SINGULAR (p, q)-EQUATIONS WITH SUPERLINEAR REACTION AND CONCAVE BOUNDARY CONDITION

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Abstract. We consider a parametric nonlinear elliptic problem driven by the sum of a p-Laplacian and of a q-Laplacian (a (p, q)-equation) with a singular and $(p-1)$-superlinear reaction and a Robin boundary condition with $(q-1)$-sublinear boundary term ($q < p$). So, the problem has the combined effects of singular, concave and convex terms. We look for positive solutions and prove a bifurcation-type theorem describing the changes in the set of positive solutions as the parameter varies.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric singular (p, q)-equation:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{-\eta} + f(z, u(z)) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} = \lambda u^{\tau-1} \quad \text{on } \partial\Omega, \quad u > 0, \lambda > 0, 1 < \tau < q \leq 2 < p, 0 < \eta < 1. \end{cases}$$

For every $r \in (1, +\infty)$ by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u) \quad \text{for all } u \in W^{1,r}(\Omega).$$

The potential function $\xi \in L^\infty(\Omega)$ and $\xi(z) \geq 0$ for a.a. $z \in \Omega$. The reaction of the problem consists of a parametric singular term and a perturbation $f(z, x)$ which is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous), which exhibits $(p-1)$-superlinear growth near $+\infty$ without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). The boundary term is parametric too and since $\tau < q$, it is $(q-1)$-sublinear. Therefore in problem (P_λ) we have the competing effects of three different kinds of nonlinearities, namely a singular, a convex (superlinear) and a concave (sublinear) nonlinearity, with the latter being in the boundary condition. Also $\dfrac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative corresponding to the (p, q)-Laplace differential operator. It is defined via the nonlinear Green's identity (see Papageorgiou-Rădulescu-Repovš [18], p. 34). In

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particular, if $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_{pq}} = \left[|\nabla u|^{p-2} + |\nabla u|^{q-2}\right] \frac{\partial u}{\partial n}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

The study of equations with competition phenomena (concave-convex problems), started with the important work of Ambrosetti-Brezis-Cerami [1] on semilinear Dirichlet problems driven by the Laplacian. Their work was extended to nonlinear Dirichlet equations driven by the p-Laplacian by Garcia Azorero-Manfredi-Peral Alonso [6] and by Guo-Zhang [8]. In the aforementioned works there is no singular term and the reaction has the special form

$$x \to \lambda x^{q-1} + x^{r-1} \quad \text{for all } x \geq 0,$$

with $1 < q < p < r < p^*$, where $p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases}$ (the critical Sobolev exponent corresponding to $p > 1$).

Recently more general versions of the concave-convex problem were considered. We mention the works of Papageorgiou-Rădulescu-Repovš [15] (semilinear equations) and by Leonardi-Papageorgiou [10], Marano-Marino-Papageorgiou [12], Papageorgiou-Vetro-Vetro [20] (nonlinear equations). Equations with the concave nonlinearity appearing in the boundary term, can be found in the works of Hu-Papageorgiou [9] (semilinear equations) and by Sabina de Lis-Segura de León [23] and by Papageorgiou-Rădulescu-Repovš [16] (nonlinear equations). However, none of these works includes in the equation a singular term. Our work here appears to be the first dealing with concave-convex singular problems, where the concave contribution comes from the boundary condition. Recently some multiplicity results for singular (p, q)-equations (but not with concave-convex nonlinearities), were proved by Papageorgiou-Rădulescu-Repovš [17] and Papageorgiou-Vetro-Vetro [19].

We mention that equations driven by a combination of differential operators of different nature (such as (p, q)-equations), arise in a variety of mathematical models of physical processes. We mention the works of Cahn-Hilliard [3] (materials science), Benci-D'Avenia-Fortunato-Pisani [2] (quantum physics) and Cherfils-Il'yasov [4] (reaction diffusion systems).

In Section 4 using variational tools based on the critical point theory, together with truncation and comparison techniques, we prove a bifurcation-type result describing the set of positive solutions of problem (P_λ) as the parameter $\lambda > 0$ varies.

2. Mathematical Background - Hypotheses

The main spaces in the study of (P_λ) are the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^s(\partial\Omega)$, $1 \leq s \leq \infty$.

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, defined by

$$\|u\| = \left[\|u\|_p^p + \|\nabla u\|_p^p\right]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

We also use another open cone in $C^1(\overline{\Omega})$ given by

$$D_+ = \left\{ u \in C^1(\overline{\Omega}) \,:\, u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the boundary Lebesgue spaces $L^s(\partial\Omega)$, $1 \leq s \leq \infty$. We know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ known as the "trace map" such that

$$\gamma_0(u) = u\Big|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map extends the notion of boundary values to all Sobolev functions. We know that $\gamma_0(\cdot)$ is compact into $L^s(\partial\Omega)$ for all $1 \leq s < \dfrac{(N-1)p}{N-p}$ if $p < N$ and into $L^s(\partial\Omega)$ for all $1 \leq s < \infty$ if $N \leq p$. Moreover, we have

$$\operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \ \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \ \text{ and } \ker \gamma_0 = W_0^{1,p}(\Omega).$$

In the sequel for the sake of notational economy, we drop the use of the map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

If $u, v \in W^{1,p}(\Omega)$ and $u(z) \leq v(z)$ for a.a. $z \in \Omega$, then we define

$$\begin{aligned} [u,v] &= \left\{ h \in W^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \quad \text{for a.a. } z \in \Omega \right\}, \\ [u) &= \left\{ h \in W^{1,p}(\Omega) : u(z) \leq h(z) \quad \text{for a.a. } z \in \Omega \right\}. \end{aligned}$$

Given $x \in \mathbb{R}$, let $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$ we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We have

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

If $h_1, h_2 \in L^\infty(\Omega)$, then we write $h_1 \prec h_2$ if and only if for every $K \subseteq \Omega$ compact we have

$$0 < c_K \leq h_2(z) - h_1(z) \quad \text{ for a.a. } z \in K.$$

Evidently if $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then $h_1 \prec h_2$.

If X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, then by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Let X^* be the topological dual of X. We say that $\varphi(\cdot)$ satisfies the "C-condition", if every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

Let $\langle\cdot,\cdot\rangle$ denote the duality brackets for the pair $(W^{1,r}(\Omega), W^{1,r}(\Omega)^*)$, $1<r<\infty$. We introduce the operator $A_r : W^{1,r}(\Omega)\to W^{1,r}(\Omega)^*$ defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u,h\in W^{1,r}(\Omega).$$

The next proposition summarizes some well-known properties of this operator (see, for example, Gasiński-Papageorgiou [7], Problem 2.192, p. 279).

Proposition 1. *The operator $A_r : W^{1,r}(\Omega)\to W^{1,r}(\Omega)^*$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (thus maximal monotone too) and of type $(S)_+$ (that is, $u_n \xrightarrow{w} u$ in $W^{1,r}(\Omega)$ and $\limsup_{n\to+\infty}\langle A_r(u_n), u_n-u\rangle\le 0$, imply $u_n\to u$ in $W^{1,r}(\Omega)$).*

The hypotheses on the data of problem (P_λ) are the following:

$\underline{H_0}$: $\xi\in L^\infty(\Omega)$, $\xi(z)\ge 0$ for a.a. $z\in\Omega$, $\xi\not\equiv 0$.

$\underline{H_1}$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$ and

(i) $0\le f(z,x)\le a(z)[1+x^{r-1}]$ for a.a. $z\in\Omega$, all $x\ge 0$, with $a\in L^\infty(\Omega)$, $p<r<p^*$;

(ii) if $F(z,x)=\int_0^x f(z,s)ds$, then

$$\lim_{x\to+\infty}\frac{F(z,x)}{x^p}=+\infty \quad \text{uniformly for a.a. } z\in\Omega;$$

(iii) there exists $\mu\in\left((r-p)\max\left\{1,\frac{N}{p}\right\}, p^*\right)$ with $\mu>\tau$ such that

$$0<c_0\le\liminf_{x\to+\infty}\frac{f(z,x)x-pF(z,x)}{x^\mu} \text{ uniformly for a.a. } z\in\Omega;$$

(iv) $\lim_{x\to 0^+}\frac{f(z,x)}{x^{q-1}}=0$ uniformly for a.a. $z\in\Omega$;

(v) for every $\rho>0$, there exists $\widehat{\xi}_\rho>0$ such that for a.a. $z\in\Omega$, the function $x\to f(z,x)+\widehat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 1. Since we look for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+=[0,+\infty)$, without any loss of generality we may assume that

$$f(z,x)=0 \quad \text{for a.a. } z\in\Omega,\ \text{all } x\le 0. \tag{1}$$

Hypotheses H_1 (ii), (iii) imply that

$$\lim_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}=+\infty \quad \text{uniformly for a.a. } z\in\Omega.$$

Therefore for a.a. $z\in\Omega$ the perturbation $f(z,\cdot)$ is $(p-1)$-superlinear. In the literature most papers dealing with superlinear nonlinearities, use the AR-condition. We recall that this condition says that there exist $\theta_0>p$ and $M_0>0$ such that

$$0<\theta_0 F(z,x)\le f(z,x)x \quad \text{for a.a. } z\in\Omega, \tag{2a}$$

$$0 < \operatorname*{essinf}_{\Omega} F(\cdot, M_0). \tag{2b}$$

In fact this is a unilateral version of the AR-condition due to (1). Integrating (2a) and using (2b), we obtain the following weaker condition

$$\begin{aligned} & c_1 x^{\theta_0} \leq F(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_0, \text{ some } c_1 > 0, \\ \Rightarrow \quad & c_1 x^{\theta_0 - 1} \leq f(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_0. \end{aligned}$$

Therefore the AR-condition dictates that $f(z,\cdot)$ eventually has at least $(\theta_0 - 1)$-polynomial growth. In the present work, we replace the AR-condition by hypotheses $H_1\,(ii)$, (iii). This way we incorporate in our framework also superlinear nonlinearities with "slower" growth near $+\infty$, which fail to satisfy the AR-condition. Consider the following two functions (for the sake of simplicity we drop the z-dependence)

$$f_1(x) = (x^+)^{r-1} \text{ with } p < r < p^* \quad \text{and} \quad f_2(x) = \begin{cases} (x^+)^{s-1} & \text{if } x \leq 1 \\ x^{p-1} \ln x + x^{\theta - 1} & \text{if } 1 < x \end{cases}, q < s,\ 1 \leq \theta < p.$$

Both functions satisfy hypothesis H_1, but only $f_1(\cdot)$ satisfies the AR-condition.

