)}] = \mu[T^{(2)}] = n+1$.

If $\mu[T^{(1)}] > n+1$ then by (13), $F[T^{(1)}]$ is a consequence of h_n. If $\mu[T^{(1)}] = \mu[T^{(2)}] = n+1$, then $F[T^{(2)}]$ is a consequence of (4) and hence $F[T^{(2)}]$ (and consequently $F[T]$) lies in I. Therefore, the result follows. □

We are now in a position to prove the main result of this section. Such theorem is a consequence of Lemmas 2.7, 2.8, 2.10 and Corollary 2.23, together with Theorem 2.4.

Theorem 2.24. *Over a field of characteristic zero, the polynomials*

$$([x,y]^2, z, t), \tag{23}$$

$$\sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)}, (x_{\sigma(2)}, x, x_{\sigma(3)}), x), \tag{24}$$

$$\sum_{\sigma \in S_{n-k+2}} (-1)^\sigma (x_{\sigma(1)}, y_1, x_{\sigma(2)}, \ldots, y_{n-k}, x_{\sigma(n-k+1)}, y_{n-k+1}, x_{\sigma(n-k+2)}), \tag{25}$$

$$\sum_{\sigma \in S_{n-k+1}} (-1)^\sigma (x_{\sigma(1)}, y_1, x_{\sigma(2)}, \ldots, y_{n-k-1}, x_{\sigma(n-k)})(y_{n-k}, x_{\sigma(n-k+1)}, y_{n-k+1}), \tag{26}$$

form a basis of the T-ideal of polynomial identities for the Jordan algebra $J_{n,k}$, with $k \geq 1$ and $1 < n < \infty$.

Proof. We denote by Cap^k_{n-k+1} the set of all Capelli-type polynomials of order $n-k+1$ that are identities for $J_{n,k}$ and by I the T-ideal generated by the poylnomials (23), (24) and (26). By Lemma 2.2, we have $I = T(J_{n,k})$, modulo Cap_{n-k+1}. As in the previous result, we will denote the identity (25) by h_{n-k}. Assuming $Q = \langle I, h_{n-k}\rangle_T$, it is clear that $Q \subseteq T(J_{n,k})$.

Conversely, let $f \in T(J_{n,k})$, then we get $f = g + h$, where $g \in I$ and $h \in Cap_{n-k+1}$. Since $h = f - g \in T(J_{n,k})$, it follows that $h \in Cap^k_{n-k+1}$. Moreover, by Corollary 2.23 any Capelli-type polynomial p of order $n-k+1$, which is an identity for $J_{n,k}$ is a consequence of h_{n-k} modulo I, i.e., $p \in Q$. In particular $h \in Q$ thus $f \in Q$ and we are done. □

We will conclude this section by establishing the Specht property for the Jordan algebra $J_{n,k}$, with $k \geq 1$ and $n > 1$. To this end, we need the following definition.

Definition 2.25. *Let A be an algebra (not necessarily associative). We say that $T(A)$ satisfies the Specht property if any T-ideal I such that $I \supset T(A)$, including $T(A)$, has a finite basis as a T-ideal.*

In other words, I has the Specht property, if I is finitely generated as a T-ideal. Moreover, we say that a variety $\mathcal{V}$ has the Specht property if the corresponding T-ideal satisfies the Specht property.

As previously highlighted in the Introduction, the Specht property was extensively investagated when considering the variety generated by B_n. Iltyakov [12] showed that the variety of unitary algebras generated by B_n satisfies the Specht property. Vasilovskii [36] described a finite generating set of the identities of the Jordan algebra B of a non-degenerate symmetric bilinear form on a vector space of infinite dimension, over any infinite field of characteristic different from 2. Combining the main results of the papers [20, 36] one deduces the Specht property for B (that is B_∞). These positive results hold in characteristic 0. Recall that Lemmas 2.7 and 2.8 implies $T(B_\infty) \subseteq T(J_{n,k})$, for all $n > 1$ and $k > 0$.

Due to these comments, we have established the following result.

Theorem 2.26. *Over a field of characteristic zero, the T-ideal of identities of $J_{n,k}$ satisfies the Specht property.*

3. Graded polynomial identities

Let $\mathbb{Z}_2$ be the cyclic group of order 2. The main goal of this section is to describe the $\mathbb{Z}_2$-graded identities of the Jordan algebras $J_{n,k}$ equipped with the so-called scalar grading. Along the way we will also compute the weak identities of these algebras.

Let G be an abelian group with identity element ϵ and let A be an algebra (not necessarily associative). A *grading by the group* G on A is a vector space decomposition $A = \oplus_{g\in G} A^{(g)}$ such that $A^{(g)}A^{(h)} \subseteq A^{(gh)}$ for every g, h in G. In this case we say that A is *G-graded*. The subspaces $A^{(g)}$ are called the *homogeneous components* of A. A non-zero element $a \in A$ is homogeneous of degree g if $a \in A^{(g)}$ and we denote it by $\|a\|_G = g$ (or simply $\|a\| = g$ when the group G is clear from the context). A vector subspace (subalgebra, ideal) B of A is said to be *graded* if $B = \oplus_{g\in G} A^{(g)} \cap B$. Moreover, if A and B are G-graded algebras, an algebra homomorphism $\varphi\colon A \to B$ is a *homomorphism of graded algebras* if $\varphi(A^{(g)}) \subseteq B^{(g)}$, for all $g \in G$.

In this section, we shall need only some facts about $\mathbb{Z}_2$-gradings and $\mathbb{Z}_2$-graded identities on Jordan algebras. We denote by $J(Y \cup Z)$ the free $\mathbb{Z}_2$-graded Jordan algebra, over K, requiring that the variables in $Y = \{y_1, y_2, \ldots\}$ are homogeneous of degree zero, and those in $Z = \{z_1, z_2, \ldots\}$ are homogeneous of degree one. This algebra has a natural $\mathbb{Z}_2$-grading, where the homogeneous component $(J(Y \cup Z))^{(0)}$ is the span of all monomials $x_{i_1} \cdots x_{i_m}$ such that $\|x_{i_1}\| + \cdots + \|x_{i_m}\| = 0$, together with the empty word 1; otherwise it has homogeneous degree 1, and this spans the homogenenous component $(J(Y \cup Z))^{(1)}$. Notice that $J(Y \cup Z) = (J(Y \cup Z))^{(0)} \oplus J(Y \cup Z)^{(1)}$ is a grading on $J(Y \cup Z)$. Actually, we have that $J(Y \cup Z)$ is free in the class of every $\mathbb{Z}_2$-graded Jordan algebra freely generated by $Y \cup Z$ over K and the elements in $J(Y \cup Z)$ are called *2-polynomials*.

