LIOUVILLE-TYPE RESULTS FOR SEMI-LINEAR INEQUALITIES INVOLVING THE DUNKL LAPLACIAN OPERATOR

MOHAMED JLELI, BESSEM SAMET, AND CALOGERO VETRO

Abstract. Let Δ_k be the Dunkl generalized Laplacian operator associated to a root system R of $\mathbb{R}^N$ and a nonnegative multiplicity function k defined on R and invariant by the finite reflection group W. In this paper, we establish Liouville-type theorems for the semilinear inequality $-\Delta_k u \geq |u|^p$ in $\mathbb{R}^N$ and the system of inequalities $-\Delta_k u \geq |v|^p$, $-\Delta_k v \geq |u|^q$ in $\mathbb{R}^N$, where $N \geq 1$ and $p, q > 1$. To the best of our knowledge, this contribution is the first work dealing with Liouville-type results for nonlinear problems involving the Dunkl Laplacian.

1. Introduction

One of the most important subjects in the theory of partial differential equations is the issue of the nonexistence of a solution or, in other words, the necessary conditions for the existence of a solution. This issue was initiated by the famous Liouville theorem for harmonic functions. Much effort has been focused on establishing Liouville-type theorems for nonlinear elliptic equations and inequalities, see e.g. the series of papers [1, 4, 5, 7–9, 12, 13, 16, 18, 19, 21–27, 31] and the references therein.

A well-known model example related to nonlinear Liouville-type theorems is given by the elliptic inequality

$$-\Delta u \geq |u|^p \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

Ni and Serrin [25] investigated the radial case of (1.1). Namely, for positive solutions, the approach based on spherical averaging shows that, if $1 < p \leq 1 + \frac{2}{N-2}$, then (1.1) has no nontrivial solution. On the other hand, without the positivity assumption, the method in [25] cannot be applied. Mitidieri and Pohozaev [21–23] (see also [1, 4, 5, 18, 27]) have developed a universal method for proving Liouville-type theorems for a large class of stationary and evolution problems. This approach is based on capacity estimates, which in turn rely on suitable choice of test functions. For instance, by means of this method, without any restriction on the sign of solutions, it was shown [23] that, if $1 < p \leq 1+\frac{2}{N-2}$, then (1.1) has no nontrivial (weak) solution. In the case of systems of the form

$$\begin{cases} -\Delta u \geq |v|^p & \text{in } \mathbb{R}^N, \\ -\Delta v \geq |u|^q & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $p, q > 1$, by the same approach, it was shown(see again [23]) that, if

$$\frac{N}{2} \leq \frac{\max\{p(q+1), q(p+1)\}}{pq-1}, \tag{1.3}$$

then (1.2) has no nontrivial solution (i.e. $u \not\equiv 0$, $v \not\equiv 0$). Notice that the above condition is sharp, in the sense that, if

$$\frac{N}{2} > \frac{\max\{p(q+1), q(p+1)\}}{pq-1},$$

2010 *Mathematics Subject Classification.* 35J61; 35A01; 35B33.
Key words and phrases. Dunkl Laplacian; semilinear inequalities; Liouville-type results.

then (1.2) admits nontrivial solutions.

The goal of this paper is to extend the above results to the Dunkl Laplacian operator. Namely, our aim is to establish Liouville-type theorems for the semi-linear problems

$$-\Delta_k u \geq |u|^p \quad \text{in } \mathbb{R}^N \tag{1.4}$$

and

$$\begin{cases} -\Delta_k u \geq |v|^p & \text{in } \mathbb{R}^N, \\ -\Delta_k v \geq |u|^q & \text{in } \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $N \geq 1$, $p, q > 1$ and Δ_k is the Dunkl Laplacian operator associated with a finite reflection group W and a multiplicity function k. Notice that in the special case $k \equiv 0$, (1.4) (resp. (1.5)) reduces to (1.1) (resp. (1.2)).

Dunkl operators have been introduced by Dunkl (1989) [6]. These operators have many applications to the study of special functions with root systems. Moreover, the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero-Sutherland-Moser models (see e.g. [14,15,17,33]). Dunkl operators also find applications in harmonic analysis and integral transforms (see e.g. [2, 3, 10, 11, 20, 28–30, 32]), as they naturally lead to the Laplace-Dunkl operators Δ_k, which are second-order differential/difference operator that generalize the standard Laplace operator. In particular, Gallardo and Godefroy [10] extended the classical Liouville theorem for harmonic functions to Dunkl harmonic functions. Namely, it was shown that, if f is a bounded Dunkl harmonic function in $\mathbb{R}^N$ ($\Delta_k f = 0$), then it is a constant. In [28], a Liouville theorem for Dunkl-polyharmonic functions in $\mathbb{R}^N$ ($(\Delta_k)^p f = 0$) has been proved. However, to the best of our knowledge, the study of Liouville-type theorems for nonlinear problems involving Dunkl opertors has not been previously considered in the literature. This fact motivated us to study problems of type (1.4) and (1.5). Namely, for each problem, the existence and nonexistence of weak solutions are investigated. The proofs of the nonexistence results make use of the so called test functions method (see e.g. [23]), which is based on the derivation of suitable a priori bounds of the weak solutions by careful selection of special test functions. The existence results are obtained by the construction of explicit solutions.

The rest of the paper is organized as follows. In Section 2, we recall some basic properties and results related to Dunkl operators. In Section 3, after defining the weak solutions to (1.4) and (1.5), we state our main results. The proofs of the obtained results are given in Section 4.

2. The Dunkl Laplacian operator

For the reader's convenience, we recall in this section some basic properties and results related to Dunkl operators. For more details, we refer to [20, 29, 30].

