AN ANISOTROPIC DIRICHLET SYSTEM INCLUDING UNBOUNDED COEFFICIENTS

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ABSTRACT. This paper focuses on an anisotropic system driven by (p,q)-Laplacian operators with unbounded coefficients depending on the solution, and whose lower order-terms exhibit full dependence on the solution and its gradient. The main results establish the existence of solutions and the uniform boundedness of the solution set. The approach is based on a suitable truncation to drop the unboundedness of the coefficients and on the solvability of the truncated system within the theory of pseudomonotone operators.

1. **Introduction.** Anisotropic phenomena represent a challenging object for mathematical physics requiring specific analysis and techniques. This is the case for example in fluid mechanics involving anisotropic media where the conductivity depends on the direction (see [1]). Anisotropic equations also appear in biology as a model for the propagation of epidemic diseases in heterogeneous domains. The interest in anisotropic problems has deeply increased recently because of many difficulties that arise in passing from the isotropic setting to the anisotropic one. In fact, some fundamental tools available for the isotropic setting(such as the strong maximum principle) cannot be extended to the anisotropic (see [2, 3, 4, 6, 12, 13, 14, 15, 16, 17] and the references therein).

Recently, Motreanu and Tornatore [10] considered a Dirichlet problem exhibiting anisotropic differential operator with unbounded coefficients and full dependence on the gradient in the lower order terms:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i(G_i(u)|\partial_i u|^{p_i-2}\partial_i u) = F(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary $\partial\Omega$, G_i are unbounded functions on $\mathbb{R}$, for $i = 1, \cdots, N$, and F is a Carathéodory function. It

is worth mentioning that anisotropic boundary value problems with full dependence on the solution and its gradient have only very recently started to be investigated (refer to [2, 3, 4, 10]). The study of problems driven by the weighted p-Laplacian operator with unbounded coefficients was initiated in [8, 11].

Here, we consider the system counterpart of equation (1), namely the Dirichlet system of coupled equations

$$\begin{cases} -\sum_{i=1}^{N}\partial_i(G_{1i}(u)|\partial_i u|^{p_i-2}\partial_i u) = F_1(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ -\sum_{i=1}^{N}\partial_i(G_{2i}(v)|\partial_i v|^{q_i-2}\partial_i v) = F_2(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is as in (1), and

(i) $p_i, q_i \in (1, +\infty)$ for $i = 1, \ldots, N$;
(ii) the functions $G_{ki:} : \mathbb{R} \to [a_{ki}, +\infty)$ are continuous with $a_{ki} > 0$ for $i = 1, \ldots, N$, $k = 1, 2$;
(iii) the functions $F_1, F_2 : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions (i.e., $F_1(\cdot, s, t, \xi, \eta)$ and $F_2(\cdot, s, t, \xi, \eta)$ are measurable on Ω for each $(s, t, \xi, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ and $F_1(x, \cdot, \cdot, \cdot, \cdot)$ and $F_2(x, \cdot, \cdot, \cdot, \cdot)$ are continuous on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ for almost all $x \in \Omega$).

We emphasize that, generally, the treatment for a system is fundamentally different with respect to an equation due to the fact that in a system problem the variables cannot be separated to deal with individual equations. In line with this, the main difficulty in handling system (2) is that the estimates obtained for equation (1) cannot cover the system case. Although we follow the reasoning plan in [10], we develop new ideas and tools to resolve the system case of (2).

Set $\overrightarrow{p} := (p_1, \ldots, p_N)$ and $\overrightarrow{q} := (q_1, \ldots, q_N)$. Denote by $W_0^{1,\overrightarrow{p}}(\Omega)$ the completion of the set $C_c^\infty(\Omega)$ of C^∞-functions with compact support in Ω with respect to the norm

$$\|u\|_{1,\overrightarrow{p}} := \sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}}$$

and by $W_0^{1,\overrightarrow{q}}(\Omega)$ the completion of the set $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,\overrightarrow{q}} := \sum_{i=1}^{N}\|\partial_i u\|_{L^{q_i}}.$$

It follows that $X := W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$ is a reflexive Banach space with respect to the norm

$$\|(u, v)\| := \|u\|_{1,\overrightarrow{p}} + \|v\|_{1,\overrightarrow{q}}.$$

To simplify the presentation, for any real number $r > 1$, we denote $r' := r/(r-1)$ (the Hölder conjugate of r).

Definition 1.1. It is said that $(u, v) \in X$ is a weak solution to problem (2) if

$$G_{1i}(u)|\partial_i u|^{p_i-2}\partial_i u\partial_i\phi, \ G_{2i}(v)|\partial_i v|^{q_i-2}\partial_i v\partial_i\psi \in L^1(\Omega), \ \forall i = 1, \ldots, N,$$

$$F_1(\cdot, u(\cdot), v(\cdot), \nabla u(\cdot), \nabla v(\cdot))\phi, \ F_2(\cdot, u(\cdot), v(\cdot), \nabla u(\cdot), \nabla v(\cdot))\psi \in L^1(\Omega),$$

and

$$\begin{cases} \sum_{i=1}^{N} \int_{\Omega} G_{1i}(u(x))|\partial_i u(x)|^{p_i-2}\partial_i u(x)\partial_i \phi(x)dx \\ \qquad\qquad = \int_{\Omega} F_1(x, u(x), v(x), \nabla u(x), \nabla v(x))\phi(x)dx, \\ \sum_{i=1}^{N} \int_{\Omega} G_{2i}(v(x))|\partial_i v(x)|^{q_i-2}\partial_i v(x)\partial_i \psi(x)dx \\ \qquad\qquad = \int_{\Omega} F_2(x, u(x), v(x), \nabla u(x), \nabla v(x))\psi(x)dx \end{cases} \tag{3}$$

for all $(\phi, \psi) \in W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega)$.

We assume that

$$\sum_{i=1}^{N} \frac{1}{p_i} > 1 \text{ and } \sum_{i=1}^{N} \frac{1}{q_i} > 1. \tag{4}$$

The critical exponents are defined as

$$p^* := \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i} - 1} \text{ and } q^* := \frac{N}{\sum_{i=1}^{N} \frac{1}{q_i} - 1}. \tag{5}$$

Notice that if $p_i = p$ for all $i = 1, \ldots, N$, then p^* in (5) becomes the ordinary Sobolev critical exponent when $N > p$. Under assumption (4), there are continuous embeddings

$$W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{\rho_1}(\Omega) \text{ and } W_0^{1,\vec{q}}(\Omega) \hookrightarrow L^{\rho_2}(\Omega) \tag{6}$$

provided $1 \leq \rho_1 \leq p^*$ and $1 \leq \rho_2 \leq q^*$, which are compact if $1 \leq \rho_1 < p^*$, $1 \leq \rho_2 < q^*$ (see [7, Theorem 1]).

We also set

$$\overline{p} := \max\{p_1, \ldots, p_N\} \quad \text{and} \quad \underline{p} := \min\{p_1, \ldots, p_N\}$$
$$\overline{q} := \max\{q_1, \ldots, q_N\} \quad \text{and} \quad \underline{q} := \min\{q_1, \ldots, q_N\}$$

and further assume

$$\overline{p} < p^* \text{ and } \overline{q} < q^*. \tag{7}$$

Consequently, by (6) there is a constant $\theta > 0$ such that

$$\|u\|_{L^{\overline{p}}}^{\underline{p}} \leq \theta\|u\|_{1,\vec{p}}^{\underline{p}} \text{ and } \|v\|_{L^{\overline{q}}}^{\underline{q}} \leq \theta\|v\|_{1,\vec{q}}^{\underline{q}}, \quad \forall (u, v) \in X. \tag{8}$$

The functions F_k, for $k = 1, 2$, are required to satisfy the following hypotheses:

(H1) There exist constants $a_i, b_i \geq 0$ for $i = 1, \cdots, 5$, and $r_1 \in (\overline{p}, p^*), r_2 \in (\overline{q}, q^*)$ such that

$$
\begin{cases}
|F_1(x, s, t, \xi, \eta)| \leq a_1|s|^{r_1-1} + a_2|t|^{\frac{r_2}{r_1'}} + a_3 \left(\sum_{i=1}^{N} |\xi_i|^{p_i} \right)^{1/r_1'} \\
\qquad\qquad\qquad\qquad\qquad + a_4 \left(\sum_{i=1}^{N} |\eta_i|^{q_i} \right)^{1/r_1'} + a_5, \\
|F_2(x, s, t, \xi, \eta)| \leq b_1|s|^{\frac{r_1}{r_2'}} + b_2|t|^{r_2-1} + b_3 \left(\sum_{i=1}^{N} |\xi_i|^{p_i} \right)^{1/r_2'} \\
\qquad\qquad\qquad\qquad\qquad + b_4 \left(\sum_{i=1}^{N} |\eta_i|^{q_i} \right)^{1/r_2'} + b_5,
\end{cases}
$$

for a.e. $x \in \Omega$, all $s, t \in \mathbb{R}$, and $\xi = (\xi_1, \ldots, \xi_N), \eta = (\eta_1, \ldots, \eta_N) \in \mathbb{R}^N$.