Let J be a Jordan algebra grading by group $\mathbb{Z}_2$, $J = J^{(0)} \oplus J^{(1)}$, a 2-polynomial $f(y_1, \ldots, y_m, z_1, \ldots, z_n)$ is a 2-identity for J if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0$ for all $a_1, \ldots, a_m \in J^{(0)}$ and for all $b_1, \ldots, b_n \in J^{(1)}$.

Clearly the set of all 2-identities of J, denoted by $T_2(J)$, is an ideal that is closed under the graded endomorphisms of $J(Y \cup Z)$. The graded ideals of $J(Y \cup Z)$ with this property are called T_2-ideal. Moreover, as was said in the ordinary case, in characteristic 0, the T_2-ideal $T_2(J)$ is generated by its multilinear 2-polynomials and when the field is infinite by the multihomogenenous 2-polynomials it contains. Let J_1 and

J_2 be two $\mathbb{Z}_2$-graded Jordan algebras, we say that J_1 and J_2 are PI-equivalent as $\mathbb{Z}_2$-graded algebras if $T_2(J_1) = T_2(J_2)$.

We are interested in the 2-polynomial identities of $J_{n,k} = K \oplus V$ equipped with the scalar grading, i.e., $J_{n,k} = J^{(0)} \oplus J^{(1)}$ where $J^{(0)} = K$ and $J^{(1)} = V$. Remark that the name "scalar" is justified by the fact that the homogeneous component $J^{(0)}$ is actually the field of scalars K. The analogous problem for B_n was solved by Diniz and Koshlukov. More precisely, in [22] the following theorem was proved.

Theorem 3.1. [22, Corollary 21] *Let K be an infinite field of characteristic different from 2. Then:*

(i) *The T_2-ideal of graded identities for the Jordan algebra B_∞ equipped with the scalar grading is generated by the 2-polynomial*

$$(y, u_1, u_2), \tag{27}$$

where $u_1, u_2 \in Y \cup Z$.

(ii) *The T_2-ideal of graded identities for the Jordan algebra B_m, equipped with the scalar grading and $1 < m < \infty$, is generated by the polynomials in (27) and*

$$g_m = \sum_{\sigma \in S_{m+1}} (-1)^\sigma z_{\sigma(1)}(z_{m+2}z_{\sigma(2)}) \cdots (z_{2m+1}z_{\sigma(m+1)}). \tag{28}$$

3.1. **Computing the weak identities and 2-identities.** From now on, let $J = J_{n,k} = J^{(0)} \oplus J^{(1)}$, where $J^{(0)} = K$ and $J^{(0)} = V$.

It is easy to verify that the 2-polynomial (27) is a identity for $J_{k,n}$, since the even elements are scalars. Notice that (z_1z_2, u_1, u_2) is a consequence of (27).

In order to reach our goal, an important tool that we will use is the so-called weak identities. As we did in the previous section in the particular case of B_m, we recall the definition of weak identity for convenience of the reader. We also refer to [12] for further details.

Let J be a Jordan algebra and V a subspace of J. A polynomial $f(x_1, \ldots, x_n) \in J(X)$ is a weak identity for the pair (J, V) if $f(v_1, \ldots, v_n) = 0$ for all $v_i \in V$. We denote by $T(J, V)$ the set formed by all weak identities for the pair (J, V). It is clear that $T(J, V)$ is an ideal of $J(X)$. Pay attention that according to V, it is natural to assume that one can substitute the variables in a weak identity by linear combinations of elements of X, i.e., if $f(x_1, \ldots, x_n) \in T(J, V)$ then $f(a_1, \ldots, a_n) \in T(J, V)$ for all $a_1, \ldots, a_n \in \operatorname{span}\{X\}$. We do not treat the most general case here but instead we refer the reader to [7, 21] for more information.

Denote by I the T_2-ideal of graded identities generated by the polynomial (27). Now consider a multi-homogeneous 2-polynomial $f = f(y_1, \ldots, y_p, z_1, \ldots, z_q)$ of $J(Y \cup Z)$. It is easy verify that modulo I, f is written as $y_1^{\alpha_1} \cdots y_p^{\alpha_p} g(z_1, \ldots, z_q)$, for some polynomial g that involves only the odd variables. Therefore f is a graded identity for $J_{k,n}$ if and only if g is. Thus f is a graded identity for $J_{k,n}$ if and only if g is a weak identity for the pair $(J_{n,k}, V)$. This motivates our next result.

Proposition 3.2. *Let $J_{n,k} = K \oplus V = B_{n-k} \oplus D_k$ be the decomposition of $J_{n,k}$ into direct sum of the non-degenerate part $B_{n-k} = K \oplus V'$ of the algebra plus the degenerate part with $k \geq 1$. A multilinear polynomial $f \in J(X)$ is a weak identity for the pair $(J_{n,k}, V)$ if and only if f is a weak identity for the pair (B_{nk}, V') and*

$$f_{x_i} = f(n_1, \ldots, n_{i-1}, m_i, n_{i+1}, \ldots, n_{\deg f}) \equiv 0, \tag{29}$$

for all $1 \leq i \leq \deg f$ and for all $n_j \in V'$, $j \neq i$, $m_i \in D_k$.

Proof. If $f \in T(J_{n,k}, V)$ then $f \in T(B_{n-k}, V')$ since $V' \subseteq V$. Moreover, condition (29) trivially holds and we are done.

Conversely, let us suppose that the multilinear polynomial $f = f(x_1, \ldots, x_r) \in T(B_{n-k}, V')$ satisfies condition (29). Since f is multilinear, one can to make the evaluation of every variable x_i by elements of $\mathcal{B}$. As in Proposition 2.5, if f is evaluated in more than one element in $\mathcal{D}$ then it is automatically zero. Therefore, it is sufficient to consider evaluations by elements of $\mathcal{B}$ where at most one element lies in $\mathcal{D}$. Hence, the result follows since that f is a weak identity for the pair (B_{n-k}, V') and satisfies the condition (29). ∎

The last proposition implies that if g is a multilinear 2-polynomial on the odd variables only, then in order to verify that g lies in $T_2(J_{n,k})$, we have only to check that g is a weak identity for the pair (B_{n-k}, V') and

$$g(b_1, \ldots, b_{i-1}, d_i, b_{i+1}, \ldots, b_{\deg g}) \equiv 0,$$

for all $1 \leq i \leq \deg g$, $b_j \in V'$, $j \neq i$, and $d_i \in D_k$.

The next theorems follow as a consequence of Theorem 3.1 and Proposition 3.2.

Theorem 3.3. *Let K be an infinite field of characteristic different from 2. The weak Jordan identities for the pair $(J_{\infty,k}, V)$ are consequences of the polynomial (x_1x_2, x_3, x_4).*

Theorem 3.4. *Let k be a positive integer. Over an infinite field of characteristic different from 2, the Jordan algebras $J_{\infty,k}$ and B_∞, both equipped with their respective scalar grading, are PI-equivalent as $\mathbb{Z}_2$-graded algebras. In particular, the identity (y, u_1, u_2) forms a basis of the T_2-ideal of 2-polynomial identities for the Jordan algebra $J_{\infty,k}$.*

Recall that if two graded algebras share the same graded identities, then they share the same ordinary identities. Thus, we can also generalize the Theorem 2.9 in the following sense.