We consider the Euclidean space $\mathbb{R}^N$ equipped with the inner product denoted by $\langle \cdot, \cdot \rangle$ and the associated Euclidean norm denoted by $|\cdot|$. For any $x \in \mathbb{R}^N$, we denote by $\langle x \rangle^\perp$ the hyperplane orthogonal to x, i.e. $\langle x \rangle^\perp = \{z \in \mathbb{R}^N : \langle z, x \rangle = 0\}$. For $\alpha \in \mathbb{R}^N \backslash \{0\}$, let σ_α be the reflection in the hyperplane $\langle \alpha \rangle^\perp$, namely

$$\sigma_\alpha(x) = x - 2\frac{\langle x, \alpha \rangle}{|\alpha|^2}\alpha, \quad x \in \mathbb{R}^N.$$

Each reflexion σ_α is an orthogonal transformation (hence preserves the inner product).

A finite subset $R \subset \mathbb{R}^N \backslash \{0\}$ is said to be a root system, if for all $\alpha \in R$, we have

(i) $\sigma_\alpha R = R$;
(ii) $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$.

For a given root system R, the subgroup of $O(N,\mathbb{R})$ (the orthogonal group w.r.t function composition) generated by $\{\sigma_\alpha, \alpha \in R\}$ is denoted by $W = W(R)$, and is called the reflection group associated with R. Notice that (see [30, Lemma 2.2]) W is finite and the set of reflections contained in W is exactly $\{\sigma_\alpha, \alpha \in R\}$.

Let R be a root system. For a given $\beta \in \mathbb{R}^N \setminus \bigcup_{\alpha\in R}\langle\alpha\rangle^\perp$, we denote by R^+ the positive subsystem

$$R^+ = \{\alpha \in R : \langle \alpha, \beta\rangle > 0\}.$$

We note that for every $\alpha \in R$, we have $\alpha \in R^+$ or $-\alpha \in R^+$.

Let $k : R \to \mathbb{R}$ be a multiplicity function on the root system R, that is, k is invariant under the natural action of W on R, i.e.

$$k(\alpha) = k(w(\alpha))$$

for all $w \in W$ and $\alpha \in R$. Let

$$\gamma = \gamma(k) = \sum_{\alpha\in R^+} k(\alpha).$$

Throughout this paper, it is assumed that

$$k \geq 0,\ \ \gamma > 0,\ \ \langle\alpha,\alpha\rangle = 2 \text{ for all } \alpha \in R.$$

We denote by ω_k the weight function

$$\omega_k(x) = \prod_{\alpha\in R^+} |\langle \alpha, x\rangle|^{2k(\alpha)}.$$

For $f \in L^1(\mathbb{R}^N, \omega_k(x)\,dx)$, we have the relation

$$\int_{\mathbb{R}^N} f(x)\omega_k(x)\,dx = \int_0^\infty \left(\int_{S^{N-1}} f(r\theta)\omega_k(r\theta)\,d\sigma(\theta)\right) r^{N-1}dr,$$

which can be written due to the homogeneity of ω_k as

$$\int_{\mathbb{R}^N} f(x)\omega_k(x)\,dx = \int_0^\infty \left(\int_{S^{N-1}} f(r\theta)\omega_k(\theta)\,d\sigma(\theta)\right) r^{2\gamma+N-1}dr,$$

where $d\sigma$ is the normalized surface measure on the unit sphere S^{N-1} of $\mathbb{R}^N$. In particular if f is radial, that is, $f(x) = F(|x|)$, then

$$\int_{\mathbb{R}^N} f(x)\omega_k(x)\,dx = c_k \int_0^\infty F(r)\, r^{2\gamma+N-1}dr, \tag{2.1}$$

where

$$c_k = \int_{S^{N-1}} \omega_k(\theta)\,d\sigma(\theta).$$

For $i = 1, 2, \cdots, N$, the Dunkl operators T_i associated with the finite reflection group W and the multiplicity function k, are given by

$$T_i f(x) = \frac{\partial f}{\partial x_i}(x) + \sum_{\alpha\in R^+} k(\alpha)\alpha_i \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad x \in \mathbb{R}^N,$$

where $f \in C^1(\mathbb{R}^N)$ and $\alpha = (\alpha_1, \cdots, \alpha_N)$. Notice that by the W-invariance of k , the definition of the Dunkl operator does not depend on the special choice of R^+. Also, the length of the roots is irrelevant in the formula for T_i. We have the following properties (see [20, Proposition 2.1]):

If $f \in C_c(\mathbb{R}^N)$ (i.e., f is continuous and compactly supported in $\mathbb{R}^N$) and $g \in C^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} T_i f(x)\, g(x)\omega_k(x)\, dx = -\int_{\mathbb{R}^N} T_i g(x)\, f(x)\omega_k(x)\, dx.$$

If $f, g \in C^1(\mathbb{R}^N)$ and g is W-invariant, then we get

$$T_i(fg) = g\, T_i f + f\, T_i g.$$

The Dunkl Laplacian Δ_k associated with the finite reflection group W and the multiplicity function k, is defined by

$$\Delta_k f = \sum_{i=1}^N T_i^2 f = \Delta f + 2 \sum_{\alpha \in R^+} k(\alpha)\delta_\alpha(f),$$

where $f \in C^2(\mathbb{R}^N)$ and

$$\delta_\alpha(f) = \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}.$$

Remark that in the special case $k \equiv 0$, the Dunkl Laplacian Δ_k reduces to the standard Laplacian. If f is a radial function, i.e. $f(x) = F(r)$, $r = |x|$, then (see [20, Proposition 4.15]) we have

$$\Delta_k f(x) = F''(r) + \frac{2\gamma + N - 1}{r} F'(r). \tag{2.2}$$

We recall below the Green's formula for Δ_k (see [20, Theorem 4.11]): Let Ω be an open bounded regular set of $\mathbb{R}^N$ which is W-invariant, and $f, g \in C^2(\overline{\Omega})$. Then

$$\int_\Omega (f\Delta_k g - g\Delta_k f)\, \omega_k(x)\, dx = \int_{\partial\Omega} \left(f\frac{\partial g}{\partial \nu} - g\frac{\partial f}{\partial \nu}\right) \omega_k(y)\, d\sigma(y), \tag{2.3}$$

where $\frac{\partial}{\partial \nu}$ denotes the normal derivative on $\partial\Omega$.

