Nonexistence of solutions to higher order evolution inequalities with nonlocal source term on Riemannian manifolds

Mohamed Jleli* Bessem Samet† Calogero Vetro‡

Abstract

We establish a set of assumptions which leads to the nonexistence of nontrivial solutions to higher order evolution inequalities, with respect to the first variable. We consider a nonlocal source term, and work on complete noncompact Riemannian manifolds. The obtained conditions depend on the parameters of the problem and the geometry of the manifold. Our main result recovers some nonexistence theorems from the literature, established in the whole Euclidean space.

2010 Mathematics Subject Classifictaion: 35B33; 35B44; 35R01
Key words: Nonexistence; nontrivial global weak solutions; higher order evolution inequalities; Riemannian manifolds.

1 Introduction

Throughout the manuscript we consider a complete noncompact Riemannian manifold $(\mathbb{M}, g)$ of dimension N, where g denotes a metric tensor. The idea is to establish a set of assumptions which lead to the nonexistence of nontrivial global weak solutions to certain higher order evolution inequalities of the form

$$\frac{\partial^k u}{\partial t^k}(t,x) - \Delta u(t,x) \geq |u(t,x)|^p \left(\int_{\mathbb{M}} a(y)|u(t,y)|^q \, d\mu(y)\right)^{\frac{r}{q}}, \quad t>0,\, x\in\mathbb{M}, \tag{1.1}$$

where Δ is the Laplace-Beltrami operator on $\mathbb{M}$ defined as $\Delta u = \operatorname{div}(\nabla u)$, $d\mu$ is the canonical Riemannian measure on $\mathbb{M}$, $p,q>0$, $r\geq 0$, $q(p+r)>q+r$, a is a measurable function with $a>0$ almost everywhere in $\mathbb{M}$. For this study we impose the initial conditions

$$\frac{\partial^i u}{\partial t^i}(t,x) = u_i(x), \quad i=0,1,\cdots,k-1, \quad x\in\mathbb{M}. \tag{1.2}$$

To motivate the study, we provide a short literature survey on evolution inequalities linked to (1.1). Starting with the case $r=0$ and $\mathbb{M}=\mathbb{R}^N$ (Euclidean setting), then (1.1) reduces to the higher order evolution inequality

$$\frac{\partial^k u}{\partial t^k}(t,x) - \Delta u(t,x) \geq |u(t,x)|^p, \quad t>0,\, x\in\mathbb{R}^N, \tag{1.3}$$

*Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia. E-mail: jleli@ksu.edu.sa
†Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia. E-mail: bsamet@ksu.edu.sa
‡Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy. E-mail: calogero.vetro@unipa.it (Corresponding author)

where $p > 1$. The class of differential inequalities of the form (1.3) has been investigated in many papers, whose results are established in respect to suitable values of the exponent p in the right-hand side of (1.3). Indeed, in the parabolic case $k = 1$, we know from Fujita [1] (see also [2, **?**, **?**, 3, 4, 5]) that, if

$$1 < p \leq 1 + \frac{2}{N} \tag{1.4}$$

and $u_0 > 0$, then (1.3) possesses no global positive solution. Notice that (1.4) is sharp, in the sense that, if $p > 1 + \frac{2}{N}$ and u_0 is smaller than a small Gaussian, then (1.3) (with "=" instead of "$\geq$") admits global positive solutions. In [6], among other problems, Kato studied (1.3) in the hyperbolic case $k = 2$. It was shown that, if the initial data satisfy some suitable positivity conditions, are compactly supported, and

$$1 < p \leq 1 + \frac{2}{N-1} \quad (N \geq 2), \tag{1.5}$$

then no global weak solution can exist in $(0, \infty) \times \mathbb{R}^N$. Notice that the behavior of solutions to the semilinear wave equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) = |u(t,x)|^p, \quad t > 0,\, x \in \mathbb{R}^N, \tag{1.6}$$

where $p > 1$, is completely different to that of solutions to (1.3) with $k = 2$. For more details about existence and nonexistence of global solutions to (1.6), see e.g. [7, 8, 9, 10, 11, 12, 13], and the references therein. In [14], among other results, Laptev proved that under suitable conditions on the initial values, if

$$1 < p \leq 1 + \frac{2}{N - 2 + \frac{2}{k}} \quad (k(N-2) + 2 > 0), \tag{1.7}$$

then (1.3) possesses no nontrivial global solution. Observe that in the case $k = 1$, (1.7) reduces to (1.4), and in the case $k = 2$, (1.7) reduces to (1.5).

