

Series expansion for the effective conductivity of a periodic dilute composite with thermal resistance at the two-phase interface

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Abstract. We study the asymptotic behavior of the effective thermal conductivity of a periodic two-phase dilute composite obtained by introducing into an infinite homogeneous matrix a periodic set of inclusions of a different material, each of them of size proportional to a positive parameter ϵ . We assume that the normal component of the heat flux is continuous at the two-phase interface, while we impose that the temperature field displays a jump proportional to the normal heat flux. For ϵ small, we prove that the effective conductivity can be represented as a convergent power series in ϵ and we determine the coefficients in terms of the solutions of explicit systems of integral equations.

Keywords: Effective conductivity, periodic dilute composite, singularly perturbed domain, asymptotic expansion, non-ideal contact condition

1. Introduction

In this paper, we study the asymptotic behavior of the effective conductivity of a two-phase composite, consisting of a matrix and of a periodic set of inclusions, each of them of size $\epsilon > 0$. Both the matrix and the inclusions are filled with two different homogeneous and isotropic heat conductor materials of conductivity λ^+ and λ^- . At the two-phase interface the thermal resistance appears: while the normal component of the heat flux is assumed to be continuous at the two-phase interface, the temperature field, instead, displays a jump proportional to the normal heat flux by means of a parameter $\rho(\epsilon) > 0$. Such a discontinuity in the temperature field is a well known phenomenon and has been largely investigated since 1941, when Kapitza carried out the first systematic study of thermal interface behavior in liquid helium (see, *e.g.*, Swartz and Pohl [49], Lipton [34] and references therein).

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We are interested in the properties of the composite in the dilute case, *i.e.*, when the singular perturbation parameter ϵ , which controls the size of the inclusions, tends to 0. In [17], the authors have proved that under suitable assumptions the effective conductivity $\lambda^{\text{eff}}(\epsilon)$ of the composite can be represented in terms of a real analytic function for ϵ close to 0. As a consequence, $\lambda^{\text{eff}}(\epsilon)$ can be expanded into a convergent power series for ϵ small. The aim of this paper is to compute explicitly *all* the coefficients of such a power series, providing therefore a constructive method to approximate the effective conductivity at *any* desired degree of precision and which is valid for *all* the shapes of the inclusions.

We recall that the expression defining the effective conductivity of a composite with imperfect contact conditions was introduced by Benveniste and Miloh in [9] by generalizing the dual theory of the effective behavior of composites with perfect contact (see also Benveniste [8] and for a review Drygaś and Miȳushev [21]). By the argument of Benveniste and Miloh, in order to evaluate the effective conductivity, one has to study the thermal distribution of the composite when so called “homogeneous conditions” are prescribed. We also note that effective properties of heat conductors with interfacial contact resistance have been studied via homogenization theory (cf. Donato and Monsurrò [20], Faella, Monsurrò, and Perugia [22], Monsurrò [42,43]).

Therefore, as in [17], we now introduce a particular transmission problem with non-ideal contact conditions where we impose that the temperature field displays a fixed jump along a certain direction and is periodic in all the other directions. For the sake of completeness, non-homogeneous boundary conditions at the two-phase interface are also investigated (cf. problem (3) below).

We fix once for all $(q_{11}, q_{22}) \in]0, +\infty[^2$. We denote the periodicity cell Q and the diagonal matrix q by setting

$$Q \equiv]0, q_{11}[\times]0, q_{22}[, \quad q \equiv \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix}.$$

We denote by $|Q|_2 \equiv q_{11}q_{22}$ the 2-dimensional measure of the fundamental cell Q and by q^{-1} the inverse matrix of q . Clearly,

$$q\mathbb{Z}^2 \equiv \{qz : z \in \mathbb{Z}^2\}$$

is the set of vertices of a periodic subdivision of \mathbb{R}^2 corresponding to the fundamental cell Q .

Let $\alpha \in]0, 1[$. We fix once and for all a subset Ω of \mathbb{R}^2 satisfying the following assumption:

Ω is a bounded open connected subset of \mathbb{R}^2 of

$$\text{class } C^{1,\alpha}, \mathbb{R}^2 \setminus \overline{\Omega} \text{ is connected, and } 0 \in \Omega. \quad (1)$$

Let $p \in Q$ be fixed. Then there exists $\epsilon_0 \in \mathbb{R}$ such that

$$\epsilon_0 \in]0, +\infty[, \quad p + \epsilon\overline{\Omega} \subseteq Q \quad \forall \epsilon \in]-\epsilon_0, \epsilon_0[. \quad (2)$$

To shorten our notation, we set

$$\Omega_{p,\epsilon} \equiv p + \epsilon\Omega \quad \forall \epsilon \in \mathbb{R}.$$

Then we introduce the periodic domains

$$\mathbb{S}_\epsilon \equiv \bigcup_{z \in \mathbb{Z}^2} (qz + \Omega_{p,\epsilon}), \quad \mathbb{T}_\epsilon \equiv \mathbb{R}^2 \setminus \overline{\mathbb{S}_\epsilon} \quad \forall \epsilon \in]-\epsilon_0, \epsilon_0[.$$

Next, we take two positive constants λ^+ , λ^- , a function f in the Schauder space $C^{0,\alpha}(\partial\Omega)$ and with zero integral on $\partial\Omega$, a function g in $C^{0,\alpha}(\partial\Omega)$, and a function ρ from $]0, \epsilon_0[$ to $]0, +\infty[$, and for each $n \in \{1, 2\}$ we consider the following transmission problem for a pair of functions $(u_n^+, u_n^-) \in C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{S}_\epsilon}) \times C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{T}_\epsilon})$:

$$\begin{cases} \Delta u_n^+ = 0 & \text{in } \mathbb{S}_\epsilon, \\ \Delta u_n^- = 0 & \text{in } \mathbb{T}_\epsilon, \\ u_n^+(x + q_{mm}e_m) = u_n^+(x) + \delta_{m,n}q_{mm} & \forall x \in \overline{\mathbb{S}_\epsilon}, \forall m \in \{1, 2\}, \\ u_n^-(x + q_{mm}e_m) = u_n^-(x) + \delta_{m,n}q_{mm} & \forall x \in \overline{\mathbb{T}_\epsilon}, \forall m \in \{1, 2\}, \\ \lambda^- \frac{\partial u_n^-}{\partial \mathbf{n}_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial u_n^+}{\partial \mathbf{n}_{\Omega_{p,\epsilon}}}(x) = f((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ \lambda^+ \frac{\partial u_n^+}{\partial \mathbf{n}_{\Omega_{p,\epsilon}}}(x) + \frac{1}{\rho(\epsilon)}(u_n^+(x) - u_n^-(x)) = g((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ \int_{\partial\Omega_{p,\epsilon}} u_n^+(x) d\sigma_x = 0, \end{cases} \quad (3)$$

for all $\epsilon \in]0, \epsilon_0[$, where $\mathbf{n}_{\Omega_{p,\epsilon}}$ denotes the outward unit normal to $\partial\Omega_{p,\epsilon}$. Here $\{e_1, e_2\}$ denotes the canonical basis of \mathbb{R}^2 .

In problem (3), the functions u_n^+ and u_n^- play the role of the temperature field in the inclusions occupying the periodic set \mathbb{S}_ϵ and in the matrix occupying \mathbb{T}_ϵ , respectively. The parameters λ^+ and λ^- represent the thermal conductivity of the materials which fill the inclusions and the matrix, respectively, while the parameter $\rho(\epsilon)$ is the interfacial thermal resistivity. The fifth and the sixth condition in (3) describe the jump of the normal heat flux and of the temperature field across the two-phase interface.

Problem (3) has been investigated in [17] under the assumption that

$$\text{the limit } \lim_{\epsilon \rightarrow 0^+} \frac{\epsilon}{\rho(\epsilon)} \text{ exists in } \mathbb{R}.$$

If $\epsilon \in]0, \epsilon_0[$, then the solution in $C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{S}_\epsilon}) \times C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{T}_\epsilon})$ of problem (3) is unique and we denote it by $(u_n^+[\epsilon], u_n^-[\epsilon])$ (see [17]). As in [17], we introduce the effective conductivity matrix $\lambda^{\text{eff}}(\epsilon)$ with (m, n) -entry $\lambda_{mn}^{\text{eff}}(\epsilon)$ defined by means of the following.

Definition 1.1. Let $\alpha \in]0, 1[$. Let $p \in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $\lambda^+, \lambda^- \in]0, +\infty[$. Let $f, g \in C^{0,\alpha}(\partial\Omega)$ and $\int_{\partial\Omega} f d\sigma = 0$. Let ρ be a function from $]0, \epsilon_0[$ to $]0, +\infty[$. Let $(m, n) \in \{1, 2\}^2$. We set

$$\begin{aligned} \lambda_{mn}^{\text{eff}}(\epsilon) &\equiv \frac{1}{|Q|_2} \left(\lambda^+ \int_{\Omega_{p,\epsilon}} \frac{\partial u_n^+[\epsilon](x)}{\partial x_m} dx + \lambda^- \int_{Q \setminus \overline{\Omega_{p,\epsilon}}} \frac{\partial u_n^-[\epsilon](x)}{\partial x_m} dx \right) \\ &\quad + \frac{1}{|Q|_2} \int_{\partial\Omega_{p,\epsilon}} f((x-p)/\epsilon) x_m d\sigma_x \quad \forall \epsilon \in]0, \epsilon_0[, \end{aligned}$$

where $(u_n^+[\epsilon], u_n^-[\epsilon])$ is the unique solution in $C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{S}_\epsilon}) \times C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{T}_\epsilon})$ of problem (3).

When $f = 0$ and $g = 0$ the above definition coincides with the standard definition of effective conductivity of the periodic composite with matrix and inclusions of conductivity λ^- and λ^+ , respectively, subject to imperfect contact conditions.

As already mentioned, under suitable assumptions, it is shown in [17, Thm. 8.1] that the effective conductivity can be continued real analytically in the parameter ϵ around the degenerate value $\epsilon = 0$. This is the case, for example, if $\rho(\epsilon) \equiv 1/r_\#$ or $\rho(\epsilon) \equiv \epsilon/r_\#$, where $r_\#$ is a positive real number. In particular, if $\rho(\epsilon)$ is as above, then $\lambda_{mn}^{\text{eff}}(\epsilon)$ can be expanded into a (convergent) power series for ϵ small. The aim of this paper is to present a fully constructive method to compute *all* the coefficients of such power series.

One of the most common approaches to study the asymptotic behavior of functionals related to the solutions of singularly perturbed boundary value problems in domains with small holes and inclusions is that of *Asymptotic Analysis*, which allows to write out asymptotic expansions for $\lambda_{mn}^{\text{eff}}(\epsilon)$. In this sense, in Ammari, Kang, and Touibi [6] the authors compute an asymptotic expansion of the effective electrical conductivity of a periodic dilute composite with ideal contact condition (see also the monograph Ammari and Kang [3]). We also mention Ammari, Kang, and Kim [4] where the authors consider anisotropic heat conductors, Ammari, Kang, and Lim [5] where effective elastic properties are investigated, and Ammari, Garapon, Kang, and Lee [1] for the analysis of effective viscosity properties. For the application of asymptotic analysis to dilute and densely packed composites we refer to Movchan, Movchan, and Poulton [44]. Concerning asymptotic methods for general elliptic problems we mention, *e.g.*, Maz'ya, Nazarov, and Plamenewskij [37,38] and Maz'ya, Movchan, and Nieves [36]. In particular, a uniform asymptotic approximation of Green's kernel for the transmission problem for domains with small inclusions has been obtained in Maz'ya, Movchan, and Nieves [35] and Nieves [45]. Boundary value problems in domains with small holes have been also analyzed with the method of multiscale asymptotic expansions (*cf.*, *e.g.*, Bonnaillie-Noël, Dambrine, Tordeux, and Vial [13] and Bonnaillie-Noël, Dambrine, and Lacave [12]). Moreover, the topological-shape sensitivity analysis of the energy shape functionals for perturbations in the form of inclusions with appropriate transmission conditions can be found in Novotny and Sokołowski [46, Ch. 5]. An anomaly detection algorithm based on problems in perforated domains can be found in Ammari, Garnier, Jugnon and Kang [2].

We note that the above mentioned technique allows to produce asymptotic expansions of the type

$$\lambda_{mn}^{\text{eff}}(\epsilon) = \sum_{j=0}^r a_j \epsilon^j + R(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

for some $r \in \mathbb{N}$, $a_0, \dots, a_r \in \mathbb{R}$, and some (small) remainder function $R(\cdot)$. This technique has revealed to be extremely versatile for a wide range of problems; on the other hand, this method usually does not allow to show that the associated power series $\sum_{j=0}^{\infty} a_j \epsilon^j$ is convergent and equal to $\lambda_{mn}^{\text{eff}}(\epsilon)$ and may not provide a constructive formula for *all* the coefficients a_j .

Another technique, the so-called *Functional Equation Method*, has shown to be very useful to express the effective conductivity in terms of convergent power series of the diameter of the inclusion (*cf.*, *e.g.*, Castro and Pesetskaya [14], Castro, Pesetskaya, and Rogosin [15], Drygaś and Mityushev [21], Kapanadze, Mishuris, and Pesetskaya [27,28], Mityushev [40]), also in the case of random composites (*cf.*, *e.g.*, Berlyand and Mityushev [10,11]). However, such a method applies to specific geometries as, for example, the cases of circular and elliptic inclusions and only in the two-dimensional case.

With this paper, we present an alternative method having some advantages. Our analysis is based on the *Functional Analytic Approach* proposed by Lanza de Cristoforis in [30] for the investigations of

singular perturbation problems in perforated domains. The main aim of such approach is to represent the solution or related functionals in terms of real analytic maps of the singular perturbation parameter. A result of this type then allows to justify representation formulas in terms of convergent power series. A preliminary step in the explicit computation of the series expansions has been performed in [18], regarding the solution of a Dirichlet problem for the Laplace equation in a bounded domain with a small hole. The present paper represents the first extension of such computation to periodic domains and to different boundary conditions (namely, transmission conditions). We also note that the computations of the present paper can be extended to the three-dimensional case. However, for the sake of simplicity, we confine here to the case of dimension two.

On the computational modeling of problems on domains with small holes we mention, for example, the paper by Babuška, Soane, and Suri [7], which presents a computational method combining analytic knowledge of the solution singularities with finite element approximation of its smooth components. Moreover, a scheme for the effective properties of unidirectional fibre-reinforced media can be found in Joyce, Parnell, Assier, and Abrahams [26].

In our analysis we investigate two specific cases: $\rho(\epsilon) \equiv 1/r_{\#}$ and $\rho(\epsilon) \equiv \epsilon/r_{\#}$. We observe that the first case corresponds to the situation where the thermal boundary resistance $\rho(\epsilon)$ is independent of ϵ , whereas in the second case the resistance is proportional to the size of the contact interface $\partial\Omega_{p,\epsilon}$. This latter case has been considered also in the works of Castro, Pesetskaya, and Rogosin [15] and of Drygaś and Mityushev [21]. In the first case, our main result is represented by Theorem 4.4, where we prove that

$$\lambda_{mn}^{\text{eff}}(\epsilon) = \lambda^{-}\delta_{m,n} + \epsilon^2 \frac{1}{|Q|_2} \sum_{k=0}^{+\infty} \frac{c_{(m,n),k}}{k!} \epsilon^k,$$

for explicitly defined coefficients $c_{(m,n),k}$. In the second case, Theorem 5.5 shows that $\lambda_{mn}^{\text{eff}}(\epsilon)$ can be expressed by means of a convergent power series of ϵ^2 , namely

$$\lambda_{mn}^{\text{eff}}(\epsilon) = \lambda^{-}\delta_{m,n} + \epsilon^2 \frac{1}{|Q|_2} \sum_{k=0}^{+\infty} \frac{d_{(m,n),2k}}{(2k)!} \epsilon^{2k}.$$

for explicitly defined coefficients $d_{(m,n),2k}$. Moreover, in both situations, we also make some considerations on the specific case when Ω is the unit ball.

The paper is organized as follows. In Section 2 we introduce some notation. Section 3 collects some preliminaries on potential theory and on the integral equation formulation of problem (3). In Section 4, we compute the power series expansion under the assumption $\rho(\epsilon) \equiv 1/r_{\#}$. Finally, Section 5 completes the paper with the analysis of the effective conductivity in the case $\rho(\epsilon) \equiv \epsilon/r_{\#}$.

