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REFLECTING ON THE BASES OF GEOMETRY: CONSTRUCTION WITH THE TRACTRIX

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(communicated by Liliana Restuccia)

ABSTRACT. Construction is, historically, the first form of geometry, and from early days the subtleties of how constructions are done have been considered important. In this paper, we examine geometric construction using a straightedge and a device for drawing tractrices. The tractrix was the first curve traced by the mechanical solution of an inverse tangent problem, the geometrical issue at the basis of Leibniz's conception of infinitesimal analysis. This nonalgebraic curve cannot be axiomatized simply, as the circle can. We show that some important constructions can be done based on a weak axiomatization that does not fully specify the curve, and that more may be done using its Cartesian representation.

1. Introduction

Circles and straight segments are at the heart of Euclid's geometry. The circle is drawn (axiomatically) given the center and a point on the circumference, not the center and the radius. An actual compass can do both of these: Euclid uses the first (Euc. I.3) to do the second. This has given rise to the myth of the "collapsible compass," a device that Euclid never mentions, though it vividly embodies his axiomatic circle construction. Again, an artist or artisan might use a straightedge to construct a tangent to a circle through a given point; Euclid does not consider this as a "primitive" construction for a line, but emulates it in Euc. III.17. Nor is Euclid's set of construction postulates the most basic possible. All the points constructed by Euclid's tools can be obtained by a compass alone (Mohr-Mascheroni) or by a straightedge and a single circle (with center) drawn on a plane (Poncelet-Steiner).

The problem of defining the ideal instruments to be adopted in "pure geometry" was strongly relevant when the foundation of mathematics relied on geometrical constructions, not only in antiquity (we may think of Pappus' distinction of the plane, solid, and linear constructions) but also in the early modern period. This problem was explicitly posed, perhaps for the first time, in 1637 Descartes' Géométrie (Bos 2001).

The first introduction of the tractrix is due to Perrault (ca. 1670). In contrast to Euclid, Perrault described this curve in terms of the device used to construct it, namely a pocket

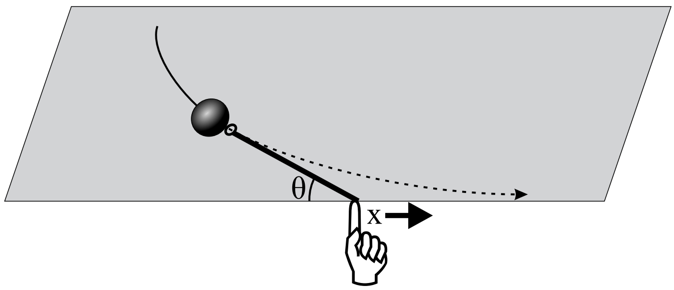

FIGURE 1. Perrault's construction of the tractrix

watch. As visible in Fig. 1, the watch is dragged across a table top by a chain of fixed length, the other end of which is pulled in a straight line (the *directrix*.)

The tractrix has many other important properties that do not follow obviously from its construction. Rotating a tractrix in space about its directrix yields a *pseudosphere*, a surface embedded in $\mathbb{R}^3$ with constant negative Gaussian curvature. The pseudosphere is a model of hyperbolic non-euclidean geometries, as the sphere is a model of elliptic geometry. Furthermore, the orthogonal trajectories of the family of congruent circles centered on a line r are translated tractrices having r as their asymptote. It is therefore interesting to consider the possibility of an axiomatic geometry of lines and tractrices.

2. Historical constructions with tractories

The tractrix is generated by the dragging of a point tied to a string or a rod. Constructions like this one are named tractional and were relevant at the turn of the 18th century to trace transcendental curves and solve inverse tangent problems (the string remains tangent to the traced curve). For historical and foundational issues cf. Bos (1988), Tournès (2009), and Milici (2015); for a deeper reflection on the underlying principles, cf. Dawson, Milici, and Plantevin (2021).