In what follows by $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ we denote the C^1-functional defined by

$$\gamma_p(u) = \|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Hypothesis H_0 and Lemma 4.11 of Mugnai-Papageorgiou [13] imply that

$$c_2 \|u\|^p \leq \gamma_p(u) \quad \text{for some } c_2 > 0, \text{ all } u \in W^{1,p}(\Omega). \tag{3}$$

Also let $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{1-\eta} + \int_\Omega (u^+)^{1-\eta} dz - \int_\Omega F(z, u^+) dz - \frac{\lambda}{\tau}\int_{\partial\Omega} (u^+)^\tau d\sigma$$
$$\text{for all } u \in W^{1,p}(\Omega).$$

On account of the third term, this functional is not C^1 on $W^{1,p}(\Omega)$ and for this reason the standard variational tools from critical point theory are not readily available. We will use truncation techniques in order to neutralize the singularity. For this reason in the next section we focus on the purely singular problem. Finally we mention that as usual by a solution of (P_λ), we mean a function $u \in W^{1,p}(\Omega)$ such that $u \geq 0$, $u \neq 0$, $u^{-\eta} h \in L^1(\Omega)$ for all $h \in W^{1,p}(\Omega)$ and

$$\begin{aligned} & \langle A_p(u), h\rangle + \langle A_q(u), h\rangle + \int_\Omega \xi(z) u^{p-1} h dz \\ = & \int_\Omega [\lambda u^{-\eta} + f(z,u)] h dz + \int_{\partial\Omega} \lambda u^{\tau - 1} h d\sigma \quad \text{for all } u \in W^{1,p}(\Omega). \end{aligned}$$

3. A Purely Singular Problem

In this section we deal with the following parametric purely singular problem

$$(Q_\lambda) \qquad \begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{-\eta} \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} = \lambda u^{\tau-1} \text{ on } \partial\Omega, \ u > 0, \ \lambda > 0, \ 1 < \tau < q \leq 2 < p, \ 0 < \eta < 1. \end{cases}$$

Proposition 2. *If hypothesis H_0 holds, then for every $\lambda > 0$ problem (Q_λ) admits a unique solution $\overline{u}_\lambda \in \operatorname{int} C_+$.*

Proof. Let $\varepsilon > 0$. We consider the following approximation of problem (Q_λ)

$$(4) \qquad \begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda [u(z)+\varepsilon]^{-\eta} \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} = \lambda u^{\tau-1} \text{ on } \partial\Omega, \ u > 0, \ \lambda > 0, \ 1 < \tau < q \leq 2 < p, \ 0 < \eta < 1. \end{cases}$$

We solve (4) using a topological approach (fixed point theory). So, let $v \in L^p(\Omega)$ and consider the following boundary value problem

$$(5) \qquad \begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda [|v(z)|+\varepsilon]^{-\eta} \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} = \lambda u^{\tau-1} \text{ on } \partial\Omega, \ u > 0, \ \lambda > 0, \ 1 < \tau < q \leq 2 < p, \ 0 < \eta < 1. \end{cases}$$

Consider the continuous maps $K_p : L^p(\Omega) \to L^{p'}(\Omega) \left(\dfrac{1}{p} + \dfrac{1}{p'} = 1\right)$ and $K_\tau^b : W^{1,p}(\Omega) \to L^{\tau'}(\partial\Omega) \left(\dfrac{1}{\tau} + \dfrac{1}{\tau'} = 1\right)$ defined by

$$K_p(u)(\cdot) = |u(\cdot)|^{p-2}u(\cdot) \quad \text{for all } u \in L^p(\Omega),$$
$$K_\tau^b(u)(\cdot) = |\gamma_0(u)(\cdot)|^{\tau-2}\gamma_0(u)(\cdot) \quad \text{for all } u \in W^{1,p}(\Omega)$$

(recall that $\gamma_0(\cdot)$ denotes the trace map). Then for every $\lambda > 0$, we introduce the operator $V_\lambda : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$V_\lambda(u) = A_p(u) + A_q(u) + \xi K_p(u) - \lambda K_\tau^b(u).$$

The maximal monotonicity of the operators $A_p(\cdot)$, $A_q(\cdot)$, together with the Sobolev embedding theorem and the compactness of the trace map, imply that $V_\lambda(\cdot)$ is pseudomonotone (see Papageorgiou-Rădulescu-Repovš [18], p. 152). Also we have

$$\langle V_\lambda(u), u\rangle \geq c_3\|u\|^p - c_4\|u\|^\tau \quad \text{for some } c_3, c_4 > 0, \text{ all } u \in W^{1,p}(\Omega) \text{ (see (3))}.$$

Since $\tau < p$, it follows that $V_\lambda(\cdot)$ is strongly coercive. But we know that a pseudomonotone, strongly coercive map on a reflexive Banach space, is surjective (see Papageorgiou-Rădulescu-Repovš [18], Theorem 2.10.10, p. 156). So, $V_\lambda(\cdot)$ is surjective. Note that $[|v(\cdot)|+\varepsilon]^{-\eta} \in L^\infty(\Omega)$. So, we can find $\widetilde{u}_\varepsilon \in W^{1,p}(\Omega)$, $\widetilde{u}_\varepsilon \neq 0$ such that

$$V_\lambda(\widetilde{u}_\varepsilon) = \lambda[|v(\cdot)| + \varepsilon]^{-\eta},$$

$$\Rightarrow \quad \langle V_\lambda(\widetilde{u}_\varepsilon), h\rangle = \lambda \int_\Omega \frac{h}{[|v|+\varepsilon]^\eta} \quad \text{for all } h \in W^{1,p}(\Omega).$$

We choose $h = -\widetilde{u}_\varepsilon^- \in W^{1,p}(\Omega)$ and obtain

$$\begin{aligned} & c_2\|\widetilde{u}_\varepsilon^-\|^p \le 0 \quad \text{(see (3))}, \\ \Rightarrow \quad & \widetilde{u}_\varepsilon \ge 0, \quad \widetilde{u}_\varepsilon \ne 0. \end{aligned}$$

Therefore $\widetilde{u}_\varepsilon$ is a solution of problem (5). Then Proposition 2.10 of Papageorgiou-Rădulescu [14] implies that $\widetilde{u}_\varepsilon \in L^\infty(\Omega)$. Applying the nonlinear regularity theory of Lieberman [11], we get $\widetilde{u}_\varepsilon \in C_+ \setminus \{0\}$. Moreover, the nonlinear maximum principle of Pucci-Serrin [22] (pp. 111, 120), implies that $\widetilde{u}_\varepsilon \in \operatorname{int} C_+$.

Next we show the uniqueness of the positive solution.

Suppose that $\widetilde{v}_\varepsilon$ is another positive solution of (5). Again we have $\widetilde{v}_\varepsilon \in \operatorname{int} C_+$. We introduce the integral functional $j_\lambda : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j_\lambda(u) = \begin{cases} \frac{1}{p}\|\nabla u^{1/p}\|_p^p + \frac{1}{q}\|\nabla u^{1/q}\|_q^q + \frac{1}{p}\int_\Omega \xi(z) u^{p/q} dz - \frac{\lambda}{\tau}\int_{\partial\Omega} u^{\tau/q} d\sigma & \text{if } u \ge 0,\ u^{1/q} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\operatorname{dom} j_\lambda = \{u \in L^1(\Omega) : j_\lambda(u) < +\infty\}$ (the effective domain of $j_\lambda(\cdot)$) and consider $u_1, u_2 \in \operatorname{dom} j_\lambda$. We define $y = [tu_1 + (1-t)u_2]^{1/q}$ with $t \in [0,1]$. From Díaz-Saá [5] (see the proof of Lemma 1), we have

$$|\nabla y(z)| \le \left[t|\nabla u_1(z)^{1/q}|^q + (1-t)|\nabla u_2(z)^{1/q}|^q\right]^{1/q} \quad \text{for a.a. } z \in \Omega.$$

Consider $G_0(t) = \frac{1}{p}t^p + \frac{1}{q}t^q$ for all $t \ge 0$. Then $G_0(\cdot)$ is increasing and $t \to G_0(t^{1/q})$ is convex. Therefore we have

$$\begin{aligned} G_0(|\nabla y(z)|) &\le G_0\left(\left(t|\nabla u_1(z)^{1/q}|^q + (1-t)|\nabla u_2(z)^{1/q}|^q\right)^{1/q}\right) \\ &\le tG_0\left(|\nabla u_1(z)^{1/q}|\right) + (1-t)G_0\left(|\nabla u_2(z)^{1/q}|\right) \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow \quad & u \to \frac{1}{p}\|\nabla u^{1/q}\|_p^p + \frac{1}{q}\|\nabla u^{1/q}\|_q^q \text{ is convex on } \operatorname{dom} j_\lambda. \end{aligned}$$

Since $1 < \tau < q \le 2 < p$, it follows that

$$u \to \frac{1}{p}\int_\Omega \xi(z) u^{p/q} dz - \frac{\lambda}{\tau}\int_{\partial\Omega} u^{\tau/q} d\sigma \text{ is convex on } \operatorname{dom} j_\lambda.$$

So, we conclude that $j_\lambda(\cdot)$ is convex.

From Proposition 4.1.22, p. 274, of Papageorgiou-Rădulescu-Repovš [18], we have $\frac{\widetilde{u}_\varepsilon}{\widetilde{v}_\varepsilon}, \frac{\widetilde{v}_\varepsilon}{\widetilde{u}_\varepsilon} \in L^\infty(\Omega)$. Let $h = \widetilde{u}_\varepsilon^q - \widetilde{v}_\varepsilon^q$. Then for $t \in [0,1]$ we have

$$\widetilde{u}_\varepsilon^q - th \in \operatorname{dom} j_\lambda \quad \text{and} \quad \widetilde{v}_\varepsilon^q + th \in \operatorname{dom} j_\lambda.$$

So, $j_\lambda(\cdot)$ is Gâteaux differentiable at $\widetilde{u}_\varepsilon^q$ and at $\widetilde{v}_\varepsilon^q$ in the direction h. Using the nonlinear Green's identity (see [18], p. 35), we obtain that

$$\begin{aligned} j'_\lambda(\widetilde{u}_\varepsilon^q)(h) =& \frac{1}{q}\int_\Omega \frac{-\Delta_p \widetilde{u}_\varepsilon - \Delta_q \widetilde{u}_\varepsilon + \xi(z)\widetilde{u}_\varepsilon^{p-1}}{\widetilde{u}_\varepsilon^{q-1}} h dz, \\ j'_\lambda(\widetilde{v}_\varepsilon^q)(h) =& \frac{1}{q}\int_\Omega \frac{-\Delta_p \widetilde{v}_\varepsilon - \Delta_q \widetilde{v}_\varepsilon + \xi(z)\widetilde{v}_\varepsilon^{p-1}}{\widetilde{v}_\varepsilon^{q-1}} h dz. \end{aligned}$$

The convexity of $j_\lambda(\cdot)$ implies the monotonicity of $j'_\lambda(\cdot)$. Therefore we have

$$\begin{aligned} & 0 \le \int_\Omega \lambda\left[|v(z)|+\varepsilon\right]^{-\eta}\left[\frac{1}{\widetilde{u}_\varepsilon^{q-1}} - \frac{1}{\widetilde{v}_\varepsilon^{q-1}}\right](\widetilde{u}_\varepsilon^q - \widetilde{v}_\varepsilon^q)dz \le 0, \\ \Rightarrow\quad & \widetilde{u}_\varepsilon = \widetilde{v}_\varepsilon. \end{aligned}$$

This proves the uniqueness of the positive solution $\widetilde{u}_\varepsilon \in \operatorname{int} C_+$ of (5).