Corollary 3.5. *Over an infinite field of characteristic different from 2, the Jordan algebras $J_{\infty,k}$, for all k, and B_∞ are PI-equivalent.*

Now, as it was done in the previous section, we focus our attention to the case n finite.

Remark 3.6. *It is easily seen that the polynomial (28), for $m = n-k$, does not lie in $T_2(J_{n,k})$. In fact, if we consider the evaluation φ such that $\varphi(z_1) = d \in D_k$ and $\varphi(z_j) = \varphi(z_{n-k+j}) = a_{j-1}$, for all $2 \leq j \leq n-k+1$, where $a_1, \ldots, a_{n-k}$ are distinct elements of $\mathcal{N}$, then*

$$\varphi(g_n) = da_1^2 \cdots a_{n-k}^2 = d \neq 0.$$

As we did for the ordinary identities, we will need ideas and methods from the invariants of the orthogonal group given in Remark 2.18. Notice that the description of these invariants is independent of characteristic of the ground field.

Definition 3.7. *Let $T = (p_1 \ldots p_r \mid q_1 \ldots q_r)$ be a double tableau consisting of one row. Then we associate to T the 2-polynomial*

$$f_T = \sum_{\sigma \in S_r} (-1)^\sigma (z_{p_1} z_{q_{\sigma(1)}})(z_{p_2} z_{q_{\sigma(2)}}) \cdots (z_{p_r} z_{q_{\sigma(r)}});$$

Moreover, if T is 0-tableau, then f_T is

$$\sum_{\sigma \in S_r} (-1)^\sigma z_{q_{\sigma(1)}} (z_{p_2} z_{q_{\sigma(2)}}) \cdots (z_{p_r} z_{q_{\sigma(r)}}).$$

Finally, let T be any double tableau and let $T^{(1)}, T^{(2)}, \ldots, T^{(s)}$ be the rows of T. Then we set $f_T = f_{T^{(1)}} f_{T^{(2)}} \cdots f_{T^{(s)}}$.

Define M as the subalgebra of $L = J(Y \cup Z)/I$ generated by the odd variables. Clearly the algebra $M = M^{(0)} \oplus M^{(1)}$ has a grading by the group $\mathbb{Z}_2$, where the subalgebra $M^{(0)}$ is spanned by all products $(z_{i_1} z_{j_1}) \cdots (z_{i_r} z_{j_r})$ while the vector space $M^{(1)}$ is spanned by all $z_{i_0}(z_{i_1} z_{j_1}) \cdots (z_{i_r} z_{j_r})$ (see [22, Lemma 17] for more details).

Proposition 3.8. [22, Proposition 19] *The vector space $M^{(0)}$ has a basis consisting of all 2-polynomials associated to doubly standard tableaux. Also, $M^{(1)}$ has a basis consisting of all 2-polynomials associated to doubly standard 0-tableaux.*

Here and in what follows, we will use a graded (weak) version of the results that we obtained in the ordinary case concerning Capelli-type polynomials and their connection to weak polynomial identities.

Proposition 3.9. *Let K be a field of characteristic zero, we then have*

$$T(J_{n,k}, V) = T(B_{n-k}, V'),$$

modulo the T-ideal generated by Capelli-type polynomials of order $n-k+1$.

Proof. By Remark 2.18, every polynomial of $T(B_{n-k}, V')$ follows from a polynomial f_T where T is a doubly standard tableau with $\mu[T^{(1)}] > n$. The assertion now follows immediately since f_T is a Capelli-type polynomial of order $n-k+1$, the polynomial (x_1x_2, x_3, x_4) is a weak identity for the pair $(J_{n,k}, V)$ and [22, Theorem 20] holds. $\square$

From now on, $Cap^{(w,k)}_{n-k+1}$ ($Cap^{(w)}_{n-k+1}$, respectively) denotes the ideal generated by the weak Capelli-type polinomials of order $n-k+1$ for $J_{n,k}$ (B_{n-k}, respectively). It is clear that $Cap^{(w,k)}_{n-k+1} \subseteq Cap^{(w)}_{n-k+1}$. The next result will be important in order to describe the weak identities of the pair $(J_{n,k}, V)$ and its proof follows step-by-step the one of Proposition 2.22.

Proposition 3.10. *Let f be a multilinear weak identity for the pair (B_{n-k}, V') and write f as a linear combination of polynomials $\{f_T\}$, where $\{T\}$'s are doubly standard tableaux. If, for all T, $T^{(1)}$ is a double 0-tableau with $\mu[(T^{(1)})] = n-k+1$ and $\mu[(T^{(2)})] < n-k+1$ (if it exists), then $f \notin T(J_{n,k}, V)$.*

Proof. The previous proposition implies that it suffices to consider that f is of order $n-k+1$. By Proposition 3.8, f can be written as linear combination of polynomials of the form

$$f_T = f_{T^{(1)}} f_{T^{(2)}} \cdots f_{T^{(h[T])}},$$

where T is a doubly standard 0-tableau with $\mu[T^{(1)}] = n-k+1$. If $h[T] = 1$, then the result follows by Remark 3.6.

Therefore, we may assume that $h[T] > 1$, for all T and we write

$$T = \begin{pmatrix} T^{(1)} \\ T^{(2)} \\ \vdots \\ T^{(s)} \end{pmatrix}.$$

Recall here that $T^{(1)}$ is a double 0-tableau.

Since f is Capelli-type polynomial of order $n-k+1$, then we must have that the set $\{1, 2, \ldots, n-k+1\}$ lies in the same row. Thus, for each T, we have $T^{(1)} = (0\ i_1 \ldots i_{n-k} \mid 1 \ldots (n-k)\ (n-k+1))$. Hence f can be expressed as a linear combination of elements of the form

$$f = f_{T'} \cdot \left(\sum \alpha_i f_{T'_i} \right),$$

where $T' = T^{(1)}$ and

$$T'_i = \begin{pmatrix} T_i^{(2)} \\ \vdots \\ T_i^{(s)} \end{pmatrix},$$

with $\mu([T_i^{(2)}]) \leq n-k$, for all i. Assume that $f \in T(J_{n,k}, V)$, then by Lemma 2.19, together with the fact that $f_{T'}$ is not a weak identity for $(J_{n,k}, V)$, we have that $\sum \alpha_i f_{T'_i}$ is a weak identity for (B_{n-k}, V') with $\mu(T_i^{(2)}) \leq n-k$. Remark 2.18 implies that $\alpha_i = 0$ for all i and so $f = 0$, a contradiction. Therefore, we have the result. $\square$

By taking into account the previous results and Proposition 3.2, if $\mathrm{char} K = 0$, the description of a basis for $T_2(J_{n,k})$ depends on the description of the weak Capelli-type polynomials of order $n-k+1$ of (B_{n-k}, V') that lie also in $T(J_{n,k}, V)$.