In [15], Chen and Huang studied the global nonexistence of nontrivial positive solutions to the parabolic inequality

$$\frac{\partial u^m}{\partial t}(t,x) - \Delta u(t,x) \geq u(t,x)^p \left(\int_{\mathbb{R}^N} a(y) u^q(t,y)\, dy \right)^{\frac{r}{q}}, \quad t > 0,\, x \in \mathbb{R}^N.$$

In [16], Xiao and Fang established nonexistence results for the homogeneous and nonhomogeneous inequalities with singular potential and weigth nonlocal source term of the form

$$\frac{\partial u}{\partial t}(t,x) \geq \Delta u^m(t,x) - V(x)u(t,x) + |x|^\alpha u^p(t,x) \left(\int_{\mathbb{R}^N} a(y) u^q(t,y)\, dy \right)^{\frac{r}{q}} + Lw(x),$$

$t > 0$, $x \in \mathbb{R}^N$, where a and V are positive functions and singular at the origin. Other generalizations and extensions of the obtained results in [15, 16] can be found in [17, 18].

Various extensions of nonexistence results from the Euclidean case to noncompact Riemannian manifolds have been obtained under suitable geometric hypotheses. In [19], Ru studied the global nonexistence of positive solutions to nonlinear wave equations with a damping term, defined on a complete noncompact Riemannian manifold $\mathbb{M}$. Notice that in the proofs, it was assumed that the distance function $\mathbb{M} \ni x \mapsto \rho(x) := d(x_0, x)$, for some fixed $x_0 \in \mathbb{M}$, is smooth. However, a such regularity is not guaranteed on general

Riemannian manifolds. In [20, 21, 22, 23], the nonexistence of nonnegative solutions to elliptic and parabolic inequalities has been considered, and the approach used to derive the nonexistence of solutions is based on an adequate choice of radial test functions (that is, depending on the distance function ρ). It is relevant to point out, in the above context, that proofs require only the manipulation of gradient of such test functions, which allows to consider general manifolds, since the gradient of the distance function is properly stated ($|\nabla\rho(x)| = 1$ a.e. in $\mathbb{M}$). For (1.1), the similar methods cannot be applied due to the presence of the term $\frac{\partial^k u}{\partial t^k}$ and the fact that u has no constant sign. Namely, in our case, we need also to estimate the second derivatives of the test functions. On the other hand, it is well-known that the distance function ρ is only C^2 on $\mathbb{M} \setminus (\{x_0\} \cup C_{x_0})$, where C_{x_0} denotes the cut locus of x_0. Consequently, the choice of a test function depending on ρ will not be helpful in our situation. To overcome this difficulty, we shall consider the class of complete noncompact Riemannian manifolds $\mathbb{M}$, for which the Ricci curvature satisfies the condition:

$$\text{Ric} \geq -C_0(N-1)\left(1+\rho^2(x)\right)^{-\frac{\sigma}{2}},$$

for some $C_0 > 0$ and $\sigma \geq -2$. This condition was adopted by Monticelli et al. [24] to establish necessary conditions for the existence of (very weak) solutions to certain semilinear hyperbolic problems on $\mathbb{M}$. For a such class of manifolds, regular test functions can be obtained thanks to some recent results of Bianchi and Setti [25], aimed to control the gradient of the test functions and ensure a certain decay for the Laplacian term (see, for example, Lemma 2.1 in Section 2). To have a more complete picture of the relevant literature about nonexistence results obtained on particular manifolds, see e.g. [26, 27, 28, 29, 30, 31, 32, 33], and the references therein.

The rest of the paper is organized as follows. In Section 2, we recall some basic notions on Riemannian geometry that will be used in the sequel. In Section 3, we state our main result, namely Theorem 3.1, and discuss deeply the involved conditions and consequences. Finally, in Section 4 we give the complete proof of Theorem 3.1.

2 Riemannian geometry

We just recall some of the closest notions and results from Riemannian geometry to the topic treated in the manuscript, with no claims to being exhaustive. We follow the usual notation in the literature (see also [24, 25]). For more details, see e.g. the monographies of Alias et al. [34] and Petersen [35]. Let $\mathbb{M}$ be a Riemannian manifold of dimension N equipped with a metric $g = (\cdot,\cdot)$. Let (U,ψ) be a chart in $\mathbb{M}$, where $\psi : U \to \psi(U) \subset \mathbb{R}^N$ and

$$\psi(x) = (x^1(x), x^2(x), \cdots, x^N(x)), \quad x \in U.$$

Let $\partial_1, \partial_2, \cdots, \partial_N$ be the corresponding vector fields on U. The local components of g are given as

$$g_{ij} = (\partial_i, \partial_j), \quad i, j = 1, 2, \cdots, N.$$

Given the matrix (g_{ij}), then (g^{ij}) means the inverse of (g_{ij}). Additionally, if $u : \mathbb{M} \to \mathbb{R}$ is smooth, then by ∇u (i.e., the gradient of u), we mean the vector field defined by

$$(\nabla u, X) = Xu,$$

for every vector field X on $\mathbb{M}$. Making use of local coordinates, we recall that

$$\nabla u = \sum_{i,j} g^{ij}(\partial_i u)\partial_j.$$

The divergence of X on $\mathbb{M}$ is defined by

$$\operatorname{div}(X)(x) = \operatorname{trace}\left(T_x\mathbb{M} \ni \xi \mapsto \widetilde{\nabla}_\xi X \in T_x\mathbb{M}\right), \quad x \in \mathbb{M},$$