2. Some notation

We denote the norm on a normed space \mathcal{X} by $\|\cdot\|_{\mathcal{X}}$. Let \mathcal{X} and \mathcal{Y} be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X} \times \mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for \mathbb{R}^2 . The symbol \mathbb{N} denotes the set of natural numbers including 0. If A is a matrix, then A_{ij} denotes the (i, j) -entry of A . For all $m, n \in \mathbb{N}$ such that $m \leq n$, we denote by $\binom{n}{m} \equiv \frac{n!}{m!(n-m)!}$ the binomial coefficient.

Let $\mathbb{D} \subseteq \mathbb{R}^2$. Then $\overline{\mathbb{D}}$ denotes the closure of \mathbb{D} and $\partial\mathbb{D}$ denotes the boundary of \mathbb{D} . We also set $\mathbb{D}^- \equiv \mathbb{R}^2 \setminus \overline{\mathbb{D}}$. For all $R > 0$, $x \in \mathbb{R}^2$, $|x|$ denotes the Euclidean modulus of x in \mathbb{R}^2 , and $\mathbb{B}_2(x, R)$ denotes the ball $\{y \in \mathbb{R}^2 : |x - y| < R\}$.

Let Ω be an open subset of \mathbb{R}^2 . Let $\phi \in C^m(\Omega)$. $D\phi$ denotes $(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2})$. For a multi-index $\eta \equiv (\eta_1, \eta_2) \in \mathbb{N}^2$ we set $|\eta| \equiv \eta_1 + \eta_2$. Then $D^\eta\phi$ denotes $\frac{\partial^{|\eta|}\phi}{\partial x_1^{\eta_1} \partial x_2^{\eta_2}}$. The subspace of $C^m(\Omega)$ of those functions ϕ whose derivatives $D^\eta\phi$ of order $|\eta| \leq m$ can be extended with continuity to $\overline{\Omega}$ is denoted $C^m(\overline{\Omega})$. The subspace of $C^m(\overline{\Omega})$ whose functions have m -th order derivatives which are uniformly Hölder continuous with exponent $\alpha \in]0, 1[$ is denoted $C^{m,\alpha}(\overline{\Omega})$. The subspace of $C^m(\overline{\Omega})$ of those functions ϕ such that $\phi|_{\overline{\Omega} \cap \mathbb{B}_2(0,R)} \in C^{m,\alpha}(\overline{\Omega} \cap \mathbb{B}_2(0,R))$ for all $R \in]0, +\infty[$ is denoted $C_{\text{loc}}^{m,\alpha}(\overline{\Omega})$.

Now let Ω be a bounded open subset of \mathbb{R}^2 . Then $C^1(\overline{\Omega})$ and $C^{1,\alpha}(\overline{\Omega})$ are endowed with their usual norm and are well known to be Banach spaces. We say that a bounded open subset Ω of \mathbb{R}^2 is of class C^1 or of class $C^{1,\alpha}$, if $\overline{\Omega}$ is a manifold with boundary imbedded in \mathbb{R}^2 of class C^1 or $C^{1,\alpha}$, respectively. We define the spaces $C^{k,\alpha}(\partial\Omega)$ for $k \in \{0, 1\}$ by exploiting the local parametrizations (cf., e.g., Gilbarg and Trudinger [24, §6.2]). The trace operator from $C^{k,\alpha}(\overline{\Omega})$ to $C^{k,\alpha}(\partial\Omega)$ is linear and continuous. For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [24] (see also Lanza de Cristoforis [29, §2, Lem. 3.1, 4.26, Thm. 4.28], Lanza de Cristoforis and Rossi [33, §2]). We denote by \mathbf{n}_Ω the outward unit normal to $\partial\Omega$ and by $d\sigma$ the length element on $\partial\Omega$. We retain the standard notation for the Lebesgue space $L^1(\partial\Omega)$ of integrable functions. By $|\partial\Omega|_1$, we denote the 1-dimensional measure of $\partial\Omega$. To shorten our notation, we denote by $f_{\partial\Omega} \phi d\sigma$ the integral mean $\frac{1}{|\partial\Omega|_1} \int_{\partial\Omega} \phi d\sigma$ for all $\phi \in L^1(\partial\Omega)$. Also, if \mathcal{X} is a vector subspace of $L^1(\partial\Omega)$, then we set $\mathcal{X}_0 \equiv \{\phi \in \mathcal{X} : \int_{\partial\Omega} \phi d\sigma = 0\}$.

For the definition and properties of real analytic operators, we refer, e.g., to Deimling [19, p. 150]. In particular, we mention that the pointwise product in Schauder spaces is bilinear and continuous, and thus real analytic (cf., e.g., Lanza de Cristoforis and Rossi [33, pp. 141–142]).

If Ω is an arbitrary open subset of \mathbb{R}^2 , $k \in \mathbb{N}$, $\beta \in]0, 1]$, we set

$$C_b^k(\overline{\Omega}) \equiv \{u \in C^k(\overline{\Omega}) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^2 \text{ such that } |\gamma| \leq k\},$$

and we endow $C_b^k(\overline{\Omega})$ with its usual norm

$$\|u\|_{C_b^k(\overline{\Omega})} \equiv \sum_{|\gamma| \leq k} \sup_{x \in \overline{\Omega}} |D^\gamma u(x)| \quad \forall u \in C_b^k(\overline{\Omega}).$$

Then we set

$$C_b^{k,\beta}(\overline{\Omega}) \equiv \{u \in C^{k,\beta}(\overline{\Omega}) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^2 \text{ such that } |\gamma| \leq k\},$$

and we endow $C_b^{k,\beta}(\overline{\Omega})$ with its usual norm

$$\|u\|_{C_b^{k,\beta}(\overline{\Omega})} \equiv \sum_{|\gamma| \leq k} \sup_{x \in \overline{\Omega}} |D^\gamma u(x)| + \sum_{|\gamma|=k} |D^\gamma u : \overline{\Omega}|_\beta \quad \forall u \in C_b^{k,\beta}(\overline{\Omega}),$$

where $|D^\gamma u : \overline{\Omega}|_\beta$ denotes the β -Hölder constant of $D^\gamma u$.

Next we turn to periodic domains. If Ω_Q is an arbitrary subset of \mathbb{R}^2 such that $\overline{\Omega_Q} \subseteq Q$, then we set

$$\mathbb{S}[\Omega_Q] \equiv \bigcup_{z \in \mathbb{Z}^2} (qz + \Omega_Q) = q\mathbb{Z}^2 + \Omega_Q, \quad \mathbb{S}[\Omega_Q]^- \equiv \mathbb{R}^2 \setminus \overline{\mathbb{S}[\Omega_Q]}.$$

Clearly,

$$\mathbb{S}[\Omega_{p,\epsilon}] = \mathbb{S}_\epsilon, \quad \mathbb{S}[\Omega_{p,\epsilon}]^- = \mathbb{T}_\epsilon.$$

Then a function u from $\overline{\mathbb{S}[\Omega_Q]}$ or from $\overline{\mathbb{S}[\Omega_Q]}^-$ to \mathbb{R} is q -periodic if $u(x + q_{hh}e_h) = u(x)$ for all x in the domain of definition of u and for all $h \in \{1, 2\}$. If Ω_Q is an open subset of \mathbb{R}^2 such that $\overline{\Omega_Q} \subseteq Q$ and if $k \in \mathbb{N}$ and $\beta \in]0, 1[$, then we denote by $C_q^k(\overline{\mathbb{S}[\Omega_Q]})$, $C_q^{k,\beta}(\overline{\mathbb{S}[\Omega_Q]})$, $C_q^k(\overline{\mathbb{S}[\Omega_Q]}^-)$, and $C_q^{k,\beta}(\overline{\mathbb{S}[\Omega_Q]}^-)$ the subsets of the q -periodic functions belonging to $C_b^k(\overline{\mathbb{S}[\Omega_Q]})$, to $C_b^{k,\beta}(\overline{\mathbb{S}[\Omega_Q]})$, to $C_b^k(\overline{\mathbb{S}[\Omega_Q]}^-)$, and to $C_b^{k,\beta}(\overline{\mathbb{S}[\Omega_Q]}^-)$, respectively. We regard the sets $C_q^k(\overline{\mathbb{S}[\Omega_Q]})$, $C_q^{k,\beta}(\overline{\mathbb{S}[\Omega_Q]})$, $C_q^k(\overline{\mathbb{S}[\Omega_Q]}^-)$, $C_q^{k,\beta}(\overline{\mathbb{S}[\Omega_Q]}^-)$ as Banach subspaces of $C_b^k(\overline{\mathbb{S}[\Omega_Q]})$, of $C_b^{k,\beta}(\overline{\mathbb{S}[\Omega_Q]})$, of $C_b^k(\overline{\mathbb{S}[\Omega_Q]}^-)$, of $C_b^{k,\beta}(\overline{\mathbb{S}[\Omega_Q]}^-)$, respectively.

3. The periodic simple layer potential and preliminaries

As in [17], our approach is based on periodic potential theory, which allows us to convert problem (3) into a system of integral equations. To do so, we need to introduce periodic layer potentials. They can be built by replacing the fundamental solution of the Laplace equation with a periodic analog in the definition of classical layer potentials. As is well known, indeed, there exists a q -periodic tempered distribution S_q such that

$$\Delta S_q = \sum_{z \in \mathbb{Z}^2} \delta_{qz} - \frac{1}{|Q|_2},$$

where δ_{qz} denotes the Dirac distribution with mass in qz . The distribution S_q is determined up to an additive constant, and we can take

$$S_q(x) \equiv - \sum_{z \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{|Q|_2 4\pi^2 |q^{-1}z|^2} e^{2\pi i (q^{-1}z) \cdot x},$$

where the series converges in the sense of distributions on \mathbb{R}^2 (cf., e.g., Ammari and Kang [3, p. 53], [31, Thm. 3.1, Thm. 3.5]). Then S_q is real analytic in $\mathbb{R}^2 \setminus q\mathbb{Z}^2$ and is locally integrable in \mathbb{R}^2 (cf., e.g., [31, Thm. 3.5]). We observe that Hasimoto [25] has introduced approximation techniques for periodic fundamental solutions by exploiting Ewald's techniques for evaluating S_q , Cichocki and Felderhof [16] have obtained expressions suitable for computations in the form of rapidly convergent series (see also Sangani, Zhang and Prosperetti [48] and Poulton, Botten, McPhedran, and Movchan [47]). Finally, we note that Mityushev and Adler [41] have proved the validity of a constructive formula for S_q via elliptic functions.

Moreover, if we denote by S the function from $\mathbb{R}^2 \setminus \{0\}$ to \mathbb{R} defined by

$$S(x) \equiv \frac{1}{2\pi} \log |x| \quad \forall x \in \mathbb{R}^2 \setminus \{0\},$$

then S is a fundamental solution of the Laplace operator and the difference $S_q - S$ admits an analytic extension to $(\mathbb{R}^2 \setminus q\mathbb{Z}^2) \cup \{0\}$. We denote such an extension by R_q , *i.e.*, we set

$$R_q \equiv S_q - S \quad \text{in } (\mathbb{R}^2 \setminus q\mathbb{Z}^2) \cup \{0\},$$

and we have that

$$\Delta R_q = \sum_{z \in \mathbb{Z}^2 \setminus \{0\}} \delta_{qz} - \frac{1}{|Q|_2} \quad (4)$$

in the sense of distributions (see, *e.g.*, [31]). Moreover, R_q is an even function and

$$(D^\gamma R_q)(0) = 0 \quad \forall \gamma \in \mathbb{N}^2 \text{ with } |\gamma| \text{ odd}, \quad (5)$$

where $(D^\gamma R_q)(0)$ means the value of the derivative of $D^\gamma R_q(\cdot)$ at the point 0.

We now introduce the classical simple layer potential for the set Ω (see (1)): for all $\theta \in C^{0,\alpha}(\partial\Omega)$ we set

$$v_\Omega[\theta](t) \equiv \int_{\partial\Omega} S(t-s)\theta(s) d\sigma_s \quad \forall t \in \mathbb{R}^2,$$

As is well known, $v_\Omega[\theta]$ is continuous in \mathbb{R}^2 , the function $v_\Omega^+[\theta] \equiv v_\Omega[\theta]_{|\overline{\Omega}}$ belongs to $C^{1,\alpha}(\overline{\Omega})$, and the function $v_\Omega^-[\theta] \equiv v_\Omega[\theta]_{|\mathbb{R}^2 \setminus \Omega}$ belongs to $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2 \setminus \Omega)$. Then we set

$$w_\Omega^*[\theta](t) \equiv \int_{\partial\Omega} DS(t-s)\mathbf{n}_\Omega(t)\theta(s) d\sigma_s \quad \forall t \in \partial\Omega,$$

and we recall that the function $w_\Omega^*[\theta]$ belongs to $C^{0,\alpha}(\partial\Omega)$ and we have

$$\frac{\partial}{\partial \mathbf{n}_\Omega} v_\Omega^\pm[\theta] = \mp \frac{1}{2}\theta + w_\Omega^*[\theta] \quad \text{on } \partial\Omega$$

(*cf.*, *e.g.*, Miranda [39], Lanza de Cristoforis and Rossi [33, Thm. 3.1]). In the sequel we shall also need the following classical result of potential theory. The proof can be deduced by Folland [23, Ch. 3].

Lemma 3.1. *The maps $\theta \mapsto \frac{1}{2}\theta + w_\Omega^*[\theta]$ and $\theta \mapsto -\frac{1}{2}\theta + w_\Omega^*[\theta]$ are bounded linear isomorphisms from $C^{0,\alpha}(\partial\Omega)_0$ to itself.*

By replacing S by S_q , we define the periodic simple layer potential. Let Ω_Q be a bounded open subset of \mathbb{R}^2 of class $C^{1,\alpha}$ such that $\overline{\Omega_Q} \subseteq Q$, and $\mu \in C^{0,\alpha}(\partial\Omega_Q)$. We set

$$v_{q,\Omega_Q}[\mu](x) \equiv \int_{\partial\Omega_Q} S_q(x-y)\mu(y) d\sigma_y \quad \forall x \in \mathbb{R}^n.$$

As is well known, $v_{q,\Omega_Q}[\mu]$ is continuous in \mathbb{R}^2 . Moreover, we recall that the function $v_{q,\Omega_Q}^+[\mu] \equiv v_{q,\Omega_Q}[\mu]_{|\mathbb{S}[\Omega_Q]}$ belongs to $C_q^{1,\alpha}(\overline{\mathbb{S}[\Omega_Q]})$, and $v_{q,\Omega_Q}^-[\mu] \equiv v_{q,\Omega_Q}[\mu]_{|\mathbb{S}[\Omega_Q]^-}$ belongs to $C_q^{1,\alpha}(\overline{\mathbb{S}[\Omega_Q]^-})$. Then we set

$$w_{q,\Omega_Q}^*[\mu](x) \equiv \int_{\partial\Omega_Q} DS_q(x-y)\mathbf{n}_{\Omega_Q}(x)\mu(y) d\sigma_y \quad \forall x \in \partial\Omega_Q.$$

The function $w_{q,\Omega_Q}^*[\mu]$ belongs to $C^{0,\alpha}(\partial\Omega_Q)$ and we have

$$\frac{\partial}{\partial \mathbf{n}_{\Omega_Q}} v_{q,\Omega_Q}^\pm[\mu] = \mp \frac{1}{2}\mu + w_{q,\Omega_Q}^*[\mu] \quad \text{on } \partial\Omega_Q$$

(cf., e.g., [31, Theorem 3.7]).