Theorem 3.11. *Over a field of characteristic zero, any weak identity of the pair $(J_{n,k}, V)$ follows from the weak identities*

$$(x_1x_2, x_3, x_4)$$

$$l_{n-k} = \sum_{\sigma \in S_{n-k+1}} (-1)^{\sigma} (y_1 x_{\sigma(1)})(y_2 x_{\sigma(2)}) \cdots (y_{n-k+1} x_{\sigma(n-k+1)}).$$

Proof. Proposition 3.10 implies that every multilinear weak identity for $J_{n,k}$ is a linear combination of polynomials f_T, where T's are doubly standard tableaux.

Let T be such a tableau. Then, by Remark 3.6 and (16), $f_{T^{(1)}}$ is a consequence of l_{n-k}. Hence, we can suppose that every tableau T is a 0-tableau such that $\mu[T^{(1)}] > n-k+1$ or $\mu[T^{(1)}] = \mu[T^{(2)}] = n-k+1$ (if $T^{(2)}$ exists). If $\mu[T^{(1)}] > n-k+1$, then $f_{T^{(1)}}$ is again a consequence of l_{n-k}. Finally, if $\mu[T^{(1)}] = \mu[T^{(2)}] = n-k+1$, then $f_{T^{(2)}}$ is consequence of l_{n-k}. Therefore, in all the cases we have f_T as a consequence of l_{n-k} and the result follows. □

We are now ready to prove the main theorem of this section.

Theorem 3.12. *Let K be a field of characteristic zero. The T_2-ideal of the 2-polynomial identities of the Jordan algebra $J_{n,k}$ with the scalar grading, where n is a positive integer and $k > 0$, is generated by the 2-polynomials from*

$$(y, u_1, u_2)$$
$$\sum_{\sigma\in S_{n-k+1}} (-1)^\sigma (z_{n-k+2}z_{\sigma(1)})(z_{n-k+3}z_{\sigma(2)})\cdots(z_{2(n-k)+3}z_{\sigma(n-k+1)}),$$

where u_1, $u_2 \in Y \cup Z$.

Proof. Let $f \in T_2(J_{k,n})$. The identity (y, u_1, u_2) implies that f can be considered as a 2-polynomial in the odd variables only. In this case, f is a weak identity for the pair $(J_{n,k}, V)$ and the result immediately follows from the previous theorem. □

The corresponding Specht property for associative algebras graded by a finite group G was proved by I. Sviridova [34] (for abelian groups) and by E. Aljadeff and A. Kanel-Belov [1], whereas in case of Jordan algebras we have experimental results, such as in [5, 30], all of them in characteristic 0. In [30], the authors proved the Specht property for B_n with the scalar grading, where n is a positive integer. The case B_∞ is open so far. Moreover, it is easy to see that $T_2(B_{n-k+1}) \subseteq T_2(J_{n,k})$.

In addition to the above comments, one can establish the next result.

Theorem 3.13. *Over a field of characteristic zero, the T_2-ideal of the 2-polynomial identities of the Jordan algebra $J_{n,k}$ with the scalar grading, where n is a positive integer and $k > 0$, satisfies the Specht property.*

4. Trace polynomial identities

This section is devoted to the study of the trace polynomial identities of $J_{n,k}$. To this end, let us now introduce some basic tools and definitions.

Let A be an algebra (not necessarily associative) over K, and let $\mathcal{Z} = \mathcal{Z}(A)$ be its associative and commutative center. A K-linear map $\tau\colon A \to \mathcal{Z}$ such that $\tau(\tau(a)b) = \tau(a)\tau(b)$ and $\tau([a, b]) = \tau((a, b, c)) = 0$, for all elements a, b and c in A, is called a *trace* on the algebra A. The pair (A, τ) will be called a *trace algebra* (or algebra with trace). The *pure trace algebra* for (A, τ) is defined as the image of A under τ, denoted by $\tau(A)$. It follows immediately that $\tau(A)$ is a subalgebra of $\mathcal{Z}$. An ideal (subalgebra, respectively) in a trace algebra (A, τ) is called a *trace ideal* (*trace subalgebra*, respectively) of (A, τ) if it is an ideal (subalgebra, respectively) that is closed under the trace map. Let (A, τ) and (B, θ) be a trace algebras, a map $\varphi\colon (A, \tau) \to (B, \theta)$ is a *homomorphism of trace algebras* if it is a homomorphism of algebras, $\varphi(1_A) = 1_B$, and $\varphi(\tau(a)) = \theta(\varphi(a))$ holds for every a in A.

In particular, the above definition implies that the pair (B_m, tr) is a Jordan algebra with trace, where $\mathrm{tr}(\alpha + v) = 2\alpha$, with $\alpha \in K$ and $v \in V$. Additionally, $(J_{n,k}, \mathrm{tr})$ is also a Jordan algebra with trace. It is important to mention that we do not require that the map trace to be non-degenerate. Furthermore, D_k is a trace ideal, while B_{n-k} is a trace subalgebra of $(J_{n,k}, \mathrm{tr})$.

Let $\mathcal{V}$ be a variety of unitary Jordan K-algebras. We recall the definition of the free trace algebra in $\mathcal{V}$, see [37] for more details. Assume $A \in \mathcal{V}$ is a Jordan algebra with trace Tr, and let $X \subseteq A$. The algebra (A, Tr) is $\mathcal{V}$-free with a set of free generators X if for all Jordan algebra with trace (B, ρ) in $\mathcal{V}$, every map $X \to B$ can be extended in unique way to a homomorphism of trace algebras $(A, \mathrm{Tr}) \to (B, \rho)$. The existence of free trace algebras in a variety follows from general arguments, see for example [39, Theorem 2]. The same arguments yield that two free trace algebras in $\mathcal{V}$ are isomorphic if and only if their free generating sets have the same cardinality. We shall denote the free Jordan trace in $\mathcal{V}$ as $J_{TR}(X)$. Recall that the trace algebras in $\mathcal{V}$ form a variety (one includes the trace in the signature of the variety).

Unless otherwise stated we shall assume that X is an infinite countable set. It is immediate that $J_{TR}(X)$ contains the ordinary free algebra $J(X)$ in the variety $\mathcal{V}$.