(H2) There exist constants $c_j, d_j \geq 0$ for $j = 1, 2, 3, 4$, with

$$
\begin{cases}
c_3 + d_3 + N^{\underline{p}-1}\theta(c_1 + d_1) < a_{1i} \text{ and} \\
c_4 + d_4 + N^{\underline{q}-1}\theta(c_2 + d_2) < a_{2i}
\end{cases}
$$

for all $i = 1, \ldots, N$, and functions $\zeta_1, \zeta_2 \in L^1(\Omega)$ such that

$$
\begin{cases}
F_1(x, s, t, \xi, \eta)s \leq c_1|s|^{\underline{p}} + c_2|t|^{\underline{q}} + c_3 \sum_{i=1}^{N} |\xi_i|^{p_i} + c_4 \sum_{i=1}^{N} |\eta_i|^{q_i} + \zeta_1(x), \\
F_2(x, s, t, \xi, \eta)t \leq d_1|s|^{\underline{p}} + d_2|t|^{\underline{q}} + d_3 \sum_{i=1}^{N} |\xi_i|^{p_i} + d_4 \sum_{i=1}^{N} |\eta_i|^{q_i} + \zeta_2(x)
\end{cases}
$$

for a.e. $x \in \Omega$, all $s, t \in \mathbb{R}$, $\xi = (\xi_1, \ldots, \xi_N), \eta = (\eta_1, \ldots, \eta_N) \in \mathbb{R}^N$.

Let us note that (H1) is a subcritical growth type assumption, whereas (H2) expresses a "sublinear" type condition.

Our main result provides the existence of a bounded weak solution to problem (2) in the sense of Definition 1.1.

Theorem 1.2. *Assume that conditions* (4) *and* (7) *hold,* $G_{ki} : \mathbb{R} \to [a_{ki}, +\infty)$ *with* $a_{ki} > 0$ *for* $i = 1, \ldots, N$ *and* $k = 1, 2$ *are continuous functions, and* $F_k : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *for* $k = 1, 2$ *are Carathéodory functions satisfying hypotheses* (H1) *and* (H2). *Then, problem* (2) *has at least a weak solution* $(u, v) \in X \cap L^\infty(\Omega)^2$.

The proof of Theorem 1.2 relies on an appropriate truncation destined to cut the unboundedness of coefficients G_{ki}. The right choice for the truncation is possible due to the uniform boundedness of the solution set by a bound that only depends on the lower bound a_{ki} of the functions G_{ki}. This is the contents of Theorem 3.2 below. In the proof of Theorem 3.2, we succeeded to find a way to reduce the estimates for the solution to system (2) to a Moser iteration separately addressing each component. The existence of solutions to the truncated system is proven in Theorem 5.1 by applying the surjectivity theorem for pseudomonotone operators. Finally, a comparison argument enables us to prove Theorem 1.2.

Remark 1.3. The existence result in Theorem 1.2 does not guarantee that the obtained solution to system (2) is nontrivial. Note that the trivial solution can occur only if $F_i(x, 0, 0, 0, 0) = 0$ for a.e. $x \in \Omega$, $i = 1, 2$. Otherwise, any solution is necessarily nontrivial. In some cases, system (2) can possess a unique solution (for instance, under appropriate monotone assumptions for the functions $F_i(x, \cdot)$). In order to have nontrivial solutions, additional hypotheses should be imposed. For variational problems, conditions of mountain-pass type are usually supposed. Problem (2) is strongly non-variational, so a different method would be effective. We plan to address this topic in a further work relying on the sub-supersolution approach. Here, we just suggest the technique in [9], where under suitable conditions the existence of positive solutions, thus nontrivial solutions, can be established.

The rest of the paper consists of the following sections. Section 2 discusses the Nemytskii operator associated to the reaction terms in system (2). Section 3 contains Theorem 3.2 and its proof establishing the uniform bound for the solution set of (2). Section 4 examines the truncated system. Section 5 presents the existence of solutions to the truncated and original systems.

2. **Associated Nemytskii operator.**

Proposition 2.1. *Assume* (4), (7), *and* (H1). *Then, the map* $\mathcal{N} : X \to L^{r'_1}(\Omega) \times L^{r'_2}(\Omega)$ *given by*

$$\mathcal{N}(u, v) = \left(F_1(x, u, v, \nabla u, \nabla v), F_2(x, u, v, \nabla u, \nabla v) \right), \quad \forall (u, v) \in X,$$

is well defined, bounded (in the sense it maps bounded sets into bounded sets), and continuous.

Proof. By (H1), through a well-known convexity inequality, we obtain

$$\int_\Omega |F_1(x, u, v, \nabla u, \nabla v)|^{r'_1} dx \leq C\left(\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i} dx + \sum_{i=1}^N \int_\Omega |\partial_i v|^{q_i} dx \right.$$
$$\left. + \int_\Omega |u|^{r_1} dx + \int_\Omega |v|^{r_2} dx + 1 \right)$$

and

$$\int_\Omega |F_2(x, u, v, \nabla u, \nabla v)|^{r'_2} dx \leq C\left(\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i} dx + \sum_{i=1}^N \int_\Omega |\partial_i v|^{q_i} dx \right.$$
$$\left. + \int_\Omega |u|^{r_1} dx + \int_\Omega |v|^{r_2} dx + 1 \right)$$

for all $(u, v) \in X$, with a constant $C > 0$. Since $\partial_i u \in L^{p_i}(\Omega)$ and $\partial_i v \in L^{q_i}(\Omega)$, we infer that the map $\mathcal{N} : X \to L^{r'_1}(\Omega)) \times L^{r'_2}(\Omega)$ is well defined and bounded.

In order to show the continuity of $\mathcal{N}$, let $(u_n, v_n) \to (u, v)$ in X. By the definition of the space X and the continuous embedding $X \hookrightarrow L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ in (6), we know that $(u_n, v_n) \to (u, v)$ in $L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ and $(\partial_i u_n, \partial_i v_n) \to (\partial_i u, \partial_i v)$ in $L^{p_i}(\Omega) \times L^{q_i}(\Omega)$ for $i = 1, \ldots, N$. Then, the growth conditions for F_1 and F_2 in assumption (H1) permit to apply Krasnoselskii's theorem, obtaining

$$(F_1(x, u_n, v_n, \nabla u_n, \nabla v_n), F_2(x, u_n, v_n, \nabla u_n, \nabla v_n))$$
$$\to (F_1(x, u, v, \nabla u, \nabla v), F_2(x, u, v, \nabla u, \nabla v))$$

in $L^{r'_1}(\Omega) \times L^{r'_2}(\Omega)$, which completes the proof. □

Corollary 2.2. *Assume that conditions* (4), (7), *and* (H1) *are fulfilled. If* $(u_n, v_n) \to (u, v)$ *in* X, *then*

$$\lim_{n \to \infty} \langle \mathcal{N}(u_n, v_n), (u_n - u, v_n - v) \rangle = 0.$$

Proof. Hölder's inequality entails the estimate

$$|\langle \mathcal{N}(u_n, v_n), (u_n - u, v_n - v) \rangle|$$

$$= \left| \int_\Omega F_1(x, u_n, v_n, \nabla u_n, \nabla v_n)(u_n - u) dx \right.$$

$$\left. + \int_\Omega F_2(x, u_n, v_n, \nabla u_n, \nabla v_n)(v_n - v) dx \right|$$

$$\leq \| F_1(x, u_n, v_n, \nabla u_n, \nabla v_n) \|_{L^{r'_1}} \| u_n - u \|_{L^{r_1}}$$

$$+ \| F_2(x, u_n, v_n, \nabla u_n, \nabla v_n) \|_{L^{r'_2}} \| v_n - v \|_{L^{r_2}}.$$

The compact embedding (6) yields $(u_n, v_n) \to (u, v)$ in $L^{r_1}(\Omega) \times L^{r_2}(\Omega)$. Since Proposition 2.1 ensures the boundedness of the sequences $F_1(x, u_n, v_n, \nabla u_n, \nabla v_n)$ and $F_2(x, u_n, v_n, \nabla u_n, \nabla v_n)$ in $L^{r'_1}(\Omega)$ and $L^{r'_2}(\Omega)$, respectively, the conclusion is achieved. $\qquad\square$

3. **Global estimates.** We first estimate the solutions to (2) in the space X.