We end this section by fixing some notations that will be used throughout this paper. For $\delta > 0$, we denote by $B(0, \delta)$ the open ball in $\mathbb{R}^N$ of center 0 and radius ρ. The notation $R \gg 1$ means that $R > 0$ is sufficiently large. By C, we mean any positive constant independent on the scaling parameter R and the solutions u and v. The value of this constant is not necessarily the same from one line to another.

3. Main results

Before stating our main results, we first define weak solutions to (1.4) and (1.5). Let $N \geq 1$, $p, q > 1$ and $k : R \to \mathbb{R}$ be a multiplicity function on the root system R, where $k \geq 0$, $\gamma = \gamma(k) > 0$ and $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R$. We consider the set

$$C_c^2(\mathbb{R}^N) = \left\{\varphi \in C^2(\mathbb{R}^N) : \varphi \geq 0, \operatorname{supp}(\varphi) \subset\subset \mathbb{R}^N\right\},$$

hence we define weak solutions to (1.4) as follows.

Definition 3.1. We say that $u \in L^p_{\text{loc}}(\mathbb{R}^N, \omega_k(x)\, dx)$ is a weak solution to (1.4), if

$$\int_{\mathbb{R}^N} |u(x)|^p \varphi(x)\omega_k(x)\, dx \leq -\int_{\mathbb{R}^N} u(x)\Delta_k\varphi(x)\omega_k(x)\, dx \tag{3.1}$$

for all $\varphi \in C_c^2(\mathbb{R}^N)$.

Remark 3.2. Any smooth solution to (1.4) is a weak solution in the sense of Definition 3.1. Namely, let u be a smooth solution to (1.4) and $\varphi \in C_c^2(\mathbb{R}^N)$. We can take $\delta > 0$ sufficiently large so that $\operatorname{supp}(\varphi) \subsetneq B(0,\delta)$. Notice that $B(0,\delta)$ is W-invariant since $W \subset O(N,\mathbb{R})$. Multiplying (1.4) by $\varphi\omega_k$ and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |u(x)|^p \varphi(x)\omega_k(x)\,dx &\leq -\int_{\mathbb{R}^N} \Delta_k u(x)\,\varphi(x)\omega_k(x)\,dx \\ &= -\int_{B(0,\delta)} \Delta_k u(x)\,\varphi(x)\omega_k(x)\,dx. \end{aligned} \tag{3.2}$$

Since $B(0,\delta)$ is W-invariant, making use of (2.3), we obtain

$$\begin{aligned} -\int_{B(0,\delta)} \Delta_k u(x)\,\varphi(x)\omega_k(x)\,dx &= -\int_{B(0,\delta)} u(x)\Delta_k\varphi(x)\omega_k(x)\,dx \\ &= -\int_{\mathbb{R}^N} u(x)\Delta_k\varphi(x)\omega_k(x)\,dx. \end{aligned} \tag{3.3}$$

Therefore, (3.1) follows from (3.2) and (3.3).

Similarly, we define weak solutions to (1.5) as follows.

Definition 3.3. We say that $(u,v) \in L^q_{\mathrm{loc}}(\mathbb{R}^N,\omega_k(x)\,dx) \times L^p_{\mathrm{loc}}(\mathbb{R}^N,\omega_k(x)\,dx)$ is a weak solution to (1.5), if

$$\int_{\mathbb{R}^N} |v(x)|^p \varphi(x)\omega_k(x)\,dx \;\leq\; -\int_{\mathbb{R}^N} u(x)\Delta_k\varphi(x)\omega_k(x)\,dx, \tag{3.4}$$

$$\int_{\mathbb{R}^N} |u(x)|^q \varphi(x)\omega_k(x)\,dx \;\leq\; -\int_{\mathbb{R}^N} v(x)\Delta_k\varphi(x)\omega_k(x)\,dx \tag{3.5}$$

for all $\varphi \in C_c^2(\mathbb{R}^N)$.

Remark 3.4. Any smooth solution to (1.5) is a weak solution in the sense of Definition 3.3.

Our first main result is stated in the following theorem.

Theorem 3.5. Let $N \geq 1$ and $k : R \to \mathbb{R}$ be a multiplicity function on the root system R, where $k \geq 0$, $\gamma = \gamma(k) > 0$ and $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$. We have the following:

(i) Let $2\gamma + N - 2 \leq 0$. Then for all $p > 1$, (1.4) admits no nontrivial weak solution.

(ii) Let $2\gamma + N - 2 > 0$. We have the following:

(a) If
$$1 < p \leq 1 + \frac{2}{2\gamma + N - 2}, \tag{3.6}$$
then (1.4) admits no nontrivial weak solution.

(b) If
$$p > 1 + \frac{2}{2\gamma + N - 2}, \tag{3.7}$$
then (1.4) admits nontrivial solutions.