We can define the solution map $S_\varepsilon : L^p(\Omega) \to L^p(\Omega)$ for problem (5) by

$$S_\varepsilon(v) = \widetilde{u}_\varepsilon.$$

Then we have

$$\langle A_p(\widetilde{u}_\varepsilon), h\rangle + \langle A_q(\widetilde{u}_\varepsilon), h\rangle + \int_\Omega \xi(z)\widetilde{u}_\varepsilon^{p-1} h dz = \lambda\int_\Omega [|v(z)|+\varepsilon]^{-\eta} h dz + \lambda\int_{\partial\Omega} \widetilde{u}_\varepsilon^{\tau-1} h d\sigma$$

for all $h \in W^{1,p}(\Omega)$. We choose $h = \widetilde{u}_\varepsilon \in W^{1,p}(\Omega)$ and using (3) we obtain

$$c_2\|\widetilde{u}_\varepsilon\|^p \le \lambda c_5\left[\frac{1}{\xi^\eta}\|\widetilde{u}_\varepsilon\| + \|\widetilde{u}_\varepsilon\|^\tau\right] \quad \text{for some } c_5 > 0.$$

Since $1 < \tau < p$, it follows that

$$\|\widetilde{u}_\varepsilon\| = \|S_\varepsilon(v)\| \le c_6 \quad \text{for some } c_6 = c_6(\varepsilon) > 0, \text{ all } v \in L^p(\Omega). \tag{6}$$

We show that $S_\varepsilon(\cdot)$ is continuous. To this end, let $v_n \to v$ in $L^p(\Omega)$ and let $\widetilde{u}_{\varepsilon,n} = S_\varepsilon(v_n)$ for all $n \in \mathbb{N}$. From (6) we see that

$$\{\widetilde{u}_{\varepsilon,n}\}_{n\ge 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$\widetilde{u}_{\varepsilon,n} \xrightarrow{w} \widetilde{u}_\varepsilon^* \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad \widetilde{u}_{\varepsilon,n} \to \widetilde{u}_\varepsilon^* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{7}$$

We have

$$\langle A_p(\widetilde{u}_{\varepsilon,n}), h\rangle + \langle A_q(\widetilde{u}_{\varepsilon,n}), h\rangle + \int_\Omega \xi(z)\widetilde{u}_{\varepsilon,n}^{p-1} h dz = \int_\Omega \lambda[|v_n(z)|+\varepsilon]^{-\eta} h dz + \int_{\partial\Omega} \lambda\widetilde{u}_{\varepsilon,n}^{\tau-1} h d\sigma \tag{8}$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$. In (8) we choose $h = \widetilde{u}_{\varepsilon,n} - \widetilde{u}_\varepsilon^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (7). We obtain

$$\begin{aligned} & \lim_{n\to+\infty} \left[\langle A_p(\widetilde{u}_{\varepsilon,n}), \widetilde{u}_{\varepsilon,n} - \widetilde{u}_\varepsilon^*\rangle + \langle A_q(\widetilde{u}_{\varepsilon,n}), \widetilde{u}_{\varepsilon,n} - \widetilde{u}_\varepsilon^*\rangle\right] = 0, \\ \Rightarrow\quad & \limsup_{n\to+\infty} \left[\langle A_p(\widetilde{u}_{\varepsilon,n}), \widetilde{u}_{\varepsilon,n} - \widetilde{u}_\varepsilon^*\rangle + \langle A_q(\widetilde{u}_{\varepsilon,n}), \widetilde{u}_{\varepsilon,n} - \widetilde{u}_\varepsilon^*\rangle\right] \le 0, \\ & \qquad\qquad\qquad\qquad \text{(since } A_q(\cdot) \text{ is monotone),} \end{aligned}$$

$$\Rightarrow \quad \limsup_{n\to+\infty}\langle A_p(\widetilde{u}_{\varepsilon,n}), \widetilde{u}_{\varepsilon,n}-\widetilde{u}_\varepsilon^*\rangle \le 0,$$

$$\Rightarrow \quad \widetilde{u}_{\varepsilon,n}\to \widetilde{u}_\varepsilon^* \text{ in } W^{1,p}(\Omega) \quad \text{(see Proposition 1).} \tag{9}$$

Then, if in (8) we pass to the limit as $n\to+\infty$ and use (9), we obtain

$$\begin{aligned} &\langle A_p(\widetilde{u}_\varepsilon^*),h\rangle + \langle A_q(\widetilde{u}_\varepsilon^*),h\rangle + \int_\Omega \xi(z)(\widetilde{u}_\varepsilon^*)^{p-1}hdz \\ &= \int_\Omega \lambda[|v(z)|+\varepsilon]^{-\eta}hdz + \int_{\partial\Omega}\lambda(\widetilde{u}_\varepsilon^*)^{\tau-1}hd\sigma \quad \text{for all } h\in W^{1,p}(\Omega), \\ &\Rightarrow \quad \widetilde{u}_\varepsilon^* = S_\varepsilon(v), \\ &\Rightarrow \quad S_\varepsilon(\cdot) \text{ is continuous.} \end{aligned}$$

On account of (6) and of the compact embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$, we can apply the Schauder-Tychonov fixed point theorem (see Papageorgiou-Winkert [21], Theorem 6.8.5, p. 581) and find $\overline{u}_\varepsilon\in W^{1,p}(\Omega)$ such that

$$S_\varepsilon(\overline{u}_\varepsilon)=\overline{u}_\varepsilon.$$

Evidently this is a positive solution of (4) and then the nonlinear regularity theory and the nonlinear maximum principle imply that

$$\overline{u}_\varepsilon\in \operatorname{int} C_+.$$

Moreover, this positive solution $\overline{u}_\varepsilon\in\operatorname{int}C_+$ of problem (4) is unique. Indeed if $\overline{u}_\varepsilon,\overline{v}_\varepsilon\in W^{1,p}(\Omega)$ are two such solutions of (4), then

$$\begin{aligned} 0 \le &\langle A_p(\overline{u}_\varepsilon)-A_p(\overline{v}_\varepsilon),(\overline{u}_\varepsilon-\overline{v}_\varepsilon)^+\rangle + \langle A_q(\overline{u}_\varepsilon)-A_q(\overline{v}_\varepsilon),(\overline{u}_\varepsilon-\overline{v}_\varepsilon)^+\rangle \\ &+\int_\Omega \xi(z)[\overline{u}_\varepsilon^{p-1}-\overline{v}_\varepsilon^{p-1}](\overline{u}_\varepsilon-\overline{v}_\varepsilon)^+dz = \lambda\int_\Omega\left[\frac{1}{\overline{u}_\varepsilon^\eta}-\frac{1}{\overline{v}_\varepsilon^\eta}\right](\overline{u}_\varepsilon-\overline{v}_\varepsilon)^+dz\le 0 \\ \Rightarrow \quad & \overline{u}_\varepsilon\le\overline{v}_\varepsilon. \end{aligned}$$

Interchanging the roles of $\overline{u}_\varepsilon$ and $\overline{v}_\varepsilon$, we also have $\overline{v}_\varepsilon\le\overline{u}_\varepsilon$, hence $\overline{u}_\varepsilon=\overline{v}_\varepsilon$. Thus the map $\varepsilon\to\overline{u}_\varepsilon$ is well-defined.

Claim: The map $\varepsilon\to\overline{u}_\varepsilon$ from $(0,+\infty)$ into C_+ is nonincreasing, that is,

$$\varepsilon'<\varepsilon\Rightarrow \overline{u}_{\varepsilon'}-\overline{u}_\varepsilon\in C_+\setminus\{0\}.$$

We consider $0<\varepsilon'<\varepsilon$. We have

$$\begin{aligned} -\Delta_p\overline{u}_{\varepsilon'}-\Delta_q\overline{u}_{\varepsilon'}+\xi(z)\overline{u}_{\varepsilon'}^{p-1} &= \lambda[\overline{u}_{\varepsilon'}+\varepsilon']^{-\eta} \\ &\ge \lambda[\overline{u}_{\varepsilon'}+\varepsilon]^{-\eta} \quad \text{for a.a. } z\in\Omega. \end{aligned} \tag{10}$$

Let $\ell_\varepsilon(z,x)$ and $\beta(z,x)$ be the Carathéodory functions defined by

$$\ell_\varepsilon(z,x)=\begin{cases}[x^++\varepsilon]^{-\eta} & \text{if } x\le\overline{u}_{\varepsilon'}(z),\\ [\overline{u}_{\varepsilon'}(z)+\varepsilon]^{-\eta} & \text{if } \overline{u}_{\varepsilon'}(z)<x,\end{cases} \quad \text{for all } (z,x)\in\Omega\times\mathbb{R}, \tag{11}$$

$$\beta(z,x)=\begin{cases}(x^+)^{\tau-1} & \text{if } x\leq \overline{u}_{\varepsilon'}(z),\\ \overline{u}_{\varepsilon'}(z)^{\tau-1} & \text{if } \overline{u}_{\varepsilon'}(z)<x,\end{cases}\quad \text{for all } (z,x)\in\partial\Omega\times\mathbb{R}. \tag{12}$$

We set $L_\varepsilon(z,x)=\int_0^x \ell_\varepsilon(z,s)ds$ and $B(z,x)=\int_0^x\beta(z,s)ds$ and consider the C^1-functional $J_\varepsilon : W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$J_\varepsilon(u)=\frac{1}{p}\gamma_p(u)+\frac{1}{q}\|\nabla u\|_q^q-\lambda\int_\Omega L_\varepsilon(z,u)dz-\lambda\int_{\partial\Omega}B(z,u)d\sigma\quad\text{for all } u\in W^{1,p}(\Omega).$$