Lemma 3.1. *Assume that conditions* (4), (7), (H1), *and* (H2) *hold. Then, the solution set to problem* (2) *is bounded in* X *with a bound that depends on the function* G_{ki} *only through its lower bound* a_{ki} *for* $i = 1, \dots, N$ *and* $k = 1, 2$.

Proof. Let $(u, v) \in X$ be a weak solution of (2). Insert $(\phi, \psi) = (u, v)$ in (3) to get

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega G_{1i}(u(x)) |\partial_i u(x)|^{p_i} dx = \int_\Omega F_1(x, u, v, \nabla u, \nabla v) u \, dx, \\[4mm] \displaystyle\sum_{i=1}^N \int_\Omega G_{2i}(v(x)) |\partial_i v(x)|^{q_i} dx = \int_\Omega F_2(x, u, v, \nabla u, \nabla v) v \, dx. \end{cases}$$

Then, from hypothesis (H2), we infer

$$\begin{cases} \displaystyle\sum_{i=1}^N a_{1i} \|\partial_i u\|_{L^{p_i}}^{p_i} \leq \|\zeta_1\|_{L^1} + c_1 \|u\|_{L^p}^p + c_2 \|v\|_{L^q}^q + c_3 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} + c_4 \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i}, \\[4mm] \displaystyle\sum_{i=1}^N a_{2i} \|\partial_i v\|_{L^{q_i}}^{q_i} \leq \|\zeta_2\|_{L^1} + d_1 \|u\|_{L^p}^p + d_2 \|v\|_{L^q}^q + d_3 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} + d_4 \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i}. \end{cases}$$

Through (8), it turns out

$$\sum_{i=1}^{N}(a_{1i}-c_3)\|\partial_i u\|_{L^{p_i}}^{p_i}$$

$$\leq c_1\|u\|_{L^{\underline{p}}}^{\underline{p}}+c_2\|v\|_{L^{\underline{q}}}^{\underline{q}}+c_4\sum_{i=1}^{N}\|\partial_i v\|_{L^{q_i}}^{q_i}+\|\zeta_1\|_{L^1}$$

$$\leq N^{\underline{p}-1}c_1\theta\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}}^{\underline{p}}+N^{\underline{q}-1}c_2\theta\sum_{i=1}^{N}\|\partial_i v\|_{L^{q_i}}^{\underline{q}}+c_4\sum_{i=1}^{N}\|\partial_i v\|_{L^{q_i}}^{q_i}+\|\zeta_1\|_{L^1}$$

$$\leq N^{\underline{p}-1}c_1\theta(N+\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}}^{p_i})+N^{\underline{q}-1}c_2\theta(N+\sum_{i=1}^{N}\|\partial_i v\|_{L^{q_i}}^{q_i})+c_4\sum_{i=1}^{N}\|\partial_i v\|_{L^{q_i}}^{q_i}$$
$$+\|\zeta_1\|_{L^1}$$

and

$$\sum_{i=1}^{N}(a_{2i}-d_4)\|\partial_i v\|_{L^{q_i}}^{q_i}$$

$$\leq d_1\theta(\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}})^{\underline{p}}+d_2\theta(\sum_{i=1}^{N}\|\partial_i v\|_{L^{q_i}})^{\underline{q}}+d_3\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}}^{p_i}+\|\zeta_2\|_{L^1}$$

$$\leq N^{\underline{p}-1}d_1\theta\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}}^{\underline{p}}+N^{\underline{q}-1}d_2\theta\sum_{i=1}^{N}\|\partial_i v\|_{L^{q_i}}^{\underline{q}}+d_3\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}}^{p_i}+\|\zeta_2\|_{L^1}$$

$$\leq N^{\underline{p}-1}d_1\theta(N+\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}}^{p_i})+N^{\underline{q}-1}d_2\theta(N+\sum_{i=1}^{N}\|\partial_i v\|_{L^{q_i}}^{q_i})+d_3\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}}^{p_i}$$
$$+\|\zeta_2\|_{L^1}.$$

These estimates imply that

$$\sum_{i=1}^{N}(a_{1i}-(c_3+d_3))\|\partial_i u\|_{L^{p_i}}^{p_i}+\sum_{i=1}^{N}(a_{2i}-(c_4+d_4))\|\partial_i v\|_{L^{q_i}}^{q_i}$$

$$\leq N^{\underline{p}-1}\theta(c_1+d_1)(N+\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}}^{p_i})+N^{\underline{q}-1}\theta(c_2+d_2)(N+\sum_{i=1}^{N}\|\partial_i v\|_{L^{q_i}}^{q_i})$$
$$+\|\zeta_1\|_{L^1}+\|\zeta_2\|_{L^1}.$$

The conditions imposed in assumption (H2) ensure

$$\begin{cases} c_3+d_3+N^{\underline{p}-1}\theta(c_1+d_1)<a_{1i} \\ c_4+d_4+N^{\underline{q}-1}\theta(c_2+d_2)<a_{2i} \end{cases}$$

for all $i=1,\ldots,N$, so the stated conclusion is valid.   □

In fact, the solution set of problem (2) is uniformly bounded.

Theorem 3.2. *If conditions* (4), (7), *(H1), and (H2) are satisfied, then the solution set of problem* (2) *is uniformly bounded, that is, there exists a constant* $C_0>0$ *such that* $\|u\|_{L^\infty}\leq C_0$ *and* $\|v\|_{L^\infty}\leq C_0$ *for all weak solutions* $(u,v)\in X$ *to problem* (2). *The uniform bound* C_0 *depends on the function* G_{ki} *only through its lower bound* a_{ki} *for* $i=1,\ldots,N$ *and* $k=1,2$.

Proof. Let $(u, v) \in X$ be a weak solution to problem (2). We can express $u = u^+ - u^-$ and $v = v^+ - v^-$, where $u^+ = \max\{u, 0\}$ (the positive part of u) and $u^- = \max\{-u, 0\}$ (the negative part of u) and similarly for v^+ and v^-. It suffices to prove the uniform boundedness for (u^+, v^+), because for (u^-, v^-) the reasoning is similar.

For an arbitrary number $h > 0$, set $u_h := \min\{u^+, h\}$ and $v_h := \min\{v^+, h\}$. Corresponding to any $k > 0$ and $1 \le i \le N$, we have $(u^+(u_h)^{kp_i}, v^+(v_h)^{kq_i}) \in X$ because

$$\begin{cases} |\partial_i(u^+(u_h)^k)| = |(u_h)^k \partial_i u^+ + k(u_h)^{k-1} u^+ \partial_i u_h| \\ \qquad\qquad \le (k+1)(u_h)^k |\partial_i u^+|, \\ \\ |\partial_i(v^+(v_h)^k)| = |(v_h)^k \partial_i v^+ + k(v_h)^{k-1} v^+ \partial_i v_h| \\ \qquad\qquad \le (k+1)(v_h)^k |\partial_i v^+|. \end{cases} \tag{9}$$

Testing (3) with $(u^+(u_h)^{kp_j}, v^+(v_h)^{kq_j})$, $1 \le j \le N$, we find

$$\begin{cases} \sum_{i=1}^N \int_\Omega G_{1i}(u) |\partial_i u|^{p_i-2} \partial_i u \partial_i(u^+(u_h)^{kp_j}) dx = \int_\Omega F_1(x, u, v, \nabla u, \nabla v) u^+(u_h)^{kp_j} dx, \\ \\ \sum_{i=1}^N \int_\Omega G_{2i}(v) |\partial_i v|^{q_i-2} \partial_i v \partial_i(v^+(v_h)^{kq_j}) dx = \int_\Omega F_2(x, u, v, \nabla u, \nabla v) v^+(v_h)^{kq_j} dx. \end{cases} \tag{10}$$

For the left-hand side of the equations in (10), we have the estimates

$$\sum_{i=1}^N \int_\Omega G_{1i}(u) |\partial_i u|^{p_i-2} \partial_i u \partial_i(u^+(u_h)^{kp_j}) dx$$

$$= \sum_{i=1}^N \int_\Omega G_{1i}(u) |\partial_i u|^{p_i-2} \partial_i u (\partial_i u^+(u_h)^{kp_j} + kp_j(u_h)^{kp_j-1} u^+ \partial_i u_h) dx \tag{11}$$

$$\ge \sum_{i=1}^N a_{1i} \int_\Omega (u_h)^{kp_j} |\partial_i u^+|^{p_i} dx$$

and

$$\sum_{i=1}^N \int_\Omega G_{2i}(v) |\partial_i v|^{q_i-2} \partial_i v \partial_i(v^+(v_h)^{kp_j}) dx$$

$$= \sum_{i=1}^N \int_\Omega G_{2i}(v) |\partial_i v|^{q_i-2} \partial_i v (\partial_i v^+(v_h)^{kq_j} + kq_j(v_h)^{kq_j-1} v^+ \partial_i v_h) dx \tag{12}$$

$$\ge \sum_{i=1}^N a_{2i} \int_\Omega (v_h)^{kq_j} |\partial_i v^+|^{q_i} dx,$$

for $j = 1, \ldots, N$.