where $T_x\mathbb{M}$ is the tangent vector space at $x \in \mathbb{M}$ and $\widetilde{\nabla}$ is the Levi-Civita connection to g. Turning to local coordinates, if $X = \sum_i X_i \partial_i$, we retrieve that

$$\operatorname{div}(X) = \frac{1}{\sqrt{\mathbf{g}}} \sum_j \partial_j \left(\mathbf{g} X_j\right),$$

where $\mathbf{g} := \det(g_{ij})$. Moreover, for the Laplacian of u we have

$$\Delta u = \frac{1}{\sqrt{\mathbf{g}}} \sum_{i,j} \partial_i \left(\sqrt{\mathbf{g}} g^{ij} \partial_j u\right).$$

Finally, we recall the formula

$$R_{ij} = R_{ji} = \sum_\ell \partial_\ell \left(\Gamma^\ell_{ij}\right) - \sum_\ell \partial_j \left(\Gamma^\ell_{i\ell}\right) + \sum_{k,\ell} \left(\Gamma^k_{ij}\Gamma^\ell_{k\ell} - \Gamma^k_{i\ell}\Gamma^\ell_{kj}\right),$$

where Γ^k_{ij} are the usual Christoffel symbols. So, the last formula means the Ricci tensor (for short, Ric) in local coordinates.

Let $|X| := \sqrt{(X, X)}$ and $f : \mathbb{M} \to \mathbb{R}$ be a given function. Thus, the notation $\operatorname{Ric} \geq f(x)$ means that

$$\operatorname{Ric}(X, X) \geq f(x)|X|^2,$$

for every vector field X on $\mathbb{M}$.

Let $x_0 \in \mathbb{M}$ be fixed. We denote by $B(x_0, \delta)$, $\delta > 0$, the geodesic ball of center x_0 and radius δ, defined by

$$B(x_0, \delta) = \{x \in \mathbb{M} : \rho(x) < \delta\},$$

where $\rho(x) = d(x_0, x)$ for all $x \in \mathbb{M}$.

As already mentioned in the Introduction, a key ingredient of our analysis is a recent result established by Bianchi and Setti [25]. Precisely, we refer to the following result (see also [25, Corollary 2.3] and [24, Proposition 4.2]).

Lemma 2.1. *Suppose that* (3.2) *holds for some* $\sigma \in [-2, 2]$. *Then there exists a family of functions* $\{\xi_R\}_{R\geq 1} \subset C^\infty_c(\mathbb{M})$ *satisfying:*

- (i) $0 \leq \xi_R \leq 1$, $\xi_R|_{B(x_0,R)} \equiv 1$,
- (ii) *there exists* $\gamma_0 > 1$ *(independent on* R*) such that* $\operatorname{supp}(\xi_R) \subset B(x_0, \gamma R)$ *for all* $\gamma > \gamma_0$,
- (iii) $|\nabla \xi_R| \leq \frac{C}{R}$,
- (iv) $|\Delta \xi_R| \leq \frac{C}{R^{1+\frac{\sigma}{2}}}$.

This lemma will play a crucial role in the proof of Theorem 3.1 (that is, the main result herein). We remark that Lemma 2.1 gives us the (σ, R)-dependence of both gradient and Laplacian term of function ξ_R whose support lies in a ball $B(x_0, \gamma R)$ with center in x_0. In general, the problem of dependence of constant C on the Ricci curvature's bounds deserves attention. For more information about that, in Lemma 2.1, we refer the reader to [25, Remark 2.5].

3 Discussion on results

As told in the Introduction, we aim to establish sufficient conditions for the nonexistence of nontrivial global weak solutions to certain higher order evolution inequalities of the form given in (1.1)–(1.2). Thus, the starting point of our discussion is the definition of solution considered herein. We denote $\Omega = [0,\infty) \times \mathbb{M}$ and

$$\mathcal{N}_q\left(a^{\frac{1}{q}}u\right) = \left(\int_{\mathbb{M}} a(y)|u(t,y)|^q \, d\mu(y)\right)^{\frac{1}{q}},$$

then we have the following key notion.

Definition 3.1. *Let $u_i \in L^1_{loc}(\mathbb{M})$ for all $i = 0, 1, \cdots, k-1$. We say that u is a global weak solution to* (1.1)–(1.2)*, if the following conditions hold:*

(i) $u, |u|^p \mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r \in L^1_{loc}(\Omega)$,

(ii) *for every nonnegative function $\varphi \in C_c^{k,2}(\Omega)$,*

$$\begin{aligned}\int_\Omega |u|^p \mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r \varphi \, d\mu(x)\, dt \leq & \sum_{i=1}^{k}(-1)^i \int_{\mathbb{M}} u_{k-i}(x) \frac{\partial^{i-1}\varphi}{\partial t^{i-1}}(0,x)\, d\mu(x) \\ & + (-1)^k \int_\Omega u \frac{\partial^k \varphi}{\partial t^k}\, d\mu(x)\, dt - \int_\Omega u \Delta\varphi \, d\mu(x)\, dt.\end{aligned} \tag{3.1}$$

Before stating our main result, we remark that throughout this paper, C denotes a generic positive constant independent on R, whose value is not necessarily the same at each occurrence.