The subalgebra $G(X)$ generated in $J_{TR}(X)$ by the set

$$\{g(x_1,\ldots,x_n), \mathrm{Tr}\,(g(x_1,\ldots,x_n)) \mid g(x_1,\ldots,x_n) \in J(X)\}$$

is the *algebra of generalized polynomials* in the variety $\mathcal{V}$. Its elements are called *trace polynomials*. It is clear that $G(X)$ is spanned by the generalized monomials of the form

$$\widehat{a}_0 \mathrm{Tr}\,(a_1)\cdots \mathrm{Tr}\,(a_t), \tag{30}$$

where $t \geq 1$ and a_0, a_1, $\ldots$, a_t are monomials in $J(X)$. Here $\widehat{a}_0$ means that a_0 can be eventually empty. The trace monomial from (30) has *degree* $\deg a_0 + \deg a_1 + \cdots + \deg a_t$ and the degree of a trace polynomial is defined as the greatest degree of a monomial that appears with non-zero coefficient in it.

Let (J,τ) be a trace algebra in the variety of all Jordan algebras with trace. A trace polynomial $f(x_1,\ldots,x_n) \in G(X)$ is a trace identity of (J,τ) if, substituting Tr by τ, we have $f(a_1,\ldots,a_n) = 0$, for all a_1, $\ldots$, $a_n \in J$. We denote by $T_{Tr}(J,\tau)$ (or simply $T_{Tr}(J)$) the set of all trace identities for trace algebra (J,τ). An ideal I of the algebra $G(X)$ is called T_{Tr}*-ideal* (or T*-ideal with trace*) if, for all trace polynomial $f(x_1,\ldots,x_n) \in I$ and for all trace polynomials $g_1,\ldots,g_n$ in $J(X)$, the trace polynomial $f(g_1,\ldots,g_n)$ is contained in I. For each Jordan algebra with trace (J,τ), the ideal of its trace identities is a T_{Tr}-ideal, in the sense that is an ideal closed under the endomorphisms of the free algebra with trace.

The trace identity $f \equiv 0$ follows from the trace identities $g_i \equiv 0$, $i \in \Lambda$, if f lies in the least T_{Tr}-ideal containing all g_i, $i \in \Lambda$. The latter ideal will be denoted as $\langle g_i \mid i \in \Lambda\rangle_{Tr}$. As in the graded and ordinary case, in characteristic 0, the T_{Tr}-ideal $T_{\mathrm{Tr}}\,(J,\tau)$ is generated, as a T_{Tr}-ideal, by its multilinear trace polynomials and when the field is infinite by this multihomogenenous ones.

Vasilovsky [37] proved that all trace identities of B_m follow from the identities

$$f_2(x) = x^2 - \mathrm{Tr}\,(x)x + (1/2)(Tr(x)^2 - \mathrm{Tr}\,(x^2)), \tag{31}$$

$$L_{m+1} = \sum_{\sigma\in S_{m+1}} (-1)^\sigma (x_{\sigma(m+1)} - (1/2)\mathrm{Tr}\,(x_{\sigma(m+1)})) \prod_{k=1}^{m} H(x_{\sigma(k)}, y_k). \tag{32}$$

Here $H(x,y) = \mathrm{Tr}\,(xy) - (1/2)\mathrm{Tr}\,(x)\mathrm{Tr}\,(y)$ and $L_{m+1} = L(x_1,\ldots,x_{m+1},y_1,\ldots,y_m)$, $m = 2, 3, \ldots$. More precisely, the author proved the following results.

Theorem 4.1. *Let K be an infinite field of characteristic different from 2.*

1. [37, Theorem 1] *All trace identities of the Jordan algebra B_m, for any positive integer $m > 1$, follow from the polynomials* (31) *and* (32).
2. [37, Theorem 2] *All trace identities of the Jordan algebra B_∞ follow from the polynomial* (31).

Remark 4.2. *It is easily seen that*

$$(x - (1/2)Tr(x))(y - (1/2)Tr(y)) = (1/2)H(x,y),$$

modulo the T-ideal with trace generated by (31).

4.1. **Computing the T-ideal of identities with trace.** We start by the following proposition that relates the trace identities of B_m with the ones of $J_{n,k}$.

Proposition 4.3. *Let $J_{n,k} = B_{n-k} \oplus D_k$ be the decomposition of $J_{n,k}$ into direct sum of the non-degenerate part of the algebra plus the degenerate part with $k > 1$. A multilinear trace polynomial $f \in G(X)$ is a trace identity of $(J_{n,k}, tr)$ if and only if $f \in T_{\mathrm{Tr}}\,(B_{n-k}, tr)$ and*

$$f_{x_i} = f(n_1,\ldots,n_{i-1},m_i,n_{i+1},\ldots,n_{degf}) \equiv 0, \tag{33}$$

for all $1 \leq i \leq \deg f$ and for all $n_j \in B_{n-k}$, $j \neq i$, $m_i \in D_k$.

Proof. If $f \in T_{Tr}(J_{n,k})$, then $f \in T_{Tr}(B_{n-k})$ since B_{n-k} is a trace subalgebra of $J_{n,k}$. Moreover, condition (33) trivially holds and we are done.

Conversely, let us suppose that the multilinear trace polynomial $f = f(x_1, \dots, x_r) \in T_{Tr}(B_{n-k})$ satisfies condition (33). Since f is multilinear, one can evaluate every variable x_i by elements from $\mathcal{B} \cup \{1\}$. Moreover, as D_k is a trace ideal of $(J_{n,k}, \mathrm{tr})$ and $D_k^2 = 0$, any monomial in f that contains some element of D_k inside of a trace or has more than one element in D_k outside is automatically zero. Hence we are done, since $f \in T_{Tr}(B_{n-k})$ and the condition (33) holds. $\square$

Let I be the T-ideal with trace generated by the trace polynomial (31). In [37], it was proved that every trace polynomial can be represented modulo $\langle f_2 \rangle_{Tr}$ in the following way:

$$\sum \alpha_{\hat{0},1,\dots,s} \hat{a}_0 \mathrm{Tr}\,(a_1) \cdots \mathrm{Tr}\,(a_s),$$

where $s \geq 0$ and a_0, a_1, …, a_s are monomials in $J(X)$ such that $\deg(a_0) \leq 1$ and $\deg(a_i) \leq 2$, $1 \leq i \leq s$.

The main goal of this section is to describe a basis for all trace identities of $(J_{n,k}, \mathrm{tr})$.

Lemma 4.4. *If K is an infinite field of characteristic different from two, then the polynomial (31) is a trace identity of the Jordan algebra $(J_{n,k}, tr)$, for any k and $n > 1$.*

Proof. The complete linearization of (31) is

$$h = 2xy - (Tr(x)y + Tr(y)x) + (Tr(xy) - Tr(x)Tr(y)). \tag{34}$$

Since h is multilinear, all variables of h can be evaluated on elements from $\mathcal{B} \cup \{1\}$. Thus, it is enough to verify that the polynomial h satisfies condition (33) of Proposition 4.3. Notice that h is symmetric with respect to the variables x and y, hence we compute h_x only.