From (3), (11), (12) it is clear that $J_\varepsilon(\cdot)$ is coercive. Also the Sobolev embedding theorem and the compactness of the trace map, imply that $J_\varepsilon(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\widetilde{u}_\varepsilon\in W^{1,p}(\Omega)$ such that

$$J_\varepsilon(\widetilde{u}_\varepsilon)=\min\left[J_\varepsilon(u):\ u\in W^{1,p}(\Omega)\right]. \tag{13}$$

Since $\tau<q<p$, if $u\in \operatorname{int} C_+$, for $t\in(0,1)$ small we will have

$$\begin{aligned}&J_\varepsilon(tu)<0,\\ \Rightarrow\quad &J_\varepsilon(\widetilde{u}_\varepsilon)<0=J_\varepsilon(0)\quad\text{(see (13))},\\ \Rightarrow\quad &\widetilde{u}_\varepsilon\neq 0.\end{aligned}$$

From (13) we have

$$\begin{aligned}&J'_\varepsilon(\widetilde{u}_\varepsilon)=0,\\ \Rightarrow\quad &\langle A_p(\widetilde{u}_\varepsilon),h\rangle+\langle A_q(\widetilde{u}_\varepsilon),h\rangle+\int_\Omega\xi(z)|\widetilde{u}_\varepsilon|^{p-2}\widetilde{u}_\varepsilon h dz\\ &=\lambda\int_\Omega \ell_\varepsilon(z,\widetilde{u}_\varepsilon)hdz+\lambda\int_{\partial\Omega}\beta(z,\widetilde{u}_\varepsilon)hd\sigma\quad\text{for all } h\in W^{1,p}(\Omega).\end{aligned} \tag{14}$$

In (14) first we choose $h=-\widetilde{u}_\varepsilon^-\in W^{1,p}(\Omega)$. Then

$$\begin{aligned}&c_2\|\widetilde{u}_\varepsilon^-\|^p\leq 0,\quad\text{(see (3), (11),(12))},\\ \Rightarrow\quad &\widetilde{u}_\varepsilon\geq 0,\ \widetilde{u}_\varepsilon\neq 0.\end{aligned}$$

Next in (14) we choose $h=(\widetilde{u}_\varepsilon-\overline{u}_{\varepsilon'})^+\in W^{1,p}(\Omega)$. We have

$$\begin{aligned}&\langle A_p(\widetilde{u}_\varepsilon),(\widetilde{u}_\varepsilon-\overline{u}_{\varepsilon'})^+\rangle+\langle A_q(\widetilde{u}_\varepsilon),(\widetilde{u}_\varepsilon-\overline{u}_{\varepsilon'})^+\rangle+\int_\Omega\xi(z)\widetilde{u}_\varepsilon^{p-1}(\widetilde{u}_\varepsilon-\overline{u}_{\varepsilon'})^+dz\\ =&\lambda\int_\Omega[\overline{u}_{\varepsilon'}+\varepsilon]^{-\eta}(\widetilde{u}_\varepsilon-\overline{u}_{\varepsilon'})^+dz+\lambda\int_{\partial\Omega}\overline{u}_{\varepsilon'}^{\tau-1}(\widetilde{u}_\varepsilon-\overline{u}_{\varepsilon'})^+d\sigma\quad\text{(see (11), (12))}\\ \leq&\langle A_p(\overline{u}_{\varepsilon'}),(\widetilde{u}_\varepsilon-\overline{u}_{\varepsilon'})^+\rangle+\langle A_q(\overline{u}_{\varepsilon'}),(\widetilde{u}_\varepsilon-\overline{u}_{\varepsilon'})^+\rangle+\int_\Omega\xi(z)\overline{u}_{\varepsilon'}^{p-1}(\widetilde{u}_\varepsilon-\overline{u}_{\varepsilon'})^+dz\quad\text{(see (10))},\\ \Rightarrow\quad&\widetilde{u}_\varepsilon\leq\overline{u}_{\varepsilon'}.\end{aligned}$$

So, we have proved that

$$\widetilde{u}_\varepsilon\in[0,\overline{u}_{\varepsilon'}],\ \widetilde{u}_\varepsilon\neq 0. \tag{15}$$

Then from (15), (11), (12) and (14), it follows that

$$\begin{aligned} & \widetilde{u}_\varepsilon = \overline{u}_\varepsilon, \\ \Rightarrow \quad & \overline{u}_\varepsilon \le \overline{u}_{\varepsilon'}. \end{aligned}$$

This proves the Claim.

Now let $\varepsilon_n \downarrow 0$ and let $\overline{u}_n = \overline{u}_{\varepsilon_n} \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} & \langle A_p(\overline{u}_n), h\rangle + \langle A_q(\overline{u}_n), h\rangle + \int_\Omega \xi(z)\overline{u}_n^{p-1} h dz \\ & = \lambda \int_\Omega [\overline{u}_n + \varepsilon_n]^{-\eta} h dz + \lambda \int_{\partial\Omega} \overline{u}_n^{\tau-1} h d\sigma \quad \text{for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{16}$$

In (16) we choose $h = \overline{u}_n \in W^{1,p}(\Omega)$ and obtain

$$c_2 \|\overline{u}_n\|^p \le \lambda \int_\Omega \frac{\overline{u}_n}{\overline{u}_1^\eta} dz + \lambda \int_{\partial\Omega} \overline{u}_n^\tau d\sigma \quad \text{for all } n \in \mathbb{N} \tag{17}$$

(see (3) and note that on account of the Claim we have $\overline{u}_1 \le \overline{u}_n$ for all $n \in \mathbb{N}$). Since $\overline{u}_1 \in \operatorname{int} C_+$, we have $0 < \overline{m}_1 = \min_{\overline{\Omega}} \overline{u}_1$. Then from (17) it follows that

$$\begin{aligned} & \|\overline{u}_n\|^p \le c_7[\|\overline{u}_n\| + \|\overline{u}_n\|^\tau] \quad \text{for some } c_7 > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow \quad & \{\overline{u}_n\}_{n\ge 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \end{aligned}$$

We may assume that

$$\overline{u}_n \xrightarrow{w} \overline{u}_\lambda \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad \overline{u}_n \to \overline{u}_\lambda \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{18}$$

On account of the Claim $\overline{u}_\lambda \ne 0$. In (16) we choose $h = \overline{u}_n - \overline{u}_\lambda \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (18). As before, we obtain

$$\limsup_{n\to+\infty} \langle A_p(\overline{u}_n), \overline{u}_n - \overline{u}_\lambda\rangle \le 0,$$

$$\Rightarrow \quad \overline{u}_n \to \overline{u}_\lambda \text{ in } W^{1,p}(\Omega) \quad \text{(see Proposition 1).} \tag{19}$$

Passing to the limit as $n \to +\infty$ in (16) and using (19), we obtain that $\overline{u}_\lambda$ is a solution of (Q_λ). Note that $\overline{u}_1 \le \overline{u}_\lambda$, hence $0 \le \overline{u}_\lambda^{-\eta} \le \overline{u}_1^{-\eta} \in L^\infty(\Omega)$. Thus the nonlinear regularity theory of Lieberman [11] implies that $\overline{u}_\lambda \in \operatorname{int} C_+$. As before (see the argument before the Claim), we show the uniqueness of $\overline{u}_\lambda \in \operatorname{int} C_+$. □

4. Positive Solutions

We introduce the following two sets

$$\begin{aligned} \mathcal{L} &= \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a positive solution}\}, \\ S_\lambda &= \text{ set of positive solutions of } (P_\lambda). \end{aligned}$$

The next proposition establishes the nonemptiness of $\mathcal{L}$ and then we will determine the regularity of the solution set S_λ.

Proposition 3. *If hypotheses H_0, H_1 hold, then $\mathcal{L} \ne \emptyset$.*

Proof. For $\lambda > 0$, let $\overline{u}_\lambda \in \operatorname{int} C_+$ be the unique positive solution of (Q_λ) produced in Proposition 2. We introduce the Carathéodory functions $g_\lambda(z,x), b_\lambda(z,x)$ defined by

$$g_\lambda(z,x) = \begin{cases} \lambda \overline{u}_\lambda(z)^{-\eta} + f(z,x^+) & \text{if } x \le \overline{u}_\lambda(z), \\ \lambda x^{-\eta} + f(z,x) & \text{if } \overline{u}_\lambda(z) < x, \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}, \tag{20}$$

$$b_\lambda(z,x) = \begin{cases} \lambda \overline{u}_\lambda(z)^{\tau-1} & \text{if } x \le \overline{u}_\lambda(z), \\ \lambda x^{\tau-1} & \text{if } \overline{u}_\lambda(z) < x, \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{21}$$

We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)ds,\ B_\lambda(z,x) = \int_0^x b_\lambda(z,s)ds$ and consider the C^1-functional $\widehat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega G_\lambda(z,u)dz - \int_{\partial\Omega} B_\lambda(z,u)d\sigma \quad \text{for all } u \in W^{1,p}(\Omega),$$

(note that $\overline{u}_\lambda^{-\eta} \in L^\infty(\Omega)$ since $\overline{u}_\lambda \in \operatorname{int} C_+$).

On account of hypotheses $H_1(i), (iv)$, given $\varepsilon > 0$, we can find $c_8 = c_8(\varepsilon) > 0$ such that

$$F(z,x) \le \varepsilon x^q + c_8 x^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{22}$$

Using (3), (22), (20) and (21), we see that for some $c_9,\ c_{10} > 0$ we have

$$\begin{aligned} \widehat{\varphi}_\lambda(u) &\ge \frac{c_2}{p}\|u\|^p - c_9[\varepsilon\|u\|^q + \|u\|^r + \lambda\|u\|^\tau] - \lambda \int_{\{u \le \overline{u}_\lambda\}} \overline{u}_\lambda^{-\eta} u dz \\ &\quad - \frac{\lambda}{1-\eta}\int_{\{u > \overline{u}_\lambda\}} [u^{1-\eta} - \overline{u}_\lambda^{1-\eta}]dz - \lambda c_{10} \\ &\ge \frac{c_2}{p}\|u\|^p - c_9[\varepsilon\|u\|^q + \|u\|^r + \lambda\|u\|^\tau] - \frac{\lambda}{1-\eta}\int_\Omega u^{1-\eta}dz - \lambda c_{10}. \end{aligned}$$

Then for $\|u\| \le 1$ and since $1-\eta < 1 < \tau < q$ we have

$$\widehat{\varphi}_\lambda(u) \ge \frac{c_2}{p}\|u\|^p - c_9\|u\|^r - c_9[\varepsilon + \lambda]\|u\|^{1-\eta} \tag{23}$$

(using the fact that $L^s(\Omega) \hookrightarrow L^\vartheta(\Omega)$ for all $0 < \vartheta < s < \infty$).