Theorem 3.1. *Let $u_i \in L^1_{loc}(\mathbb{M})$ for all $i = 0, 1, \cdots, k-1$ and let $u_{k-1} \geq 0$. Suppose that for some $C_0 > 0$, $\sigma > -2$ one has*

$$\mathrm{Ric} \geq -C_0(N-1)\left(1+\rho^2(x)\right)^{-\frac{\sigma}{2}}. \tag{3.2}$$

Assume that

$$\int_{B(x_0,R)} [a(x)]^{-\varsigma}\, d\mu(x) = O(R^\lambda), \quad \text{as } R \to \infty, \tag{3.3}$$

where

$$\varsigma = \frac{r}{q(p+r)-(q+r)} \quad \text{and} \quad \lambda = \frac{q}{k}\min\left\{1+\frac{\sigma}{2}, 2\right\} \frac{(p+r)(k-1)+1}{q(p+r)-(q+r)}.$$

Then (1.1)–(1.2) *admits no nontrivial global weak solution.*

Remark 3.1. *Observe that under condition* (3.2)*, a regularity assumption for function a to guarantee the validity of our condition* (3.3) *is*

$$a^{-\varsigma} \in L^1(\mathbb{M}).$$

We deduce such a assumption is crucial for the nonexistence of nontrivial global weak solution to (1.1)–(1.2)*, by Theorem 3.1.*

Now, let us discuss some special cases of Theorem 3.1. Then we give more details about the role of σ in obtaining our main result. Assume that (3.2) holds for some $C_0 > 0$ and $\sigma \in (-2, 2)$. From Grigor'yan [36], appealing to volume comparison theorems (which give us upper bounds of the volume growth), we deduce that

$$\mu(B(x_0, R)) \le \widetilde{C} \exp\left(\mathcal{B}(N-1)R^{1-\frac{\sigma}{2}}\right), \tag{3.4}$$

for some $\mathcal{B} > 0$ and $\widetilde{C} > 0$ (see also [24, Remark 2.4]). Consider the class of functions a satisfying the inequality

$$a(x) \ge C(1+\rho(x))^{\alpha} \exp\left(\beta\rho^{1-\frac{\sigma}{2}}(x)\right), \quad \text{a.e. } x \in \mathbb{M}, \tag{3.5}$$

for some $\alpha \in \mathbb{R}$ and $\beta > 0$. For sufficiently large R, we write $R = 2^L\tau_0$, where $\tau_0 \in [1, 2]$ and L is a positive integer. Then, by (3.4) and (3.5), there holds

$$\begin{aligned}
&\int_{B(x_0,R)} [a(x)]^{-\varsigma}\, d\mu(x) \\
\le\ & C\int_{B(x_0,R)} (1+\rho(x))^{-\alpha\varsigma} \exp\left(-\varsigma\beta\rho^{1-\frac{\sigma}{2}}(x)\right)\, d\mu(x) \\
=\ & \int_{B(x_0,\tau_0)} (1+\rho(x))^{-\alpha\varsigma} \exp\left(-\varsigma\beta\rho^{1-\frac{\sigma}{2}}(x)\right)\, d\mu(x) \\
& + \sum_{j=1}^{L} \int_{B(x_0,2^j\tau_0)\setminus B(x_0,2^{j-1}\tau_0)} (1+\rho(x))^{-\alpha\varsigma} \exp\left(-\varsigma\beta\rho^{1-\frac{\sigma}{2}}(x)\right)\, d\mu(x) \\
\le\ & C + C\sum_{j=1}^{L} (2^j\tau_0)^{-\alpha\varsigma} \exp\left(-\varsigma\beta(2^{j-1}\tau_0)^{1-\frac{\sigma}{2}} + \mathcal{B}(N-1)(2^j\tau_0)^{1-\frac{\sigma}{2}}\right) \\
=\ & C + C\sum_{j=1}^{L} (2^j\tau_0)^{-\alpha\varsigma} \exp\left[(2^j\tau_0)^{1-\frac{\sigma}{2}}\left(\mathcal{B}(N-1) - \frac{\varsigma\beta}{2^{1-\frac{\sigma}{2}}}\right)\right].
\end{aligned}$$