Remark 3.6. By a trivial solution to (1.4), we mean that $u \equiv 0$.

Remark 3.7. From Theorem 3.5, we deduce that, if $2\gamma + N - 2 > 0$, then the dividing line with respect to existence or nonexistence of weak solutions to (1.4) is given by the critical exponent

$$p^*(N,\gamma) = 1 + \frac{2}{2\gamma + N - 2}.$$

In the special case $k \equiv 0$ (so $\gamma = 0$), we have $p^*(N,0) = 1 + \frac{2}{N-2}$, which is the critical exponent for (1.1).

Remark 3.8. Let us consider (1.4) with $N = 1$. In this case, the only choice of the root system is $R = \{\pm\sqrt{2}\}$ and the corresponding reflection group is $W = \{\mathrm{id}, \sigma\}$ acting on $\mathbb{R}$ by $\sigma(x) = -x$. Furthermore, the multiplicity function $k : R \to \mathbb{R}$ satisfies

$$k(\sqrt{2}) = k(-\sqrt{2}) := k_0$$

and

$$\gamma(k) = \gamma = k_0.$$

On the other hand, (since $\gamma > 0$) $2\gamma + N - 2 \leq 0$ if and only if $N = 1$ and $0 < \gamma \leq \frac{1}{2}$. Hence, in the one-dimensional case, we deduce from Theorem 3.5 the following results:

(i) Let $0 < k_0 \leq \frac{1}{2}$. Then for all $p > 1$, (1.4) admits no nontrivial weak solution.

(ii) Let $k_0 > \frac{1}{2}$. We distinguish the following cases:

(a) If
$$1 < p \leq 1 + \frac{2}{2k_0 - 1},$$
then (1.4) admits no nontrivial weak solution.

(b) If
$$p > 1 + \frac{2}{2k_0 - 1},$$
then (1.4) admits nontrivial solutions.

We now consider problem (1.5). Our second main result is stated below.

Theorem 3.9. Let $N \geq 1$, $p, q > 1$ and $k : R \to \mathbb{R}$ be a multiplicity function on the root system R, where $k \geq 0$, $\gamma = \gamma(k) > 0$ and $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$. We have the following:

(i) If
$$\frac{N}{2} + \gamma \leq \frac{\max\{p(q+1), q(p+1)\}}{pq - 1}, \tag{3.8}$$
then (1.5) admits no nontrivial weak solution.

(ii) If
$$\frac{N}{2} + \gamma > \frac{\max\{p(q+1), q(p+1)\}}{pq - 1}, \tag{3.9}$$
then (1.5) admits nontrivial solutions.

Remark 3.10. By a trivial solution to (1.5), we mean that $u \equiv 0$ or $v \equiv 0$.

Remark 3.11. Observe that (3.8) is equivalent to

$$2\gamma + N - 2 \leq \frac{2}{pq - 1}\left(1 + \max\{p, q\}\right).$$

So, if $2\gamma + N - 2 \leq 0$, then (3.8) is always satisfied. Hence, from Theorem 3.9, we deduce that in this case, for all $p, q > 1$, (1.5) admits no nontrivial weak solution.

Remark 3.12. In the special case $k \equiv 0$, (3.8) reduces to (1.3).

4. Proofs of the main results

4.1. **Preliminaries.** Let $N \geq 1$, $p, q > 1$ and $k : R \to \mathbb{R}$ be a multiplicity function on the root system R, where $k \geq 0$, $\gamma = \gamma(k) > 0$ and $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$.

The following proposition is crucial in the proofs of our main results.

Proposition 4.1. There exists a family of radial functions $\{\xi_R\}_{R>1} \subset C_c^2(\mathbb{R}^N)$ satisfying the following properties:

(i) $0 \leq \xi_R \leq 1$, $\xi_R \equiv 1$ in $B(0, \frac{R}{2})$;

(ii) $\text{supp}(\xi_R) = \overline{B(0, R)}$;

(iii) For all $x \in \mathbb{R}^N$ with $\frac{R}{2} \leq |x| \leq R$, it holds that

$$|\nabla \xi_R(x)| \leq CR^{-1}, \ |\Delta_k \xi_R(x)| \leq CR^{-2}.$$

Proof. Let us consider a cut-off function $\xi \in C^\infty([0, \infty))$ satisfying the conditions

$$0 \leq \xi \leq 1, \ \xi \equiv 1 \text{ in } \left[0, \frac{1}{2}\right], \ \text{supp}(\xi) = [0, 1].$$

For all $R > 1$, we introduce the function

$$\xi_R(x) = \xi\left(\frac{|x|}{R}\right), \quad x \in \mathbb{R}^N.$$

It is clear that $\{\xi_R\}_{R>1} \subset C_c^2(\mathbb{R}^N)$ and satisfies properties (i)–(ii). On the other hand, since ξ_R is a radial function, it follows from (2.2) and the properties of ξ that

$$\text{supp}(\Delta_k \xi_R) \subset \left\{x \in \mathbb{R}^N : \frac{R}{2} \leq |x| \leq R\right\}$$

and

$$\Delta_k \xi_R(x) = R^{-2}\xi''\left(\frac{|x|}{R}\right) + \frac{2\gamma + N - 1}{|x|}R^{-1}\xi'\left(\frac{|x|}{R}\right), \quad \frac{R}{2} \leq |x| \leq R.$$