Since the first publication on tractrices, made by Huygens in 1693, many other properties of the curve have been discovered. Several of these are familiar exercises in calculus: it is possible to rectify tractrix arcs, to find the area under the curve and the area and volume of the solid generated by turning the tractrix around its directrix. All of these results yield straightedge-and-compass constructions. In some cases purely geometric demonstrations are known: see, for instance, section 15.4 of Apostol and Mnatsakanian (2013).

Perrault's watch can construct at most half of a tractrix (less than half, if the chain is not originally orthogonal to the directrix.) This problem is avoided if the chain is replaced by a rigid rod, which can be both pushed and pulled. A further improvement, in the 19th century, was the "tractoriograph" of Fig. 2. A sharp-edged wheel T is inked by the buffer F, and both are part of a carriage: after fixing the length of the "arm" (the distance between the carriage and the pointer H), the motion of H along the directrix makes the wheel's contact point t draw the tractrix.

Especially with a rigid tractoriograph, the linear directrix can be replaced by any other curve, although the resulting "tractory" may well not be expressible in terms of elementary

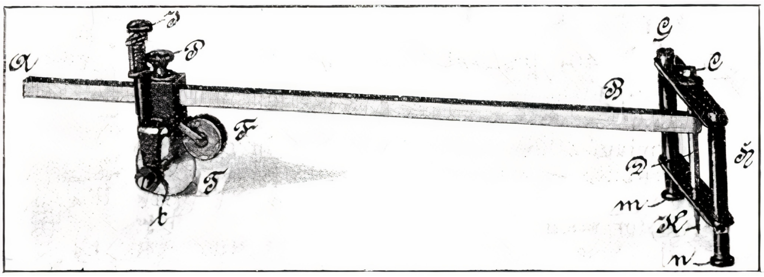

FIGURE 2. A tractoriograph (Kleritj 1897, p. 234).

functions. If the directing curve is a closed contour Γ, it does not follow that the tractory will also be closed. Indeed, if the rod is long compared with the diameter of Γ, the distance between the endpoints of the tractory after one circuit of Γ is approximately proportional to the area inside Γ, regardless of its shape. This is the principle behind the famous Prytz hatchet planimeter, an instrument used widely before the introduction of computers to compute the area of irregularly-shaped regions. The tractoriograph, in combination with straightedge and compass, can also be used to construct π, e, and any regular polygon. Most of these constructions make use of the "circular tractrix": the tractory directed by a circle, with the arm of the tractoriograph set equal to the radius of the circle, see D. Tournès, "Instruments for impossible problems: Around the work of Ljubomir Klerić (1844-1910)" (Gessner *et al.* 2017, pp. 3517–3520).

But what if we consider geometric construction by a tractoriograph and a straightedge, without the compass? To allow readers to experiment, we provide a GeoGebra file that introduces a button to trace the tractrix given 3 points: the dragging one (A), the dragged one (B), and the one that defines the direction of the baseline (the point C, where the baseline is AC). The file is available at https://www.geogebra.org/classic/dxbfhark.

3. The axiomatic tractrix

Euclid defines a circle (Definitions 15, 16) as the set of points lying at a fixed distance from the center. Unfortunately, he does not define distance, and we now know that there are infinitely many ways to define distance in the real plane. On its own, then, Euclid's definition isn't very helpful. But he slips in the very powerful axioms that the metric is homogeneous and isotropic when he assumes in Prop. I.4 that one configuration consisting of two segments and an included angle can be "applied" to another if the three parts are congruent. (Today we would say that there exists an isomorphism of the plane taking one configuration to the other.) We are then at least close to a full axiomatic characterization of the circle.

But the circle is one of the simplest curves: it seems impossible to capture the entire nature of a more complicated curve such as a tractrix axiomatically. In this note we will not attempt to do this, but will note some properties of tractrices that, taken as axioms, allow us to prove certain constructions to be possible.