Reasoning on the argument of the last exponential above, if we assume the sign condition

$$\mathcal{B}(N-1) - \frac{\varsigma\beta}{2^{1-\frac{\sigma}{2}}} \le 0,$$

then we obtain the estimate

$$\int_{B(x_0,R)} [a(x)]^{-\varsigma}\, d\mu(x) \le C + C\sum_{j=1}^{L} (2^j\tau_0)^{-\alpha\varsigma},$$

which can be particularized for different occurrences of α, as follows.
If $\alpha > 0$, then

$$\int_{B(x_0,R)} [a(x)]^{-\varsigma}\, d\mu(x) \le C.$$

If $\alpha = 0$, then

$$\int_{B(x_0,R)} [a(x)]^{-\varsigma}\, d\mu(x) \le C(L+1) \le C\ln R.$$

Finally, if $\alpha < 0$ we obtain

$$\begin{aligned}
\int_{B(x_0,R)} [a(x)]^{-\varsigma}\, d\mu(x) &\le C + C\tau_0^{-\alpha\varsigma}\frac{2^{-\alpha\varsigma L} - 1}{2^{-\alpha\varsigma} - 1} \\
&\le CR^{-\alpha\varsigma}.
\end{aligned}$$

These estimates imply that (3.3) holds, whenever

$$\alpha \geq 0 \text{ or } -\frac{\lambda}{\varsigma} \leq \alpha < 0.$$

Now, returning to Theorem 3.1, we easily deduce the following result.

Corollary 3.1. *Let $u_i \in L^1_{loc}(\mathbb{M})$ for all $i = 0, 1, \cdots, k-1$, and $u_{k-1} \geq 0$. Suppose that* (3.2) *holds for some $\sigma \in (-2, 2)$, and the function a satisfies* (3.5)*. Assume that*

$$\alpha \geq 0 \quad \textit{and} \quad \beta \geq \frac{2^{1-\frac{\sigma}{2}}\mathcal{B}(N-1)}{\varsigma},$$

or

$$-\frac{\lambda}{\varsigma} \leq \alpha < 0 \quad \textit{and} \quad \beta \geq \frac{2^{1-\frac{\sigma}{2}}\mathcal{B}(N-1)}{\varsigma}.$$

Then (1.1)–(1.2) *admits no nontrivial global weak solution.*

Next, assume that (3.2) holds for some $C_0 > 0$ and $\sigma = 2$. We can refer again to the volume comparison theorems in [36], to obtain the upper bound

$$\mu(B(x_0, R)) \leq \widetilde{C} R^{(N-1)\tau+1}, \tag{3.6}$$

for some $\widetilde{C} > 0$, where $\tau = \frac{1+\sqrt{1+\frac{4C_0}{N-1}}}{2}$. This time, we consider the class of functions a satisfying the bound from below

$$a(x) \geq C(1+\rho(x))^{\alpha}, \quad \text{a.e. } x \in \mathbb{M}, \tag{3.7}$$

for some $\alpha \in \mathbb{R}$. As previously, for sufficiently large R, we write $R = 2^L\tau_0$, where $\tau_0 \in [1, 2]$ and L is a positive integer. Then, by (3.6) and (3.7), there holds

$$\begin{aligned}
\int_{B(x_0,R)} [a(x)]^{-\varsigma}\, d\mu(x) &\leq C\int_{B(x_0,R)} (1+\rho(x))^{-\alpha\varsigma}\, d\mu(x) \\
&= \int_{B(x_0,\tau_0)} (1+\rho(x))^{-\alpha\varsigma}\, d\mu(x) \\
&\quad + \sum_{j=1}^{L} \int_{B(x_0,2^j\tau_0)\setminus B(x_0,2^{j-1}\tau_0)} (1+\rho(x))^{-\alpha\varsigma}\, d\mu(x) \\
&\leq C + C\sum_{j=1}^{L} (2^j\tau_0)^{-\alpha\varsigma+(N-1)\tau+1}.
\end{aligned}$$

The right-hand side of the last inequality can now be particularized for different occurrences of the exponent $-\alpha\varsigma + (N-1)\tau + 1$, as follows.
If $-\alpha\varsigma + (N-1)\tau + 1 < 0$, then

$$\int_{B(x_0,R)} [a(x)]^{-\varsigma}\, d\mu(x) \leq C.$$

If $-\alpha\varsigma + (N-1)\tau + 1 = 0$, then

$$\int_{B(x_0,R)} [a(x)]^{-\varsigma}\, d\mu(x) \leq C(L+1) \leq C\ln R.$$

If $-\alpha\varsigma + (N-1)\tau + 1 > 0$, then

$$\int_{B(x_0,R)} [a(x)]^{-\varsigma}\, d\mu(x) \leq CR^{-\alpha\varsigma+(N-1)\tau+1}.$$

Considering these three estimates, we conclude that condition (3.3) holds, whenever

$$\alpha \geq \frac{(N-1)\tau+1}{\varsigma} \quad \text{or} \quad \frac{(N-1)\tau+1-\lambda}{\varsigma} \leq \alpha < \frac{(N-1)\tau+1}{\varsigma}$$

Thus, by Theorem 3.1, we deduce the following result.