For the right-hand side of the equations in (10), hypothesis (H1) enables to derive the estimates

$$\int_\Omega F_1(x, u, v, \nabla u, \nabla v) u^+(u_h)^{kp_j} dx$$

$$\leq a_1 \int_\Omega |u|^{r_1-1}(u_h)^{kp_j} u^+ dx + a_2 \int_\Omega |v|^{\frac{r_2}{r_1'}} (u_h)^{kp_j} u^+ dx$$

$$+ a_3 \int_\Omega \left(\left(\sum_{i=1}^N |\partial_i u|^{p_i} \right)^{1/r_1'} (u_h)^{\frac{kp_j}{r_1'}} \right) \left((u_h)^{\frac{kp_j}{r_1}} u^+ \right) dx$$

$$+ a_4 \int_\Omega \left(\sum_{i=1}^N |\partial_i v|^{q_i} \right)^{1/r_1'} (u_h)^{kp_j} u^+ dx + a_5 \int_\Omega (u_h)^{kp_j} u^+ dx$$

and

$$\int_\Omega F_2(x,u,v,\nabla u,\nabla v)v^+(v_h)^{kq_j} dx$$

$$\leq b_1 \int_\Omega |u|^{\frac{r_1}{r_2'}} (v_h)^{kq_j} v^+ dx$$

$$+ b_2 \int_\Omega |v|^{r_2-1}(v_h)^{kq_j} v^+ dx + b_3 \int_\Omega \left(\sum_{i=1}^N |\partial_i u|^{p_i} \right)^{1/r_2'} (v_h)^{kq_j} v^+ dx$$

$$+ b_4 \int_\Omega \left(\left(\sum_{i=1}^N |\partial_i v|^{q_i} \right)^{1/r_2'} (v_h)^{\frac{kq_j}{r_2'}} \right) \left((v_h)^{\frac{kq_j}{r_2}} v^+ \right) dx + b_5 \int_\Omega (v_h)^{kq_j} v^+ dx.$$

For any $\varepsilon > 0$, Young's inequality provides constants $c(\varepsilon), d(\varepsilon) > 0$ such that

$$\int_\Omega \left(\left(\sum_{i=1}^N |\partial_i u|^{p_i} \right)^{1/r_1'} (u_h)^{\frac{kp_j}{r_1'}} \right) \left((u_h)^{\frac{kp_j}{r_1}} u^+ \right) dx$$

$$\leq \varepsilon \sum_{i=1}^N \int_\Omega |\partial_i(u^+)|^{p_i}(u_h)^{kp_j} dx + c(\varepsilon) \int_\Omega (u_h)^{kp_j}(u^+)^{r_1} dx$$

and

$$\int_\Omega \left(\left(\sum_{i=1}^N |\partial_i v|^{q_i} \right)^{1/r_2'} (v_h)^{\frac{kq_j}{r_2'}} \right) \left((v_h)^{\frac{kq_j}{r_2}} v^+ \right) dx$$

$$\leq \varepsilon \sum_{i=1}^N \int_\Omega |\partial_i(v^+)|^{q_i}(v_h)^{kq_j} dx + d(\varepsilon) \int_\Omega (v_h)^{kq_j}(v^+)^{r_2} dx.$$

By Hölder's inequality, Lemma 3.1, and (5), there exists a constant $C_1 > 0$ such that

$$\int_\Omega |v|^{\frac{r_2}{r_1'}}(u_h)^{kp_j} u^+ dx \leq \left(\int_\Omega |v|^{r_2} dx \right)^{\frac{1}{r_1'}} \|(u_h)^{kp_j} u^+\|_{L^{r_1}}$$

$$\leq C_1 \|(u_h)^{kp_j} u^+\|_{L^{r_1}}$$

and

$$\int_\Omega |u|^{\frac{r_1}{r_2'}}(v_h)^{kq_j} v^+ dx \leq \left(\int_\Omega |u|^{r_1} dx \right)^{\frac{1}{r_2'}} \|(v_h)^{kq_j} v^+\|_{L^{r_2}}$$

$$\leq C_1 \|(v_h)^{kq_j} v^+\|_{L^{r_2}}.$$

Again, by Hölder's inequality, Lemma 3.1, and (5), we obtain a constant $C_2 > 0$ with

$$\int_\Omega \Big(\sum_{i=1}^N |\partial_i v|^{q_i} \Big)^{1/r_1'} (u_h)^{kp_j} u^+ dx \leq \Big(\sum_{i=1}^N \int_\Omega |\partial_i v|^{q_i} dx \Big)^{1/r_1'} \|(u_h)^{kp_j} u^+\|_{L^{r_1}}$$
$$\leq C_2 \|(u_h)^{kp_j} u^+\|_{L^{r_1}}$$

and

$$\int_\Omega \Big(\sum_{i=1}^N |\partial_i u|^{p_i} \Big)^{1/r_2'} (v_h)^{kq_j} v^+ dx \leq \Big(\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i} dx \Big)^{1/r_2'} \|(v_h)^{kq_j} v^+\|_{L^{r_2}}$$
$$\leq C_2 \|(v_h)^{kq_j} v^+\|_{L^{r_2}}.$$

There are also the estimates

$$\int_\Omega (u_h)^{kp_j} u^+ dx \leq \int_\Omega (u_h)^{kp_j} (u^+)^{r_1} dx + |\Omega|$$

and

$$\int_\Omega (v_h)^{kq_j} v^+ dx \leq \int_\Omega (v_h)^{kq_j} (v^+)^{r_2} dx + |\Omega|,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

The preceding estimates lead to

$$\int_\Omega F_1(x, u, v, \nabla u, \nabla v) u^+ (u_h)^{kp_j} dx$$
$$\leq a_1 \int_\Omega (u_h)^{kp_j} (u^+)^{r_1} dx + a_2 C_1 \|(u_h)^{kp_j} u^+\|_{L^{r_1}} + a_3 \varepsilon \sum_{i=1}^N \int_\Omega |\partial_i u^+|^{p_i} (u_h)^{kp_j} dx$$
$$+ a_3 c(\varepsilon) \int_\Omega (u_h)^{kp_j} (u^+)^{r_1} dx + a_4 C_2 \|(u_h)^{kp_j} u^+\|_{L^{r_1}}$$
$$+ a_5 \Big(\int_\Omega (u_h)^{kp_j} (u^+)^{r_1} dx + |\Omega| \Big)$$

and

$$\int_\Omega F_2(x, u, v, \nabla u, \nabla v) v^+ (v_h)^{kq_j} dx$$
$$\leq b_1 C_1 \|(v_h)^{kq_j} v^+\|_{L^{r_2}} + b_2 \int_\Omega (v_h)^{kq_j} (v^+)^{r_2} dx + b_3 C_2 \|(v_h)^{kq_j} v^+\|_{L^{r_2}}$$
$$+ b_4 \varepsilon \sum_{i=1}^N \int_\Omega |\partial_i(v^+)|^{q_i} (v_h)^{kq_j} dx + b_4 d(\varepsilon) \int_\Omega (v_h)^{kq_j} (v^+)^{r_2} dx$$
$$+ b_5 \Big(\int_\Omega (v_h)^{kq_j} (v^+)^{r_2} dx + |\Omega| \Big).$$