As shown in [17], by means of the periodic simple layer potential, we can convert problem (3) into an equivalent system of integral equations. To do so, we first introduce the maps Λ and Λ_n from the space $]-\epsilon_0, \epsilon_0[\times C^{0,\alpha}(\partial\Omega)_0$ to $C^{1,\alpha}(\partial\Omega)$ and to $C^{0,\alpha}(\partial\Omega)$, respectively, defined by

$$\Lambda[\epsilon, \theta](t) \equiv \int_{\partial\Omega} R_q(\epsilon(t-s))\theta(s) d\sigma_s \quad \forall t \in \partial\Omega,$$

and by

$$\Lambda_n[\epsilon, \theta](t) \equiv \int_{\partial\Omega} DR_q(\epsilon(t-s))\mathbf{n}_\Omega(t)\theta(s) d\sigma_s \quad \forall t \in \partial\Omega,$$

for all $(\epsilon, \theta) \in]-\epsilon_0, \epsilon_0[\times C^{0,\alpha}(\partial\Omega)_0$. Now, let $n \in \{1, 2\}$ and let $\rho(\cdot)$ be either $\epsilon \mapsto 1/r_\#$ or $\epsilon \mapsto \epsilon/r_\#$. To provide an integral equation formulation of problem (3), we define the map $M_n \equiv (M_{n,1}, M_{n,2})$ from $]-\epsilon_0, \epsilon_0[\times (C^{0,\alpha}(\partial\Omega)_0)^2$ to $(C^{0,\alpha}(\partial\Omega)_0)^2$ by setting

$$\begin{aligned} M_{n,1}[\epsilon, \theta^i, \theta^o](t) &\equiv \lambda^- \left(\frac{1}{2}\theta^o(t) + w_\Omega^*[\theta^o](t) + \epsilon \Lambda_n[\epsilon, \theta^o](t) \right) \\ &\quad - \lambda^+ \left(-\frac{1}{2}\theta^i(t) + w_\Omega^*[\theta^i](t) + \epsilon \Lambda_n[\epsilon, \theta^i](t) \right) \\ &\quad - f(t) + (\lambda^- - \lambda^+)(\mathbf{n}_\Omega(t))_n \quad \forall t \in \partial\Omega, \\ M_{n,2}[\epsilon, \theta^i, \theta^o](t) &\equiv \lambda^+ \left(-\frac{1}{2}\theta^i(t) + w_\Omega^*[\theta^i](t) + \epsilon \Lambda_n[\epsilon, \theta^i](t) \right) \\ &\quad + \frac{\epsilon}{\rho(\epsilon)} \left(v_\Omega^+[\theta^i](t) + \Lambda[\epsilon, \theta^i](t) - \int_{\partial\Omega} (v_\Omega^+[\theta^i] + \Lambda[\epsilon, \theta^i]) d\sigma \right) \\ &\quad - v_\Omega^-[\theta^o](t) - \Lambda[\epsilon, \theta^o](t) + \int_{\partial\Omega} (v_\Omega^-[\theta^o] + \Lambda[\epsilon, \theta^o]) d\sigma \\ &\quad - g(t) + \int_{\partial\Omega} g d\sigma + \lambda^+(\mathbf{n}_\Omega(t))_n \quad \forall t \in \partial\Omega, \end{aligned}$$

for all $(\epsilon, \theta^i, \theta^o) \in]-\epsilon_0, \epsilon_0[\times (C^{0,\alpha}(\partial\Omega)_0)^2$.

By means of the operator M_n , we can convert problem (3) into a system of integral equations, as the following proposition shows (for a proof we refer to [17, Prop. 6.1]).

Proposition 3.2. *Let either $\rho(\epsilon) \equiv 1/r_\#$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$ or $\rho(\epsilon) \equiv \epsilon/r_\#$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$. Let $\epsilon \in]0, \epsilon_0[$. Let $n \in \{1, 2\}$. Then the unique solution $(u_n^+[\epsilon], u_n^-[\epsilon])$ in $C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{S}_\epsilon}) \times C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{T}_\epsilon})$ of problem (3) is delivered by*

$$\begin{aligned} u_n^+[\epsilon](x) &\equiv \int_{\partial\Omega_{p,\epsilon}} S_q(x-y)\hat{\theta}_n^i[\epsilon]((y-p)/\epsilon) d\sigma_y \\ &\quad - \int_{\partial\Omega_{p,\epsilon}} \int_{\partial\Omega_{p,\epsilon}} S_q(z-y)\hat{\theta}_n^i[\epsilon]((y-p)/\epsilon) d\sigma_y d\sigma_z \\ &\quad + x_n - \int_{\partial\Omega_{p,\epsilon}} y_n d\sigma_y \quad \forall x \in \overline{\mathbb{S}_\epsilon}, \\ u_n^-[\epsilon](x) &\equiv \int_{\partial\Omega_{p,\epsilon}} S_q(x-y)\hat{\theta}_n^o[\epsilon]((y-p)/\epsilon) d\sigma_y \\ &\quad - \int_{\partial\Omega_{p,\epsilon}} \int_{\partial\Omega_{p,\epsilon}} S_q(z-y)\hat{\theta}_n^o[\epsilon]((y-p)/\epsilon) d\sigma_y d\sigma_z \\ &\quad - \rho(\epsilon) \int_{\partial\Omega_{p,\epsilon}} g((y-p)/\epsilon) d\sigma_y + x_n - \int_{\partial\Omega_{p,\epsilon}} y_n d\sigma_y \quad \forall x \in \overline{\mathbb{T}_\epsilon}, \end{aligned}$$

where $(\hat{\theta}_n^i[\epsilon], \hat{\theta}_n^o[\epsilon])$ denotes the unique solution (θ^i, θ^o) in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of

$$M_n[\epsilon, \theta^i, \theta^o] = 0.$$

In order to investigate the asymptotic behavior of the (m, n) -entry $\lambda_{mn}^{\text{eff}}(\epsilon)$ of the effective conductivity tensor as $\epsilon \rightarrow 0^+$, we need to study the functions $u_n^+[\epsilon]$ and $u_n^-[\epsilon]$ for ϵ close to the degenerate value 0. On the other hand, Proposition 3.2 tells us how to represent $u_n^+[\epsilon]$ and $u_n^-[\epsilon]$ in terms of the densities $\hat{\theta}_n^i[\epsilon]$ and $\hat{\theta}_n^o[\epsilon]$. Therefore, the analysis of $\lambda_{mn}^{\text{eff}}(\epsilon)$ for ϵ close to 0 can be deduced by the asymptotic behavior of $\hat{\theta}_n^i[\epsilon]$ and $\hat{\theta}_n^o[\epsilon]$. Accordingly, as a first step, in the following theorem we present a regularity result for $\hat{\theta}_n^i[\epsilon]$ and $\hat{\theta}_n^o[\epsilon]$ for ϵ small and positive (cf. [17, Prop. 5.2, Thm. 6.2 and Thm. 6.3]).

Proposition 3.3. *Let either $\rho(\epsilon) \equiv 1/r_\#$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$ or $\rho(\epsilon) \equiv \epsilon/r_\#$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$. Let $n \in \{1, 2\}$. The following statements hold.*

- (i) M_n is a real analytic map from $]-\epsilon_0, \epsilon_0[\times (C^{0,\alpha}(\partial\Omega)_0)^2$ to $(C^{0,\alpha}(\partial\Omega)_0)^2$.
- (ii) There exists a unique pair $(\tilde{\theta}_n^i, \tilde{\theta}_n^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ such that $M_n[0, \tilde{\theta}_n^i, \tilde{\theta}_n^o] = 0$.
- (iii) There exist $\epsilon_1 \in]0, \epsilon_0[$ and a real analytic map $\epsilon \mapsto (\theta_n^i[\epsilon], \theta_n^o[\epsilon])$ from $]-\epsilon_1, \epsilon_1[$ to $(C^{0,\alpha}(\partial\Omega)_0)^2$ such that

$$M_n[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]] = 0 \quad \forall \epsilon \in]-\epsilon_1, \epsilon_1[. \quad (6)$$

In particular,

$$(\theta_n^i[\epsilon], \theta_n^o[\epsilon]) = (\hat{\theta}_n^i[\epsilon], \hat{\theta}_n^o[\epsilon]) \quad \forall \epsilon \in]0, \epsilon_1[\quad \text{and} \quad (\theta_n^i[0], \theta_n^o[0]) = (\tilde{\theta}_n^i, \tilde{\theta}_n^o),$$

where the pair $(\hat{\theta}_n^i[\epsilon], \hat{\theta}_n^o[\epsilon])$ is defined in Proposition 3.2.

Now we observe that the real analyticity result of Proposition 3.3(iii) implies that there exists $\epsilon_2 \in]0, \epsilon_1[$ small enough such that we can expand $\theta_n^i[\epsilon]$ and $\theta_n^o[\epsilon]$ into power series of ϵ , i.e.,

$$\theta_n^i[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_{n,k}^i}{k!} \epsilon^k, \quad \theta_n^o[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_{n,k}^o}{k!} \epsilon^k, \quad (7)$$

for some $\{\theta_{n,k}^i\}_{k \in \mathbb{N}}, \{\theta_{n,k}^o\}_{k \in \mathbb{N}}$ and for all $\epsilon \in]-\epsilon_2, \epsilon_2[$. Moreover,

$$\theta_{n,k}^i = (\partial_\epsilon^k \theta_n^i[\epsilon])|_{\epsilon=0}, \quad \theta_{n,k}^o = (\partial_\epsilon^k \theta_n^o[\epsilon])|_{\epsilon=0},$$

for all $k \in \mathbb{N}$. As a consequence,

$$\hat{\theta}_n^i[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_{n,k}^i}{k!} \epsilon^k, \quad \hat{\theta}_n^o[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_{n,k}^o}{k!} \epsilon^k,$$

for all $\epsilon \in]0, \epsilon_2[$. Therefore, in order to obtain a power series expansion for $\lambda_{mn}^{\text{eff}}(\epsilon)$ for ϵ close to 0, we want to exploit the expansion of $(\hat{\theta}_n^i[\epsilon], \hat{\theta}_n^o[\epsilon])$ (or equivalently of $(\theta_n^i[\epsilon], \theta_n^o[\epsilon])$). Since the coefficients of the expansions in (7) are given by the derivatives with respect to ϵ of $\theta_n^i[\epsilon]$ and $\theta_n^o[\epsilon]$, we would like to obtain some equations identifying $(\partial_\epsilon^k \theta_n^i[\epsilon])|_{\epsilon=0}$ and $(\partial_\epsilon^k \theta_n^o[\epsilon])|_{\epsilon=0}$. The plan is to obtain such equations by differentiating with respect to ϵ equality (6), which then leads to

$$\partial_\epsilon^k (M_n[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]]) = 0 \quad \forall \epsilon \in]-\epsilon_1, \epsilon_1[, \forall k \in \mathbb{N}. \quad (8)$$

Then by taking $\epsilon = 0$ in (8), we will obtain integral equations identifying $(\partial_\epsilon^k \theta_n^i[\epsilon])|_{\epsilon=0}$ and $(\partial_\epsilon^k \theta_n^o[\epsilon])|_{\epsilon=0}$.

In order to compute the derivative in (8), we recall that

$$\partial_\epsilon^j (F(\epsilon t)) = \sum_{h=0}^j \binom{j}{h} t_1^h t_2^{j-h} (\partial_1^h \partial_2^{j-h} F)(\epsilon t) \quad (9)$$

for all $j \in \mathbb{N}$, $\epsilon \in \mathbb{R}$, $t \equiv (t_1, t_2) \in \mathbb{R}^2$, and for all functions F analytic in a neighborhood of ϵt . Here, if $l \in \{1, 2\}$, then $(\partial_l F)(s)$ denotes the partial derivative with respect to t_l of the function $F(t) \equiv F(t_1, t_2)$ evaluated at $s \equiv (s_1, s_2) \in \mathbb{R}^2$.

We begin with a preliminary lemma.

Lemma 3.4. Let $\epsilon'_0 \in]0, \epsilon_0[$. Let $\epsilon \mapsto \theta[\epsilon]$ be a real analytic map from $]-\epsilon'_0, \epsilon'_0[$ to $C^{0,\alpha}(\partial\Omega)_0$. Possibly shrinking ϵ'_0 , assume that $\{\theta_k\}_{k \in \mathbb{N}}$ is a sequence in $C^{0,\alpha}(\partial\Omega)_0$ such that

$$\theta[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_k}{k!} \epsilon^k \quad \forall \epsilon \in]-\epsilon'_0, \epsilon'_0[,$$

where the series converges uniformly for $\epsilon \in]-\epsilon'_0, \epsilon'_0[$ in $C^{0,\alpha}(\partial\Omega)_0$. Then the following statements hold.

(i) The map from $]-\epsilon'_0, \epsilon'_0[$ to $C^{1,\alpha}(\partial\Omega)$ which takes ϵ to $\Lambda[\epsilon, \theta[\epsilon]]$ is real analytic and we have

$$\Lambda[0, \theta[0]] = 0, \quad (\partial_\epsilon (\Lambda[\epsilon, \theta[\epsilon]]))|_{\epsilon=0} = 0,$$

and

$$\begin{aligned} & (\partial_\epsilon^k (\Lambda[\epsilon, \theta[\epsilon]])(t))|_{\epsilon=0} \\ &= \sum_{j=2}^k \binom{k}{j} \sum_{h=0}^j \binom{j}{h} (\partial_1^h \partial_2^{j-h} R_q)(0) \int_{\partial\Omega} (t_1 - s_1)^h (t_2 - s_2)^{j-h} \theta_{k-j}(s) d\sigma_s \\ & \quad \forall t \in \partial\Omega, \forall k \in \mathbb{N} \setminus \{0, 1\}. \end{aligned} \tag{10}$$

(ii) The map from $]-\epsilon'_0, \epsilon'_0[$ to $C^{0,\alpha}(\partial\Omega)$ which takes ϵ to $\Lambda_n[\epsilon, \theta[\epsilon]]$ is real analytic and we have

$$\Lambda_n[0, \theta[0]] = 0,$$

and

$$\int_{\partial\Omega} \partial_\epsilon^k (\epsilon \Lambda_n[\epsilon, \theta[\epsilon]]) d\sigma = 0 \quad \forall \epsilon \in]-\epsilon'_0, \epsilon'_0[, \forall k \in \mathbb{N}. \tag{11}$$

Moreover,

$$(\partial_\epsilon (\epsilon \Lambda_n[\epsilon, \theta[\epsilon]]))|_{\epsilon=0} = 0,$$

and

$$\begin{aligned} & (\partial_\epsilon^k (\epsilon \Lambda_n[\epsilon, \theta[\epsilon]])(t))|_{\epsilon=0} \\ &= k \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{h=0}^j \binom{j}{h} (\partial_1^h \partial_2^{j-h} D R_q)(0) \mathbf{n}_\Omega(t) \\ & \quad \times \int_{\partial\Omega} (t_1 - s_1)^h (t_2 - s_2)^{j-h} \theta_{k-1-j}(s) d\sigma_s \quad \forall t \in \partial\Omega, \forall k \in \mathbb{N} \setminus \{0, 1\}. \end{aligned} \tag{12}$$

Proof. We first consider statement (i). The analyticity of the map

$$\epsilon \mapsto \Lambda[\epsilon, \theta[\epsilon]]$$

from $]-\epsilon'_0, \epsilon'_0[$ to $C^{1,\alpha}(\partial\Omega)$ follows by the analyticity of $\epsilon \mapsto \theta[\epsilon]$ and by standard properties of integral operators with real analytic kernels and with no singularity (cf., e.g., [32, §4]). Clearly,

$$\Lambda[0, \theta[0]](t) = R_q(0) \int_{\partial\Omega} \theta[0](s) d\sigma_s = 0 \quad \forall t \in \partial\Omega.$$