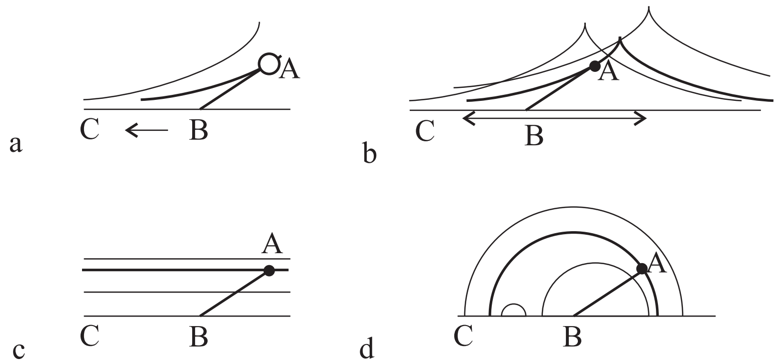

FIGURE 3. Some examples of P-families

Note that we adopt the following notation: angles $\angle ABC$, lines $\overleftrightarrow{AB}$, rays $\overrightarrow{AB}$, segments $\overline{AB}$, distance $|AB|$. Furthermore, a point P between A and B is represented by $A-P-B$.

4. First approximation: homothetic families of curves

Let $T(A,B,C)$ be the partial ("Perrault") tractrix in which the watch starts at A and the other end of the chain is pulled along the ray $\overrightarrow{BC}$. (We assume that the chain is initially taut and that $\angle ABC$ is not acute.) We call B the *foot*, $\overline{AB}$ the *leg*, and the ray $\overrightarrow{BC}$ the *directrix*.

Remark 1. $A \in T(A,B,C)$.

Remark 2. *Any two partial tractrices T,T' with the same directrix $\overrightarrow{BC}$, and legs $\overline{AB}$, $\overline{A'B}$ with $A-A'-B$, are homothetic with center B.*

Remark 3. *If a partial tractrix has directrix $\overrightarrow{BC}$ and leg $\overline{AB}$, it intersects every ray $\overrightarrow{BX}$ on the interior of $\angle ABC$.*

We call any family of plane curves indexed by noncollinear ordered triples of points (A,B,C), and for which Remarks 1-3 hold, a *P-family*. This definition is very broad.

Example 1. *The family of partial tractrices as generated by a watch tractoriograph, with the watch initially at A and the chain dragged along the ray $\overrightarrow{BC}$ is a P-family (Fig. 3a).*

Example 2. *The family of tractrices as generated by a hatchet tractoriograph with the tip initially at A and the other end dragged back and forth along the line $\overleftrightarrow{BC}$, starting at B, is a P-family (Fig. 3b).*

Example 3. *The family of lines through A parallel to $\overrightarrow{BC}$ is a P-family (Fig. 3c).*

Example 4. *The family of semicircular arcs centered at B, passing through A, and beginning and ending on $\overleftrightarrow{BC}$, is a P-family (Fig. 3d).*

By a *P-graph* we will understand a device that can construct a curve from a fixed *but unspecified* P-family given the triple (A,B,C). We encapsulate Remarks 1-3 in the following set of construction postulates. (We are assuming the geometry of the plane to be Euclidean.)

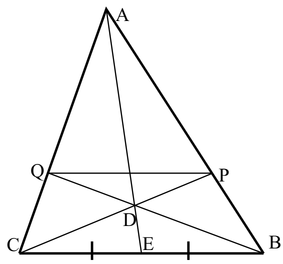

FIGURE 4. Steiner's lemma – a straightedge-only construction

Postulate 1. *To draw a straight line through two given points.*

Postulate 2. *Given three points A,B,C, not all on one line, to construct a P-curve $P(A,B,C)$ with leg $\overline{AB}$ and directrix $\overrightarrow{BC}$.*

Postulate 3. *To find the intersection of two nonparallel lines.*

Postulate 4. *To find the intersections of a line and a P-curve if they exist.*

We call a construction *affine* if it is preserved under all affine transformations of the plane. Affine straightedge-and-compass constructions can parallel-translate any segment to an arbitrary location, add and subtract parallel lengths, and find fourth proportionals $p = xy/z$ where $x||z$, $y||p$. These affine constructions are precisely those that can be carried out with a parallel rule (which is a specific form of P-graph, as in example 3.)