Choose $\rho \in (0,1)$ small such that

$$\frac{c_2}{p}\rho^p - c_9\rho^r \ge \eta_1 > 0 \quad (\text{recall that } r > p). \tag{24}$$

Having chosen $\rho \in (0,1)$ this way, we then choose $\varepsilon_0 > 0$ and $\lambda_0 > 0$ such that

$$c_9[\varepsilon + \lambda]\rho^{1-\eta} + \lambda c_{10} \le \frac{\eta_1}{2} \quad \text{for all } \varepsilon \in (0,\varepsilon_0], \text{ all } \lambda \in (0,\lambda_0]. \tag{25}$$

Then using (24) and (25) in (23), we conclude that

$$\widehat{\varphi}_\lambda(u) \ge \frac{\eta_1}{2} \quad \text{for all } \|u\| = \rho, \text{ all } \lambda \in (0,\lambda_0]. \tag{26}$$

Let $B_\rho = \{u \in W^{1,p}(\Omega) : \|u\| < \rho\}$. Clearly $\widehat{\varphi}_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. Also by the Alaoglu and Eberlein-Smulian theorems (see Papageorgiou-Winkert [21], pp. 215, 221), we have that $\overline{B}_\rho$ is sequentially weakly compact. So, by the Weierstrass-Tonelli theorem, for every $\lambda \in (0, \lambda_0]$, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\lambda(u_\lambda) = \min[\widehat{\varphi}_\lambda(u) : u \in \overline{B}_\rho]. \tag{27}$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0, 1)$ small so that $tu \leq \overline{u}_\lambda$. Then using hypothesis $H_1(i)$ and (21), we have

$$\begin{aligned} \widehat{\varphi}_\lambda(tu) &\leq \frac{t^p}{p}\gamma_p(u) + \frac{t^q}{q}\|\nabla u\|_q^q - \lambda t \int_\Omega \overline{u}_\lambda^{\tau-1} u dz \\ &\leq c_{11} t^q - \lambda t^\tau \|u\|_{L^\tau(\partial\Omega)}^\tau \quad (\text{since } q < p,\ 0 < t < 1 \text{ and } tu \leq \overline{u}_\lambda). \end{aligned}$$

Since $\tau < q$ choosing, $t \in (0, 1)$ even small if necessary, we obtain

$$\begin{aligned} & \widehat{\varphi}_\lambda(tu) < 0, \\ \Rightarrow\ & \widehat{\varphi}_\lambda(u_\lambda) < 0 = \widehat{\varphi}_\lambda(0), \\ \Rightarrow\ & u_\lambda \neq 0. \end{aligned} \tag{28}$$

From (28) and (26) we see that $u_\lambda \in B_\rho \setminus \{0\}$ and so

$$\begin{aligned} & \widehat{\varphi}'_\lambda(u_\lambda) = 0 \quad (\text{see } (27)) \\ \Rightarrow\ & \langle A_p(u_\lambda), h\rangle + \langle A_q(u_\lambda), h\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h dz \\ & = \int_\Omega g_\lambda(z, u_\lambda) h dz + \int_{\partial\Omega} b_\lambda(z, u_\lambda) h d\sigma \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{29}$$

In (29), we choose $h = (\overline{u}_\lambda - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A_p(u_\lambda), (\overline{u}_\lambda - u_\lambda)^+\rangle + \langle A_q(u_\lambda), (\overline{u}_\lambda - u_\lambda)^+\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda(\overline{u}_\lambda - u_\lambda)^+ dz \\ & = \int_\Omega \left[\lambda \overline{u}_\lambda^{-\eta} + f(z, u_\lambda)\right](\overline{u}_\lambda - u_\lambda)^+ dz + \lambda \int_{\partial\Omega} \overline{u}_\lambda^{\tau-1}(\overline{u}_\lambda - u_\lambda)^+ d\sigma \quad (\text{see } (20), (21)) \\ & \geq \int_\Omega \lambda \overline{u}_\lambda^{-\eta}(\overline{u}_\lambda - u_\lambda)^+ dz + \lambda \int_{\partial\Omega} \overline{u}_\lambda^{\tau-1}(\overline{u}_\lambda - u_\lambda)^+ d\sigma \quad (\text{see hypothesis } H_1(i)) \\ & = \langle A_p(\overline{u}_\lambda), (\overline{u}_\lambda - u_\lambda)^+\rangle + \langle A_q(\overline{u}_\lambda), (\overline{u}_\lambda - u_\lambda)^+\rangle + \int_\Omega \xi(z)\overline{u}_\lambda^{p-1}(\overline{u}_\lambda - u_\lambda)^+ d\sigma \ (\text{see Proposition 2}), \end{aligned}$$

$\Rightarrow \overline{u}_\lambda \leq u_\lambda$.

From (20), (21) and (29) we conclude that $u_\lambda \in S_\lambda$, that is, $(0, \lambda_0] \subseteq \mathcal{L}$ and so $\mathcal{L} \neq \emptyset$. ☐

Proposition 4. *If hypotheses H_0, H_1 hold and $\lambda \in \mathcal{L}$, then $\overline{u}_\lambda \leq u$ for all $u \in S_\lambda$.*

Proof. Let $u \in S_\lambda$. Then on $\Omega \times (0, +\infty)$ and on $\partial\Omega \times \mathbb{R}$ respectively, we introduce the following Carathéodory functions:

$$\widehat{e}(z, x) = \begin{cases} x^{-\eta} & \text{if } 0 < x \leq u(z), \\ u(z)^{-\eta} & \text{if } u(z) < x \end{cases} \quad \text{and} \quad \widehat{b}(z, x) = \begin{cases} (x^+)^{\tau-1} & \text{if } x \leq u(z), \\ u(z)^{\tau-1} & \text{if } u(z) < x. \end{cases} \tag{30}$$

We consider the following purely singular problem

$$(Q'_\lambda) \qquad \begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda\widehat{e}(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} = \lambda\widehat{b}(z,u) \text{ on } \partial\Omega, \ u > 0, \ \lambda > 0. \end{cases}$$

Using a fixed point argument as in the proof of Proposition 2 we infer that for every $\lambda > 0$ problem (Q'_λ) admits a solution $\widetilde{u}_\lambda \in \operatorname{int} C_+$. Then we have

$$\begin{aligned} &\langle A_p(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - u)^+\rangle + \langle A_q(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - u)^+\rangle + \int_\Omega \xi(z)\widetilde{u}_\lambda^{p-1}(\widetilde{u}_\lambda - u)^+ dz \\ &= \lambda \int_\Omega u^{-\eta}(\widetilde{u}_\lambda - u)^+ dz + \lambda \int_{\partial\Omega} u^{\tau-1}(\widetilde{u}_\lambda - u)^+ d\sigma \quad \text{(see (30))} \\ &\leq \int_\Omega [\lambda u^{-\eta} + f(z,u)](\widetilde{u}_\lambda - u)^+ dz + \int_{\partial\Omega} \lambda u^{\tau-1}(\widetilde{u}_\lambda - u)^+ d\sigma \quad \text{(see hypothesis } H_1\,(i)), \\ &= \langle A_p(u), (\widetilde{u}_\lambda - u)^+\rangle + \langle A_q(u), (\widetilde{u}_\lambda - u)^+\rangle + \int_\Omega \xi(z)u^{p-1}(\widetilde{u}_\lambda - u)^+ dz \ \text{(since } u \in S_\lambda), \\ \Rightarrow \quad &\widetilde{u}_\lambda \leq u. \end{aligned}$$

So, we have

$$\widetilde{u}_\lambda \in [0,u], \quad \widetilde{u}_\lambda \neq 0. \tag{31}$$

From (31) and (30) it follows that

$$\begin{aligned} &\widetilde{u}_\lambda = \overline{u}_\lambda \quad \text{(see Proposition 2)}, \\ \Rightarrow \quad &\overline{u}_\lambda \leq u \quad \text{for all } u \in S_\lambda,\, \lambda \in \mathcal{L}. \end{aligned}$$

□

On account of this proposition, we see that for every $\lambda \in \mathcal{L}$ and $u \in S_\lambda$, we have that $u^{-\eta} \in L^\infty(\Omega)$. From Proposition 2.10 of Papageorgiou-Rădulescu [4], it follows that $u \in L^\infty(\Omega)$ and then the regularity theory of Lieberman [11] implies that $u \in C_+ \setminus \{0\}$. Finally the nonlinear maximum principle of Pucci-Serrin [22] (pp. 111, 120) implies that $u \in \operatorname{int} C_+$. So, we can state the following regularity result for the solution set S_λ.

Proposition 5. *If hypotheses H_0, H_1 hold and $\lambda \in \mathcal{L}$, then $S_\lambda \subseteq \operatorname{int} C_+$.*

Next we will determine the structure of the set $\mathcal{L}$. We will show that $\mathcal{L}$ is an interval. To this end first we prove a monotonicity property of the map $\lambda \to \overline{u}_\lambda$ from $(0,+\infty)$ into $C^1(\overline{\Omega})$.

Proposition 6. *If hypothesis H_0 holds and $0 < \vartheta < \lambda$, then $\overline{u}_\vartheta \leq \overline{u}_\lambda$.*

Proof. We introduce the following Carathéodory functions:

$$k(z,x) = \begin{cases} x^{-\eta} & \text{if } 0 < x \leq \overline{u}_\lambda(z), \\ \overline{u}_\lambda(z)^{-\eta} & \text{if } \overline{u}_\lambda(z) < x, \end{cases} \quad \text{for all } (z,x) \in \Omega \times (0,+\infty), \tag{32}$$

$$d(z,x)=\begin{cases}(x^+)^{\tau-1} & \text{if } x\le \overline{u}_\lambda(z),\\ \overline{u}_\lambda(z)^{\tau-1} & \text{if } \overline{u}_\lambda(z)<x,\end{cases}\quad \text{for all } (z,x)\in\partial\Omega\times\mathbb{R}. \tag{33}$$

We consider the following boundary value problem

$$\begin{cases}-\Delta_p u(z)-\Delta_q u(z)+\xi(z)u(z)^{p-1}=\vartheta k(z,u(z)) \quad \text{in } \Omega,\\ \dfrac{\partial u}{\partial n_{pq}}=\vartheta u^{\tau-1} \text{ on } \partial\Omega,\ u>0.\end{cases} \tag{34}$$

Problem (34) has a solution $\widetilde{u}_\vartheta\in \operatorname{int} C_+$ with $\widetilde{u}_\vartheta\in[0,\overline{u}_\vartheta]$ (see the proof of Proposition 3). Then from (32), (33) and Proposition 2 it follows that

$$\begin{aligned}&\widetilde{u}_\vartheta=\overline{u}_\vartheta,\\ \Rightarrow\quad &\overline{u}_\vartheta\le\overline{u}_\lambda.\end{aligned}$$

□

Using this monotonicity property, we can show that $\mathcal{L}$ is an interval.