The validity of equality (10) follows by standard calculus in Banach spaces and by formula (9). Moreover, by formula (9) and equality (5), one verifies that $(\partial_\epsilon(\Lambda[\epsilon, \theta[\epsilon]]))|_{\epsilon=0} = 0$. We now turn to prove (ii). Again, by the analyticity of $\epsilon \mapsto \theta[\epsilon]$ and by standard properties of integral operators with real analytic kernels and with no singularity (cf., e.g., [32, §4]), we deduce the analyticity of the map

$$\epsilon \mapsto \Lambda_n[\epsilon, \theta[\epsilon]]$$

from $]-\epsilon'_0, \epsilon'_0[$ to $C^{0,\alpha}(\partial\Omega)$. Moreover, by standard calculus in Banach spaces and by formula (9), we deduce the validity of (12). Clearly,

$$\Lambda_n[0, \theta[0]](t) = (DR_q)(0)\mathbf{n}_\Omega(t) \int_{\partial\Omega} \theta[0](s) d\sigma_s = 0 \quad \forall t \in \partial\Omega,$$

which also implies $(\partial_\epsilon(\epsilon\Lambda_n[\epsilon, \theta[\epsilon]])(t))|_{\epsilon=0} = 0$ for all $t \in \partial\Omega$. To prove (11), we note that the map H from $]-\epsilon'_0, \epsilon'_0[$ to $C^{1,\alpha}(\overline{\Omega})$ which takes ϵ to the function

$$H[\epsilon](t) \equiv \int_{\partial\Omega} R_q(\epsilon(t-s))\theta[\epsilon](s) d\sigma_s \quad \forall t \in \overline{\Omega},$$

is real analytic. Moreover,

$$\begin{aligned} \Delta_t \int_{\partial\Omega} R_q(\epsilon(t-s))\theta[\epsilon](s) d\sigma_s &= \epsilon^2 \int_{\partial\Omega} (\Delta R_q)(\epsilon(t-s))\theta[\epsilon](s) d\sigma_s \\ &= -\frac{\epsilon^2}{|Q|_2} \int_{\partial\Omega} \theta[\epsilon](s) d\sigma_s = 0 \quad \forall t \in \overline{\Omega}, \end{aligned}$$

and thus $H[\epsilon]$ is harmonic in Ω for all $\epsilon \in]-\epsilon'_0, \epsilon'_0[$. Therefore,

$$\int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}_\Omega} H[\epsilon] d\sigma = 0 \quad \forall \epsilon \in]-\epsilon'_0, \epsilon'_0[.$$

On the other hand, a straightforward computation shows that

$$\frac{\partial}{\partial \mathbf{n}_\Omega} H[\epsilon] = \epsilon \Lambda_n[\epsilon, \theta[\epsilon]] \quad \forall \epsilon \in]-\epsilon'_0, \epsilon'_0[.$$

As a consequence,

$$\int_{\partial\Omega} \epsilon \Lambda_{\mathbf{n}}[\epsilon, \theta[\epsilon]] d\sigma = 0 \quad \forall \epsilon \in]-\epsilon'_0, \epsilon'_0[. \quad (13)$$

By differentiating equality (13) with respect to ϵ , we deduce that

$$0 = \partial_{\epsilon}^k \left(\int_{\partial\Omega} \epsilon \Lambda_{\mathbf{n}}[\epsilon, \theta[\epsilon]] d\sigma \right) = \int_{\partial\Omega} \partial_{\epsilon}^k (\epsilon \Lambda_{\mathbf{n}}[\epsilon, \theta[\epsilon]]) d\sigma \quad \forall \epsilon \in]-\epsilon'_0, \epsilon'_0[.$$

Thus the proof is complete. \square

In view of Lemma 3.4, we find convenient to introduce the following notation:

$$\Lambda^k[\theta_0, \dots, \theta_{k-2}](t) \equiv \sum_{j=2}^k \binom{k}{j} \sum_{h=0}^j \binom{j}{h} (\partial_1^h \partial_2^{j-h} R_q)(0) \int_{\partial\Omega} (t_1 - s_1)^h (t_2 - s_2)^{j-h} \theta_{k-j}(s) d\sigma_s$$

$$\forall t \in \partial\Omega, \forall (\theta_0, \dots, \theta_{k-2}) \in (C^{0,\alpha}(\partial\Omega)_0)^{k-1},$$

for all $k \in \mathbb{N} \setminus \{0, 1\}$, and

$$\Lambda_{\mathbf{n}}^k[\theta_0, \dots, \theta_{k-2}](t) \equiv k \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{h=0}^j \binom{j}{h} (\partial_1^h \partial_2^{j-h} DR_q)(0) \mathbf{n}_{\Omega}(t)$$

$$\times \int_{\partial\Omega} (t_1 - s_1)^h (t_2 - s_2)^{j-h} \theta_{k-1-j}(s) d\sigma_s$$

$$\forall t \in \partial\Omega, \forall (\theta_0, \dots, \theta_{k-2}) \in (C^{0,\alpha}(\partial\Omega)_0)^{k-1},$$

for all $k \in \mathbb{N} \setminus \{0, 1\}$. We observe that by (5), $\Lambda^k[\theta_0, \dots, \theta_{k-2}]$ and $\Lambda_{\mathbf{n}}^k[\theta_0, \dots, \theta_{k-2}]$ depend only on the θ_j 's with j odd if k is odd, and only on the θ_j 's with j even if k is even. Then, by a linearity argument one deduces the validity of the following technical result on Λ^{2k+1} and $\Lambda_{\mathbf{n}}^{2k+1}$ for $k \in \mathbb{N} \setminus \{0\}$.

Lemma 3.5. *Let $k \in \mathbb{N} \setminus \{0\}$. If $\theta_0, \dots, \theta_{2k-1} \in C^{0,\alpha}(\partial\Omega)_0$ are such that $\theta_{2j-1} = 0$ for all $j \in \{1, \dots, k\}$, then $\Lambda^{2k+1}[\theta_0, \dots, \theta_{2k-1}](t) = 0$ and $\Lambda_{\mathbf{n}}^{2k+1}[\theta_0, \dots, \theta_{2k-1}](t) = 0$ for all $t \in \partial\Omega$.*

4. Power series expansion for $\rho(\epsilon) \equiv 1/r_{\#}$

4.1. Series expansions for $\theta_{\mathbf{n}}^i[\epsilon]$ and $\theta_{\mathbf{n}}^o[\epsilon]$

Throughout this section, we consider the case where

$$\rho(\epsilon) \equiv 1/r_{\#} \quad \forall \epsilon \in]-\epsilon_0, \epsilon_0[. \quad (14)$$

In order to compute the asymptotic expansion of the effective conductivity under assumption (14), we start with the following proposition where we identify the coefficients of the power series expansions of $\theta_{\mathbf{n}}^i[\epsilon]$ and of $\theta_{\mathbf{n}}^o[\epsilon]$ in terms of the solutions of systems of integral equations.

Proposition 4.1. *Let $n \in \{1, 2\}$. Let $\epsilon_1, \epsilon \mapsto \theta_n^i[\epsilon]$, and $\epsilon \mapsto \theta_n^o[\epsilon]$ be as in Proposition 3.3. Then there exist $\epsilon_2 \in]0, \epsilon_1[$ and a sequence $\{(\theta_{n,k}^i, \theta_{n,k}^o)\}_{k \in \mathbb{N}}$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$ such that*

$$\theta_n^i[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_{n,k}^i}{k!} \epsilon^k \quad \text{and} \quad \theta_n^o[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_{n,k}^o}{k!} \epsilon^k \quad \forall \epsilon \in]-\epsilon_2, \epsilon_2[, \quad (15)$$

where the two series converge uniformly for $\epsilon \in]-\epsilon_2, \epsilon_2[$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$. Moreover, the following statements hold.

(i) *The pair of functions $(\theta_{n,0}^i, \theta_{n,0}^o)$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations*

$$-\frac{1}{2}\theta_{n,0}^i(t) + w_{\Omega}^*[\theta_{n,0}^i](t) = \frac{1}{\lambda^+} \left(g(t) - \int_{\partial\Omega} g \, d\sigma \right) - (\mathbf{n}_{\Omega}(t))_n \quad \forall t \in \partial\Omega, \quad (16)$$

$$\frac{1}{2}\theta_{n,0}^o(t) + w_{\Omega}^*[\theta_{n,0}^o](t) = \frac{1}{\lambda^-} \left(g(t) - \int_{\partial\Omega} g \, d\sigma + f(t) \right) - (\mathbf{n}_{\Omega}(t))_n \quad \forall t \in \partial\Omega. \quad (17)$$

(ii) *The pair of functions $(\theta_{n,1}^i, \theta_{n,1}^o)$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations*

$$\begin{aligned} -\frac{1}{2}\theta_{n,1}^i(t) + w_{\Omega}^*[\theta_{n,1}^i](t) = & -\frac{r_{\#}}{\lambda^+} \left(v_{\Omega}^+[\theta_{n,0}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,0}^i] \, d\sigma \right. \\ & \left. - v_{\Omega}^-[\theta_{n,0}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,0}^o] \, d\sigma \right) \quad \forall t \in \partial\Omega, \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{1}{2}\theta_{n,1}^o(t) + w_{\Omega}^*[\theta_{n,1}^o](t) = & -\frac{r_{\#}}{\lambda^-} \left(v_{\Omega}^+[\theta_{n,0}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,0}^i] \, d\sigma \right. \\ & \left. - v_{\Omega}^-[\theta_{n,0}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,0}^o] \, d\sigma \right) \quad \forall t \in \partial\Omega. \end{aligned} \quad (19)$$

(iii) *The pair of functions $(\theta_{n,2}^i, \theta_{n,2}^o)$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations*

$$\begin{aligned} -\frac{1}{2}\theta_{n,2}^i(t) + w_{\Omega}^*[\theta_{n,2}^i](t) = & -\Lambda_{\mathbf{n}}^2[\theta_{n,0}^i](t) - \frac{2r_{\#}}{\lambda^+} \left(v_{\Omega}^+[\theta_{n,1}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,1}^i] \, d\sigma \right. \\ & \left. - v_{\Omega}^-[\theta_{n,1}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,1}^o] \, d\sigma \right) \quad \forall t \in \partial\Omega, \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{1}{2}\theta_{n,2}^o(t) + w_{\Omega}^*[\theta_{n,2}^o](t) = & -\Lambda_{\mathbf{n}}^2[\theta_{n,0}^o](t) - \frac{2r_{\#}}{\lambda^-} \left(v_{\Omega}^+[\theta_{n,1}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,1}^i] \, d\sigma \right. \\ & \left. - v_{\Omega}^-[\theta_{n,1}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,1}^o] \, d\sigma \right) \quad \forall t \in \partial\Omega. \end{aligned} \quad (21)$$

(iv) For all $k \in \mathbb{N} \setminus \{0, 1, 2\}$ the pair of functions $(\theta_{n,k}^i, \theta_{n,k}^o)$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations which involves $\{(\theta_{n,h}^i, \theta_{n,h}^o)\}_{h=0}^{k-1}$

$$\begin{aligned}
& -\frac{1}{2}\theta_{n,k}^i(t) + w_{\Omega}^*[\theta_{n,k}^i](t) \\
& = -\Lambda_{\mathbf{n}}^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i](t) \\
& \quad - \frac{kr_{\#}}{\lambda^+} \left(v_{\Omega}^+[\theta_{n,k-1}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,k-1}^i] d\sigma - v_{\Omega}^-[\theta_{n,k-1}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,k-1}^o] d\sigma \right) \\
& \quad - \frac{kr_{\#}}{\lambda^+} \left(\Lambda^{k-1}[\theta_{n,0}^i, \dots, \theta_{n,k-3}^i](t) - \int_{\partial\Omega} \Lambda^{k-1}[\theta_{n,0}^i, \dots, \theta_{n,k-3}^i] d\sigma \right. \\
& \quad \left. - \Lambda^{k-1}[\theta_{n,0}^o, \dots, \theta_{n,k-3}^o](t) + \int_{\partial\Omega} \Lambda^{k-1}[\theta_{n,0}^o, \dots, \theta_{n,k-3}^o] d\sigma \right) \quad \forall t \in \partial\Omega, \quad (22)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\theta_{n,k}^o(t) + w_{\Omega}^*[\theta_{n,k}^o](t) \\
& = -\Lambda_{\mathbf{n}}^k[\theta_{n,0}^o, \dots, \theta_{n,k-2}^o](t) \\
& \quad - \frac{kr_{\#}}{\lambda^-} \left(v_{\Omega}^+[\theta_{n,k-1}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,k-1}^i] d\sigma - v_{\Omega}^-[\theta_{n,k-1}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,k-1}^o] d\sigma \right) \\
& \quad - \frac{kr_{\#}}{\lambda^-} \left(\Lambda^{k-1}[\theta_{n,0}^i, \dots, \theta_{n,k-3}^i](t) - \int_{\partial\Omega} \Lambda^{k-1}[\theta_{n,0}^i, \dots, \theta_{n,k-3}^i] d\sigma \right. \\
& \quad \left. - \Lambda^{k-1}[\theta_{n,0}^o, \dots, \theta_{n,k-3}^o](t) + \int_{\partial\Omega} \Lambda^{k-1}[\theta_{n,0}^o, \dots, \theta_{n,k-3}^o] d\sigma \right) \quad \forall t \in \partial\Omega. \quad (23)
\end{aligned}$$

Proof. We first note that Proposition 3.3(iii) implies the existence of ϵ_2 and of a family $\{(\theta_{n,k}^i, \theta_{n,k}^o)\}_{k \in \mathbb{N}}$ such that (15) holds. By standard properties of real analytic maps, one has

$$(\theta_{n,k}^i, \theta_{n,k}^o) = (\partial_{\epsilon}^k \theta_n^i[0], \partial_{\epsilon}^k \theta_n^o[0]) \quad \forall k \in \mathbb{N}.$$

By equality $\rho(\epsilon) \equiv 1/r_{\#}$ and by taking $\epsilon = 0$, equation (6) can be immediately written as the system of integral equations (16)–(17). The existence and uniqueness of solution for this system are then ensured by Proposition 3.3(ii). Then observe that $M_n[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]] = 0$ for all $\epsilon \in]-\epsilon_2, \epsilon_2[$ (cf. Proposition 3.3(iii)). Accordingly, the map which takes ϵ to $M_n[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]]$ has derivatives which are equal to zero, i.e., $\partial_{\epsilon}^k(M_n[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]]) = 0$ for all $\epsilon \in]-\epsilon_2, \epsilon_2[$ and all $k \in \mathbb{N} \setminus \{0\}$. Keeping equality $\rho(\epsilon) \equiv 1/r_{\#}$ in mind, a straightforward calculation shows that

$$\begin{aligned}
& \partial_{\epsilon}^k(M_{n,1}[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]])(t) \\
& = \lambda^- \left(\frac{1}{2} \partial_{\epsilon}^k \theta_n^o[\epsilon](t) + w_{\Omega}^*[\partial_{\epsilon}^k \theta_n^o[\epsilon]](t) + \partial_{\epsilon}^k(\epsilon \Lambda_{\mathbf{n}}[\epsilon, \theta_n^o[\epsilon]])(t) \right) \\
& \quad - \lambda^+ \left(-\frac{1}{2} \partial_{\epsilon}^k \theta_n^i[\epsilon](t) + w_{\Omega}^*[\partial_{\epsilon}^k \theta_n^i[\epsilon]](t) + \partial_{\epsilon}^k(\epsilon \Lambda_{\mathbf{n}}[\epsilon, \theta_n^i[\epsilon]])(t) \right) \\
& = 0 \quad \forall t \in \partial\Omega, \quad (24)
\end{aligned}$$

$$\begin{aligned}
& \partial_\epsilon^k (M_{n,2}[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]])(t) \\
&= \lambda^+ \left(-\frac{1}{2} \partial_\epsilon^k \theta_n^i[\epsilon](t) + w_\Omega^* [\partial_\epsilon^k \theta_n^i[\epsilon]](t) + \partial_\epsilon^k (\epsilon \Lambda_n[\epsilon, \theta_n^i[\epsilon]])(t) \right) \\
&+ \epsilon r_\# \left(v_\Omega^+ [\partial_\epsilon^k \theta_n^i[\epsilon]](t) + \partial_\epsilon^k (\Lambda[\epsilon, \theta_n^i[\epsilon]])(t) - \int_{\partial\Omega} (v_\Omega^+ [\partial_\epsilon^k \theta_n^i[\epsilon]] + \partial_\epsilon^k (\Lambda[\epsilon, \theta_n^i[\epsilon]])) d\sigma \right. \\
&- v_\Omega^- [\partial_\epsilon^k \theta_n^o[\epsilon]](t) - \partial_\epsilon^k (\Lambda[\epsilon, \theta_n^o[\epsilon]])(t) + \int_{\partial\Omega} (v_\Omega^- [\partial_\epsilon^k \theta_n^o[\epsilon]] + \partial_\epsilon^k (\Lambda[\epsilon, \theta_n^o[\epsilon]])) d\sigma \left. \right) \\
&+ kr_\# \left(v_\Omega^+ [\partial_\epsilon^{k-1} \theta_n^i[\epsilon]](t) + \partial_\epsilon^{k-1} (\Lambda[\epsilon, \theta_n^i[\epsilon]])(t) - \int_{\partial\Omega} (v_\Omega^+ [\partial_\epsilon^{k-1} \theta_n^i[\epsilon]] \right. \\
&+ \partial_\epsilon^{k-1} (\Lambda[\epsilon, \theta_n^i[\epsilon]])) d\sigma - v_\Omega^- [\partial_\epsilon^{k-1} \theta_n^o[\epsilon]](t) - \partial_\epsilon^{k-1} (\Lambda[\epsilon, \theta_n^o[\epsilon]])(t) \\
&\left. + \int_{\partial\Omega} (v_\Omega^- [\partial_\epsilon^{k-1} \theta_n^o[\epsilon]] + \partial_\epsilon^{k-1} (\Lambda[\epsilon, \theta_n^o[\epsilon]])) d\sigma \right) \\
&= 0 \quad \forall t \in \partial\Omega, \tag{25}
\end{aligned}$$

for all $\epsilon \in]-\epsilon_2, \epsilon_2[$ and all $k \in \mathbb{N} \setminus \{0\}$. Then, by Lemma 3.4, one verifies that system (24)–(25) with $\epsilon = 0$ can be rewritten as (18)–(19) if $k = 1$, as (20)–(21) if $k = 2$, and as (22)–(23) for all $k \in \mathbb{N} \setminus \{0, 1, 2\}$.