Let φ be a substitution such that $\varphi(x) = d \in \mathcal{D}$ and $\varphi(y) = 1$, then

$$\varphi(h) = 2d - (\mathrm{tr}(d)1 + \mathrm{tr}(1)d) + (\mathrm{tr}(d \cdot 1) - \mathrm{tr}(d)\mathrm{tr}(1)) = 2d - 2d = 0.$$

Now let φ' be a substitution such that $\varphi'(x) = d$ and $\varphi'(y) = e \in \mathcal{N} \setminus \{1\}$, then we get easily that $\varphi(h) = 0$. Therefore condition (33) holds and the proof this lemma is complete. $\square$

The following result is a immediate consequence of the above lemma together with Theorem 4.1.

Theorem 4.5. *Over an infinite field of characteristic different from 2, all trace identities of the Jordan algebra $J_{\infty,k}$, for any integer $k > 0$, follow from the polynomial*

$$f_2(x) = x^2 - \mathrm{Tr}\,(x)x + (1/2)(Tr(x)^2 - \mathrm{Tr}\,(x^2)).$$

We now restrict our attention to the case n finite. Consider the evaluation φ such that $\varphi(x_{n-k+1}) = d$ and $\varphi(x_j) = \varphi(y_j) = e_j$ for all $1 \leq j \leq n$, where $e_1, e_2, \dots, e_{n-k}$ are distinct elements of $\mathcal{N}$, then the only monomial in L_{n-k+1} which is non-zero under this evaluation is $x_{n-k+1}Tr(x_1y_1) \cdots Tr(x_{n-k}y_{n-k})$ and its evaluation is $2^{n-k}d$. We conclude that the polynomial (32), where $m = n-k$, is not a trace identity for $J_{n,k}$.

Also in the case of trace polynomials, we can generalize the definition of Capelli-type polynomial of order l. We say that a trace polynomial $f \in G(X)$ is Capelli-type of order l if f is linear and alternating in l variables. It is clear that f vanishes whenever the set of alternating variables is substituted for linearly dependent elements. Hence if J is a Jordan algebra with trace of dimension r then all Capelli-type polynomials with trace of order m where $m \geq r+1$ are trace identities of J.

Remark 4.6. *As in Lemma 2.2, one can deduce the following fact. Two T_{Tr}-ideals I and Q are equal modulo the Capelli-type polynomials of order $r+1$ in $G(X)$ if and only if $I^{(m)} = Q^{(m)}$, where $I^{(m)} = I \cap G(x_1, \dots, x_m)$ and $Q^{(m)} = Q \cap G(x_1, \dots, x_m)$. See for example* [8, Remark 5.4].

Due to Remark 4.6 and Proposition 4.3, over a field of characteristic zero, a complete description of a basis for $T_{Tr}(J_{n,k}, \mathrm{tr})$ as T_{Tr}-ideal depends on the Capelli-type trace polynomials of order $n-k+1$ in $T_{Tr}(B_{n-k}, \mathrm{tr})$ which are also in $T_{Tr}(J_{n,k}, \mathrm{tr})$. Thus, from now on K denotes a field of characteristic zero.

Lemma 4.7. *Every Capelli-type polynomial of order $n-k+1$ which is a multilinear trace identity for $(J_{n,k}, tr)$ is a consequence of*

$$N_{n-k+1} = \sum_{\sigma \in S_{n-k+1}} (-1)^\sigma \prod_{i=1}^{n-k+1} H(x_{\sigma(i)}, y_i).$$

Proof. Clearly N_{n-k+1} is a trace identity Capelli-type for $(J_{n,k}, \mathrm{tr})$. Now let f be a multilinear Capelli-type polynomial of order $n-k+1$ in $G(X)$ which is a trace identity for $T_{Tr}(J_{n,k}, \mathrm{tr})$. Proposition 4.3 implies that we may assume $f \in T_{Tr}(B_{n-k})$ satisfying condition (33).

If f has even degree, then by the proof of [37, Theorem 1], we have that f is a consequence of N_{n-k+1}. Therefore, we will now focus on the case where f has odd degree, denoted as $2l+1$. Moreover, we can suppose $l > n$, since L_{n-k+1} is not an element of $T_{Tr}(J_{n,k}, \mathrm{tr})$. By using again the proof of [37, Theorem 1], we can write

$$f = \sum_{i_0,k,l} \alpha_{i_0}^{k,l}(x_{i_0} - (1/2)Tr(x_{i_0})) \prod_{r,l\geq 1} H(x_r, x_l).$$

Notice that if x_0 is a new variable, then

$$g = (x_0 - (1/2)Tr(x_0))f \equiv \sum_{i_0,k,l} \alpha_{i_0}^{k,l} H(x_0, x_{i_0}) \prod_{r,l\geq 1} H(x_k, x_l),$$

modulo I. In this case, the polynomial g is still a trace identity of $(J_{n,k}, \mathrm{tr})$. Since this identity holds in B_{n-k}, we have

$$\sum_{i_0,r,l} \alpha_{i_0}^{r,l}(v_r, v_l) = 0,$$

where $(v_r, v_l) = v_r \circ v_l$ is the inner product on V' defined before in Theorem 2.16. Recall also that by Remark 2.18, the algebra of invariants of the orthogonal group $K[(v_i, v_j)]$ is generated by polynomials of the form

$$f_T = f_{T^{(1)}} f_{T^{(2)}} \cdots f_{T^{(r)}},$$

where T is a doubly standard tableau with rows $T^{(1)}, \ldots, T^{(r)}$ and

$$f_{T^{(i)}} = \sum_{\sigma \in S_{m_i}} (-1)^\sigma (v_{p_{i1}}, v_{q_{i\sigma(1)}}) \cdots (v_{p_{im_i}}, v_{q_{i\sigma(m_i)}}),$$

with $T^{(i)} = (p_{i1} \ldots p_{im_i} \mid q_{i1} \ldots q_{im_i})$, for all $1 \leq i \leq r$. Using the same arguments as in Proposition 3.10, if $T^{(1)}$ is a double 0-tableau with $\mu[T^{(1)}] = n+1$ and $\mu[T^{(2)}] < n-k+1$ (if it exists), then g is zero in $G(X)$. Hence the polynomial g is a linear combination of polynomials of the form

(i) $(x_0 - (1/2)Tr(x_0))L_{n-k+1}N_{n-k+1} \prod H(z_k, z_j)$,
(ii) $(x_0 - (1/2)Tr(x_0))(x_i - (1/2)Tr(x_i))N_{n-k+1} \prod H(z_r, z_j)$,

since in a standard tableau the leftmost entry of the first row must correspond to x_0. Furthermore, modulo the trace identities of B_{n-k}, we have that f is a linear combination of trace polynomials of the form

(i) $L_{n-k+1}N_{n-k+1} \prod H(z_r, z_j)$;
(ii) $(x_i - (1/2)Tr(x_i))N_{n-k+1} \prod H(z_r, z_j)$,

since the form tr is non-degenerate on B_{n-k}. Then the result follows, since f is a polynomial in $G(X)$. ☐

Now we have the ingredients to prove the main result of this section.