Theorem 1. *With construction postulates 1-4 we can perform exactly the affine constructions listed above.*

Certainly the generic P-graph and straightedge can do no more than the parallel rule. We will show that with any P-graph and a straightedge we can construct a parallel to a given line through a given point (Euc. I.31) thus emulating any parallel-rule construction. We begin with a lemma that Steiner used similarly in his fixed-circle-and-straightedge construction (Steiner 1833).

Lemma 1 (Steiner)**.** *Given A,B,C,P,Q with $A-P-B$, $A-Q-C$, let $D := \overline{BQ} \cap \overline{CP}$, and $E := \overrightarrow{AD} \cap \overline{BC}$ (Fig. 4). The following are equivalent:*

(1) $|AB|/|AP| = |AC|/|AQ|$;
(2) $\overline{BC}||\overline{PQ}$;
(3) *E is the midpoint of $\overline{BC}$.*

Proof. (1)$\Leftrightarrow$(2) by familiar properties of similar triangles. (1)$\Leftrightarrow$(3) by Ceva's theorem applied to $\triangle ABC$. □

Problem 1. *Using a straightedge and P-graph, to construct a bisected segment with one endpoint on a given line ℓ.*

Solution: Construct an arbitrary second line λ intersecting ℓ at a point A. Construct an arbitrary segment $\overline{AB}$ not coincident with ℓ or λ, a point $P \in \overline{AB}$ distinct from A and B, and a point $D \in \lambda$ such that one ray of ℓ is interior to $\angle BAD$.

Now construct the P-curves $T := P(B,A,D)$ and $T' := P(P,A,D)$ (Fig. 5*a*). By hypothesis, $C := \ell \cap T$ and $Q := \ell \cap T'$ exist, and $|AB|/|AP| = |AC|/|AQ|$. We can thus use Steiner's lemma to construct the midpoint E of $\overline{BC}$ (Fig. 5*b*). □

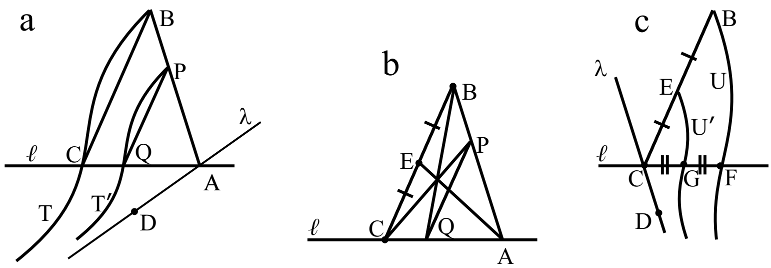

FIGURE 5. Bisected segment constructions using P-curves

Problem 2. *Using a straightedge and P-graph, to construct a bisected segment with both ends on a given line ℓ.*

Solution: Using the result of Problem 1, construct a line segment $\overline{BC}$ with $C \in \ell$ and with midpoint E. Select an arbitrary point D such that one ray of ℓ is interior to $\angle BCD$, and construct the P-curves $U := P(B,C,D)$ and $U' := P(E,C,D)$. Then by hypothesis $F := \ell \cap U$ and $G := \ell \cap U'$ exist, and $|CF|/|CG| = |CB|/|CE| = 2$ (figure 5*c*). □

Problem 3. *Using a straightedge, and given the midpoint E of $\overline{BC}$, to construct a line parallel to $\overline{BC}$ through a given point P.*

Solution: Construct $\overrightarrow{BP}$ and choose an arbitrary point A on $\overrightarrow{BP}$ with $B-P-A$ (see figure 4.) Let $D := \overline{CP} \cap \overline{AE}$ and $Q := \overrightarrow{BD} \cap \overline{CA}$. Then, by Steiner's lemma, $\overleftrightarrow{PQ} \| \ell$. □

This concludes the proof of Theorem 1. It follows immediately that using a straightedge, and a P-graph such as Perrault's watch, we can perform any rational construction (adding, subtracting, or finding fourth proportionals xy/z) provided the data are given as parallel segments.