Let us notice that

$$
\begin{aligned}
\|(u_h)^{kp_j}u^+\|_{L^{r_1}} &= \left(\int_\Omega ((u_h)^{kp_j}u^+)^{r_1}dx \right)^{\frac{1}{r_1}} \\
&\leq \left(\int_\Omega ((u_h)^{kp_jr_1}(u^+)^{p_jr_1}+1)dx \right)^{\frac{1}{r_1}} \\
&= \left(\int_\Omega ((u_h)^k u^+)^{p_jr_1}dx + |\Omega| \right)^{\frac{1}{r_1}} \\
&\leq \left(\int_\Omega ((u_h)^k u^+)^{p_jr_1}dx \right)^{\frac{1}{r_1}} + |\Omega|^{\frac{1}{r_1}} = \|(u_h)^k u^+\|_{L^{p_jr_1}}^{p_j} + |\Omega|^{\frac{1}{r_1}}.
\end{aligned}
$$

and similarly

$$
\|(v_h)^{kq_j}v^+\|_{L^{r_2}} \leq \|(v_h)^k v^+\|_{L^{q_jr_2}}^{q_j} + |\Omega|^{\frac{1}{r_2}}.
$$

Consequently, we obtain

$$
\begin{cases}
\displaystyle \int_\Omega F_1(x,u,v,\nabla u,\nabla v)u^+(u_h)^{kp_j}dx \leq a_3\varepsilon \sum_{i=1}^N \int_\Omega |\partial_i(u^+)|^{p_i}(u_h)^{kp_j}dx \\
\qquad\qquad +C\left(\int_\Omega (u_h)^{kp_j}(u^+)^{r_1}dx + \|(u_h)^k u^+\|_{L^{p_jr_1}}^{p_j} + 1 \right), \\
\\
\displaystyle \int_\Omega F_2(x,u,v,\nabla u,\nabla v)v^+(v_h)^{kq_j}dx \leq b_4\varepsilon \sum_{i=1}^N \int_\Omega |\partial_i(v^+)|^{q_i}(v_h)^{kq_j}dx \\
\qquad\qquad +C\left(\int_\Omega (v_h)^{kq_j}(v^+)^{r_2}dx + \|(v_h)^k v^+\|_{L^{q_jr_2}}^{q_j} + 1 \right),
\end{cases}
\tag{13}
$$

with a constant $C > 0$.

Then, (10), (11), (12), and (13) show that

$$
\begin{cases}
\displaystyle \sum_{i=1}^N (a_{1i}-a_3\varepsilon) \int_\Omega (u_h)^{kp_j}|\partial_i u^+|^{p_i}dx \\
\qquad\qquad \leq C\left(\int_\Omega (u_h)^{kp_j}(u^+)^{r_1}dx + \|(u_h)^k u^+\|_{L^{p_jr_1}}^{p_j} + 1 \right), \\
\displaystyle \sum_{i=1}^N (a_{2i}-b_4\varepsilon) \int_\Omega (v_h)^{kq_j}|\partial_i v^+|^{q_i}dx \\
\qquad\qquad \leq C\left(\int_\Omega (v_h)^{kq_j}(v^+)^{r_2}dx + \|(v_h)^k v^+\|_{L^{q_jr_2}}^{q_j} + 1 \right).
\end{cases}
$$

For any $\varepsilon > 0$ small enough, we get

$$
\begin{cases}
\displaystyle \sum_{i=1}^N \int_\Omega (u_h)^{kp_j}|\partial_i u^+|^{p_i}dx \leq C\left(\int_\Omega (u_h)^{kp_j}(u^+)^{r_1}dx + \|(u_h)^k u^+\|_{L^{p_jr_1}}^{p_j} + 1 \right), \\
\displaystyle \sum_{i=1}^N \int_\Omega (v_h)^{kq_j}|\partial_i v^+|^{q_i}dx \leq C\left(\int_\Omega (v_h)^{kq_j}(v^+)^{r_2}dx + \|(v_h)^k v^+\|_{L^{q_jr_2}}^{q_j} + 1 \right),
\end{cases}
$$

with a new constant $C > 0$.

From (9), we infer that

$$\begin{cases} \|\partial_j(u^+(u_h)^k)\|_{L^{p_j}} \leq C^{\frac{1}{p_j}}(k+1)\Big(\int_\Omega (u_h)^{kp_j}(u^+)^{r_1}dx + \|(u_h)^k u^+\|_{L^{p_j r_1}}^{p_j} + 1\Big)^{\frac{1}{p_j}}, \\[2mm] \|\partial_j(v^+(v_h)^k)\|_{L^{q_j}} \leq C^{\frac{1}{p_j}}(k+1)\Big(\int_\Omega (v_h)^{kq_j}(v^+)^{r_2}dx + \|(v_h)^k v^+\|_{L^{q_j r_2}}^{q_j} + 1\Big)^{\frac{1}{q_j}}. \end{cases}$$
(14)

Using that $r_1 \in (\bar{p}, p^*)$ and $r_2 \in (\bar{q}, q^*)$ as postulated in hypothesis (H1), we can find $\alpha \in (p_j, r_1)$ and $\beta \in (q_j, r_2)$ satisfying

$$\frac{(r_1 - p_j)\alpha}{\alpha - p_j} < p^* \text{ and } \frac{(r_2 - q_j)\beta}{\beta - q_j} < q^*. \tag{15}$$

Hölder's inequality, (6), (15), and Lemma 3.1 provide the existence of constants $K_1 > 0$ and $K_2 > 0$ such that

$$\begin{aligned} \int_\Omega (u_h)^{kp_j}(u^+)^{r_1}dx &= \int_\Omega (u^+)^{r_1-p_j}((u_h)^k u^+)^{p_j}dx \\ &\leq \Big(\int_\Omega (u^+)^{\frac{(r_1-p_j)\alpha}{\alpha-p_j}}dx\Big)^{\frac{\alpha-p_j}{\alpha}}\Big(\int_\Omega (u^+(u_h)^k)^\alpha dx\Big)^{\frac{p_j}{\alpha}} \\ &\leq K_1 \|u^+(u_h)^k\|_{L^\alpha}^{p_j} \end{aligned}$$

and

$$\begin{aligned} \int_\Omega (v_h)^{kq_j}(v^+)^{r_2}dx &= \int_\Omega (v^+)^{r_2-q_j}((v_h)^k v^+)^{q_j}dx \\ &\leq \Big(\int_\Omega (v^+)^{\frac{(r_2-q_j)\beta}{\beta-q_j}}dx\Big)^{\frac{\beta-q_j}{\beta}}\Big(\int_\Omega (v^+(v_h)^k)^\beta dx\Big)^{\frac{q_j}{\beta}} \\ &\leq K_2 \|v^+(v_h)^k\|_{L^\beta}^{q_j}. \end{aligned}$$

Note that $p_j r_1 > \alpha$ and $q_j r_2 > \beta$. From (14), we arrive at

$$\begin{cases} \|\partial_j(u^+(u_h)^k)\|_{L^{p_j}} \leq K(k+1)(\|u^+(u_h)^k\|_{L^\alpha} + 1), \\ \|\partial_j(v^+(v_h)^k)\|_{L^{q_j}} \leq K(k+1)(\|v^+(v_h)^k\|_{L^\beta} + 1), \end{cases}$$

with a constant $K > 0$. As $j = 1, ..., N$ was arbitrary, this results in

$$\begin{cases} \|u^+(u_h)^k\|_{W_0^{1,\vec{p}}(\Omega)} \leq KN(k+1)(\|u^+(u_h)^k\|_{L^\alpha} + 1), \\ \|v^+(v_h)^k\|_{W_0^{1,\vec{q}}(\Omega)} \leq KN(k+1)(\|v^+(v_h)^k\|_{L^\beta} + 1). \end{cases}$$

Then, the continuous embedding (6) implies

$$\begin{cases} \|u^+(u_h)^k\|_{L^{p^*}} \leq C(k+1)(\|u^+\|_{L^{\alpha(k+1)}}^{k+1} + 1), \\ \|v^+(v_h)^k\|_{L^{q^*}} \leq C(k+1)(\|v^+\|_{L^{\beta(k+1)}}^{k+1} + 1), \end{cases}$$

with a constant $C > 0$. Letting $h \to 0$, Fatou's lemma yields

$$\begin{aligned} \|u^+\|_{L^{p^*(k+1)}}^{k+1} &= \|(u^+)^{k+1}\|_{L^{p^*}} \leq C(k+1)(\|u^+\|_{L^{\alpha(k+1)}}^{k+1} + 1), \\ \|v^+\|_{L^{q^*(k+1)}}^{k+1} &= \|(v^+)^{k+1}\|_{L^{q^*}} \leq C(k+1)(\|v^+\|_{L^{\beta(k+1)}}^{k+1} + 1). \end{aligned} \tag{16}$$

At this point, since now the variables u^+ and v^+ are decoupled, the problem is reduced to the setting of [10], so we can proceed as therein. For the sake of completeness, we carry out the proof. It suffices to argue in the case of u^+. If for a sequence $k_n \to +\infty$ we have $\|u^+\|_{L^{p^*(k_n+1)}} \leq 1$ for all n, then $\|u^+\|_{L^\infty} \leq 1$, which

ends the proof. Two situations remain to be discussed: (a) $\|u^+\|_{L^{p^*(k+1)}} > 1$ for all $k \geq 0$; (b) there is $k_0 > 0$ such that $\|u^+\|_{L^{p^*(k_0+1)}} \leq 1$ and $\|u^+\|_{L^{p^*(k+1)}} > 1$ for all $k > k_0$.