Then, we deduce that

$$\begin{aligned}
|\Delta_k \xi_R(x)| &\leq R^{-2}\left(\left|\xi''\left(\frac{|x|}{R}\right)\right| + CR|x|^{-1}\left|\xi'\left(\frac{|x|}{R}\right)\right|\right) \\
&\leq R^{-2}\left(\left|\xi''\left(\frac{|x|}{R}\right)\right| + 2C\left|\xi'\left(\frac{|x|}{R}\right)\right|\right) \\
&\leq CR^{-2}.
\end{aligned}$$

Furthermore, we get

$$|\nabla \xi_R(x)| = R^{-1}\left|\xi'\left(\frac{|x|}{R}\right)\right| \leq CR^{-1},$$

hence (iii) is established. □

Now, for $R, \ell \gg 1$ we introduce the function

$$\varphi(x) = \xi_R^\ell(x), \quad x \in \mathbb{R}^N, \tag{4.1}$$

where the family $\{\xi_R\}_{R>1}$ is given by Proposition 4.1. By the same Proposition, we know that $\varphi \in C_c^2(\mathbb{R}^N)$. For $\tau > 1$, let

$$A(\tau, \varphi) = \int_{\text{supp}(\Delta_k \varphi)} |\Delta_k \varphi(x)|^{\frac{\tau}{\tau-1}} \varphi^{\frac{-1}{\tau-1}}(x)\omega_k(x)\, dx,$$

where φ is given by (4.1). We can now establish the following estimate.

Lemma 4.2. Let $\tau > 1$. We have

$$A(\tau,\varphi) \le CR^{2\gamma+N-\frac{2\tau}{\tau-1}}.$$

Proof. By (4.1) and Proposition 4.1, we have

$$A(\tau,\varphi) = \int_{\frac{R}{2}<|x|<R} |\Delta_k\varphi(x)|^{\frac{\tau}{\tau-1}} \varphi^{\frac{-1}{\tau-1}}(x)\omega_k(x)\,dx. \tag{4.2}$$

On the other hand, since ξ_R is radial, if follows from (2.2) that

$$\Delta_k\varphi(x) = \ell(\ell-1)\xi_R^{\ell-2}(x)|\nabla\xi_R(x)|^2 + \ell\xi_R^{\ell-1}(x)\Delta\xi_R(x), \quad \frac{R}{2} < |x| < R.$$

Then, making use of Proposition 4.1 (iii), we obtain

$$|\Delta_k\varphi(x)| \le CR^{-2}\xi_R^{\ell-2}(x),$$

which implies that (since $0 \le \xi_R \le 1$)

$$|\Delta_k\varphi(x)|^{\frac{\tau}{\tau-1}}\varphi^{\frac{-1}{\tau-1}}(x)\omega_k(x) \le CR^{\frac{-2\tau}{\tau-1}}\xi_R^{\ell-\frac{2\tau}{\tau-1}}(x)\omega_k(x) \le CR^{\frac{-2\tau}{\tau-1}}\omega_k(x).$$

Hence, by (2.1) and (4.2), we conclude that

$$\begin{aligned} A(\tau,\varphi) &\le CR^{\frac{-2\tau}{\tau-1}} \int_{\frac{R}{2}<|x|<R} \omega_k(x)\,dx \\ &= CR^{\frac{-2\tau}{\tau-1}} \int_{r=\frac{R}{2}}^{R} r^{2\gamma+N-1}\,dr \\ &= CR^{\frac{-2\tau}{\tau-1}+2\gamma+N}, \end{aligned}$$

which proves the desired estimate. □

Wehave developed sufficient materials to conclude the proofs of our main results.

4.2. **Proof of Theorem 3.5.** To finalize the nonexistence result stated in item (i), we follow the contradiction argument. Namely, we suppose that $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N, \omega_k(x)\,dx)$ is a nontrivial weak solution to (1.4). Hence, from (3.1) we have the inequality

$$\int_{\mathbb{R}^N} |u(x)|^p\varphi(x)\omega_k(x)\,dx \le \int_{\mathbb{R}^N} |u(x)||\Delta_k\varphi(x)|\omega_k(x)\,dx, \tag{4.3}$$

where φ is given by (4.1) (we recall that $\varphi \in C_c^2(\mathbb{R}^N)$). Making use of Young's inequality, we get

$$\begin{aligned} \int_{\mathbb{R}^N} |u(x)||\Delta_k\varphi(x)|\omega_k(x)\,dx &= \int_{\mathbb{R}^N} \left(|u(x)|\varphi^{\frac{1}{p}}(x)\omega_k^{\frac{1}{p}}(x)\right)\left(\varphi^{\frac{-1}{p}}(x)|\Delta_k\varphi(x)|\omega_k^{\frac{p-1}{p}}(x)\right)dx \\ &\le \frac{1}{p}\int_{\mathbb{R}^N} |u(x)|^p\varphi(x)\omega_k(x)\,dx + \frac{p-1}{p}A(p,\varphi). \end{aligned} \tag{4.4}$$

Hence, in view of (4.3) and (4.4), we obtain

$$\int_{\mathbb{R}^N} |u(x)|^p\varphi(x)\omega_k(x)\,dx \le A(p,\varphi). \tag{4.5}$$

On the other hand, by Proposition 4.1 (i) and (4.1), we have

$$\int_{\mathbb{R}^N} |u(x)|^p\varphi(x)\omega_k(x)\,dx \ge \int_{B\left(0,\frac{R}{2}\right)} |u(x)|^p\omega_k(x)\,dx. \tag{4.6}$$

Furthermore, making use of Lemma 4.2 with $\tau = p$, we obtain

$$A(p,\varphi) \leq CR^{2\gamma+N-\frac{2p}{p-1}}. \tag{4.7}$$

Next, if follows from (4.5), (4.6) and (4.7) that

$$\int_{B\left(0,\frac{R}{2}\right)} |u(x)|^p \omega_k(x)\,dx \leq CR^{\frac{(2\gamma+N-2)p-(2\gamma+N)}{p-1}}. \tag{4.8}$$

Assume now that $2\gamma + N - 2 \leq 0$. In this case, one has

$$\frac{(2\gamma+N-2)p-(2\gamma+N)}{p-1} < 0. \tag{4.9}$$

Hence, passing to the limit as $R \to \infty$ in (4.8), we obtain

$$\int_{\mathbb{R}^N} |u(x)|^p \omega_k(x)\,dx = 0, \tag{4.10}$$

reaching to a contradiction. This completes the proof of item (i).