5. Second approximation: symmetry

We now consider two properties of full tractrices that are not shared by partial tractrices (figure 6).

Remark 4. *A (full) tractrix has mirror symmetry with respect to the perpendicular to its directrix through its cusp.*

Remark 5. *If a line λ is parallel to the directrix of a tractrix T and intersects T, it does so in two points or at the cusp.*

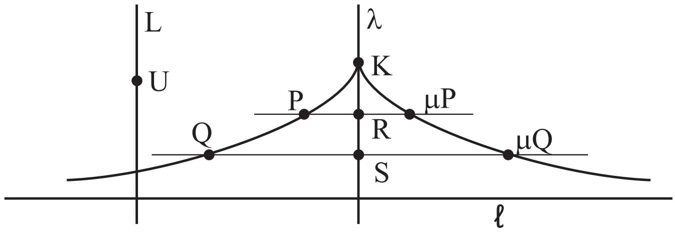

FIGURE 6. Lines of symmetry and perpendiculars

Suppose that every P-curve in a family has mirror symmetry μ with respect to some perpendicular to its directrix, and if it intersects a line parallel to its directrix, does so in at most two points. We call such curves *M-curves*, the family an *M-family*, and a P-graph that generates them an *M-graph*. The families of curves in examples 2 and 4 are M-families; the families in examples 1 and 3 are not. We strengthen our construction postulates as follows:

Postulate. **(2^m)** *Given three points A,B,C, to construct a M-curve T(A,B,C) with leg $\overline{AB}$ and directrix $\overleftrightarrow{BC}$.*

With postulates (1, 2^m, 3, 4) we can construct perpendiculars.

Problem 4. *Given an M-curve T and its directrix ℓ, to construct the line of symmetry using a straightedge and P-graph.*

Solution: Let $P,Q \in T$ be arbitrary. Construct parallels to ℓ through these: they meet T again at $\mu P, \mu Q$. Let R,S be the midpoints of $\overline{P\,\mu P}$ and $\overline{Q\,\mu Q}$; then $\overleftrightarrow{RS}$ is the line of symmetry (Fig. 6). □

(Note that this also constructs the point $T \cap \overleftrightarrow{RS}$; if T is a tractrix, this is the cusp.)

Problem 5. *Using a straightedge and M-graph, to construct a perpendicular to a line ℓ through a given point U.*

Solution: Construct an M-curve T on ℓ and (as above) its line of symmetry λ; then construct a parallel to λ through U. □

The next result is an impossibility proof, showing that a *generic* M-graph and straightedge are not sufficient for certain standard Euclidean constructions, such as mean proportionals and Euc. I.3 (nonparallel transfer of length data.) Of course, a specific M-graph (*e.g.*, the common compass) may be able to do one or both of these. The following result is similar to the main result of Dawson (2007).

Theorem 2. *The construction of mean proportionals, and the nonparallel length transfer of Euc. I.3, cannot be carried out with a generic M-graph and straightedge.*

Proof. Given a line ℓ and a segment $\overline{AB}$ with B (and not A) on ℓ, let A' be the orthogonal projection of A onto ℓ. Let D, E be points on ℓ at distance $|AA'| + |A'B|$ from A'. Call the