Theorem 4.8. *Let K be a field of characteristic 0 and let n and k be positive integers, where $1 < n < \infty$. Every trace identity of the Jordan algebra $(J_{n,k}, tr)$ follows from the polynomials*

$$\begin{aligned} f_2(x) &= x^2 - \mathrm{Tr}\,(x)x + (1/2)(Tr(x)^2 - \mathrm{Tr}\,(x^2)), \\ N_{n-k+1} &= \sum_{\sigma \in S_{n-k+1}} (-1)^\sigma \prod_{i=1}^{n-k+1} H(x_{\sigma(i)}, y_i). \end{aligned}$$

Here $H(x, y) = \mathrm{Tr}\,(xy) - (1/2)\mathrm{Tr}\,(x)\mathrm{Tr}\,(y)$.

Proof. We denote by Cap^{Tr}_{n-k+1} the trace ideal of all Capelli-type trace polynomials of order $n-k+1$ that are trace identities for $(J_{n,k}, \mathrm{tr})$ and by I the T-ideal with trace generated by the trace polynomial f_2. Theorem 4.1 implies $I = T_{Tr}(J_{n,k}, \mathrm{tr})$, modulo Cap^{Tr}_{n-k+1}.

Assuming that Q is the T_{Tr}-ideal generated by f_2 and N_{n-k+1}, we have $Q \subseteq T_{Tr}(J_{n,k}, \mathrm{tr})$. On the other hand, as $\mathrm{char}K = 0$, for any multilinear polynomial $f \in T_{Tr}(J_{n,k}, \mathrm{tr})$, we get $f = g + h$, where $g \in I$ and $h \in Cap^{Tr}_{n-k+1}$. Lemma 4.7 implies that any Capelli-type trace polynomial p of order $n-k+1$, which lies in

$T_{Tr}(J_{k,n})$, is a consequence of N_{n-k+1}, modulo I, i.e., $p \in Q$. In particular, $h \in Q$, and so $f \in Q$. Thus, we get the desired conclusion. □

We can also generalize the Specht problem to classes of trace algebras. In other words, the ideal of trace identities of (A, τ) satisfies the Specht property if each T-ideal with trace containing $T_{Tr}(A, \tau)$ is finitely generated, as a T_{Tr}-ideal.

Let us mention that the variety of associative algebras with trace generated by the $n \times n$ matrices as well as the variety of the Jordan algebras with trace generated by B_n both have the Specht property. The former result was deduced by Razmyslov [29], and Fidelis, Diniz, and Koshlukov established the latter in [8]. These are the only known examples so far, although in [8] it was proved that under some additional restrictions on the algebra A, each ascending chain of T_{Tr}-ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ of the algebra $G(X)$, containing $T_{Tr}(A)$, stabilizes. Finally, as in the graded case, it is easy to see that $T_{Tr}(B_{n-k+1}) \subseteq T_{Tr}(J_{n,k})$ holds, for all $n > 1$ and $k > 0$.

Due to these comments we have just established the following result.

Theorem 4.9. *Over a field of characteristic zero, the T_{Tr}-ideal $T_{Tr}(J_{n,k}, tr)$ satisfies the Specht property, for all positive integers n and k satisfying $1 \leq n < \infty$ and $k > 0$.*

We conclude this paper by making some short consideration about the so-called embedding problem. To better understand it, we have to make a brief comment on the theory of invariants.

Let (A, tr) be an algebra with trace, let $A^k = A \oplus \cdots \oplus A$, k times, be the direct sum of k copies of A and let $G \subseteq Aut_K A$ be an algebraic group acting diagonally on A^k. If one considers the algebra $\mathcal{A}$ of polynomial functions $A^k \to A$, then G acts also on $\mathcal{A}$ as follows. If $f \in \mathcal{A}$, then

$$f^g(a_1, \dots, a_k) = g \cdot f(g^{-1} \cdot a_1, \dots, g^{-1} \cdot a_k),$$

for all $g \in G$ and for all $a_1, \dots, a_k \in A$. Therefore, $f = f^g$ for all $g \in G$ means that f is a G-equivariant map.

Denote by R_a, respectively L_a, the operator of right, respectively left, multiplication on A by the element $a \in A$ and denote, as usual, by $End_K A$ the K-algebra of all linear transformations on the vector space A. Every element $b = b(x_1, \dots, x_k) \in K(x_1, \dots, x_k)$ defines the following functions on A^k :

$$R_b,\ L_b : A^k \to End_K A,$$

by means of $R_b(a_1, \dots, a_k) = R_{b(a_1,\dots,a_k)}$ and $L_b(a_1, \dots, a_k) = L_{b(a_1,\dots,a_k)}$. Finally, denote by $K[A^k]$ the algebra of polynomial functions $A^k \to K$ and let Tr_k be the subalgebra of $K[A^k]$ generated by the set

$$\{tr(T_{h_1} \cdots T_{h_n}) \mid T \in \{R, L\},\ h_1, \dots, h_n \in K(X)\}.$$

The description of the generators of the algebra of invariants

$$K[A^k]^G = \{f \in K[A^k] \mid f^g = f \text{ for all } g \in G\}$$

is one of the main problems of the classical Invariant Theory. It is often referred to as the First Fundamental Theorem of Invariant theory. We also recall for completeness that the Second Fundamental Theorem deals with the description of the relations among the generators. If we consider G as the automorphism group of A, it is clear that $Tr_k \subseteq K[A^k]^G$. It is natural to ask whether $Tr_k = K[A^k]^G$. Such equality is not valid in general, although holds for many important cases. For instance, the equality holds for the algebra $M_n(K)$ of $n \times n$ matrices over K (see [26, 28]). Popov in [25, subsection 4.7] constructed a simple algebra A such that $Tr_k \subsetneq K[A^k]^G$, a proper inclusion.

In [8], the authors consider A as being a central simple finite dimensional associative or Jordan algebra over K. They take the generic trace Trd on A and denote by Tr'_k the subalgebra of $K[A^k]$ generated by the elements $\mathrm{Trd}\,(v) = \mathrm{Trd} \circ v$, where $v \in K(y_1, \dots, y_k)$, the free algebra on the set of k free generators y_1, $\dots$, y_k. Notice that $Tr'_k \subseteq K[A^k]^G$. In [8], Fidelis, Diniz and Koshlukov considered algebras that satisfy the following condition:

(C1) Let G be an algebraic linear reductive non-trivial group. If G is the group of automorphisms of the central simple finite dimensional algebra A then $Tr'_k = K[A^k]^G$.

In the same paper, the authors obtained the following theorem.

Theorem 4.10. [8, Theorem 4.7] *Let A be a central simple finite dimensional associative or Jordan algebra that satisfies the conditions (C1). Assume that R is an algebra with trace belonging to the same variety of algebras as A, and that satisfies all trace identities of A. Then there exists a commutative–associative algebra C such that R can be embedded into $A \otimes_K C$ as a K-algebra.*

Here an algebra will be called *commutative–associative* if it is a commutative algebra in the variety of all associative algebras over K.