When (a) holds, (16) gives

$$\|u^+\|_{L^{p^*(k+1)}} \leq (2C)^{\frac{1}{k+1}}(k+1)^{\frac{1}{k+1}}\|u^+\|_{L^{\alpha(k+1)}}. \quad \forall k \geq 0.$$

The function $k \mapsto (k+1)^{1/\sqrt{k+1}}$ is bounded on $(0, +\infty)$, so one has

$$\|u^+\|_{L^{p^*(k+1)}} \leq C^{1/\sqrt{k+1}}\|u^+\|_{L^{\alpha(k+1)}}, \quad \forall k \geq 0, \tag{17}$$

with a new constant $C > 0$. We can build the Moser iteration

$$(k_n + 1)\alpha = (k_{n-1} + 1)p^*, \quad \forall n \geq 2,$$

starting with $(k_1 + 1)\alpha = p^*$. Repeatedly applying (17) gives

$$\|u^+\|_{L^{p^*(k_n+1)}} \leq C^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}}\|u^+\|_{L^{p^*}}.$$

The inductive definition of (k_n) ensures that $k_n \to +\infty$ and the series $\sum_{n=1}^\infty 1/\sqrt{k_n + 1}$ converges. Then, Lemma 3.1 shows that letting $n \to \infty$, we obtain $\|u^+\|_{L^\infty} \leq C_0$, with a constant $C_0 > 0$ independent of the solution (u, v).

For case (b), a similar reasoning can be developed giving

$$\|u^+\|_{L^{p^*(k+1)}} \leq (2C)^{\frac{1}{k+1}}(k+1)^{\frac{1}{k+1}}\|u^+\|_{L^{\alpha(k+1)}}, \quad \forall k \geq k_0,$$

The above argument shows

$$\|u^+\|_{L^{p^*(k+1)}} \leq C^{1/\sqrt{k+1}}\|u^+\|_{L^{\alpha(k+1)}}, \quad \forall k \geq k_0. \tag{18}$$

with a constant $C > 0$. The same Moser iteration applies, this time starting with $(k_1 + 1)\alpha = p^*(k_0 + 1)$. Then, (18) entails

$$\|u^+\|_{L^{p^*(k_n+1)}} \leq C^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}}\|u^+\|_{L^{p^*(k_0+1)}}.$$

As in case (a), we prove that $\|u^+\|_{L^\infty(\Omega)} \leq C_0$ with a constant $C_0 > 0$ independent of the solution (u, v). Similarly, we can establish the bound $\|v^+\|_{L^\infty(\Omega)} \leq C_0$.

A careful inspection of the proof reveals that the constant C_0 depends on G_{1i} and G_{2i} only through their lower bounds a_{ki} for $i = 1, \ldots, N$, $k = 1, 2$, respectively. This completes the proof. $\quad\square$

4. **Auxiliary truncated system.** Here, we extend to system (2) the idea of truncation used in the case of isotropic equations (see [8] and [11]) to overcome the difficulty of unbounded coefficients G_{ki} for $i = 1, \cdots, N$ and $k = 1, 2$.

Corresponding to any real number $R > 0$, we truncate the functions G_{ki} in the following way:

$$G_{1i,R}(t) = \begin{cases} G_{1i}(t) & \text{if } |t| \leq R, \\ G_{1i}(R) & \text{if } t > R, \\ G_{1i}(-R) & \text{if } t < -R. \end{cases} \quad \text{and} \quad G_{2i,R}(t) = \begin{cases} G_{2i}(t) & \text{if } |t| \leq R, \\ G_{ki}(R) & \text{if } t > R, \\ G_{ki}(-R) & \text{if } t < -R. \end{cases}$$

$$\tag{19}$$

In accordance with (19), we state the auxiliary system

$$
\begin{cases}
-\displaystyle\sum_{i=1}^{N} \partial_i(G_{1i,R}(u)|\partial_i u|^{p_i-2}\partial_i u) = F_1(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\
-\displaystyle\sum_{i=1}^{N} \partial_i(G_{2i,R}(v)|\partial_i v|^{q_i-2}\partial_i v) = F_2(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{20}
$$

The driving operator in problem (20) is the map $\mathcal{A}_R : X \to X^*$ defined by

$$
\begin{aligned}
&\langle \mathcal{A}_R(u,v),(\phi,\psi)\rangle \\
&= \sum_{i=1}^{N} \int_{\Omega} G_{1i,R}(u)|\partial_i u|^{p_i-2}\partial_i u \partial_i \phi \, dx + \sum_{i=1}^{N} \int_{\Omega} G_{2i,R}(v)|\partial_i v|^{q_i-2}\partial_i v \partial_i \psi \, dx
\end{aligned}
\tag{21}
$$

for all $(u,v),(\phi,\psi) \in X$.

Proposition 4.1. *Assume conditions* (4) *and* (7). *Then, for each* $R > 0$, *the map* $\mathcal{A}_R : X \to X^*$ *in* (21) *is well defined, bounded, continuous, and fulfills the* S_+*-property, that is,* $(u_n,v_n) \rightharpoonup (u,v)$ *in* X, *and*

$$
\limsup_{n\to\infty}\langle \mathcal{A}_R(u_n,v_n),(u_n-u,v_n-v)\rangle \le 0
\tag{22}
$$

imply $(u_n,v_n) \to (u,v)$ *in* X.

Proof. By Hölder's inequality, for all $(u,v),(\phi,\psi) \in X$ and $i = 1,\ldots,N$, we get

$$
\begin{aligned}
&\left|\int_{\Omega} G_{1i,R}(u)|\partial_i u|^{p_i-2}\partial_i u \partial_i \phi \, dx\right| + \left|\int_{\Omega} G_{2i,R}(v)|\partial_i v|^{q_i-2}\partial_i v \partial_i \psi \, dx\right| \\
&\le \max_{|t|\le R} G_{1i,R}(t)\|\partial_i u\|_{L^{p_i}}^{p_i-1}\|\partial_i \phi\|_{L^{p_i}} + \max_{|t|\le R} G_{2i,R}(t)\|\partial_i v\|_{L^{q_i}}^{q_i-1}\|\partial_i \psi\|_{L^{q_i}}.
\end{aligned}
\tag{23}
$$

We infer for all $(u,v) \in X$ that $\mathcal{A}_R(u,v) \in X^*$, so $\mathcal{A}_R$ is well defined. Moreover, (23) shows that the mapping $\mathcal{A}_R$ is bounded.