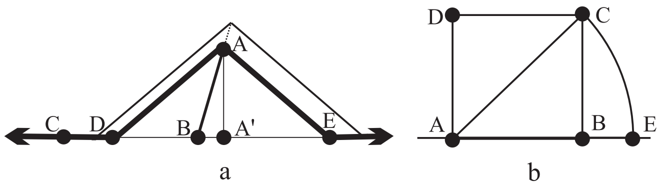

FIGURE 7. An M-family that preserves rationality

union $(\ell \setminus \overline{DE}) \cup \overline{DA} \cup \overline{AE})$ a Λ-*curve* with leg $\overline{AB}$ and directrix ℓ. We may specify such a curve by the points A, B, and any other point $C \in \ell$ with $\angle ABC \geq \pi/2$ (Fig. 7a).

If the points defining a Λ-curve have rational coordinates, then the rays and segments comprising it have rational slopes and intercepts; it follows that this model of postulates (1, 2^m, 3, 4) is rational-closed. Thus, given a line segment, no construction using these postulates can construct a parallel segment in the ratio $1 : \sqrt{2}$, from which both claims follow. In particular, given problem 5 and Euc. I.3, we could construct a square $\Box ABCD$ on a given base $\overline{AB}$, and then construct a segment $\overline{AE}$ on the ray $\overrightarrow{AB}$ with $|AE| = |AC| = \sqrt{2}|AB|$ (Fig. 7b). $\Box$

We do not know if a generic M-graph and straightedge can perform all rational-closed Euclidean constructions. In particular, we ask: is it possible to rotate a line segment through $\pi/2$ and construct a square? (We know we cannot rotate it though $\pi/4$.)

6. The tractrix itself

A specific M-graph, such as a tractoriograph, may be able to do more. How much more appears to depend upon how we view the use of the device. This is analogous to distinctions implicit in Euclid's *Elements*. In Postulate 3, the construction of a circle, Euclid does not even mention compasses: only the circle and the data from which it is constructed are mentioned. Contrary to popular belief, the question of whether the compasses may "legally" be used as dividers is never raised in the *Elements*, and Euc. I.3 makes it moot.

The basic tractrix-construction postulate is the following.

Postulate. **(2^t)** *Given three noncollinear points A, B, C, to construct a tractrix with leg $\overline{AB}$ and directrix $\overleftrightarrow{BC}$.*

With this postulate, we can carry out the construction of Euc. I.3 and thus relate nonparallel lengths. In light of problem 3 it suffices to consider the case where one end of the segment is on the line.

Problem 6. *Given three noncollinear points A, B, C, to construct D on $\overrightarrow{AC}$ such that $|AD| = |AB|$, using construction postulates 1, 2^t, 3, and 4.*

Solution: We construct ℓ through A perpendicular to $\overline{AC}$. If B doesn't belong to ℓ, choose an arbitrary $E \in \ell$ on the far side of $\overline{AC}$ from B. We then construct the tractrix $T(B,A,E)$ with leg $\overline{AB}$ and directrix ℓ. Let F be its cusp, let $F \in \lambda \| \ell$, and let $D := \lambda \cap \overrightarrow{AC}$.
If B belongs to ℓ, choose an arbitrary C' s.t. $\overline{AC'}$ is not perpendicular to $\overline{AB}$ and $\overline{AC}$. By

the previous construction construct D' on $\overrightarrow{AC'}$ such that $|AD'| = |AB|$, and finally on $\overrightarrow{AC}$ construct D s.t. $|AD| = |AD'|$

$\square$

Given two legs of a right triangle, we can construct the hypotenuse. However, this in itself is not enough to emulate all Euclidean constructions: addition, subtraction, the fourth-proportional partial operation, and the operation

$$H : (x,y) \mapsto \sqrt{x^2+y^2}$$

cannot be composed to compute the mean proportional $\sqrt{xy}$. To see this, note that the three operations