Corollary 3.2. *Let $u_i \in L^1_{loc}(\mathbb{M})$ for all $i = 0, 1, \cdots, k-1$, and $u_{k-1} \geq 0$. Suppose that* (3.2) *holds for $\sigma = 2$, and the function a satisfies* (3.7). *Assume that*

$$\alpha\varsigma \geq (N-1)\tau + 1$$

or

$$(N-1)\tau + 1 - \lambda \leq \alpha\varsigma < (N-1)\tau + 1.$$

Then (1.1)–(1.2) *admits no nontrivial global weak solution.*

The last situation to consider is when (3.2) holds for some $C_0 > 0$ and $\sigma > 2$. This time, from volume comparison theorems in [36], it follows the growth bound

$$\mu(B(x_0, R)) \leq \widetilde{C} R^N, \tag{3.8}$$

for some $\widetilde{C} > 0$. Hence, taking $\tau = 1$ in Corollary 3.2, we deduce the following result.

Corollary 3.3. *Let $u_i \in L^1_{loc}(\mathbb{M})$ for all $i = 0, 1, \cdots, k-1$, and $u_{k-1} \geq 0$. Suppose that* (3.2) *holds for $\sigma > 2$, and the function a satisfies* (3.7). *Assume that*

$$\alpha\varsigma \geq N$$

or

$$N - \lambda \leq \alpha\varsigma < N. \tag{3.9}$$

Then (1.1)–(1.2) *admits no nontrivial global weak solution.*

Remark 3.2. *Consider the Euclidean case $\mathbb{M} = \mathbb{R}^N$. Then,* (3.2) *holds for $\sigma = 2$, and $\mu(B(x_0, R)) = \widetilde{C} R^N$, for some $\widetilde{C} > 0$. Taking $\tau = 1$ in Corollary 3.2, we deduce that Corollary 3.3 holds also in the Euclidean case. In particular, for $r = 0$ (then $\varsigma = 0$),* (3.9) *(with $\sigma = 2$) is equivalent to*

$$\left(N - 2 + \frac{2}{k}\right) p \leq N + \frac{2}{k}.$$

Thus, we recover the result obtained by Laptev in [14].

4 Proof of Theorem 3.1

For readears convenience, we first consider the case $\sigma \in (-2, 2]$. The presented approach is a proof by contradiction, So, we assume that (1.1)–(1.2) admits a global weak solution u. Using item (ii) of Definition 3.1, that is, by inequality (3.1) for every nonnegative function $\varphi \in C_c^{k,2}(\Omega)$, we have

$$\begin{aligned} &\int_\Omega |u|^p \mathcal{N}_q\left(a^{\frac{1}{q}} u\right)^r \varphi \, d\mu(x)\, dt \\ &\leq \sum_{i=1}^{k} (-1)^i \int_{\mathbb{M}} u_{k-i}(x) \frac{\partial^{i-1}\varphi}{\partial t^{i-1}}(0, x)\, d\mu(x) + \int_\Omega |u| \left|\frac{\partial^k \varphi}{\partial t^k}\right| d\mu(x)\, dt + \int_\Omega |u||\Delta\varphi|\, d\mu(x)\, dt. \end{aligned} \tag{4.1}$$

Assuming the following choice of parameters

$$\eta_1 = p + r > 1, \quad \eta_2 = \frac{q(p+r)}{q(p+r) - (q+r)} > 1, \quad \eta_1' = \frac{\eta_1}{\eta_1 - 1}, \tag{4.2}$$

we apply the Hölder's inequality to obtain

$$\begin{aligned}
&\int_{\mathbb{M}} |u| \left| \frac{\partial^k \varphi}{\partial t^k} \right| d\mu(x) \\
&\leq \left(\int_{\mathbb{M}} |u|^p \varphi \, d\mu(x) \right)^{\frac{1}{\eta_1}} \left(\int_{\mathbb{M}} a(x)|u|^q \, d\mu(x) \right)^{\frac{r}{q\eta_1}} \left(\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x) \left| \frac{\partial^k \varphi}{\partial t^k} \right|^{\eta_2} \varphi^{\frac{-\eta_2}{\eta_1}} \, d\mu(x) \right)^{\frac{1}{\eta_2}}.
\end{aligned} \tag{4.3}$$