One can easily verify that the integral on $\partial\Omega$ of the functions on the right hand sides of (18) and (19) vanishes. Then, by Lemma 3.1 one proves that the solution $(\theta_{n,1}^i, \theta_{n,1}^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ of (18)–(19) exists and is unique (we have already observed that the existence is granted by Proposition 3.3(iii)). By Lemma 3.4(ii) one also has that

$$\int_{\partial\Omega} \Lambda_n^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i] d\sigma = 0 \quad \text{and} \quad \int_{\partial\Omega} \Lambda_n^k[\theta_{n,0}^o, \dots, \theta_{n,k-2}^o] d\sigma = 0$$

for all $k \in \mathbb{N} \setminus \{0, 1\}$. Then a straightforward computation shows that the right hand sides of (20)–(23) belong to $C^{0,\alpha}(\partial\Omega)_0$ for all $k \in \mathbb{N} \setminus \{0, 1\}$. Hence, Lemma 3.1 ensures that the solution $(\theta_{n,k}^i, \theta_{n,k}^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ of system (20)–(21) for $k = 2$ and of system (22)–(23) for all $k \in \mathbb{N} \setminus \{0, 1, 2\}$ exists and is unique (the existence also follows by Proposition 3.3(iii)). The proof is now complete. \square

4.2. Series expansion of the effective conductivity

The aim of this subsection is to compute the series expansion for the effective conductivity under assumption (14). To do so, we need the following two lemmas where we compute the power series expansions of two auxiliary maps.

Lemma 4.2. *Let $m, n \in \{1, 2\}$. Let $\epsilon \mapsto \theta_n^i[\epsilon]$ and $\epsilon \mapsto \theta_n^o[\epsilon]$ be as in Proposition 3.3. Let ϵ_2 and $\{(\theta_{n,k}^i, \theta_{n,k}^o)\}_{k \in \mathbb{N}}$ be as in Proposition 4.1. Let U_n^+ be the map from $]-\epsilon_2, \epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned}
U_n^+[\epsilon](t) &\equiv v_\Omega^+[\theta_n^i[\epsilon]](t) + \Lambda[\epsilon, \theta_n^i[\epsilon]](t) \\
&- \int_{\partial\Omega} (v_\Omega^+[\theta_n^i[\epsilon]] + \Lambda[\epsilon, \theta_n^i[\epsilon]]) d\sigma + t_n - \int_{\partial\Omega} s_n d\sigma_s \quad \forall t \in \partial\Omega.
\end{aligned}$$

Then U_n^+ is real analytic and there exists $\epsilon_3 \in]0, \epsilon_2[$ such that

$$\begin{aligned}
& \int_{\partial\Omega} U_n^+[\epsilon](t)(\mathbf{n}_\Omega(t))_m d\sigma_t \\
&= |\Omega|_2 \delta_{m,n} + \int_{\partial\Omega} v_\Omega^+[\theta_{n,0}^i](t)(\mathbf{n}_\Omega(t))_m d\sigma_t \\
&+ \epsilon \int_{\partial\Omega} v_\Omega^+[\theta_{n,1}^i](t)(\mathbf{n}_\Omega(t))_m d\sigma_t \\
&+ \sum_{k=2}^{+\infty} \frac{1}{k!} \left(\int_{\partial\Omega} (v_\Omega^+[\theta_{n,k}^i](t) + \Lambda^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i](t))(\mathbf{n}_\Omega(t))_m d\sigma_t \right) \epsilon^k
\end{aligned} \tag{26}$$

for all $\epsilon \in]-\epsilon_3, \epsilon_3[$, where the series converges uniformly for $\epsilon \in]-\epsilon_3, \epsilon_3[$.

Proof. We first note that by [17, Thm. 7.1] U_n^+ is a real analytic map from $]-\epsilon_2, \epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$. Therefore, there exist $\epsilon_3 \in]0, \epsilon_2[$ and a sequence $\{a_{n,k}\}_{k \in \mathbb{N}}$ in $C^{1,\alpha}(\partial\Omega)$ such that

$$U_n^+[\epsilon](t) = \sum_{k=0}^{+\infty} \frac{a_{n,k}(t)}{k!} \epsilon^k \quad \forall \epsilon \in]-\epsilon_3, \epsilon_3[\quad \forall t \in \partial\Omega,$$

where the series converges uniformly for $\epsilon \in]-\epsilon_3, \epsilon_3[$. By taking $\epsilon = 0$ and by Lemma 3.4(i), we verify that

$$a_{n,0}(t) \equiv v_\Omega^+[\theta_{n,0}^i](t) - \int_{\partial\Omega} v_\Omega^+[\theta_{n,0}^i] d\sigma + t_n - \int_{\partial\Omega} s_n d\sigma_s \quad \forall t \in \partial\Omega.$$

In order to compute the other coefficients, we take the derivative of order $k \in \mathbb{N} \setminus \{0\}$ of the map $\epsilon \mapsto U_n^+[\epsilon]$ and we obtain

$$\begin{aligned}
\partial_\epsilon^k (U_n^+[\epsilon])(t) &= v_\Omega^+[\partial_\epsilon^k \theta_n^i[\epsilon]](t) + \partial_\epsilon^k (\Lambda[\epsilon, \theta_n^i[\epsilon]])(t) \\
&- \int_{\partial\Omega} (v_\Omega^+[\partial_\epsilon^k \theta_n^i[\epsilon]] + \partial_\epsilon^k (\Lambda[\epsilon, \theta_n^i[\epsilon]])) d\sigma \quad \forall t \in \partial\Omega.
\end{aligned}$$

Again, by taking $\epsilon = 0$ and by Lemma 3.4(i), we find

$$a_{n,1}(t) = v_\Omega^+[\theta_{n,1}^i](t) - \int_{\partial\Omega} v_\Omega^+[\theta_{n,1}^i] d\sigma \quad \forall t \in \partial\Omega,$$

$$a_{n,k}(t) = v_\Omega^+[\theta_{n,k}^i](t) + \Lambda^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i](t) - \int_{\partial\Omega} (v_\Omega^+[\theta_{n,k}^i] + \Lambda^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i]) d\sigma$$

$$\forall t \in \partial\Omega, \quad \forall k \in \mathbb{N} \setminus \{0, 1\}.$$

As a consequence, possibly shrinking ϵ_3 , we have

$$\begin{aligned} \int_{\partial\Omega} U_n^+[\epsilon](t)(\mathbf{n}_\Omega(t))_m d\sigma_t &= \int_{\partial\Omega} \sum_{k=0}^{+\infty} \frac{a_{n,k}(t)}{k!} \epsilon^k (\mathbf{n}_\Omega(t))_m d\sigma_t \\ &= \sum_{k=0}^{+\infty} \frac{1}{k!} \left(\int_{\partial\Omega} a_{n,k}(t)(\mathbf{n}_\Omega(t))_m d\sigma_t \right) \epsilon^k, \end{aligned}$$

where the series converges uniformly for $\epsilon \in]-\epsilon_3, \epsilon_3[$. Then we consider separately the cases $k = 0$, $k = 1$, and $k \in \mathbb{N} \setminus \{0, 1\}$, and we have

$$\begin{aligned} \int_{\partial\Omega} a_{n,0}(t)(\mathbf{n}_\Omega(t))_m d\sigma_t &= \int_{\partial\Omega} \left(v_\Omega^+[\theta_{n,0}^i](t) - \int_{\partial\Omega} v_\Omega^+[\theta_{n,0}^i] d\sigma + t_n - \int_{\partial\Omega} s_n d\sigma_s \right) (\mathbf{n}_\Omega(t))_m d\sigma_t \\ &= \int_{\partial\Omega} v_\Omega^+[\theta_{n,0}^i](t)(\mathbf{n}_\Omega(t))_m d\sigma_t - \int_{\partial\Omega} v_\Omega^+[\theta_{n,0}^i] d\sigma \int_{\partial\Omega} (\mathbf{n}_\Omega(t))_m d\sigma_t \\ &\quad + \int_{\partial\Omega} t_n (\mathbf{n}_\Omega(t))_m d\sigma_t - \int_{\partial\Omega} s_n d\sigma_s \int_{\partial\Omega} (\mathbf{n}_\Omega(t))_m d\sigma_t \\ &= \int_{\partial\Omega} v_\Omega^+[\theta_{n,0}^i](t)(\mathbf{n}_\Omega(t))_m d\sigma_t + \int_{\partial\Omega} t_n (\mathbf{n}_\Omega(t))_m d\sigma_t, \end{aligned}$$

and

$$\begin{aligned} \int_{\partial\Omega} a_{n,1}(t)(\mathbf{n}_\Omega(t))_m d\sigma_t &= \int_{\partial\Omega} \left(v_\Omega^+[\theta_{n,1}^i](t) - \int_{\partial\Omega} v_\Omega^+[\theta_{n,1}^i] d\sigma \right) (\mathbf{n}_\Omega(t))_m d\sigma_t \\ &= \int_{\partial\Omega} v_\Omega^+[\theta_{n,1}^i](t)(\mathbf{n}_\Omega(t))_m d\sigma_t, \end{aligned}$$

and

$$\begin{aligned} \int_{\partial\Omega} a_{n,k}(t)(\mathbf{n}_\Omega(t))_m d\sigma_t &= \int_{\partial\Omega} \left(v_\Omega^+[\theta_{n,k}^i](t) + \Lambda^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i](t) \right. \\ &\quad \left. - \int_{\partial\Omega} (v_\Omega^+[\theta_{n,k}^i] + \Lambda^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i]) d\sigma \right) (\mathbf{n}_\Omega(t))_m d\sigma_t \\ &= \int_{\partial\Omega} (v_\Omega^+[\theta_{n,k}^i](t) + \Lambda^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i](t)) (\mathbf{n}_\Omega(t))_m d\sigma_t. \end{aligned}$$

Moreover, by the Divergence Theorem one verifies that

$$\int_{\partial\Omega} t_n (\mathbf{n}_\Omega(t))_m d\sigma_t = \int_{\Omega} \frac{\partial t_n}{\partial t_m} dt = |\Omega|_2 \delta_{m,n}.$$

Accordingly, the validity of (26) follows. \square

Lemma 4.3. Let $m, n \in \{1, 2\}$. Let $\epsilon \mapsto \theta_n^i[\epsilon]$ and $\epsilon \mapsto \theta_n^o[\epsilon]$ be as in Proposition 3.3. Let ϵ_2 and $\{(\theta_{n,k}^i, \theta_{n,k}^o)\}_{k \in \mathbb{N}}$ be as in Proposition 4.1. Let V_n^- be the map from $]-\epsilon_2, \epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$ defined by

$$V_n^-[\epsilon](t) \equiv v_\Omega^-[\theta_n^o[\epsilon]](t) + \Lambda[\epsilon, \theta_n^o[\epsilon]](t) + t_n \quad \forall t \in \partial\Omega.$$

Then V_n^- is real analytic and there exists $\epsilon_4 \in]0, \epsilon_2[$ such that

$$\begin{aligned} & \int_{\partial\Omega} V_n^-[\epsilon](t) (\mathbf{n}_\Omega(t))_m d\sigma_t \\ &= |\Omega|_2 \delta_{m,n} + \int_{\partial\Omega} v_\Omega^-[\theta_{n,0}^o](t) (\mathbf{n}_\Omega(t))_m d\sigma_t + \epsilon \int_{\partial\Omega} v_\Omega^-[\theta_{n,1}^o](t) (\mathbf{n}_\Omega(t))_m d\sigma_t \\ &+ \sum_{k=2}^{+\infty} \frac{1}{k!} \left(\int_{\partial\Omega} (v_\Omega^-[\theta_{n,k}^o](t) + \Lambda^k[\theta_{n,0}^o, \dots, \theta_{n,k-2}^o](t)) (\mathbf{n}_\Omega(t))_m d\sigma_t \right) \epsilon^k, \end{aligned} \quad (27)$$

where the series converges uniformly for $\epsilon \in]-\epsilon_4, \epsilon_4[$.

Proof. We first note that by [17, Thm. 7.2 (ii)] V_n^- is a real analytic map from $]-\epsilon_2, \epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$. Therefore, there exist $\epsilon_4 \in]0, \epsilon_2[$ and a sequence $\{b_{n,k}\}_{k \in \mathbb{N}}$ in $C^{1,\alpha}(\partial\Omega)$ such that

$$V_n^-[\epsilon](t) = \sum_{k=0}^{+\infty} \frac{b_{n,k}(t)}{k!} \epsilon^k \quad \forall \epsilon \in]-\epsilon_4, \epsilon_4[\quad \forall t \in \partial\Omega,$$

where the series converges uniformly for $\epsilon \in]-\epsilon_4, \epsilon_4[$. By taking $\epsilon = 0$ and by Lemma 3.4(i), we verify that

$$b_{n,0}(t) = v_\Omega^-[\theta_{n,0}^o](t) + t_n \quad \forall t \in \partial\Omega.$$

In order to compute the other coefficients, we take the derivative of order $k \in \mathbb{N} \setminus \{0\}$ of the map $\epsilon \mapsto V_n^-[\epsilon]$ and we obtain

$$\partial_\epsilon^k (V_n^-[\epsilon])(t) = v_\Omega^-[\partial_\epsilon^k \theta_n^o[\epsilon]](t) + \partial_\epsilon^k (\Lambda[\epsilon, \theta_n^o[\epsilon]])(t) \quad \forall t \in \partial\Omega.$$

Again, by taking $\epsilon = 0$ and by Lemma 3.4(i), we find

$$b_{n,1}(t) = v_\Omega^-[\theta_{n,1}^o](t) \quad \forall t \in \partial\Omega,$$

$$b_{n,k}(t) = v_\Omega^-[\theta_{n,k}^o](t) + \Lambda^k[\theta_{n,0}^o, \dots, \theta_{n,k-2}^o](t) \quad \forall t \in \partial\Omega$$

and for all $k \in \mathbb{N} \setminus \{0, 1\}$.