We can now focus on the second item of the proof. So, according to (ii) we assume now that $2\gamma + N - 2 > 0$. Then, we distinguish two cases.
(a) If

$$1 < p < 1 + \frac{2}{2\gamma+N-2},$$

then (4.9) holds, and the same conclusion as above follows easily. On the other hand, if we consider the case when

$$p = 1 + \frac{2}{2\gamma+N-2}, \tag{4.11}$$

then we note that

$$\frac{(2\gamma+N-2)p-(2\gamma+N)}{p-1} = 0.$$

Consequently, by (4.8) we know that $u \in L^p\left(\mathbb{R}^N, \omega_k(x)\,dx\right)$. Furthermore, making use of Hölder's inequality, (4.7) and (4.11), it follows from the properties of the function φ that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u(x)||\Delta_k\varphi(x)|\omega_k(x)\,dx \\
&= \int_{\frac{R}{2}<|x|<R} \left(|u(x)|\varphi^{\frac{1}{p}}(x)\omega_k^{\frac{1}{p}}(x)\right)\left(\varphi^{\frac{-1}{p}}(x)|\Delta_k\varphi(x)|\omega_k^{\frac{p-1}{p}}(x)\right)dx \\
&\leq \left(\int_{\frac{R}{2}<|x|<R} |u(x)|^p\varphi(x)\omega_k(x)\,dx\right)^{\frac{1}{p}} [A(p,\varphi)]^{\frac{p-1}{p}} \\
&\leq C\left(\int_{\frac{R}{2}<|x|<R} |u(x)|^p\xi_R^\ell(x)\omega_k(x)\,dx\right)^{\frac{1}{p}},
\end{aligned}$$

which implies by (4.3) and (4.6) that

$$\int_{B\left(0,\frac{R}{2}\right)} |u(x)|^p\omega_k(x)\,dx \leq C\left(\int_{\frac{R}{2}<|x|<R} |u(x)|^p\xi_R^\ell(x)\omega_k(x)\,dx\right)^{\frac{1}{p}}.$$

By the properties of the function ξ_R (See Proposition 4.1) and since $u \in L^p\left(\mathbb{R}^N, \omega_k(x)\,dx\right)$, passing to the limit as $R \to \infty$ in the above inequality and using the dominated convergence

theorem, we get the contradiction (4.10). Consequently, if (3.6) holds, then (1.4) admits no nontrivial weak solution. This proves (ii) (a). It remains to establish the existence result ((ii) (b)).
(b) Assume now that (3.7) holds. For

$$\frac{2}{p-1} \leq \delta < 2\gamma + N - 2 \tag{4.12}$$

and

$$0 < \varepsilon < \left[\delta\left(2\gamma + N - 2 - \delta\right)\right]^{\frac{1}{p-1}}, \tag{4.13}$$

let

$$u_{\delta,\varepsilon}(x) = \varepsilon(1+r^2)^{-\frac{\delta}{2}}, \quad x \in \mathbb{R}^N,\ r = |x|.$$

Notice that by (3.7), the set of values δ satisfying (4.12) is nonempty. On the other hand, since $u_{\delta,\varepsilon}$ is a radial function, making use of (2.2), we obtain

$$-\Delta_k u_{\delta,\varepsilon}(x) = \varepsilon\delta(1+r^2)^{-\frac{\delta+4}{2}}\left[(2\gamma+N-2-\delta)r^2 + 2\gamma + N\right],$$

which implies by (4.12) and (4.13) that

$$\begin{aligned} -\Delta_k u_{\delta,\varepsilon}(x) &\geq \varepsilon\delta(2\gamma+N-2-\delta)(1+r^2)^{-\frac{\delta+2}{2}} \\ &= \varepsilon^p(1+r^2)^{-\frac{\delta p}{2}}\left[\varepsilon^{1-p}\delta(2\gamma+N-2-\delta)(1+r^2)^{\frac{\delta(p-1)-2}{2}}\right] \\ &\geq \varepsilon^p(1+r^2)^{-\frac{\delta p}{2}} \\ &= u^p_{\delta,\varepsilon}(x) \end{aligned}$$

for all $x \in \mathbb{R}^N$. This shows that for all δ and ε satisfying respectively (4.12) and (4.13), the function $u_{\delta,\varepsilon}$ is a solution to (1.4). Then (ii) (b) is proved. □