Now, we can integrate over $(0,\infty)$ and use again Hölder's inequality. So, we obtain the following chain of inequalities

$$\begin{aligned}
&\int_{\Omega} |u| \left| \frac{\partial^k \varphi}{\partial t^k} \right| d\mu(x)\, dt \\
&\leq \int_0^\infty \left(\int_{\mathbb{M}} |u|^p \varphi \, d\mu(x) \right)^{\frac{1}{\eta_1}} \left(\int_{\mathbb{M}} a(x)|u|^q \, d\mu(x) \right)^{\frac{r}{q\eta_1}} \left(\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x) \left| \frac{\partial^k \varphi}{\partial t^k} \right|^{\eta_2} \varphi^{\frac{-\eta_2}{\eta_1}} \, d\mu(x) \right)^{\frac{1}{\eta_2}} dt \\
&\leq \left(\int_{\Omega} |u|^p \mathcal{N}_q \left(a^{\frac{1}{q}} u \right)^r \varphi \, d\mu(x)\, dt \right)^{\frac{1}{\eta_1}} \left(\int_0^\infty \left(\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x) \left| \frac{\partial^k \varphi}{\partial t^k} \right|^{\eta_2} \varphi^{\frac{-\eta_2}{\eta_1}} \, d\mu(x) \right)^{\frac{\eta_1'}{\eta_2}} dt \right)^{\frac{1}{\eta_1'}}.
\end{aligned}$$

Additionally, the Young's inequality leads to the following estimate

$$\begin{aligned}
&\int_{\Omega} |u| \left| \frac{\partial^k \varphi}{\partial t^k} \right| d\mu(x)\, dt \\
&\leq \frac{1}{4} \int_{\Omega} |u|^p \mathcal{N}_q \left(a^{\frac{1}{q}} u \right)^r \varphi \, d\mu(x)\, dt + C \int_0^\infty \left(\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x) \left| \frac{\partial^k \varphi}{\partial t^k} \right|^{\eta_2} \varphi^{\frac{-\eta_2}{\eta_1}} \, d\mu(x) \right)^{\frac{\eta_1'}{\eta_2}} dt.
\end{aligned} \tag{4.4}$$

A similar argument yields to the new estimate involving the Laplacian

$$\begin{aligned}
&\int_{\Omega} |u||\Delta \varphi| \, d\mu(x)\, dt \\
&\leq \frac{1}{4} \int_{\Omega} |u|^p \mathcal{N}_q \left(a^{\frac{1}{q}} u \right)^r \varphi \, d\mu(x)\, dt + C \int_0^\infty \left(\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x) |\Delta \varphi|^{\eta_2} \varphi^{\frac{-\eta_2}{\eta_1}} \, d\mu(x) \right)^{\frac{\eta_1'}{\eta_2}} dt.
\end{aligned} \tag{4.5}$$

Combining (4.1), (4.4), and (4.5), we deduce that

$$\begin{aligned}
&\int_{\Omega} |u|^p \mathcal{N}_q \left(a^{\frac{1}{q}} u \right)^r \varphi \, d\mu(x)\, dt + 2 \sum_{i=1}^{k} (-1)^{i+1} \int_{\mathbb{M}} u_{k-i}(x) \frac{\partial^{i-1} \varphi}{\partial t^{i-1}}(0,x) \, d\mu(x) \\
&\leq C \left(I_1 + I_2 \right),
\end{aligned} \tag{4.6}$$

where we put

$$I_1 := \int_0^\infty \left(\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x) \left| \frac{\partial^k \varphi}{\partial t^k} \right|^{\eta_2} \varphi^{\frac{-\eta_2}{\eta_1}} \, d\mu(x) \right)^{\frac{\eta_1'}{\eta_2}} dt$$

and

$$I_2 := \int_0^\infty \left(\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x) |\Delta \varphi|^{\eta_2} \varphi^{\frac{-\eta_2}{\eta_1}} \, d\mu(x) \right)^{\frac{\eta_1'}{\eta_2}} dt.$$

Based on the test function method, we propose now the construction of appropriate test function for our problem, with a special choice of cut-off function. Consequently, we are able to obtain some useful estimates to develop the proof of Theorem 3.1. Let $\chi \in C^\infty([0,\infty))$ be a cut-off function satisfying

$$0 \le \chi \le 1, \quad \chi \equiv 1 \text{ in } [0,1], \quad \chi \equiv 0 \text{ in } [2,\infty).$$

Consider also the family of functions $\{\xi_R\}_{R\ge 1} \subset C_c^\infty(\mathbb{M})$ provided by Lemma 2.1. For ℓ and R sufficiently large and $\theta > 0$, we introduce the test function

$$\varphi(t,x) = \chi^\ell\left(\frac{t}{R^\theta}\right)\xi_R^\ell(x) := F^\ell(t)G^\ell(x), \quad (t,x) \in \Omega.$$

Then I_1 can be written as

$$I_1 = \left(\int_0^\infty F^{\frac{-\ell\eta_1'}{\eta_1}}(t)\left|(F^\ell)^{(k)}(t)\right|^{\eta_1'}\,dt\right)\left(\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x)G^{\frac{\ell\eta_2}{\eta_1'}}(x)\,d\mu\right)^{\frac{\eta_1'}{\eta_2}}, \tag{4.7}$$

where $(\cdot)^{(j)} = \frac{d^j\cdot}{dt^j}$. In view of the properties of the cut-off function χ, there holds

$$\begin{aligned}\int_0^\infty F^{\frac{-\ell\eta_1'}{\eta_1}}(t)\left|(F^\ell)^{(k)}(t)\right|^{\eta_1'}\,dt &\le CR^{-k\theta\eta_1'}\int_{R^\theta}^{2R^\theta}\chi^{\ell-k\eta_1'}\left(\frac{t}{R^\theta}\right)\,dt\\ &\le CR^{\theta(1-k\eta_1')}.\end{aligned} \tag{4.8}$$