As a consequence, possibly shrinking ϵ_4 , we have

$$\begin{aligned} \int_{\partial\Omega} V_n^-[\epsilon](t) (\mathbf{n}_\Omega(t))_m d\sigma_t &= \int_{\partial\Omega} \sum_{k=0}^{+\infty} \frac{b_{n,k}(t)}{k!} \epsilon^k (\mathbf{n}_\Omega(t))_m d\sigma_t \\ &= \sum_{k=0}^{+\infty} \frac{1}{k!} \left(\int_{\partial\Omega} b_{n,k}(t) (\mathbf{n}_\Omega(t))_m d\sigma_t \right) \epsilon^k \end{aligned}$$

where the series converges uniformly for $\epsilon \in]-\epsilon_4, \epsilon_4[$. Then we consider cases $k = 0$, $k = 1$, and $k \in \mathbb{N} \setminus \{0, 1\}$ separately and we have

$$\begin{aligned} \int_{\partial\Omega} b_{n,0}(t)(\mathbf{n}_\Omega(t))_m d\sigma_t &= \int_{\partial\Omega} (v_\Omega^-[\theta_{n,0}^o](t) + t_n)(\mathbf{n}_\Omega(t))_m d\sigma_t \\ &= \int_{\partial\Omega} v_\Omega^-[\theta_{n,0}^o](t)(\mathbf{n}_\Omega(t))_m d\sigma_t + |\Omega|_2 \delta_{m,n}, \\ \int_{\partial\Omega} b_{n,1}(t)(\mathbf{n}_\Omega(t))_m d\sigma_t &= \int_{\partial\Omega} v_\Omega^-[\theta_{n,1}^o](t)(\mathbf{n}_\Omega(t))_m d\sigma_t, \\ \int_{\partial\Omega} b_{n,k}(t)(\mathbf{n}_\Omega(t))_m d\sigma_t &= \int_{\partial\Omega} (v_\Omega^-[\theta_{n,k}^o](t) + \Lambda^k[\theta_{n,0}^o, \dots, \theta_{n,k-2}^o](t))(\mathbf{n}_\Omega(t))_m d\sigma_t. \end{aligned}$$

Accordingly, the validity of (27) follows. \square

We are now ready to prove the main result of this section, where we expand $\lambda_{mn}^{\text{eff}}(\epsilon)$ as a power series and we provide explicit and constructive expressions for the coefficients of the series.

Theorem 4.4. *Let $m, n \in \{1, 2\}$. Let ϵ_2 and $\{\theta_{n,k}^i, \theta_{n,k}^o\}_{k \in \mathbb{N}}$ be as in Proposition 4.1. Then there exists $\epsilon_5 \in]0, \epsilon_2[$ such that*

$$\lambda_{mn}^{\text{eff}}(\epsilon) = \lambda^- \delta_{m,n} + \epsilon^2 \frac{1}{|\mathcal{Q}|_2} \sum_{k=0}^{+\infty} \frac{c_{(m,n),k}}{k!} \epsilon^k \quad (28)$$

for all $\epsilon \in]0, \epsilon_5[$, where

$$\begin{aligned} c_{(m,n),0} &= \lambda^+ \int_{\partial\Omega} v_\Omega^+[\theta_{n,0}^i](t)(\mathbf{n}_\Omega(t))_m d\sigma_t + (\lambda^+ - \lambda^-) |\Omega|_2 \delta_{m,n} \\ &\quad - \lambda^- \int_{\partial\Omega} v_\Omega^-[\theta_{n,0}^o](t)(\mathbf{n}_\Omega(t))_m d\sigma_t + \int_{\partial\Omega} f(t) t_m d\sigma_t, \\ c_{(m,n),1} &= \lambda^+ \int_{\partial\Omega} v_\Omega^+[\theta_{n,1}^i](t)(\mathbf{n}_\Omega(t))_m d\sigma_t - \lambda^- \int_{\partial\Omega} v_\Omega^-[\theta_{n,1}^o](t)(\mathbf{n}_\Omega(t))_m d\sigma_t, \\ c_{(m,n),k} &= \lambda^+ \int_{\partial\Omega} (v_\Omega^+[\theta_{n,k}^i](t) + \Lambda^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i](t))(\mathbf{n}_\Omega(t))_m d\sigma_t \\ &\quad - \lambda^- \int_{\partial\Omega} (v_\Omega^-[\theta_{n,k}^o](t) + \Lambda^k[\theta_{n,0}^o, \dots, \theta_{n,k-2}^o](t))(\mathbf{n}_\Omega(t))_m d\sigma_t, \end{aligned}$$

for all $k \in \mathbb{N} \setminus \{0, 1\}$.

Proof. By [17, Thm. 8.1], if we set

$$\begin{aligned} \Lambda_{mn}(\epsilon) &= \frac{\lambda^+}{|\mathcal{Q}|_2} \int_{\partial\Omega} U_n^+[\epsilon](t)(\mathbf{n}_\Omega(t))_m d\sigma_t - \frac{\lambda^-}{|\mathcal{Q}|_2} \int_{\partial\Omega} V_n^-[\epsilon](t)(\mathbf{n}_\Omega(t))_m d\sigma_t \\ &\quad + \frac{1}{|\mathcal{Q}|_2} \int_{\partial\Omega} f(t) t_m d\sigma_t, \end{aligned}$$

for all $\epsilon \in]-\epsilon_2, \epsilon_2[$, then we have

$$\lambda_{mn}^{\text{eff}}(\epsilon) = \lambda^- \delta_{m,n} + \epsilon^2 \Lambda_{mn}(\epsilon) \quad \forall \epsilon \in]0, \epsilon_2[.$$

Then the definition of $\{c_{(m,n),k}\}_{k \in \mathbb{N}}$ and Lemmas 4.2, 4.3 imply the validity of the statement. \square

4.3. Application to the effective conductivity in the composite with inclusions in the form of a disc

Introducing more restrictive assumptions, it is possible to obtain simpler expressions for the coefficients $c_{(m,n),k}$. For example, in this subsection we will assume that

$$Q \equiv]0, 1[\times]0, 1[, \quad f \equiv 0, \quad g \text{ is a real constant, and } \Omega \equiv \mathbb{B}_2(0, 1), \quad (29)$$

and we will write the first five coefficients as simple functions of $r_\#, \lambda^+$, and λ^- . We begin by observing that, under assumptions (29), the system of integral equations (16)–(17) takes the following form

$$-\frac{1}{2}\theta_{n,0}^i(t) + w_\Omega^*[\theta_{n,0}^i](t) = -(\mathbf{n}_\Omega(t))_n \quad \forall t \in \partial\Omega, \quad (30)$$

$$\frac{1}{2}\theta_{n,0}^o(t) + w_\Omega^*[\theta_{n,0}^o](t) = -(\mathbf{n}_\Omega(t))_n \quad \forall t \in \partial\Omega, \quad (31)$$

which is equivalent to the system of the following equations

$$\frac{\partial v_\Omega^+[\theta_{n,0}^i]}{\partial \mathbf{n}_\Omega}(t) = -(\mathbf{n}_\Omega(t))_n \quad \forall t \in \partial\Omega,$$

$$\frac{\partial v_\Omega^-[\theta_{n,0}^o]}{\partial \mathbf{n}_\Omega}(t) = -(\mathbf{n}_\Omega(t))_n \quad \forall t \in \partial\Omega.$$

Hence, one can verify that if $(\theta_{n,0}^i, \theta_{n,0}^o)$ is the solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of system (30)–(31), then there exists a constant $c_0 \in \mathbb{R}$ such that

$$\begin{aligned} v_\Omega^+[\theta_{n,0}^i](t) &= -t_n + c_0 \quad \forall t \in \overline{\Omega}, \\ v_\Omega^-[\theta_{n,0}^o](t) &= \frac{t_n}{|t|^2} \quad \forall t \in \mathbb{R}^2 \setminus \Omega. \end{aligned} \quad (32)$$

We now note that, since Ω is a ball, we have

$$w_\Omega^*[\theta] = 0 \quad \forall \theta \in C^{0,\alpha}(\partial\Omega)_0. \quad (33)$$

Then by (30)–(31) and (33) we have that

$$\theta_{n,0}^i(t) = 2t_n, \quad \theta_{n,0}^o(t) = -2t_n \quad \forall t \in \partial\Omega. \quad (34)$$

Next, taking into account equalities (32) and since $t_n = (\mathbf{n}_\Omega(t))_n$ on $\partial\Omega$, the system of integral equations (18)–(19) takes the following form

$$-\frac{1}{2}\theta_{n,1}^i(t) + w_\Omega^*[\theta_{n,1}^i](t) = \frac{2r_\#}{\lambda^+}(\mathbf{n}_\Omega(t))_n \quad \forall t \in \partial\Omega, \quad (35)$$

$$\frac{1}{2}\theta_{n,1}^o(t) + w_\Omega^*[\theta_{n,1}^o](t) = \frac{2r_\#}{\lambda^-}(\mathbf{n}_\Omega(t))_n \quad \forall t \in \partial\Omega, \quad (36)$$

or, equivalently,

$$\frac{\partial v_\Omega^+[\theta_{n,1}^i]}{\partial \mathbf{n}_\Omega}(t) = \frac{2r_\#}{\lambda^+}(\mathbf{n}_\Omega(t))_n \quad \forall t \in \partial\Omega,$$

$$\frac{\partial v_\Omega^-[\theta_{n,1}^o]}{\partial \mathbf{n}_\Omega}(t) = \frac{2r_\#}{\lambda^-}(\mathbf{n}_\Omega(t))_n \quad \forall t \in \partial\Omega.$$

If $(\theta_{n,1}^i, \theta_{n,1}^o)$ is the solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of system (35)–(36), then one can verify that there exists a constant $c_1 \in \mathbb{R}$ such that

$$\begin{aligned} v_\Omega^+[\theta_{n,1}^i](t) &= \frac{2r_\#}{\lambda^+}t_n + c_1 \quad \forall t \in \bar{\Omega}, \\ v_\Omega^-[\theta_{n,1}^o](t) &= -\frac{2r_\#}{\lambda^-} \frac{t_n}{|t|^2} \quad \forall t \in \mathbb{R}^2 \setminus \Omega, \end{aligned} \quad (37)$$

and, moreover, by (35)–(36) and (33), one has

$$\theta_{n,1}^i(t) = -\frac{4r_\#}{\lambda^+}t_n, \quad \theta_{n,1}^o(t) = \frac{4r_\#}{\lambda^-}t_n \quad \forall t \in \partial\Omega. \quad (38)$$

Next, we rewrite the system of integral equations (20)–(21) as follows

$$\begin{aligned} \frac{\partial v_\Omega^+[\theta_{n,2}^i]}{\partial \mathbf{n}_\Omega}(t) &= -\Lambda_{\mathbf{n}}^2[\theta_{n,0}^i](t) - \frac{2r_\#}{\lambda^+} \left(v_\Omega^+[\theta_{n,1}^i](t) - \int_{\partial\Omega} v_\Omega^+[\theta_{n,1}^i] d\sigma \right. \\ &\quad \left. - v_\Omega^-[\theta_{n,1}^o](t) + \int_{\partial\Omega} v_\Omega^-[\theta_{n,1}^o] d\sigma \right) \quad \forall t \in \partial\Omega, \end{aligned} \quad (39)$$

$$\begin{aligned} \frac{\partial v_\Omega^-[\theta_{n,2}^o]}{\partial \mathbf{n}_\Omega}(t) &= -\Lambda_{\mathbf{n}}^2[\theta_{n,0}^o](t) - \frac{2r_\#}{\lambda^-} \left(v_\Omega^+[\theta_{n,1}^i](t) - \int_{\partial\Omega} v_\Omega^+[\theta_{n,1}^i] d\sigma \right. \\ &\quad \left. - v_\Omega^-[\theta_{n,1}^o](t) + \int_{\partial\Omega} v_\Omega^-[\theta_{n,1}^o] d\sigma \right) \quad \forall t \in \partial\Omega, \end{aligned} \quad (40)$$

and, moreover, for $k = 3$ we rewrite the system of integral equations (22)–(23) as follows

$$\begin{aligned} \frac{\partial v_{\Omega}^{+}[\theta_{n,3}^i]}{\partial \mathbf{n}_{\Omega}}(t) &= -\Lambda_{\mathbf{n}}^3[\theta_{n,0}^i, \theta_{n,1}^i](t) - \frac{3r_{\#}}{\lambda^{+}} \left(v_{\Omega}^{+}[\theta_{n,2}^i](t) - \int_{\partial\Omega} v_{\Omega}^{+}[\theta_{n,2}^i] d\sigma - v_{\Omega}^{-}[\theta_{n,2}^o](t) \right. \\ &\quad \left. + \int_{\partial\Omega} v_{\Omega}^{-}[\theta_{n,2}^o] d\sigma \right) - \frac{3r_{\#}}{\lambda^{+}} \left(\Lambda^2[\theta_{n,0}^i](t) - \int_{\partial\Omega} \Lambda^2[\theta_{n,0}^i] d\sigma \right. \\ &\quad \left. - \Lambda^2[\theta_{n,0}^o](t) + \int_{\partial\Omega} \Lambda^2[\theta_{n,0}^o] d\sigma \right) \quad \forall t \in \partial\Omega, \end{aligned} \quad (41)$$

$$\begin{aligned} \frac{\partial v_{\Omega}^{-}[\theta_{n,3}^o]}{\partial \mathbf{n}_{\Omega}}(t) &= -\Lambda_{\mathbf{n}}^3[\theta_{n,0}^o, \theta_{n,1}^o](t) - \frac{3r_{\#}}{\lambda^{-}} \left(v_{\Omega}^{+}[\theta_{n,2}^i](t) - \int_{\partial\Omega} v_{\Omega}^{+}[\theta_{n,2}^i] d\sigma - v_{\Omega}^{-}[\theta_{n,2}^o](t) \right. \\ &\quad \left. + \int_{\partial\Omega} v_{\Omega}^{-}[\theta_{n,2}^o] d\sigma \right) - \frac{3r_{\#}}{\lambda^{-}} \left(\Lambda^2[\theta_{n,0}^i](t) - \int_{\partial\Omega} \Lambda^2[\theta_{n,0}^i] d\sigma \right. \\ &\quad \left. - \Lambda^2[\theta_{n,0}^o](t) + \int_{\partial\Omega} \Lambda^2[\theta_{n,0}^o] d\sigma \right) \quad \forall t \in \partial\Omega. \end{aligned} \quad (42)$$

We now exploit (39)–(40) and (41)–(42) to add other two explicit terms in our expansion. To do so, we have to compute the Λ^2 , Λ^3 , $\Lambda_{\mathbf{n}}^2$, and $\Lambda_{\mathbf{n}}^3$ terms which appear in (39)–(40) and (41)–(42). In view of the definitions of Λ^k and $\Lambda_{\mathbf{n}}^k$ we need to know the value of $(\partial_1^2 R_q)(0)$ and $(\partial_2^2 R_q)(0)$. Hence we observe that by (4) we have $(\Delta R_q)(0) = -1$, thus $(\partial_1^2 R_q)(0) + (\partial_2^2 R_q)(0) = -1$. Since Q is a square, by a symmetry argument we deduce that

$$(\partial_1^2 R_q)(0) = (\partial_2^2 R_q)(0) = -\frac{1}{2}. \quad (43)$$

Also, one verifies that $R_q(x_1, x_2) = R_q(-x_1, x_2)$ for all $x \in (\mathbb{R}^2 \setminus q\mathbb{Z}^2) \cup \{0\}$. Then it follows that

$$(\partial_1 \partial_2 R_q)(0) = 0. \quad (44)$$

Hence, by the definitions of Λ^k and $\Lambda_{\mathbf{n}}^k$ and by equalities (5), (34), (38), (43), and (44) one can show that

$$\begin{aligned} \Lambda^2[\theta_{n,0}^i](t) &= 2\pi t_n, & \Lambda^2[\theta_{n,0}^o](t) &= -2\pi t_n \quad \forall t \in \partial\Omega, \\ \Lambda_{\mathbf{n}}^2[\theta_{n,0}^i](t) &= 2\pi t_n, & \Lambda_{\mathbf{n}}^2[\theta_{n,0}^o](t) &= -2\pi t_n \quad \forall t \in \partial\Omega, \\ \Lambda_{\mathbf{n}}^3[\theta_{n,0}^i, \theta_{n,1}^i](t) &= -\frac{12\pi r_{\#}}{\lambda^{+}} t_n, & \Lambda_{\mathbf{n}}^3[\theta_{n,0}^o, \theta_{n,1}^o](t) &= \frac{12\pi r_{\#}}{\lambda^{-}} t_n \quad \forall t \in \partial\Omega. \end{aligned} \quad (45)$$

Again, if $(\theta_{n,2}^i, \theta_{n,2}^o)$ and $(\theta_{n,3}^i, \theta_{n,3}^o)$ are the solutions in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the systems (39)–(40) and (41)–(42), respectively, then, taking into account equalities (37) and (45), one can verify that there exists a

1 real constant c_2 such that

$$\begin{aligned}
 2 \quad v_{\Omega}^+[\theta_{n,2}^i](t) &= -2 \left(\frac{2(r_{\#})^2}{\lambda^+} \left(\frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right) + \pi \right) t_n + c_2 \quad \forall t \in \overline{\Omega}, \\
 3 \quad & \\
 4 \quad & \\
 5 \quad v_{\Omega}^-[\theta_{n,2}^o](t) &= 2 \left(\frac{2(r_{\#})^2}{\lambda^-} \left(\frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right) - \pi \right) \frac{t_n}{|t|^2} \quad \forall t \in \mathbb{R}^2 \setminus \Omega. \\
 6 \quad & \\
 7 \quad &
 \end{aligned} \tag{46}$$

8 Similarly, by (46) one verifies that there exists a real constant c_3 such that

$$\begin{aligned}
 10 \quad v_{\Omega}^+[\theta_{n,3}^i](t) &= \frac{12(r_{\#})^3}{\lambda^+} \left(\frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right)^2 t_n + c_3 \quad \forall t \in \overline{\Omega}, \\
 11 \quad & \\
 12 \quad & \\
 13 \quad v_{\Omega}^-[\theta_{n,3}^o](t) &= -\frac{12r_{\#}}{\lambda^-} \left((r_{\#})^2 \left(\frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right)^2 - 2\pi \right) \frac{t_n}{|t|^2} \quad \forall t \in \mathbb{R}^2 \setminus \Omega. \\
 14 \quad & \\
 15 \quad &
 \end{aligned} \tag{47}$$

16 Then, by Theorem 4.4 and equalities (32), (37), (46), and (47) we deduce that if $m, n \in \{1, 2\}$
 17 then

$$\begin{aligned}
 19 \quad \lambda_{mn}^{\text{eff}}(\epsilon) &= \left(\lambda^- - 2\pi\lambda^-\epsilon^2 + 4\pi r_{\#}\epsilon^3 - 4\pi \left((r_{\#})^2 \left(\frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right) - \frac{1}{2}\pi\lambda^- \right) \epsilon^4 \right. \\
 20 \quad & \\
 21 \quad & \\
 22 \quad & \left. + 4\pi r_{\#} \left((r_{\#})^2 \left(\frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right)^2 - 2\pi \right) \epsilon^5 \right) \delta_{m,n} + O(\epsilon^6) \\
 23 \quad & \\
 24 \quad &
 \end{aligned}$$

25 as $\epsilon \rightarrow 0^+$.