In order to prove that $\mathcal{A}_R$ is continuous, let $(u_n,v_n) \to (u,v)$ in X. We note that

$$
\begin{aligned}
&\|\mathcal{A}_R(u_n,v_n) - \mathcal{A}_R(u,v)\|_{X^*} \\
&\le \sum_{i=1}^{N} \left\|(G_{1i,R}(u_n) - G_{1i,R}(u))|\partial_i u_n|^{p_i-2}\partial_i u_n\right\|_{L^{\frac{p_i}{p_i-1}}} \\
&\quad + \sum_{i=1}^{N} \left\|G_{1i,R}(u)(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i u|^{p_i-2}\partial_i u)\right\|_{L^{\frac{p_i}{p_i-1}}} \\
&\quad + \sum_{i=1}^{N} \left\|(G_{2i,R}(v_n) - G_{2i,R}(v))|\partial_i v_n|^{q_i-2}\partial_i v_n\right\|_{L^{\frac{q_i}{q_i-1}}} \\
&\quad + \sum_{i=1}^{N} \left\|G_{2i,R}(v)(|\partial_i v_n|^{q_i-2}\partial_i v_n - |\partial_i v|^{q_i-2}\partial_i v)\right\|_{L^{\frac{q_i}{q_i-1}}}.
\end{aligned}
\tag{24}
$$

The following inequalities hold:

$$
\begin{cases}
\left\| (G_{1i,R}(u_n) - G_{1i,R}(u)) |\partial_i u_n|^{p_i-2} \partial_i u_n \right\|_{L^{\frac{p_i}{p_i-1}}}^{\frac{p_i}{p_i-1}} \\
\qquad \leq \int_\Omega |G_{1i,R}(u_n) - G_{1i,R}(u)|^{\frac{p_i}{p_i-1}} |\partial_i u_n|^{p_i} dx, \\[2mm]
\left\| (G_{2i,R}(v_n) - G_{2i,R}(v)) |\partial_i v_n|^{q_i-2} \partial_i v_n \right\|_{L^{\frac{q_i}{q_i-1}}}^{\frac{q_i}{q_i-1}} \\
\qquad \leq \int_\Omega |G_{2i,R}(v_n) - G_{2i,R}(v)|^{\frac{q_i}{q_i-1}} |\partial_i v_n|^{q_i} dx.
\end{cases}
$$

The continuity and boundedness of the functions $G_{1i,R}$ and $G_{2i,R}$ enable us to apply Lebesgue's dominated convergence theorem on the basis of the strong convergence $(u_n, v_n) \to (u, v)$ in X that provides

$$
\begin{cases}
\lim_{n\to\infty} \left\| (G_{1i,R}(u_n) - G_{1i,R}(u)) |\partial_i u_n|^{p_i-2} \partial_i u_n \right\|_{L^{\frac{p_i}{p_i-1}}} = 0, \\[2mm]
\lim_{n\to\infty} \left\| (G_{2i,R}(v_n) - G_{2i,R}(v)) |\partial_i v_n|^{q_i-2} \partial_i v_n \right\|_{L^{\frac{q_i}{q_i-1}}} = 0,
\end{cases}
$$

and

$$
\begin{cases}
\lim_{n\to\infty} \left\| G_{1i,R}(u)(|\partial_i u_n|^{p_i-2} \partial_i u_n - |\partial_i u|^{p_i-2} \partial_i u) \right\|_{L^{\frac{p_i}{p_i-1}}} = 0, \\[2mm]
\lim_{n\to\infty} \left\| G_{2i,R}(v)(|\partial_i v_n|^{q_i-2} \partial_i v_n - |\partial_i v|^{q_i-2} \partial_i v) \right\|_{L^{\frac{q_i}{q_i-1}}} = 0.
\end{cases}
$$

Then, from (24), we derive that $\mathcal{A}_R(u_n, v_n) \to \mathcal{A}_R(u, v)$ in X^*, whence the continuity of $\mathcal{A}_R$.

Now we focus on the S_+-property of $\mathcal{A}_R$. Let $(u_n, v_n) \rightharpoonup (u, v)$ in X and (22) be satisfied, which takes the form

$$
\limsup_{n\to\infty} \langle \mathcal{A}_R(u_n, v_n) - \mathcal{A}_R(u, v), (u_n - u, v_n - v) \rangle \leq 0. \tag{25}
$$

It is seen that

$$
\begin{aligned}
\langle \mathcal{A}_R&(u_n, v_n) - \mathcal{A}_R(u, v), (u_n - u, v_n - v) \rangle \\
\geq & \sum_{i=1}^N a_{1i} \int_\Omega (|\partial_i u_n|^{p_i-2} \partial_i u_n - |\partial_i u|^{p_i-2} \partial_i u) \partial_i (u_n - u) dx \\
& + \sum_{i=1}^N \int_\Omega (G_{1i,R}(u_n) - G_{1i,R}(u)) |\partial_i u|^{p_i-2} \partial_i u \partial_i (u_n - u) dx \\
& + \sum_{i=1}^N a_{2i} \int_\Omega (|\partial_i v_n|^{q_i-2} \partial_i v_n - |\partial_i v|^{q_i-2} \partial_i v) \partial_i (v_n - v) dx \\
& + \sum_{i=1}^N \int_\Omega (G_{2i,R}(v_n) - G_{2i,R}(v)) |\partial_i v|^{q_i-2} \partial_i v \partial_i (v_n - v) dx.
\end{aligned} \tag{26}
$$

As before, we can prove that

$$
\begin{cases}
\lim_{n\to\infty} \int_\Omega (G_{1i,R}(u_n) - G_{1i,R}(u)) |\partial_i(u)|^{p_i-2} \partial_i u \partial_i (u_n - u) dx = 0, \\[2mm]
\lim_{n\to\infty} \int_\Omega (G_{2i,R}(v_n) - G_{2i,R}(v)) |\partial_i(v)|^{q_i-2} \partial_i v \partial_i (v_n - v) dx = 0.
\end{cases} \tag{27}
$$

On the other hand, using Hölder's inequality, we observe that

$$\int_\Omega (|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i u|^{p_i-2}\partial_i u)\partial_i(u_n - u)dx$$

$$\geq \int_\Omega |\partial_i u_n|^{p_i}dx + \int_\Omega |\partial_i u|^{p_i}dx$$

$$- \int_\Omega |\partial_i u_n|^{p_i-1}|\partial_i u|dx - \int_\Omega |\partial_i u|^{p_i-1}|\partial_i u_n|dx$$

$$\geq \|u_n\|_{L^{p_i}}^{p_i} + \|u\|_{L^{p_i}}^{p_i} - \left(\int_\Omega |\partial_i u_n|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}\left(\int_\Omega |\partial_i u|^{p_i}dx\right)^{\frac{1}{p_i}}$$

$$- \left(\int_\Omega |\partial_i u|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}\left(\int_\Omega |\partial_i u_n|^{p_i}dx\right)^{\frac{1}{p_i}}$$

$$= \|u_n\|_{L^{p_i}}^{p_i} + \|u\|_{L^{p_i}}^{p_i} - \|u_n\|_{L^{p_i}}^{p_i-1}\|u\|_{L^{p_i}} - \|u\|_{L^{p_i}}^{p_i-1}\|u_n\|_{L^{p_i}}$$

$$= (\|u_n\|_{L^{p_i}} - \|u\|_{L^{p_i}})(\|u_n\|_{L^{p_i}}^{p_i-1} - \|u\|_{L^{p_i}}^{p_i-1}).$$

Then, from (25), (26), (27), and Hölder's inequality, it turns out

$$\begin{cases} \lim\limits_{n\to\infty}(\|\partial_i u_n\|_{L^{p_i}} - \|\partial_i u\|_{L^{p_i}})\Big(\|\partial_i u_n\|_{L^{p_i}}^{p_i-1} - \|\partial_i u\|_{L^{p_i}}^{p_i-1}\Big) = 0, \\ \lim\limits_{n\to\infty}(\|\partial_i v_n\|_{L^{q_i}} - \|\partial_i v\|_{L^{q_i}})\Big(\|\partial_i v_n\|_{L^{q_i}}^{q_i-1} - \|\partial_i v\|_{L^{q_i}}^{q_i-1}\Big) = 0, \end{cases}$$

for all $i = 1,\dots,N$. This results in

$$\begin{cases} \lim\limits_{n\to\infty}\|\partial_i u_n\|_{L^{p_i}} = \|\partial_i u\|_{L^{p_i}} \quad \text{and} \\ \lim\limits_{n\to\infty}\|\partial_i v_n\|_{L^{q_i}} = \|\partial_i v\|_{L^{q_i}}, \quad \text{for all } i = 1,\dots,N. \end{cases}$$

Since the spaces $L^{p_i}(\Omega)$ and $L^{q_i}(\Omega)$ are uniformly convex, we infer the strong convergence $u_n \to u$ and $v_n \to v$ in $W_0^{1,\overrightarrow{p}}(\Omega)$, which completes the proof. $\qquad\square$

The properties of the operator $\mathcal{A}_R - \mathcal{N}$, with $\mathcal{A}_R$ and $\mathcal{N}$ introduced in (21) and Proposition 2.1, respectively, are listed in the next statement.