A question that seems to be interesting is whether the latter result holds in the case of infinite dimension. Moreover, an important tool was the construction of universal maps. More precisely, let A and R be algebras in a same variety $\mathcal{V}$ such that A is a central simple power associative algebra. In [8], it was proved that if A is of finite dimension, then there is an universal map

$$j\colon R \to A \otimes_K S,$$

where S is a commutative–associative algebra, satisfying the following universal property: given a homomorphism $\varsigma\colon R \to A \otimes_K F$, for some commutative–associative algebra F, there exists an unique homomorphism $\eta\colon S \to F$ making the diagram

$$\begin{tikzcd} R \arrow[r, "\varsigma"] \arrow[d, "j"'] & A \otimes_K F \\ A \otimes_K S \arrow[ur, "\eta_A"'] & \end{tikzcd},$$

commutes, where η_A is the map induced by η. The pair (S, j) is called *A-universal map for R*. This pair has an important function: the entries of the elements $j(r)$, for all $r \in R$, together with the element 1, generate the algebra S. In particular, assuming the projection map $\pi_{n-k}\colon J_{n,k} \to B_{n-k} \simeq (J_{n,k}/D_k)$, it is easy to verify that the pair (K, π_n) is B_{n-k}-universal map for $J_{n,k}$ for all $n > 1$ (in variety of all Jordan algebra). Such property is even obtained for $n = \infty$.

Our final goal, in this work, is to exhibit an example for which the embedding problem solved in Theorem 4.10 is not valid in general. Notice that Theorem 4.5 implies that the algebras $J_{\infty,k}$ and B_∞ are PI-equivalent as trace algebras.

In order to simplify the notation, from now on let $B = B_\infty$.

Clearly Remark 2.18 also applies to B. Hence $K[B^l]^G$ can be expressible in terms of scalar products, i.e., it is generated by $\operatorname{Tr} X_i$ and $\operatorname{Tr}(X_i X_j)$, where each X_l denotes the projection given by $(a_1, a_2, \ldots) \mapsto a_l$ on the l-th coordinate. We conclude that B satisfies (C1).

Now we suppose that the trace algebra (B, tr) satisfies Theorem 4.10. By Theorem 4.5, there exists a commutative–associative algebra C_k (which depends on the integer $k > 1$) such that $J_{\infty,k}$ can be embedded into $B \otimes_K C_k$ as a K-algebra. In particular, $J_{\infty,1} \subseteq B \otimes_K C_1$, if $k = 1$. On the other hand, if we let F be the field of fractions of C_1, then we get

$$J_{\infty,1} \otimes_K F \supseteq B \otimes_K F = B \otimes_K (C_1 \otimes_{C_1} F) \simeq (B \otimes_K C_1) \otimes_{C_1} F \supseteq (B \otimes_K C_1) \otimes_K F.$$

Here the first inclusion is valid because $B \subseteq J_{\infty,1}$ and the last one follows from the restriction of scalars induced by the inclusion map $K \hookrightarrow C_1$. Therefore,

$$J_{\infty,1} \otimes_K F \subseteq (B \otimes_K C_1) \otimes_K F \subseteq B \otimes_K F \subseteq J_{\infty,1} \otimes_K F.$$

We conclude $J_{\infty,1} \otimes_K F = B \otimes_K F$, which is a contradiction.

Hence we can summarize the previous arguments in the following theorem.

Theorem 4.11. *In the variety of unitary Jordan K-algebras, the finiteness hypothesis of Theorem 4.10 is essential.*

5. Open problems

Throughout this section we relate some possible generalizations of the results presented in this paper. Here our intention is to motivate future research on the subject. We believe we have grounds to raise the following conjectures.

Conjecture 5.1. *Let K be an infinite field of characteristic different from 2,3,5 and 7. The polynomials*

$$([x,y]^2,z,t),$$
$$\sum_{\sigma\in S_3}(-1)^\sigma(x_{\sigma(1)},(x_{\sigma(2)},x,x_{\sigma(3)}),x),$$
$$\sum_{\sigma\in S_{n-k+2}}(-1)^\sigma(x_{\sigma(1)},y_1,x_{\sigma(2)},\ldots,y_n,x_{\sigma(n-k+1)},y_{n-k+1},x_{\sigma(n-k+2)}),$$
$$\sum_{\sigma\in S_{n-k+1}}(-1)^\sigma(x_{\sigma(1)},y_1,x_{\sigma(2)},\ldots,y_{n-1},x_{\sigma(n-k)})(y_{n-k},x_{\sigma(n+1)},y_{n-k+1}),$$

form a basis of the T-ideal of polynomial identities for the Jordan algebra $J_{n,k}$, with $k\geq 1$ and $1<n<\infty$.

Conjecture 5.2. *Let K be an infinite field of characteristic different from 2, then any weak identity of the pair $(J_{n,k},V)$ follows from the weak identiies*

$$(x_1x_2,x_3,x_4)$$
$$\sum_{\sigma\in S_{n-k+1}}(-1)^\sigma(y_1x_{\sigma(1)})(y_2x_{\sigma(2)})\cdots(y_{n-k+1}x_{\sigma(n-k+1)}).$$

Conjecture 5.3. *Let K be an infinite field of characteristic different from 2. The T_2-ideal of the 2-polynomial identities of the Jordan algebra $J_{n,k}$ with the scalar grading, where n is a positive integer and $k>0$, is generated by the 2-polynomials from*

$$(y,u_1,u_2)$$
$$\sum_{\sigma\in S_{n-k+1}}(-1)^\sigma(z_{n-k+2}z_{\sigma(1)})(z_{n-k+3}z_{\sigma(2)})\cdots(z_{2(n-k)+3}z_{\sigma(n-k+1)}),$$

where u_1, $u_2\in Y\cup Z$.

Conjecture 5.4. *Let K be an infinite field of characteristic different from 2 and let n and k be positive integers, where $1<n<\infty$ and $k>0$. Every trace identity of the Jordan algebra $(J_{n,k},tr)$ follows from the polynomials*

$$f_2(x) = x^2-\mathrm{Tr}\,(x)x+(1/2)(Tr(x)^2-\mathrm{Tr}\,(x^2)),$$
$$\sum_{\sigma\in S_{n-k+1}}(-1)^\sigma\prod_{i=1}^{n-k+1}(\mathrm{Tr}\,(x_{\sigma(i)}y_i)-(1/2)\mathrm{Tr}\,(x_{\sigma(i)})\mathrm{Tr}\,(y_i)).$$

Although we have enough evidence to believe that these results may be true, the techniques developed in this paper are not enough to prove such conjectures, since the use of multilinear polynomials was essential in order to get our main theorems.

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