28 5. Power series expansions for $\rho(\epsilon) \equiv \epsilon/r_{\#}$

30 5.1. Series expansions for $\theta_n^i[\epsilon]$ and $\theta_n^o[\epsilon]$

32 Throughout this section we consider the case where

$$34 \quad \rho(\epsilon) \equiv \epsilon/r_{\#} \quad \forall \epsilon \in]-\epsilon_0, \epsilon_0[. \tag{48}$$

36 As done in Section 4, in order to compute the asymptotic expansion of the effective conductivity under assumption (48), we start with the following proposition, where we identify the coefficients of the power series expansions of $\theta_n^i[\epsilon]$ and of $\theta_n^o[\epsilon]$ in terms of the solutions of systems of integral equations.

41 **Proposition 5.1.** *Let $\epsilon_1, \epsilon \mapsto \theta_n^i[\epsilon]$, and $\epsilon \mapsto \theta_n^o[\epsilon]$ be as in Proposition 3.3. Then there exist $\epsilon_2 \in]0, \epsilon_1[$ and a sequence $\{(\theta_{n,k}^i, \theta_{n,k}^o)\}_{k \in \mathbb{N}}$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$ such that*

$$44 \quad \theta_n^i[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_{n,k}^i}{k!} \epsilon^k \quad \text{and} \quad \theta_n^o[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_{n,k}^o}{k!} \epsilon^k \quad \forall \epsilon \in]-\epsilon_2, \epsilon_2[, \tag{49}$$

where the two series converge uniformly for $\epsilon \in]-\epsilon_2, \epsilon_2[$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$. Moreover, the following statements hold.

- (i) The pair of functions $(\theta_{n,0}^i, \theta_{n,0}^o)$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations

$$\begin{aligned} & \lambda^- \left(\frac{1}{2} \theta_{n,0}^o(t) + w_{\Omega}^*[\theta_{n,0}^o](t) \right) - \lambda^+ \left(-\frac{1}{2} \theta_{n,0}^i(t) + w_{\Omega}^*[\theta_{n,0}^i](t) \right) \\ & - f(t) + (\lambda^- - \lambda^+) (\mathbf{n}_{\Omega}(t))_n = 0 \quad \forall t \in \partial\Omega, \end{aligned} \quad (50)$$

$$\begin{aligned} & \lambda^+ \left(-\frac{1}{2} \theta_{n,0}^i(t) + w_{\Omega}^*[\theta_{n,0}^i](t) \right) + r_{\#} \left(v_{\Omega}^+[\theta_{n,0}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,0}^i] d\sigma - v_{\Omega}^-[\theta_{n,0}^o](t) \right. \\ & \left. + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,0}^o] d\sigma \right) - g(t) + \int_{\partial\Omega} g d\sigma + \lambda^+ (\mathbf{n}_{\Omega}(t))_n = 0 \quad \forall t \in \partial\Omega. \end{aligned} \quad (51)$$

- (ii) For all $k \in \mathbb{N}$, we have $(\theta_{n,2k+1}^i, \theta_{n,2k+1}^o) = (0, 0)$.

- (iii) For all $k \in \mathbb{N} \setminus \{0\}$ the pair of functions $(\theta_{n,2k}^i, \theta_{n,2k}^o)$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations which involves $\{(\theta_{n,2h}^i, \theta_{n,2h}^o)\}_{h=0}^{k-1}$

$$\begin{aligned} & \lambda^- \left(\frac{1}{2} \theta_{n,2k}^o(t) + w_{\Omega}^*[\theta_{n,2k}^o](t) \right) - \lambda^+ \left(-\frac{1}{2} \theta_{n,2k}^i(t) + w_{\Omega}^*[\theta_{n,2k}^i](t) \right) \\ & = \lambda^+ \Lambda_{\mathbf{n}}^{2k}[\theta_{n,0}^i, \dots, \theta_{n,2k-2}^i](t) - \lambda^- \Lambda_{\mathbf{n}}^{2k}[\theta_{n,0}^o, \dots, \theta_{n,2k-2}^o](t) \quad \forall t \in \partial\Omega, \end{aligned} \quad (52)$$

$$\begin{aligned} & \lambda^+ \left(-\frac{1}{2} \theta_{n,2k}^i(t) + w_{\Omega}^*[\theta_{n,2k}^i](t) \right) + r_{\#} \left(v_{\Omega}^+[\theta_{n,2k}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,2k}^i] d\sigma \right. \\ & \left. - v_{\Omega}^-[\theta_{n,2k}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,2k}^o] d\sigma \right) \\ & = -\lambda^+ \Lambda_{\mathbf{n}}^{2k}[\theta_{n,0}^i, \dots, \theta_{n,2k-2}^i](t) \\ & - r_{\#} \left(\Lambda^{2k}[\theta_{n,0}^i, \dots, \theta_{n,2k-2}^i](t) - \int_{\partial\Omega} \Lambda^{2k}[\theta_{n,0}^i, \dots, \theta_{n,2k-2}^i] d\sigma \right. \\ & \left. - \Lambda^{2k}[\theta_{n,0}^o, \dots, \theta_{n,2k-2}^o](t) + \int_{\partial\Omega} \Lambda^{2k}[\theta_{n,0}^o, \dots, \theta_{n,2k-2}^o] d\sigma \right) \quad \forall t \in \partial\Omega. \end{aligned} \quad (53)$$

Proof. The existence of ϵ_2 and $\{(\theta_{n,k}^i, \theta_{n,k}^o)\}_{k \in \mathbb{N}}$ for which (49) holds true is granted by Proposition 3.3(iii). By equality $\rho(\epsilon) \equiv \epsilon/r_{\#}$ and by taking $\epsilon = 0$, equation (6) can be written as the system of integral equations (50)–(51). The uniqueness of the solution for this system is then ensured by Proposition 3.3(ii) (see also [17, Prop. 5.2]). Next, we observe that $M_n[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]] = 0$ for all $\epsilon \in]-\epsilon_2, \epsilon_2[$ (cf. Proposition 3.3(iii)). Accordingly, the map which takes ϵ to $M_n[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]]$ has all the derivatives equal to zero, i.e., $\partial_{\epsilon}^k(M_n[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]]) = 0$ for all $\epsilon \in]-\epsilon_2, \epsilon_2[$ and all $k \in \mathbb{N} \setminus \{0\}$. Keeping

equality $\rho(\epsilon) \equiv \epsilon/r_{\#}$ in mind, a straightforward calculation shows that

$$\begin{aligned} & \partial_{\epsilon}^k (M_{n,1}[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]])(t) \\ &= \lambda^{-} \left(\frac{1}{2} \partial_{\epsilon}^k \theta_n^o[\epsilon](t) + w_{\Omega}^* [\partial_{\epsilon}^k \theta_n^o[\epsilon]](t) + \partial_{\epsilon}^k (\epsilon \Lambda_{\mathbf{n}}[\epsilon, \theta_n^o[\epsilon]])(t) \right) \\ & \quad - \lambda^{+} \left(-\frac{1}{2} \partial_{\epsilon}^k \theta_n^i[\epsilon](t) + w_{\Omega}^* [\partial_{\epsilon}^k \theta_n^i[\epsilon]](t) + \partial_{\epsilon}^k (\epsilon \Lambda_{\mathbf{n}}[\epsilon, \theta_n^i[\epsilon]])(t) \right) \\ &= 0, \end{aligned} \tag{54}$$

$$\begin{aligned} & \partial_{\epsilon}^k (M_{n,2}[\epsilon, \theta_n^i[\epsilon], \theta_n^o[\epsilon]])(t) \\ &= \lambda^{+} \left(-\frac{1}{2} \partial_{\epsilon}^k \theta_n^i[\epsilon](t) + w_{\Omega}^* [\partial_{\epsilon}^k \theta_n^i[\epsilon]](t) + \partial_{\epsilon}^k (\epsilon \Lambda_{\mathbf{n}}[\epsilon, \theta_n^i[\epsilon]])(t) \right) \\ & \quad + r_{\#} \left(v_{\Omega}^{+} [\partial_{\epsilon}^k \theta_n^i[\epsilon]](t) + \partial_{\epsilon}^k (\Lambda[\epsilon, \theta_n^i[\epsilon]])(t) - \int_{\partial\Omega} (v_{\Omega}^{+} [\partial_{\epsilon}^k \theta_n^i[\epsilon]] + \partial_{\epsilon}^k (\Lambda[\epsilon, \theta_n^i[\epsilon]])) d\sigma \right. \\ & \quad \left. - v_{\Omega}^{-} [\partial_{\epsilon}^k \theta_n^o[\epsilon]](t) - \partial_{\epsilon}^k (\Lambda[\epsilon, \theta_n^o[\epsilon]])(t) + \int_{\partial\Omega} (v_{\Omega}^{-} [\partial_{\epsilon}^k \theta_n^o[\epsilon]] + \partial_{\epsilon}^k (\Lambda[\epsilon, \theta_n^o[\epsilon]])) d\sigma \right) \\ &= 0 \end{aligned} \tag{55}$$

for all $t \in \partial\Omega$, all $\epsilon \in]-\epsilon_2, \epsilon_2[$, and all $k \in \mathbb{N} \setminus \{0\}$. By taking $\epsilon = 0$ in (54)–(55) and noting that $(\theta_{n,k}^i, \theta_{n,k}^o) = (\partial_{\epsilon}^k \theta_n^i[0], \partial_{\epsilon}^k \theta_n^o[0])$ for all $k \in \mathbb{N}$, we deduce that $(\theta_{n,1}^i, \theta_{n,1}^o)$ is a solution of the following system

$$\lambda^{-} \left(\frac{1}{2} \theta_{n,1}^o(t) + w_{\Omega}^* [\theta_{n,1}^o](t) \right) - \lambda^{+} \left(-\frac{1}{2} \theta_{n,1}^i(t) + w_{\Omega}^* [\theta_{n,1}^i](t) \right) = 0 \quad \forall t \in \partial\Omega, \tag{56}$$

$$\begin{aligned} & \lambda^{+} \left(-\frac{1}{2} \theta_{n,1}^i(t) + w_{\Omega}^* [\theta_{n,1}^i](t) \right) + r_{\#} \left(v_{\Omega}^{+} [\theta_{n,1}^i](t) - \int_{\partial\Omega} v_{\Omega}^{+} [\theta_{n,1}^i] d\sigma \right. \\ & \quad \left. - v_{\Omega}^{-} [\theta_{n,1}^o](t) + \int_{\partial\Omega} v_{\Omega}^{-} [\theta_{n,1}^o] d\sigma \right) = 0 \quad \forall t \in \partial\Omega, \end{aligned} \tag{57}$$

(see also Lemma 3.4), and that, for $k \in \mathbb{N} \setminus \{0, 1\}$, the pair $(\theta_{n,k}^i, \theta_{n,k}^o)$ is a solution of the following system

$$\begin{aligned} & \lambda^{-} \left(\frac{1}{2} \theta_{n,k}^o(t) + w_{\Omega}^* [\theta_{n,k}^o](t) \right) - \lambda^{+} \left(-\frac{1}{2} \theta_{n,k}^i(t) + w_{\Omega}^* [\theta_{n,k}^i](t) \right) \\ &= \lambda^{+} \Lambda_{\mathbf{n}}^k [\theta_{n,0}^i, \dots, \theta_{n,k-2}^i](t) - \lambda^{-} \Lambda_{\mathbf{n}}^k [\theta_{n,0}^o, \dots, \theta_{n,k-2}^o](t) \quad \forall t \in \partial\Omega, \end{aligned} \tag{58}$$

$$\begin{aligned}
& \lambda^+ \left(-\frac{1}{2} \theta_{n,k}^i(t) + w_{\Omega}^*[\theta_{n,k}^i](t) \right) + r_{\#} \left(v_{\Omega}^+[\theta_{n,k}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,k}^i] d\sigma \right. \\
& \quad \left. - v_{\Omega}^-[\theta_{n,k}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,k}^o] d\sigma \right) \\
& = -\lambda^+ \Lambda_{\mathbf{n}}^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i](t) \\
& \quad - r_{\#} \left(\Lambda^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i](t) - \int_{\partial\Omega} \Lambda^k[\theta_{n,0}^i, \dots, \theta_{n,k-2}^i] d\sigma \right. \\
& \quad \left. - \Lambda^k[\theta_{n,0}^o, \dots, \theta_{n,k-2}^o](t) + \int_{\partial\Omega} \Lambda^k[\theta_{n,0}^o, \dots, \theta_{n,k-2}^o] d\sigma \right) \quad \forall t \in \partial\Omega. \tag{59}
\end{aligned}$$

Since the right hand sides of equalities (56)–(57) and (58)–(59) belong to the space $C^{0,\alpha}(\partial\Omega)_0$, the uniqueness of the solution to (56)–(57) and of the solution to (58)–(59) follows by [17, Prop. 5.2] (we have already observed that the existence is granted by Proposition 3.3(iii)). Moreover, $(\theta_{n,1}^i, \theta_{n,1}^o) = (0, 0)$. Also, by Lemma 3.5 and by the uniqueness of the solution to system (58)–(59), one can verify that $(\theta_{2k+1}^i, \theta_{2k+1}^o) = (0, 0)$ for all $k \in \mathbb{N} \setminus \{0\}$. The validity of the proposition is now proved. \square

By Proposition 5.1, we immediately deduce the validity of the following.

Corollary 5.2. *Let the assumptions of Proposition 5.1 hold. Then*

$$\theta_n^i[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_{n,2k}^i}{(2k)!} \epsilon^{2k} \quad \text{and} \quad \theta_n^o[\epsilon] = \sum_{k=0}^{+\infty} \frac{\theta_{n,2k}^o}{(2k)!} \epsilon^{2k} \quad \forall \epsilon \in]-\epsilon_2, \epsilon_2[,$$

where $\{(\theta_{n,2k}^i, \theta_{n,2k}^o)\}_{k \in \mathbb{N}}$ are as in Proposition 5.1.