Proposition 4.2. *Assume* (4), (7), (H1), *and* (H2). *Then, for each $R > 0$, the map $\mathcal{A}_R - \mathcal{N} : X \to X^*$ satisfies:*

(i) $\mathcal{A}_R - \mathcal{N}$ *is bounded;*

(ii) $\mathcal{A}_R - \mathcal{N}$ *is pseudomonotone, that is, if $(u_n, v_n) \rightharpoonup (u, v)$ in X and*

$$\limsup_{n\to\infty}\langle(\mathcal{A}_R - \mathcal{N})(u_n, v_n), (u_n - u, v_n - v)\rangle \leq 0, \tag{28}$$

then

$$\liminf_{n\to\infty}\langle\mathcal{A}_R - \mathcal{N})(u_n, v_n), (u_n - \phi, v_n - \psi)\rangle \tag{29}$$

$$\geq \liminf_{n\to\infty}\langle(\mathcal{A}_R - \mathcal{N})(u, v), (u, v) - (\phi, \psi)\rangle, \quad \forall (\phi, \psi) \in X;$$

(iii) $\mathcal{A}_R - \mathcal{N}$ *is coercive, that is,*

$$\lim_{\|(u,v)\|\to\infty}\frac{\langle(\mathcal{A}_R - \mathcal{N})(u, v), (u, v)\rangle}{\|(u, v)\|} = +\infty. \tag{30}$$

Proof. (i) The assertion follows from Propositions 2.1 and 4.1.

(ii) Let $u_n \rightharpoonup u$ and $v_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ and $W_0^{1,\overrightarrow{q}}(\Omega)$, respectively, and suppose that (28) is satisfied. From Corollary 2.2 and (28), we derive (22). Thanks to Proposition 4.1, the S_+-property of the operator $\mathcal{A}_R$ implies $u_n \to u$ and

$v_n \to v$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ and $W_0^{1,\overrightarrow{q}}(\Omega)$, respectively. Then, Proposition 2.1 ensures that $\mathcal{N}(u_n, v_n) \to \mathcal{N}(u, v)$ in X^*, while Proposition 4.1 entails $\mathcal{A}_R(u_n, v_n) \to \mathcal{A}_R(u, v)$ in X^*. Therefore, (29) holds true.

(iii) By assumption (H2) and (8), we infer that

$$\langle (\mathcal{A}_R - \mathcal{N})(u, v), (u, v) \rangle$$

$$\geq \sum_{i=1}^{N} \int_{\Omega} G_{1i,R}(u)|\partial_i u|^{p_i} dx - c_1 \|u\|_{L^{\underline{p}}}^{\underline{p}} - c_2 \|v\|_{L^{\underline{q}}}^{\underline{q}} - c_3 \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}}^{p_i}$$

$$- c_4 \sum_{i=1}^{N} \|\partial_i v\|_{L^{q_i}}^{q_i} - \|\zeta_1\|_{L^1}$$

$$+ \sum_{i=1}^{N} \int_{\Omega} G_{2i,R}(v)|\partial_i v|^{q_i} dx - d_1 \|u\|_{L^{\underline{p}}}^{\underline{p}} - d_2 \|v\|_{L^{\underline{q}}}^{\underline{q}} - d_3 \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}}^{p_i}$$

$$- d_4 \sum_{i=1}^{N} \|\partial_i v\|_{L^{q_i}}^{q_i} - \|\zeta_2\|_{L^1}$$

$$\geq \sum_{i=1}^{N} (a_{1i} - (c_3 + d_3) - N^{\underline{p}-1}(c_1 + d_1)\theta)\|\partial_i u\|_{L^{p_i}}^{p_i}$$

$$+ \sum_{i=1}^{N} (a_{2i} - (c_4 + d_4) - N^{\underline{q}-1}(c_2 + d_2)\theta)\|\partial_i v\|_{L^{q_i}}^{q_i}$$

$$- N^{\underline{p}}(c_1 + d_1)\theta - N^{\underline{p}}(c_2 + d_2)\theta - \|\zeta_1\|_{L^1} - \|\zeta_2\|_{L^1}.$$

From hypothesis (H2), we know that $a_{1i} - (c_3 + d_3) - N^{\underline{p}-1}(c_1 + d_1)\theta > 0$ and $a_{2i} - (c_4 + d_4) - N^{\underline{q}-1}(c_2 + d_2)\theta > 0$. Taking into account that $p_i > 1$ and $q_i > 1$ for all $i = 1, \ldots, N$, the preceding inequality implies (30), thus completing the proof. $\square$

5. **Solutions to the anisotropic systems.** Now we are able to prove the existence of solutions to auxiliary system (20).

Theorem 5.1. *Assume that conditions* (4) *and* (7) *hold. The functions* $G_{ki} : \mathbb{R} \to [a_{ki}, +\infty)$, *with* $a_{ki} > 0$ *for* $i = 1, \ldots, N$, $k = 1, 2$, *are continuous, and* $F_k : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $k = 1, 2$, *are Carathéodory functions satisfying hypotheses (H1) and (H2). Then, for every* $R > 0$, *problem* (20) *has at least a weak solution* $(u_R, v_R) \in X$, *which means*

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_{\Omega} G_{1i,R}(u_R(x))|\partial_i u(x)|^{p_i-2}\partial_i u(x)\partial_i \phi(x)dx = \int_{\Omega} F_1(x, u, v, \nabla u, \nabla v)\phi dx, \\ \displaystyle\sum_{i=1}^{N} \int_{\Omega} G_{2i,R}(v_R(x))|\partial_i v(x)|^{q_i-2}\partial_i v(x)\partial_i \psi(x)dx = \int_{\Omega} F_2(x, u, v, \nabla u, \nabla v)\psi dx \end{cases}$$

(31)

for all $(\phi, \psi) \in X$. *Moreover, the solution set of system* (20) *is uniformly bounded with the bound* $C_0 > 0$ *in Theorem* 3.2. *In particular, one has* $\|(u_R, v_R)\|_{L^\infty} \leq C_0$.

Proof. For a fixed $R > 0$, the auxiliary system (20) is equivalent to resolving in X the operator equation

$$(\mathcal{A}_R - \mathcal{N})(u, v) = 0. \tag{32}$$

By Proposition 4.2, it is known that the operator $\mathcal{A}_R - \mathcal{N} : X \to X^*$ is pseudomonotone, bounded, and coercive. Hence, we are entitled to apply the main theorem for pseudomonotone operators (see, e.g., [5, Theorem 2.99]) from which we deduce that equation (32) possesses at least a solution $(u_R, v_R) \in X$. Due to the mentioned equivalence, (u_R, v_R) is a weak solution of auxiliary problem (20) in the sense of (31).

On the other hand, Theorem 3.2 is applicable with $G_{ki,R}$ in place of G_{ki} for every $i = 1, \ldots, N$ and $k = 1, 2$. This is true because the range of $G_{ki,R}$ is contained in the interval $[a_{ki}, +\infty)$, as it is the case for G_{ki}. Consequently, the solution (u_R, v_R) of (20) fulfills the a priori estimate $\|(u_R, v_R)\|_{L^\infty} \leq C_0$, where $C_0 > 0$ is the uniform bound given in Theorem 3.2. This completes the proof. $\qquad\square$

Remark 1.3 can be applied to Theorem 5.1 in place of Theorem 1.2.

Now we prove that $(u_R, v_R) \in X$ obtained in Theorem 5.1 is a weak solution of the original problem (2) provided $R > 0$ is sufficiently large.

Proof of Theorem 1.2. Theorem 3.2 provides the existence of a positive constant C_0 such that $\|(u, v)\|_{L^\infty} \leq C_0$ for all weak solutions $(u, v) \in X$ to problem (2). As known from the statement of Theorem 3.2, the constant C_0 depends on the function G_{ki} in (2) only though its lower bound a_{ki} for $i = 1, \ldots, N$ and $k = 1, 2$. From (19), it is clear that a_{ki} is a lower bound for each truncation $G_{ki,R}$. Therefore, the same constant C_0 bounds the solution set of each auxiliary problem (20) whenever $R > 0$, so for the solution $(u_R, v_R) \in X$ to problem (20) given by Theorem 5.1, it holds that $\|(u_R, v_R)\|_{L^\infty} \leq C_0$.

In view of the fact that C_0 does not depend on R, we are allowed to choose $R \geq C_0$, getting that $|u_R(x)| \leq R$ and $|v_R(x)| \leq R$ almost everywhere on Ω. Owing to (19), we have

$$G_{1i,R}(u_R(x)) = G_{1i}(u_R(x)) \text{ and } G_{2i,R}(v_R(x)) = G_{2i}(v_R(x))$$

for a.e. $x \in \Omega$, and $i = 1, \ldots, N$. It follows that $(u_R, v_R) \in X$ is a bounded weak solution for the original problem (2), thus completing the proof of Theorem 1.2. $\quad\square$

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