$$+,-,H : \mathbb{R}^2 \to \mathbb{R}$$

and the partial operation

$$FP : \mathbb{R}^3 \rightharpoonup \mathbb{R}$$

map into $\mathbb{R}$ wherever they are defined. Thus $(x,y) \mapsto \sqrt{xy}$, which takes imaginary values for some (x,y), cannot be a composition of $+,-,H$, and FP. By the same argument, those four operations cannot be composed to compute one leg of a right triangle given the hypotenuse and the other leg: for if we could do this, we could compute

$$\frac{1}{2}\sqrt{(x+y)^2-(x-y)^2} = \sqrt{xy}.$$

Now, in the presence of postulates 1, 3, and 4, problem 6 does not represent the full strength of postulate 2^t. For instance, the tractrix directed by the x axis and with cusp $(0,a)$ has Cartesian formula

$$x = \pm a \ln \frac{a+\sqrt{a^2-y^2}}{y} - \sqrt{a^2-y^2}. \tag{1}$$

Let $a = u^2/v + v$, $y = 2u$. Then

$$\sqrt{a^2-y^2} = \sqrt{u^4/v^2+2u^2+v^2-4u^2} = u^2/v - v$$

and substituting this into (1) we get

$$x = \left(u^2/v+v\right)\ln(u/v) - \left(u^2/v-v\right). \tag{2}$$

Thus if we use postulate 2^t to construct a tractrix with cusp at $(0,u^2/v+v)$ directed by the x axis, a horizontal line through $(0,2u)$ will meet the curve at

$$\left(\left(u^2/v+v\right)\ln(u/v) - \left(u^2/v-v\right), 2u\right).$$

Straightforward arithmetic constructions yield the solution to the following.

Problem 7. *Given lengths x,y,and z, to construct a segment of length* $x\ln(y/z)$ *using construction postulates 1,* 2^t*, 3, and 4.* [1]

[1] Dimensional analysis tells us that any transcendental function must take a dimensionless argument and return a dimensionless value; thus, if our data are to be given as segment lengths, the logarithm function must be given in the form that we've used. Of course, if we arbitrarily declare a segment to be of length 1 (a sort of "standard meter") we can set x and z to that length and "construct $\ln(y)$," but this is contrary to the spirit of Euclidean construction.

Nonetheless, we conjecture that constructing a tractrix as in postulate 2^t, where we "present three points and get a completed tractrix back," does not allow us to construct mean proportionals. While there are various unexplored ways in which a tractrix can intersect with a line or another tractrix, most of these yield algebraically intractable results. However, if we consider the tractrix to be constructed one point at a time (as by Perrault's watch or Prytz's planimeter), we can pause the construction when the "leading end" reaches a specific point on the directrix, or when the "following end" crosses a specified curve, and note the position of the other end. We can, of course, axiomatize these actions without reference to the tractoriograph.

Postulate 5. *Given a tractrix T and its directrix, to construct the tangent to T at a point $B \in T$.*

Postulate 6. *Given a tractrix T with directrix ℓ, to construct the tangent to T through a point $C \in \ell$.*

In each case, by the defining property of the tractrix, the tangent may also be identified by the distance between its intersections with T and ℓ. These construction postulates are thus equivalent to finding a point on the directrix (resp. tractrix) whose distance from a given point on the tractrix (resp. directrix) is the height of the tractrix. Both may, of course, easily be performed with compasses.

Problem 8. *Given segments of lengths x and y, to construct a segment of length $\sqrt{x^2-y^2}$ using construction postulates 1, 2^t, 3, 4 and 5.*

Solution: Suppose $\overline{AB}$ to have length x. Construct the line ℓ through A perpendicular to $\overline{AB}$, and the tractrix T with leg AB and directrix ℓ. Now construct a parallel λ to ℓ at distance y, with C one of the points where λ cuts T, and C' the foot of its perpendicular to ℓ. Using postulate 5, construct a point $D \in \ell$ with $|DC| = |AB|$. Then $|DC'| = \sqrt{x^2-y^2}$.