4.3. **Proof of Theorem 3.9.** We consider separately the two possibilities:
(i) Similar to previous proof, we argue by contradiction. Let us suppose that $(u,v) \in L^q_{\text{loc}}(\mathbb{R}^N, \omega_k(x)\,dx) \times L^p_{\text{loc}}(\mathbb{R}^N, \omega_k(x)\,dx)$ is a nontrivial weak solution to (1.5). From (3.4), we have

$$\begin{aligned} \int_{\mathbb{R}^N} |v(x)|^p\varphi(x)\omega_k(x)\,dx &\leq \int_{\mathbb{R}^N} |u(x)||\Delta_k\varphi(x)|\omega_k(x)\,dx \\ &= \int_{\mathbb{R}^N} \left(|u(x)|\varphi^{\frac{1}{q}}(x)\omega_k^{\frac{1}{q}}(x)\right)\left(\varphi^{\frac{-1}{q}}(x)|\Delta_k\varphi(x)|\omega_k^{\frac{q-1}{q}}(x)\right)dx, \end{aligned}$$

where φ is given by (4.1). Making use of Hölder's inequality, we obtain

$$\int_{\mathbb{R}^N} |v(x)|^p\varphi(x)\omega_k(x)\,dx \leq \left(\int_{\mathbb{R}^N} |u(x)|^q\varphi(x)\omega_k(x)\,dx\right)^{\frac{1}{q}} \left[A(q,\varphi)\right]^{\frac{q-1}{q}}. \tag{4.14}$$

Similarly, using (3.5) and Hölder's inequality, we get

$$\int_{\mathbb{R}^N} |u(x)|^q\varphi(x)\omega_k(x)\,dx \leq \left(\int_{\mathbb{R}^N} |v(x)|^p\varphi(x)\omega_k(x)\,dx\right)^{\frac{1}{p}} \left[A(p,\varphi)\right]^{\frac{p-1}{p}}. \tag{4.15}$$

Then, it follows from (4.14) and (4.15) that

$$\int_{\mathbb{R}^N} |v(x)|^p\varphi(x)\omega_k(x)\,dx \leq \left(\int_{\mathbb{R}^N} |v(x)|^p\varphi(x)\omega_k(x)\,dx\right)^{\frac{1}{pq}} \left[A(p,\varphi)\right]^{\frac{p-1}{pq}} \left[A(q,\varphi)\right]^{\frac{q-1}{q}},$$

which yields

$$\left(\int_{\mathbb{R}^N}|v(x)|^p\varphi(x)\omega_k(x)\,dx\right)^{pq-1}\leq [A(p,\varphi)]^{p-1}\,[A(q,\varphi)]^{p(q-1)}. \tag{4.16}$$

Similarly, we have

$$\left(\int_{\mathbb{R}^N}|u(x)|^q\varphi(x)\omega_k(x)\,dx\right)^{pq-1}\leq [A(q,\varphi)]^{q-1}\,[A(p,\varphi)]^{q(p-1)}. \tag{4.17}$$

Thanks to Lemma 4.2, we deduce from (4.16) and (4.17) that

$$\left(\int_{\mathbb{R}^N}|v(x)|^p\varphi(x)\omega_k(x)\,dx\right)^{pq-1}\leq CR^{\lambda_1} \tag{4.18}$$

and

$$\left(\int_{\mathbb{R}^N}|u(x)|^q\varphi(x)\omega_k(x)\,dx\right)^{pq-1}\leq CR^{\lambda_2}, \tag{4.19}$$

where

$$\lambda_1=(2\gamma+N)(pq-1)-2p(q+1)$$

and

$$\lambda_2=(2\gamma+N)(pq-1)-2q(p+1).$$

Assume now that

$$\frac{N}{2}+\gamma<\frac{p(q+1)}{pq-1}.$$

In this case, one has $\lambda_1<0$. Then, passing to the limit as $R\to\infty$ in (4.18), we reach a contradiction. Similarly, if we consider the case when

$$\frac{N}{2}+\gamma<\frac{q(p+1)}{pq-1},$$

then $\lambda_2<0$. Hence, passing to the limit as $R\to\infty$ in (4.19), we reach a contradiction. Consequently, if

$$\frac{N}{2}+\gamma<\frac{\max\{p(q+1),q(p+1)\}}{pq-1},$$

then (1.5) admits no nontrivial weak solution. The critical case

$$\frac{N}{2}+\gamma=\frac{\max\{p(q+1),q(p+1)\}}{pq-1},$$

can be treated in a similar way as in the proof of Theorem 3.5 (ii) (a), we omit the details.
(ii) Let

$$\delta=\left[\frac{2(p+1)}{pq-1}\left(2\gamma+N-2-\frac{2(p+1)}{pq-1}\right)\right]^{\frac{1}{p-1}}$$

and

$$\rho=\left[\frac{2(q+1)}{pq-1}\left(2\gamma+N-2-\frac{2(q+1)}{pq-1}\right)\right]^{\frac{1}{q-1}}.$$

For

$$0<\varepsilon<\min\{\delta,\rho\}, \tag{4.20}$$

let

$$u_\varepsilon(x)=\varepsilon(1+r^2)^{-\frac{p+1}{pq-1}},\ v_\varepsilon(x)=\varepsilon(1+r^2)^{-\frac{q+1}{pq-1}},\quad x\in\mathbb{R}^N,\ r=|x|.$$

Notice that by (3.9), we have $\delta,\gamma>0$, so the set of values ε satisfying (4.20) is nonempty. Using (2.2), elementary calculations show that $(u_\varepsilon,v_\varepsilon)$ is a solution to (1.5). This proves (ii). □

Acknowledgement

The second author is supported by Researchers Supporting Project number (RSP2023R4), King Saud University, Riyadh, Saudi Arabia.