Proposition 7. *If hypotheses H_0, H_1 hold, $\lambda\in\mathcal{L}$ and $0<\vartheta<\lambda$, then $\vartheta\in\mathcal{L}$.*

Proof. From Proposition 6, we have that $\overline{u}_\vartheta\le\overline{u}_\lambda$. Since $\lambda\in\mathcal{L}$ we can find $u\in S_\lambda\subseteq \operatorname{int} C_+$. From Proposition 4 we know that $\overline{u}_\lambda\le u$, hence $\overline{u}_\vartheta\le u$. Therefore we can define the following truncations of the data of problem (34):

$$\widehat{f}(z,x)=\begin{cases}f(z,\overline{u}_\vartheta(z)) & \text{if } x<\overline{u}_\vartheta(z),\\ f(z,x) & \text{if } \overline{u}_\vartheta(z)\le x\le u(z),\\ f(z,u(z)) & \text{if } u(z)<x,\end{cases}\quad \text{for all } (z,x)\in\Omega\times\mathbb{R}, \tag{35}$$

$$\widehat{k}_\vartheta(z,x)=\begin{cases}\vartheta\overline{u}_\vartheta(z)^{-\eta} & \text{if } x<\overline{u}_\vartheta(z),\\ \vartheta x^{-\eta} & \text{if } \overline{u}_\vartheta(z)\le x\le u(z),\\ \vartheta u(z)^{-\eta} & \text{if } u(z)<x,\end{cases}\quad \text{for all } (z,x)\in\Omega\times\mathbb{R}, \tag{36}$$

$$\widehat{d}_\vartheta(z,x)=\begin{cases}\vartheta\overline{u}_\vartheta(z)^{\tau-1} & \text{if } x<\overline{u}_\vartheta(z),\\ \vartheta x^{\tau-1} & \text{if } \overline{u}_\vartheta(z)\le x\le u(z),\\ \vartheta u(z)^{\tau-1} & \text{if } u(z)<x,\end{cases}\quad \text{for all } (z,x)\in\partial\Omega\times\mathbb{R}, \tag{37}$$

All three are Carathéodory functions. We set

$$\widehat{F}(z,x)=\int_0^x\widehat{f}(z,s)ds,\quad \widehat{K}_\vartheta(z,x)=\int_0^x\widehat{k}_\vartheta(z,s)ds,\quad \widehat{D}_\vartheta(z,x)=\int_0^x\widehat{d}_\vartheta(z,s)ds.$$

We consider the C^1-functional $\widehat{\mathcal{J}}_\vartheta: W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\mathcal{J}}_\vartheta(u)=\frac{1}{p}\gamma_p(u)+\frac{1}{q}\|\nabla u\|_q^q-\int_\Omega\widehat{F}(z,u)dz-\int_\Omega\widehat{K}_\vartheta(z,u)dz-\int_{\partial\Omega}\widehat{D}_\vartheta(z,u)d\sigma\quad \text{for all } u\in W^{1,p}(\Omega).$$

From (3), (35), (36), (37) it follows that $\widehat{\mathcal{J}}(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
& \widehat{\mathcal{J}}_\vartheta(\widehat{u}) = \min\left[\widehat{\mathcal{J}}_\vartheta(u) \,:\, u \in W^{1,p}(\Omega)\right], \\
\Rightarrow \quad & \widehat{\mathcal{J}}'_\vartheta(\widehat{u}) = 0, \\
\Rightarrow \quad & \langle A_p(\widehat{u}), h\rangle + \langle A_q(\widehat{u}), h\rangle + \int_\Omega \xi(z)|\widehat{u}|^{p-2}\widehat{u} h dz \\
& = \int_\Omega \left[\widehat{k}_\vartheta(z,\widehat{u}) + \widehat{f}(z,\widehat{u})\right] h dz + \int_{\partial\Omega} \widehat{d}_\vartheta(z,\widehat{u}) h d\sigma \quad \text{for all } h \in W^{1,p}(\Omega).
\end{aligned} \tag{38}$$

In (38), first we choose $h = (\overline{u}_\vartheta - \widehat{u})^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}
& \langle A_p(\widehat{u}), (\overline{u}_\vartheta - \widehat{u})^+\rangle + \langle A_q(\widehat{u}), (\overline{u}_\vartheta - \widehat{u})^+\rangle + \int_\Omega \xi(z)|\widehat{u}|^{p-2}\widehat{u}(\overline{u}_\vartheta - \widehat{u})^+ dz \\
& = \int_\Omega \left[\vartheta \overline{u}_\vartheta^{-\eta} + f(z,\overline{u}_\vartheta)\right](\overline{u}_\vartheta - \widehat{u})^+ dz + \int_{\partial\Omega} \vartheta \overline{u}_\vartheta^{\tau-1}(\overline{u}_\vartheta - \widehat{u})^+ d\sigma \quad \text{(see (35), (36), (37))} \\
& \geq \int_\Omega \vartheta \overline{u}_\vartheta^{-\eta}(\overline{u}_\vartheta - \widehat{u})^+ dz + \int_{\partial\Omega} \vartheta \overline{u}_\vartheta^{\tau-1}(\overline{u}_\vartheta - \widehat{u})^+ d\sigma \quad \text{(see hypothesis } H_1\,(i)) \\
& = \langle A_p(\overline{u}_\vartheta), (\overline{u}_\vartheta - \widehat{u})^+\rangle + \langle A_q(\overline{u}_\vartheta), (\overline{u}_\vartheta - \widehat{u})^+\rangle + \int_\Omega \xi(z)\overline{u}_\vartheta^{p-1}(\overline{u}_\vartheta - \widehat{u})^+ dz, \\
\Rightarrow \quad & \overline{u}_\vartheta \leq \widehat{u}.
\end{aligned}$$

Next in (38) we choose $h = (\widehat{u} - u)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
& \langle A_p(\widehat{u}), (\widehat{u} - u)^+\rangle + \langle A_q(\widehat{u}), (\widehat{u} - u)^+\rangle + \int_\Omega \xi(z) u^{p-1}(\widehat{u} - u)^+ dz \\
& = \int_\Omega \left[\vartheta u^{-\eta} + f(z,u)\right](\widehat{u} - u)^+ dz + \int_{\partial\Omega} \vartheta u^{\tau-1}(\widehat{u} - u)^+ d\sigma \quad \text{(see (35), (36), (37))} \\
& \leq \int_\Omega [\lambda u^{-\eta} + f(z,u)](\widehat{u} - u)^+ dz + \int_{\partial\Omega} \lambda u^{\tau-1}(\widehat{u} - u)^+ d\sigma \quad (\text{since } \vartheta < \lambda) \\
& = \langle A_p(u), (\widehat{u} - u)^+\rangle + \langle A_q(u), (\widehat{u} - u)^+\rangle + \int_\Omega \xi(z) u^{p-1}(\widehat{u} - u)^+ dz \quad (\text{since } u \in S_\lambda), \\
\Rightarrow \quad & \widehat{u} \leq u.
\end{aligned}$$

So, we have proved that

$$\widehat{u} \in [\overline{u}_\vartheta, u]. \tag{39}$$

From (39), (35), (36), (37) and (38), we infer that

$$\begin{aligned}
& \widehat{u} \in S_\vartheta \subseteq \operatorname{int} C_+, \\
\Rightarrow \quad & \vartheta \in \mathcal{L}.
\end{aligned}$$

∎

The solution multifunction $\lambda \to S_\lambda$ has the following strict monotonicity type property.

Proposition 8. *If hypotheses H_0, H_1 hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ and $0 < \vartheta < \lambda$, then $\vartheta \in \mathcal{L}$ and there exists $u_\vartheta \in S_\vartheta \subseteq \operatorname{int} C_+$ such that $u_\lambda - u_\vartheta \in D_+$.*

Proof. From Proposition 7 and its proof, we already know that $\vartheta \in \mathcal{L}$ and that there exists $u_\vartheta \in S_\vartheta \subseteq \operatorname{int} C_+$ such that

$$u_\vartheta \leq u_\lambda. \tag{40}$$

Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be the map defined by

$$a(y) = |y|^{p-2}y + |y|^{q-2}y \quad \text{for all } y \in \mathbb{R}^N.$$

Note that $a \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ and for every $y \in \mathbb{R}^N \setminus \{0\}$ we have

$$\nabla a(y) = |y|^{p-2}\left[I + (p-2)\frac{y \otimes y}{|y|^2}\right] + |y|^{q-2}\left[I + (q-2)\frac{y \otimes y}{|y|^2}\right].$$

Then for all $\xi' \in \mathbb{R}^N$, we have

$$(\nabla a(y)\xi', \xi')_{\mathbb{R}^N} \geq \frac{q-1}{|y|^{2-q}}|\xi'|^2. \tag{41}$$

We see that

$$\operatorname{div} a(\nabla u) = \Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (41) and since $u_\lambda \in \operatorname{int} C_+$, we see that $\nabla a(u_\lambda)$ is positive definite. Then the tangency principle of Pucci-Serrin [22] (Theorem 2.5.2, p. 35) implies that

$$u_\vartheta(z) < u_\lambda(z) \quad \text{for all } z \in \Omega \text{ (see (40))}. \tag{42}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H_1\,(v)$. For $\widetilde{\xi}_\rho > \widehat{\xi}_\rho$, we have

$$\begin{aligned}
& -\Delta_p u_\vartheta - \Delta_q u_\vartheta + [\xi(z) + \widetilde{\xi}_\rho]u_\vartheta^{p-1} - \lambda u_\vartheta^{-\eta} \\
& \leq f(z, u_\vartheta) + \widetilde{\xi}_\rho u_\vartheta^{p-1} \quad \text{(since } \vartheta < \lambda) \\
& = f(z, u_\vartheta) + \widehat{\xi}_\rho u_\vartheta^{p-1} + [\widetilde{\xi}_\rho - \widehat{\xi}_\rho]u_\vartheta^{p-1} \\
& \leq f(z, u_\lambda) + \widehat{\xi}_\rho u_\lambda^{p-1} + [\widetilde{\xi}_\rho - \widehat{\xi}_\rho]u_\lambda^{p-1} \quad \text{(see hypothesis } H_1\,(v) \text{ and (40))} \\
& = -\Delta_p u_\lambda - \Delta_q u_\lambda + \xi(z)u_\lambda^{p-1} - \lambda u_\lambda^{-\eta} \quad \text{(since } u_\lambda \in S_\lambda).
\end{aligned} \tag{43}$$