Problem 9. *Given segments of lengths x and y, to construct a segment of length $\sqrt{xy}$ using construction postulates 1, 2^t, 3, 4, and 5.*

Solution: As observed above, $\sqrt{xy} = \frac{1}{2}\sqrt{(x+y)^2-(x-y)^2}$; so this follows from problem 8. □

Theorem 3. *All Euclidean constructions can be emulated using construction postulates 1, 2^t, 3, 4, and 5.*

Problem 10. *Given segments of lengths p and x, to construct segments of length $pe^{x/p}$ and $pe^{-x/p}$ using construction postulates 1, 2^t, 3, 4, and 6.*

Solution: We construct a tractrix directed by the x axis with cusp at $(0,p)$, and construct the point $X := (x,0)$ with $x > 0$. Using postulate 6 we find the point $U \in T$ with coordinates (u,v), $0 < u < x$, from which the tangent contains X. Let $U' = (u,0)$ be the projection of U onto the directrix. Then

$$u = p\ln\left(\frac{p+\sqrt{p^2-v^2}}{v}\right) - \sqrt{p^2-v^2}$$

and

$$x = u + \sqrt{p^2 - v^2} = p \ln \left(\frac{p + \sqrt{p^2 - v^2}}{v} \right).$$

Construct the point W on the x axis with $|XW| = p$ and $U' - X - W$. Construct ℓ perpendicular to the x-axis at X, and let $Z := \ell \cap \overline{UW}$. Then $\triangle UU'W$ and $\triangle ZXW$ are similar, so

$$|ZX| = |UU'||XW|/|U'W| = \frac{vp}{p + \sqrt{p^2 - v^2}} = pe^{-x/p}.$$

We use fourth proportionals:

$$pe^{x/p} = \frac{p \times p}{pe^{-x/p}}$$

to complete the construction. □

Given the constructions of Problems 7 and 10, and the ability to compute fourth proportionals, the solution of the next problem is straightforward.

Problem 11. *Given segments of length p, q, x, y, to construct a segment of length z such that $z/x = (y/x)^{p/q}$ using construction postulates 1, 2^t, 3, 4, and 6.*

□

Construction postulate 5 (finding the point on the directrix corresponding to a given point on the tractrix) is equivalent to problem 9; both of these constructions may be carried out using the result of problem 11 in the special case $p/q = 1/2$. The special case $p/q = 1/3$ allows us to construct cube roots, and thus find one real root of any cubic equation. We thus have:

Theorem 4. *Construction postulates 1, 2^t, 3, 4, and 6 suffice to trisect the angle, duplicate the cube, and construct the regular heptagon.*

7. Conclusions and open questions

We have identified various ways in which the construction of a tractrix can be axiomatized, ranging from a very generic axiomatization that still allows some important constructions to axiomatizations that allow all Euclidean constructions, and others beside, to be performed.

Many open questions remain. In what follows, use of a straightedge and finding points of intersection will be assumed. The authors conjecture the answers to be in the negative in each case.

- A generic P-graph can, as we've seen, do rather little. But a (full) tractoriograph can do much more than a generic M-graph. So what about the Perrault watch tractoriograph, that can construct a minor arc or (at most) a half-tractrix, given the directrix and leg? Given a watch tractoriograph, the construction of a perpendicular to a given line and the construction of a full tractrix with given directrix and cusp are equivalent constructions: are they possible?

- Suppose that (as above) we can construct a full tractrix given its directrix and cusp. With only this construction and a straightedge, can we construct the cusp, and full tractrix, corresponding to a given minor arc? (Compare Euc. III.25, where a circle is reconstructed from an arc.)
- Our set of construction postulates does not include finding the intersections (there can be at least seven) of two tractrices. Can they be found using postulates 1, 2^t, 3, 4, and 6?
- Construction postulate 6 implies construction postulate 5. Is the converse true?
- Given a tractrix, can we construct its directrix using a tractoriograph? (Construction postulates 5 and 6 cannot be used on a tractrix with unknown directrix.)

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