5.2. Series expansion of the effective conductivity

By exploiting Proposition 5.1, we can prove the following Lemmas 5.3 and 5.4. The proofs can be effected by a straightforward modification of the proofs of the analogous Lemmas 4.2 and 4.3 and it is accordingly omitted.

Lemma 5.3. *Let $m, n \in \{1, 2\}$. Let $\epsilon \mapsto \theta_n^i[\epsilon]$ and $\epsilon \mapsto \theta_n^o[\epsilon]$ be as in Proposition 3.3. Let ϵ_2 and $\{(\theta_{n,k}^i, \theta_{n,k}^o)\}_{k \in \mathbb{N}}$ be as in Proposition 5.1. Let U_n^+ be the map from $]-\epsilon_2, \epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned}
U_n^+[\epsilon](t) & \equiv v_{\Omega}^+[\theta_n^i[\epsilon]](t) + \Lambda[\epsilon, \theta_n^i[\epsilon]](t) \\
& \quad - \int_{\partial\Omega} (v_{\Omega}^+[\theta_n^i[\epsilon]] + \Lambda[\epsilon, \theta_n^i[\epsilon]]) d\sigma + t_n - \int_{\partial\Omega} s_n d\sigma_s \quad \forall t \in \partial\Omega.
\end{aligned}$$

Then U_n^+ is real analytic and there exists $\epsilon_3 \in]0, \epsilon_2[$ such that

$$\begin{aligned} & \int_{\partial\Omega} U_n^+[\epsilon](t)(\mathbf{n}_\Omega(t))_m d\sigma_t \\ &= \int_{\partial\Omega} v_\Omega^+[\theta_{n,0}^i](t)(\mathbf{n}_\Omega(t))_m d\sigma_t + |\Omega|_2 \delta_{m,n} \\ &+ \sum_{k=1}^{+\infty} \frac{1}{(2k)!} \left(\int_{\partial\Omega} (v_\Omega^+[\theta_{n,2k}^i](t) + \Lambda^{2k}[\theta_{n,0}^i, \dots, \theta_{n,2k-2}^i](t))(\mathbf{n}_\Omega(t))_m d\sigma_t \right) \epsilon^{2k} \\ & \forall \epsilon \in]-\epsilon_3, \epsilon_3[, \end{aligned}$$

where the series converges uniformly for $\epsilon \in]-\epsilon_3, \epsilon_3[$.

Lemma 5.4. Let $m, n \in \{1, 2\}$. Let $\epsilon \mapsto \theta_n^i[\epsilon]$ and $\epsilon \mapsto \theta_n^o[\epsilon]$ be as in Proposition 3.3. Let ϵ_2 and $\{(\theta_{n,k}^i, \theta_{n,k}^o)\}_{k \in \mathbb{N}}$ be as in Proposition 5.1. Let V_n^- be the map from $]-\epsilon_2, \epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$ defined by

$$V_n^-[\epsilon](t) \equiv v_\Omega^-[\theta_n^o[\epsilon]](t) + \Lambda[\epsilon, \theta_n^o[\epsilon]](t) + t_n \quad \forall t \in \partial\Omega.$$

Then V_n^- is real analytic and there exists $\epsilon_4 \in]0, \epsilon_2[$ such that

$$\begin{aligned} & \int_{\partial\Omega} V_n^-[\epsilon](t)(\mathbf{n}_\Omega(t))_m d\sigma_t \\ &= \int_{\partial\Omega} v_\Omega^-[\theta_{n,0}^o](t)(\mathbf{n}_\Omega(t))_m d\sigma_t + |\Omega|_2 \delta_{m,n} \\ &+ \sum_{k=1}^{+\infty} \frac{1}{(2k)!} \left(\int_{\partial\Omega} (v_\Omega^-[\theta_{n,2k}^o](t) + \Lambda^{2k}[\theta_{n,0}^o, \dots, \theta_{n,2k-2}^o](t))(\mathbf{n}_\Omega(t))_m d\sigma_t \right) \epsilon^{2k} \\ & \forall \epsilon \in]-\epsilon_4, \epsilon_4[, \end{aligned}$$

where the series converges uniformly for $\epsilon \in]-\epsilon_4, \epsilon_4[$.

By Lemmas 5.3 and 5.4 and by arguing as in the proof of Theorem 4.4, one deduces the validity of the following result concerning the expansion of $\lambda_{mn}^{\text{eff}}(\epsilon)$.

Theorem 5.5. Let $m, n \in \{1, 2\}$. Let ϵ_2 and $\{(\theta_{n,k}^i, \theta_{n,k}^o)\}_{k \in \mathbb{N}}$ be as in Proposition 5.1. Then there exists $\epsilon_5 \in]0, \epsilon_2[$ such that

$$\lambda_{mn}^{\text{eff}}(\epsilon) = \lambda^- \delta_{m,n} + \epsilon^2 \frac{1}{|Q|_2} \sum_{k=0}^{+\infty} \frac{d_{(m,n),2k}}{(2k)!} \epsilon^{2k} \quad (60)$$

for all $\epsilon \in]0, \epsilon_5[$, where

$$d_{(m,n),0} = \lambda^+ \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,0}^i](t)(\mathbf{n}_{\Omega}(t))_m d\sigma_t + (\lambda^+ - \lambda^-)|\Omega|_2 \delta_{m,n} - \lambda^- \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,0}^o](t)(\mathbf{n}_{\Omega}(t))_m d\sigma_t + \int_{\partial\Omega} f(t)t_m d\sigma_t, \quad (61)$$

$$d_{(m,n),2k} = \lambda^+ \int_{\partial\Omega} (v_{\Omega}^+[\theta_{n,2k}^i](t) + \Lambda^{2k}[\theta_{n,0}^i, \dots, \theta_{n,2k-2}^i](t))(\mathbf{n}_{\Omega}(t))_m d\sigma_t - \lambda^- \int_{\partial\Omega} (v_{\Omega}^-[\theta_{n,2k}^o](t) + \Lambda^{2k}[\theta_{n,0}^o, \dots, \theta_{n,2k-2}^o](t))(\mathbf{n}_{\Omega}(t))_m d\sigma_t \quad (62)$$

for all $k \in \mathbb{N} \setminus \{0\}$.

5.3. Application to the effective conductivity in the composite with inclusions in the form of a disc

As in Section 4.3 we consider assumption (29), but this time with $\rho(\epsilon) \equiv \epsilon/r_{\#}$. We will write the first three terms of the series expansion of $\lambda_{mn}^{\text{eff}}(\epsilon)$ in terms of simple functions of $r_{\#}$, λ^+ , and λ^- . We begin by noting that under assumption (29) the system of integral equations (50)–(51) takes the following form

$$\begin{aligned} & \lambda^- \left(\frac{1}{2} \theta_{n,0}^o(t) + w_{\Omega}^*[\theta_{n,0}^o](t) \right) - \lambda^+ \left(-\frac{1}{2} \theta_{n,0}^i(t) + w_{\Omega}^*[\theta_{n,0}^i](t) \right) \\ & + (\lambda^- - \lambda^+) (\mathbf{n}_{\Omega}(t))_n = 0 \quad \forall t \in \partial\Omega, \\ & \lambda^+ \left(-\frac{1}{2} \theta_{n,0}^i(t) + w_{\Omega}^*[\theta_{n,0}^i](t) \right) + \lambda^+ (\mathbf{n}_{\Omega}(t))_n + r_{\#} \left(v_{\Omega}^+[\theta_{n,0}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,0}^i] d\sigma \right. \\ & \left. - v_{\Omega}^-[\theta_{n,0}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,0}^o] d\sigma \right) = 0 \quad \forall t \in \partial\Omega, \end{aligned}$$

which are equivalent to the following equations

$$\lambda^- \frac{\partial v_{\Omega}^-[\theta_{n,0}^o]}{\partial \mathbf{n}_{\Omega}}(t) - \lambda^+ \frac{\partial v_{\Omega}^+[\theta_{n,0}^i]}{\partial \mathbf{n}_{\Omega}}(t) + (\lambda^- - \lambda^+) (\mathbf{n}_{\Omega}(t))_n = 0 \quad \forall t \in \partial\Omega, \quad (63)$$

$$\begin{aligned} & \lambda^+ \frac{\partial v_{\Omega}^+[\theta_{n,0}^i]}{\partial \mathbf{n}_{\Omega}}(t) + \lambda^+ (\mathbf{n}_{\Omega}(t))_n + r_{\#} \left(v_{\Omega}^+[\theta_{n,0}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,0}^i] d\sigma \right. \\ & \left. - v_{\Omega}^-[\theta_{n,0}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,0}^o] d\sigma \right) = 0 \quad \forall t \in \partial\Omega. \quad (64) \end{aligned}$$

If $(\theta_{n,0}^i, \theta_{n,0}^o)$ is the solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of system (63)–(64), one can verify that there exists a constant $c_0 \in \mathbb{R}$ such that

$$\begin{aligned} v_{\Omega}^+[\theta_{n,0}^i](t) &= -\left(1 - \frac{2\lambda^- r_{\#}}{\lambda^- \lambda^+ + \lambda^+ r_{\#} + \lambda^- r_{\#}}\right) t_n + c_0 \quad \forall t \in \overline{\Omega}, \\ v_{\Omega}^-[\theta_{n,0}^o](t) &= \left(1 - \frac{2\lambda^+ r_{\#}}{\lambda^- \lambda^+ + \lambda^+ r_{\#} + \lambda^- r_{\#}}\right) \frac{t_n}{|t|^2} \quad \forall t \in \mathbb{R}^2 \setminus \Omega. \end{aligned} \quad (65)$$

Then we recall that by the jump formula for the normal derivative of the single layer potential we deduce that if $\Omega = \mathbb{B}_2(0, 1)$, then

$$\theta(t) = -2 \frac{\partial v_{\Omega}^+[\theta]}{\partial \mathbf{n}_{\Omega}}(t) = 2 \frac{\partial v_{\Omega}^-[\theta]}{\partial \mathbf{n}_{\Omega}}(t) \quad \forall t \in \partial\Omega, \quad (66)$$

for all $\theta \in C^{0,\alpha}(\partial\Omega)_0$. Therefore, by (65) and (66), one has

$$\begin{aligned} \theta_{n,0}^i(t) &= 2 \left(1 - \frac{2\lambda^- r_{\#}}{\lambda^- \lambda^+ + \lambda^+ r_{\#} + \lambda^- r_{\#}}\right) t_n \quad \forall t \in \partial\Omega, \\ \theta_{n,0}^o(t) &= -2 \left(1 - \frac{2\lambda^+ r_{\#}}{\lambda^- \lambda^+ + \lambda^+ r_{\#} + \lambda^- r_{\#}}\right) t_n \quad \forall t \in \partial\Omega. \end{aligned} \quad (67)$$

Now if $k = 1$, the system of integral equations (52)–(53) takes the form

$$\begin{aligned} \lambda^- \frac{\partial v_{\Omega}^-[\theta_{n,2}^o]}{\partial \mathbf{n}_{\Omega}}(t) - \lambda^+ \frac{\partial v_{\Omega}^+[\theta_{n,2}^i]}{\partial \mathbf{n}_{\Omega}}(t) &= \lambda^+ \Lambda_{\mathbf{n}}^2[\theta_{n,0}^i](t) - \lambda^- \Lambda_{\mathbf{n}}^2[\theta_{n,0}^o](t) \quad \forall t \in \partial\Omega, \\ \lambda^+ \frac{\partial v_{\Omega}^+[\theta_{n,2}^i]}{\partial \mathbf{n}_{\Omega}}(t) + r_{\#} \left(v_{\Omega}^+[\theta_{n,2}^i](t) - \int_{\partial\Omega} v_{\Omega}^+[\theta_{n,2}^i] d\sigma - v_{\Omega}^-[\theta_{n,2}^o](t) + \int_{\partial\Omega} v_{\Omega}^-[\theta_{n,2}^o] d\sigma \right) \\ &= -\lambda^+ \Lambda_{\mathbf{n}}^2[\theta_{n,0}^i](t) - r_{\#} \left(\Lambda^2[\theta_{n,0}^i](t) - \int_{\partial\Omega} \Lambda^2[\theta_{n,0}^i] d\sigma \right. \\ &\quad \left. - \Lambda^2[\theta_{n,0}^o](t) + \int_{\partial\Omega} \Lambda^2[\theta_{n,0}^o] d\sigma \right) \quad \forall t \in \partial\Omega, \end{aligned} \quad (68)$$

since $\theta_{n,1}^i \equiv 0$ and $\theta_{n,1}^o \equiv 0$. Then, by the definitions of Λ^k and $\Lambda_{\mathbf{n}}^k$, by equalities (5), (43), and (67), one can show that

$$\begin{aligned} \Lambda^2[\theta_{n,0}^i](t) &= \Lambda_{\mathbf{n}}^2[\theta_{n,0}^i](t) = 2\pi \left(1 - \frac{2\lambda^- r_{\#}}{\lambda^- \lambda^+ + \lambda^+ r_{\#} + \lambda^- r_{\#}}\right) t_n \quad \forall t \in \partial\Omega, \\ \Lambda^2[\theta_{n,0}^o](t) &= \Lambda_{\mathbf{n}}^2[\theta_{n,0}^o](t) = -2\pi \left(1 - \frac{2\lambda^+ r_{\#}}{\lambda^- \lambda^+ + \lambda^+ r_{\#} + \lambda^- r_{\#}}\right) t_n \quad \forall t \in \partial\Omega. \end{aligned} \quad (70)$$

If $(\theta_{n,2}^i, \theta_{n,2}^o)$ is the solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of system (68)–(69), then, taking into account equalities (70), one verifies that there exists a constant $c_1 \in \mathbb{R}$ such that

$$\begin{aligned} v_{\Omega}^+[\theta_{n,2}^i](t) &= -2\pi \left(1 - \frac{4\lambda^+\lambda^-r_{\#}^2}{(\lambda^-\lambda^+ + \lambda^+r_{\#} + \lambda^-r_{\#})^2} \right) t_n + c_1 \quad \forall t \in \overline{\Omega}, \\ v_{\Omega}^-[\theta_{n,2}^o](t) &= -2\pi \left(1 - \frac{2\lambda^+r_{\#}}{\lambda^-\lambda^+ + \lambda^+r_{\#} + \lambda^-r_{\#}} \right)^2 \frac{t_n}{|t|^2} \quad \forall t \in \mathbb{R}^2 \setminus \Omega. \end{aligned} \quad (71)$$

Then by Theorem 5.5 and equations (65), (71), we have that if $m, n \in \{1, 2\}$, then

$$\begin{aligned} \lambda_{mn}^{\text{eff}}(\epsilon) &= \lambda^- \left(1 - 2\pi \left(1 - \frac{2\lambda^+r_{\#}}{\lambda^-\lambda^+ + \lambda^+r_{\#} + \lambda^-r_{\#}} \right) \epsilon^2 \right. \\ &\quad \left. + 2\pi^2 \left(1 - \frac{2\lambda^+r_{\#}}{\lambda^-\lambda^+ + \lambda^+r_{\#} + \lambda^-r_{\#}} \right)^2 \epsilon^4 \right) \delta_{m,n} + O(\epsilon^6) \end{aligned} \quad (72)$$

as $\epsilon \rightarrow 0^+$. Taking $\lambda^- = 1$, we observe that series expansion (72) agrees with the first terms in the series expansion of the effective conductivity obtained in Drygaś and Mityushev [21] for the case where the unit cell Q contains only one inclusion.

6. Conclusions

In this paper, we have investigated the asymptotic behavior of the effective thermal conductivity of a periodic two-phase dilute composite. The composite is obtained by introducing into an infinite homogeneous matrix a periodic set of inclusions of a different material, each of them of size proportional to a positive parameter ϵ . At the two-phase interface, we have assumed that the temperature field displays a jump proportional to the normal heat flux. By [17], one knows that the effective conductivity can be represented as a convergent power series in ϵ for ϵ small. Here we have determined the coefficients in terms of the solutions of explicit systems of integral equations. The method developed in the present paper provides a constructive formula for all the coefficients of the power series expansion and for very general shapes of the inclusions. Our techniques are based on the reduction of the problem defining the effective conductivity into integral equations. Such formulation is obtained by exploiting a periodic version of layer potentials operators built with a periodic analog of the fundamental solution of the Laplace operator. The authors plan to extend the approach proposed in the papers to other properties of periodic materials.

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