On account of (42), we have

$$[\widetilde{\xi}_\rho - \widehat{\xi}_\rho]u_\vartheta^{p-1} \prec [\widetilde{\xi}_\rho - \widehat{\xi}_\rho]u_\lambda^{p-1}.$$

Then from (41) and Proposition 7 of Papageorgiou-Rădulescu-Repovš [17], we have

$$u_\lambda - u_\vartheta \in D_+.$$

∎

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 9. *If hypotheses H_0, H_1 hold, then $\lambda^* < +\infty$.*

Proof. We can find $\lambda_0 > 0$ such that

$$\lambda_0 x^{-\eta} + f(z,x) \geq \|\xi\|_\infty x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{44}$$

Let $\lambda > \lambda_0$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ (see Proposition 5). Let $m_\lambda = \min_{\overline{\Omega}} u_\lambda > 0$. For $\delta \in (0,1]$ small we define $m_\lambda^\delta = m_\lambda + \delta$. Let $\rho = \max\{m_\lambda^1, \|u_\lambda\|_\infty\}$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H_1(v)$. Then we have

$$\begin{aligned}
& -\Delta_p m_\lambda^\delta - \Delta_q m_\lambda^\delta + \left[\xi(z) + \widehat{\xi}_\rho\right] (m_\lambda^\delta)^{p-1} - \lambda m_\lambda^{-\eta} \\
& \leq \left[\xi(z) + \widehat{\xi}_\rho\right] m_\lambda^{p-1} + \chi(\delta) - \lambda m_\lambda^{-\eta} \quad \text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
& = \left[\xi(z) + \widehat{\xi}_\rho\right] m_\lambda^{p-1} - \lambda_0 m_\lambda^{-\eta} - (\lambda - \lambda_0) m_\lambda^{-\eta} + \chi(\delta) \\
& \leq f(z, m_\lambda) + \widehat{\xi}_\rho m_\lambda^{p-1} \quad \text{for } \delta \in (0,1) \text{ small (see (44))} \\
& \leq f(z, u_\lambda) + \widehat{\xi}_\rho u_\lambda^{p-1} \quad \text{(see hypothesis } H_1(v)) \\
& = -\Delta_p u_\lambda - \Delta_q u_\lambda + \left[\xi(z) + \widehat{\xi}_\rho\right] u_\lambda^{p-1} - \lambda u_\lambda^{-\eta}, \\
\Rightarrow \quad & u_\lambda - m_\lambda^\delta \in D_+ \quad \text{for all } \delta \in (0,1) \text{ small} \\
& \text{(see Papageorgiou-Rădulescu-Repovš [17], Proposition 6).}
\end{aligned}$$

This is a contradiction to the definition of m_λ.

Therefore we have $\lambda^* \leq \lambda_0$ and so $\lambda^* < +\infty$. ∎

Proposition 10. *If hypotheses H_0, H_1 hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) admits at least two positive solutions $u_0, \widehat{u} \in \operatorname{int} C_+$, $u_0 \neq \widehat{u}$.*

Proof. Let $\beta \in (\lambda, \lambda^*)$. On account of Proposition 8, we can find $u_\beta \in S_\beta \subseteq \operatorname{int} C_+$ and $u_0 \in S_\lambda \subseteq \operatorname{int} C_+$ such that

$$u_\beta - u_0 \in D_+. \tag{45}$$

We introduce the following truncations of the data of (P_λ):

$$k_\lambda^*(z,x) = \begin{cases} \lambda \overline{u}_\lambda(z)^{-\eta} + f(z, \overline{u}_\lambda(z)) & \text{if } x \leq \overline{u}_\lambda(z), \\ \lambda x^{-\eta} + f(z,x) & \text{if } \overline{u}_\lambda(z) < x, \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}, \tag{46}$$

$$b_\lambda^*(z,x) = \begin{cases} \lambda \overline{u}_\lambda(z)^{\tau-1} & \text{if } x \leq \overline{u}_\lambda(z), \\ \lambda x^{\tau-1} & \text{if } \overline{u}_\lambda(z) < x, \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{47}$$

These are Carathéodory functions. We set $K_\lambda^*(z,x) = \int_0^x k_\lambda^*(z,s)ds$, $B_\lambda^*(z,x) = \int_0^x b_\lambda^*(z,s)ds$ and consider the C^1-functional $\psi_\lambda^* : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda^*(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega K_\lambda^*(z,u)dz - \int_{\partial\Omega} B_\lambda^*(z,u)d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

Using (46), (47), we can easily show that

$$K_{\psi_\lambda^*} \subseteq [\overline{u}_\lambda) \cap \operatorname{int} C_+. \tag{48}$$

On account of (45) and Propositions 4 and 6, we have $\overline{u}_\lambda \leq u_\theta$.

Then from (48) we see that without any loss of generality, we may assume that

$$K_{\psi_\lambda^*} \subseteq [\overline{u}_\lambda, u_\beta] = \{u_0\}. \tag{49}$$

Otherwise we already have a second positive solution of (P_λ) distinct from u_0. Using $u_\beta \in \operatorname{int} C_+$, we introduce the following truncations of $k_\lambda^*(z,x)$ and of $b_\lambda^*(z,x)$:

$$\widehat{k}_\lambda^*(z,x) = \begin{cases} k_\lambda^*(z,x) & \text{if } x \leq u_\beta(z), \\ k_\lambda^*(z,u_\beta(z)) & \text{if } u_\beta(z) < x, \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}, \tag{50}$$

$$\widehat{b}_\lambda^*(z,x) = \begin{cases} b_\lambda^*(z,x) & \text{if } x \leq u_\beta(z), \\ b_\lambda^*(z,u_\beta(z)) & \text{if } u_\beta(z) < x, \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{51}$$

These are Carathéodory functions. We set $\widehat{K}_\lambda^*(z,x) = \int_0^x \widehat{k}_\lambda^*(z,s)ds$, $\widehat{B}_\lambda^*(z,x) = \int_0^x \widehat{b}_\lambda^*(z,s)ds$ and consider the C^1-functional $\widehat{\psi}_\lambda^* : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda^*(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \widehat{K}_\lambda^*(z,u)dz - \int_{\partial\Omega} \widehat{B}_\lambda^*(z,u)d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

Using (50), (51), we can show that

$$K_{\widehat{\psi}_\lambda^*} \subseteq [\overline{u}_\lambda, u_\beta] \cap \operatorname{int} C_+. \tag{52}$$

It is clear from (3), (50), (51) that $\widehat{\psi}_\lambda^*(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\widehat{\psi}_\lambda^*(\widetilde{u}_0) = \inf\left[\widehat{\psi}_\lambda^*(u) \,:\, u \in W^{1,p}(\Omega)\right], \\ \Rightarrow\quad &\widetilde{u}_0 \in K_{\widehat{\psi}_\lambda^*} \subseteq [\overline{u}_\lambda, u_\beta] \cap \operatorname{int} C_+ \quad \text{(see (52))}. \end{aligned} \tag{53}$$

From (46), (47), (50), (51), we see that

$$\psi_\lambda^*\Big|_{[0,u_\beta]} = \widehat{\psi}_\lambda^*\Big|_{[0,u_\beta]} \quad \text{and} \quad (\psi_\lambda^*)'\Big|_{[0,u_\beta]} = (\widehat{\psi}_\lambda^*)'\Big|_{[0,u_\beta]}. \tag{54}$$

Then on account of (53) and (49), we have

$$\begin{aligned} &\widetilde{u}_0 = u_0, \\ \Rightarrow\quad &u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \psi_\lambda^*(\cdot) \text{ (see (45) and (54))}, \\ \Rightarrow\quad &u_0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \psi_\lambda^*(\cdot) \\ &\text{(see Papageorgiou-Rădulescu [14], Proposition 2.12)}. \end{aligned}$$

From (48) we see that we may assume that $K_{\psi_\lambda^*}$ is finite (otherwise we already have an infinity of positive smooth solutions of problem (P_λ)). Then Theorem 5.7.6, p. 449, of Papageorgiou-Rădulescu-Repovš [18] says that we can find $\rho \in (0,1)$ small such that

$$\psi_\lambda^*(u_0) < \inf\left[\psi_\lambda^*(u) : \|u - u_0\| = \rho\right] = m_\lambda^*. \tag{55}$$

Note that (46), (47) and hypothesis $H_1(ii)$ imply that if $u \in \operatorname{int} C_+$, then

$$\psi_\lambda^*(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{56}$$

Claim: $\psi_\lambda^*(\cdot)$ satisfies the C-condition.

We consider a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$|\psi_\lambda^*(u_n)| \leq c_{12} \quad \text{for some } c_{12} > 0, \text{ all } n \in \mathbb{N}, \tag{57}$$

$$(1 + \|u_n\|)(\psi_\lambda^*)'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to +\infty. \tag{58}$$

From (58) we have

$$\begin{aligned}&\Big|\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n h dz - \int_\Omega k_\lambda^*(z, u_n) h dz \\ &- \int_{\partial\Omega} b_\lambda^*(z, u_n) h d\sigma\Big| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+.\end{aligned} \tag{59}$$

First we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}&c_2\|u_n^-\|^p \leq c_{13} \quad \text{for some } c_{13} > 0, \text{ all } n \in \mathbb{N} \text{ (see (3), (46), (47))}, \\ \Rightarrow \quad &\{u_n^-\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}\end{aligned} \tag{60}$$

From (57) and (60), we have

$$\begin{aligned}&\gamma_p(u_n^+) + \frac{p}{q}\|\nabla u_n^+\|_q^q - \int_\Omega pK_\lambda^*(z, u_n^+)dz - \int_{\partial\Omega} B_\lambda^*(z, u_n^+)d\sigma \leq c_{14} \text{ for some } c_{14} > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow \quad &\gamma_p(u_n^+) + \frac{p}{q}\|\nabla u_n^+\|_q^q - \lambda p\int_\Omega (u_n^+)^{1-\eta}dz - \int_\Omega pF(z, u_n^+)dz - \frac{\lambda p}{\tau}\int_{\partial\Omega}(u_n^+)^\tau d\sigma \leq c_{15} \\ &\qquad \text{for some } c_{15} > 0, \text{ all } n \in \mathbb{N} \text{ (see (46), (47))}.\end{aligned} \tag{61}$$