References

[1] G. Caristi, E. Mitidieri, Nonexistence of positive solutions of quasilinear equations, Adv. Differential Equations. 2 (3) (1997) 319–359.
[2] F. Dai, Y. Xu, Approximation Theory and Harmonic Analysis on Spheres and Balls, Springer, New York, 2013.
[3] F. Dai, Y. Xu, Analysis on h-harmonics and Dunkl transforms, Advanced Courses in Mathematics-CRM Barcelona, Birkhaüser/Springer, Basel, 2015.
[4] L. D'Ambrosio, Liouville type theorems for anisotropic quasilinear inequalities, Nonlinear Anal. 70 (2009) 2855–2869.
[5] L. D'Ambrosio, E. Mitidieri, A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities, Adv. Math. 224 (2010) 967–1020.
[6] C.F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc. 311 (1989) 167–183.
[7] A. Farina, Liouville-type theorems of elliptic problems, M. Chipot (Ed.), Handbook of Differential Equations-Stationary Partial Differential Equations, vol. 4, Elsevier BV (2007), pp. 61–116.
[8] R. Filippucci, P. Pucci, V. Rădulescu, Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions, Comm. Partial Differential Equations. 33 (2008) 706–717.
[9] R. Filippucci, P. Pucci, M. Rigoli, On weak solutions of nonlinear weighted p-Laplacian elliptic inequalities, Nonlinear Anal. 70 (8) (2009) 3008–3019.
[10] L. Gallardo, L. Godefroy, Propriété de Liouville et équation de Poisson pour le Laplacien généralisé de Dunkl, C. R. Math. Acad. Sci. Paris. 337 (2003) 639–644.
[11] L. Gallardo, C. Rejeb, A new mean value property for harmonic functions relative to the Dunkl Laplacian operator and applications, Trans. Amer. Math. Soc. 368 (5) (2015) 3727–3753.
[12] M. Ghergu, Partial Differential Inequalities with Nonlinear Convolution Terms, SpringerBriefs in Mathematics, 2022.
[13] M. Ghergu, V. Rădulescu, Nonradial blow-up solutions of sublinear elliptic equations with gradient term, Commun. Pure Appl. Anal. 3 (3) (2004) 465–474.
[14] G.J. Heckman, An elementary approach to the hypergeometric shift operators of Opdam, Invent. Math. 103 (1991) 341–350.
[15] K. Hikami, Dunkl operator formalism for quantum many-body problems associated with classical root systems, J . Phys. Soc. Japan. 65 (1996) 394–401.
[16] M. Jleli, M.A. Ragusa, B. Samet, Nonlinear Liouville-type theorems for generalized Baouendi-Grushin operator on Riemannian manifolds, Adv. Differ. Equ. 28 (1–2) (2023) 143–168.
[17] S. Kakei, Common algebraic structure for the Calogero-Sutherland models, J . Phys. A29 (1996) 619–624.
[18] M. Kirane, E. Nabana, S.I. Pohozaev, The absence of solutions of elliptic systems with dynamic boundary conditions, J. Differ. Equations 38 (6) (2002) 808–815.
[19] V.V. Kurta, On the absence of positive solutions to semilinear elliptic equations, Tr. Mat. Inst. Steklova. 227 (1999) 162–169.
[20] H. Mejjaoli, K. Trimèche, On a mean value property associated with the Dunkl Laplacian operator and applications, Integral Transform. Spec. Funct. 12 (3) (2001) 279–302.
[21] E. Mitidieri, S.I. Pohozaev, Absence of global positive solutions of quasilinear elliptic inequalities, Dokl. Akad. Nauk. 359 (4) (1998) 456–460.
[22] E. Mitidieri, S.I. Pohozaev, Nonexistence of positive solutions for quasilinear elliptic problems on $\mathbb{R}^N$, Proc. Steklov Inst. Math. 227 (1999) 186–216.
[23] E. Mitidieri, S.I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, Tr. Mat. Inst. Steklova. 234 (2001) 1–384.

[24] W. Ni, J. Serrin, Nonexistence theorems for quasilinear partial differential equations, Rend. Circ. Mat. Palermo, Suppl. no. 8, (1985) 171–185.

[25] W. Ni, J. Serrin, Existence and nonexistence theorems for ground states of quasilinear partial differential equations. The anomalous case, Accad. Naz. Lincei, Atti dei Convegni 77 (1986) 231–257.

[26] R. Osserman, On the inequality $\Delta u \geq f(u)$, Pacific J. Math. 7 (1957) 1641–1647.

[27] S.I. Pohozaev, L. Veron, Nonexistence results of solutions of semilinear differential inequalities on the Heisenberg group, Manuscripta Math. 102 (2000) 85–99.

[28] G. Ren, L. Liu, Liouville theorem for Dunkl polyharmonic functions, SIGMA Symmetry Integrability Geom. Methods Appl.. 4 (2008), 076, 6 pages.

[29] M. Rösler, Contributions to the theory of Dunkl operators, Ph.D. thesis, Technische Universität München (1999).

[30] M. Rösler, Dunkl operators: theory and applications, E. Koelink, W. Van Assche (Eds.), Orthogonal Polynomials and Special Functions, Lect. Notes Math., Vol. 1817, Leuven, 2002, Springer-Verlag (2003), pp. 93–135.

[31] Y. Sun, Uniqueness results for nonnegative solutions of semilinear inequalities on Riemannian manifolds, J. Math. Anal. Appl. 419 (2014) 646–661.

[32] K. Trimèche, Harmonic analysis on measures spaces attached to some Dunkl operators on $\mathbb{R}^d$ and applications, J. Inequal. Spec. Funct. 8(2017) 104–117.

[33] J.F. Van Diejen, Confluent hypergeometric orthogonal polynomials related to the rational quantum Calogero system with harmonic confinement, Comm. Math. Phys. 188 (1997) 467–497.

(M. Jleli) Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

Email address: jleli@ksu.edu.sa

(B. Samet) Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

Email address: bsamet@ksu.edu.sa

(C. Vetro) Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy

Email address: calogero.vetro@unipa.it