Also, if in (59) we choose $h = u_n^+ \in W^{1,p}(\Omega)$, then

$$\begin{aligned}&-\gamma_p(u_n^+) - \|\nabla u_n^+\|_q^q + \int_\Omega k_\lambda^*(z, u_n^+)u_n^+ dz + \int_{\partial\Omega} b_\lambda^*(z, u_n^+)u_n^+ d\sigma \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad &-\gamma_p(u_n^+) - \|\nabla u_n^+\|_q^q + \lambda\int_\Omega (u_n^+)^{1-\eta}dz + \int_\Omega f(z, u_n^+)u_n^+ dz + \lambda\int_{\partial\Omega}(u_n^+)^\tau d\sigma \leq c_{16} \\ &\qquad \text{for some } c_{16} > 0, \text{ all } n \in \mathbb{N}.\end{aligned} \tag{62}$$

We add (61) and (62). Since $q < p$, we obtain

$$\begin{aligned}\int_\Omega \left[f(z, u_n^+)u_n^+ - pF(z, u_n^+)\right] dz \leq \lambda[p-1]\int_\Omega (u_n^+)^{1-\eta}dz + \lambda\left[\frac{p}{\tau} - 1\right]\int_{\partial\Omega}(u_n^+)^\tau d\sigma + c_{17} \\ \text{for some } c_{17} > 0, \text{ all } n \in \mathbb{N}.\end{aligned} \tag{63}$$

Hypotheses $H_1(i), (iii)$ imply that we can find $c_{18} > 0$ such that

$$\frac{c_0}{2}x^\mu - c_{18} \leq f(z, x)x - pF(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0,$$

$$\Rightarrow \quad \frac{c_0}{2}\|u_n^+\|_\mu^\mu - c_{19} \leq \int_\Omega \left[f(z,u_n^+)u_n^+ - pF(z,u_n^+)\right] dz \quad \text{for some } c_{19}>0, \text{ all } n\in\mathbb{N}. \tag{64}$$

We use (64) in (63). Since $\tau < \mu$, we obtain

$$\|u_n^+\|_\mu^\mu \leq c_{20}\left[\|u_n^+\|_\mu + \|u_n^+\|_\mu^\tau + 1\right] \quad \text{for some } c_{20}>0, \text{ all } n\in\mathbb{N},$$

$$\Rightarrow \quad \{u_n^+\}_{n\geq 1} \subseteq L^\mu(\Omega) \text{ is bounded.} \tag{65}$$

First assume that $N \neq p$. From hypothesis $H_1(iii)$ it is clear that we may assume that $\mu < r < p^*$. Then let $t\in(0,1)$ be such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{p^*}. \tag{66}$$

using the interpolation inequality (see Papageorgiou-Winkert [21], Proposition 2.3.17, p. 116), we have

$$\|u_n^+\|_r \leq \|u_n^+\|_\mu^{1-t}\|u_n^+\|_{p^*}^t,$$

$$\Rightarrow \quad \|u_n^+\|_r^r \leq c_{21}\|u_n^+\|^{tr} \quad \text{for some } c_{21}>0, \text{ all } n\in\mathbb{N} \text{ (see (65))}. \tag{67}$$

In (59) we choose $h = u_n^+ \in W^{1,p}(\Omega)$ and using (46) and (47), we have

$$\gamma_p(u_n^+) + \|\nabla u_n^+\|_q^q - \lambda\int_\Omega (u_n^+)^{1-\eta}dz - \int_\Omega f(z,u_n^+)u_n^+dz - \lambda\int_{\partial\Omega}(u_n^+)^\tau d\sigma \leq c_{22}$$
$$\text{for some } c_{22}>0, \text{ all } n\in\mathbb{N},$$

$$\Rightarrow \quad \|u_n^+\|^p \leq c_{23}\left[\|u_n^+\|^{tr}+1\right] \quad \text{for some } c_{23}=c_{23}(\lambda)>0, \text{ all } n\in\mathbb{N} \tag{68}$$
$$\text{(see hypothesis } H_1(i)\text{, (66) and recall that } 1-\eta<1<\tau<p<r).$$

The condition on μ (see hypothesis $H_1(iii)$) and (66) imply that $tr < p$.
So, from (68) it follows that

$$\{u_n^+\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded,}$$

$$\Rightarrow \quad \{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded (see (60))}. \tag{69}$$

If $N = p$, then $p^* = +\infty$, while the Sobolev embedding theorem says that $W^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \leq s < +\infty$. Then for the above argument to work we replace p^* by $s > r > \mu$. Let $t\in(0,1)$ be such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{s},$$

$$\Rightarrow \quad tr = \frac{s(r-\mu)}{s-\mu}.$$

If $s \to +\infty$, then $\dfrac{s(r-\mu)}{s-\mu} \to r-\mu < p$ (see hypothesis $H_1(iii)$). So, we choose $s > r$ big so that $tr = \dfrac{s(r-\mu)}{s-\mu} < p$. Then again we have (69).

On account of (65) we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{70}$$

In (59) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (70). Then

$$\begin{aligned}
&\lim_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle] = 0,\\
\Rightarrow\quad &\limsup_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle] \le 0 \text{ (since } A_q(\cdot) \text{ is monotone)},\\
\Rightarrow\quad &\limsup_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle \le 0 \text{ (see (70))},\\
\Rightarrow\quad &u_n \to u \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1)}.
\end{aligned}$$

Therefore $\psi_\lambda^*(\cdot)$ satisfies the C-condition. This proves the Claim. Then (55), (56) and the Claim permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\psi_\lambda^*} \subseteq [\overline{u}_\lambda) \cap \operatorname{int} C_+ \text{ (see (48)) and } m_\lambda^* \le \psi_\lambda^*(\widehat{u}) \quad \text{(see (55))}. \tag{71}$$

From (71), (55), (46), (47) we infer that $\widehat{u} \in \operatorname{int} C_+$ is a second positive solution of (P_λ), $\widehat{u} \neq u_0$. ☐

Next we show that the critical parameter λ^* is admissible.

Proposition 11. *If hypotheses H_0, H_1 hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Let $\lambda_n \to (\lambda^*)^-$. We slightly modify the proof of Proposition 7 (replacing $\overline{u}_\theta$ with $\overline{u}_{\lambda_1}$) and introduce the following Carathéodory functions:

$$\widetilde{f}(z,x) = \begin{cases} f(z, \overline{u}_{\lambda_1}(z)) & \text{if } x \le \overline{u}_{\lambda_1}(z),\\ f(z,x) & \text{if } \overline{u}_{\lambda_1}(z) < x, \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R},$$

$$\widetilde{k}_{\lambda_n}(z,x) = \begin{cases} \lambda_n \overline{u}_{\lambda_1}(z)^{-\eta} & \text{if } x \le \overline{u}_{\lambda_1}(z),\\ \lambda_n x^{-\eta} & \text{if } \overline{u}_{\lambda_1}(z) < x, \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R},$$

$$\widetilde{d}_{\lambda_n}(z,x) = \begin{cases} \lambda_n \overline{u}_{\lambda_1}(z)^{\tau-1} & \text{if } x \le \overline{u}_{\lambda_1}(z),\\ \lambda_n x^{\tau-1} & \text{if } \overline{u}_{\lambda_1}(z) < x, \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}.$$

We set $\widetilde{F}(z,x) = \int_0^x \widetilde{f}(z,s)ds$, $\widetilde{K}_{\lambda_n}(z,x) = \int_0^x \widetilde{k}_{\lambda_n}(z,s)ds$ and $\widetilde{D}_{\lambda_n}(z,x) = \int_0^x \widetilde{d}_{\lambda_n}(z,s)ds$ and consider the C^1-functional $\widetilde{\mathcal{J}}_{\lambda_n} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\mathcal{J}}_{\lambda_n}(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \widetilde{F}(z,u)dz - \int_\Omega \widetilde{K}_{\lambda_n}(z,u)dz - \int_{\partial\Omega} \widetilde{D}_{\lambda_n}(z,u)d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

From the proof of Proposition 7, we know that we can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ such that

$$\begin{aligned}
\widetilde{\mathcal{J}}_{\lambda_n}(u_n) &\le \widetilde{\mathcal{J}}_{\lambda_n}(\overline{u}_{\lambda_1})\\
&\le \frac{1}{p}\gamma_p(\overline{u}_{\lambda_1}) + \frac{1}{q}\|\nabla \overline{u}_{\lambda_1}\|_q^q - \lambda_1 \int_\Omega \overline{u}_{\lambda_1}^{1-\eta} dz - \int_\Omega f(z, \overline{u}_{\lambda_1})\overline{u}_{\lambda_1} dz - \lambda_1 \int_{\partial\Omega} \overline{u}_{\lambda_1}^\tau d\sigma\\
&\qquad \text{(since } \lambda_1 \le \lambda_n \text{ for all } n \in \mathbb{N}),
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{p}\gamma_p(\overline{u}_{\lambda_1}) + \frac{1}{q}\|\nabla \overline{u}_{\lambda_1}\|_q^q - \lambda_1 \int_\Omega \overline{u}_{\lambda_1}^{1-\eta} dz \\
&= \left[\frac{1}{p} - 1\right]\gamma_p(\overline{u}_{\lambda_1}) + \left[\frac{1}{q} - 1\right]\|\nabla \overline{u}_{\lambda_1}\|_q^q \quad (\text{since } \overline{u}_{\lambda_1} \in \operatorname{int} C_+ \text{ solves } (Q_{\lambda_1})) \\
&< 0. \qquad (72)
\end{aligned}
$$

We have $K_{\widetilde{\mathcal{J}}_{\lambda_n}} \subseteq [\overline{u}_{\lambda_1}) \cap \operatorname{int} C_+$ and from (72) we see that

$$
\widetilde{\mathcal{J}}_{\lambda_n}(u_n) < 0, \quad \widetilde{\mathcal{J}}'_{\lambda_n}(u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{73}
$$

From (73) and reasoning as in the Claim in the proof of Proposition 10, we obtain that

$$
\begin{aligned}
& u_n \to u^* \text{ in } W^{1,p}(\Omega), \\
\Rightarrow\ & u^* \in S_{\lambda^*} \subseteq \operatorname{int} C_+ \quad (\text{see } (73)), \\
\Rightarrow\ & \lambda^* \in \mathcal{L}.
\end{aligned}
$$

□

Concluding, we can state the following bifurcation-type theorem describing the set of positive solutions of problem (P_λ) as the parameter $\lambda > 0$ varies.

Theorem 1. *If hypotheses H_0, H_1 hold, then there exists $\lambda^* > 0$ such that*

(a) *for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions u_0, $\widehat{u} \in \operatorname{int} C_+$, $u_0 \neq \widehat{u}$;*
(b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution $u^* \in \operatorname{int} C_+$;*
(c) *for all $\lambda > \lambda^*$ problem (P_λ) has no positive solutions.*

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