Moreover, by (3.2) and Lemma 2.1, there exists $\gamma > 1$ such that

$$\begin{aligned}\left(\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x)G^{\frac{\ell\eta_2}{\eta_1'}}(x)\,d\mu(x)\right)^{\frac{\eta_1'}{\eta_2}} &= \left(\int_{B(x_0,\gamma R)} a^{\frac{-r\eta_2}{q\eta_1}}(x)\xi_R^{\frac{\ell\eta_2}{\eta_1'}}(x)\,d\mu(x)\right)^{\frac{\eta_1'}{\eta_2}}\\ &\le \left(\int_{B(x_0,\gamma R)} a^{\frac{-r\eta_2}{q\eta_1}}(x)\,d\mu(x)\right)^{\frac{\eta_1'}{\eta_2}}.\end{aligned} \tag{4.9}$$

We remark that γ herein is independent on R. Next, from (4.7), (4.8), and (4.9) we obtain the estimate

$$I_1 \le CR^{\theta(1-k\eta_1')}\left(\int_{B(x_0,\gamma R)} a^{\frac{-r\eta_2}{q\eta_1}}(x)\,d\mu(x)\right)^{\frac{\eta_1'}{\eta_2}}, \quad \gamma > \gamma_0. \tag{4.10}$$

As previously, by the definition of the function φ, I_2 can be written as

$$I_2 = \left(\int_0^\infty F^\ell(t)\,dt\right)\left(\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x)G^{\frac{-\eta_2\ell}{\eta_1}}(x)\left|\Delta\left(G^\ell\right)(x)\right|^{\eta_2}\,d\mu(x)\right)^{\frac{\eta_1'}{\eta_2}}. \tag{4.11}$$

Using the properties of the cut-off function χ, we obtain

$$\begin{aligned}\int_0^\infty F^\ell(t)\,dt &= \int_0^\infty \chi^\ell\left(\frac{t}{R^\theta}\right)\,dt\\ &= \int_0^{2R^\theta}\chi^\ell\left(\frac{t}{R}\right)\,dt\\ &\le 2R^\theta.\end{aligned} \tag{4.12}$$

On the other hand, by Lemma 2.1, we have

$$\begin{aligned}
&\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x) G^{\frac{-\eta_2\ell}{\eta_1}}(x)\left|\Delta\left(G^{\ell}\right)(x)\right|^{\eta_2}\,d\mu(x)\\
&=\int_{B(x_0,\gamma R)\backslash B(x_0,R)} a^{\frac{-r\eta_2}{q\eta_1}}(x)\xi_R^{\frac{-\eta_2\ell}{\eta_1}}(x)\left|\Delta\left(\xi_R^{\ell}\right)(x)\right|^{\eta_2}\,d\mu(x).
\end{aligned} \tag{4.13}$$

Since

$$\Delta\left(\xi_R^{\ell}\right)(x)=\xi_R^{\ell-2}(x)\left[\ell\xi_R(x)\Delta\xi_R(x)+\ell(\ell-1)|\nabla\xi_R(x)|^2\right],$$

we deduce by Lemma 2.1 that

$$\begin{aligned}
\left|\Delta\left(\xi_R^{\ell}\right)(x)\right| &\leq C\xi_R^{\ell-2}(x)\left(\frac{1}{R^{1+\frac{\sigma}{2}}}+\frac{1}{R^2}\right)\\
&\leq CR^{-1-\frac{\sigma}{2}}\xi_R^{\ell-2}(x).
\end{aligned} \tag{4.14}$$

Hence, by (4.13) and (4.14), we get the chain of inequalities

$$\begin{aligned}
&\int_{\mathbb{M}} a^{\frac{-r\eta_2}{q\eta_1}}(x) G^{\frac{-\eta_2\ell}{\eta_1}}(x)\left|\Delta\left(G^{\ell}\right)(x)\right|^{\eta_2}\,d\mu(x)\\
&\leq CR^{-\eta_2\left(1+\frac{\sigma}{2}\right)}\int_{B(x_0,\gamma R)\backslash B(x_0,R)} a^{\frac{-r\eta_2}{q\eta_1}}(x)\xi_R^{\frac{\eta_2}{\eta_1'}\left(\ell-2\eta_1'\right)}(x)\,d\mu(x)\\
&\leq CR^{-\eta_2\left(1+\frac{\sigma}{2}\right)}\int_{B(x_0,\gamma R)\backslash B(x_0,R)} a^{\frac{-r\eta_2}{q\eta_1}}(x)\,d\mu(x)\\
&\leq CR^{-\eta_2\left(1+\frac{\sigma}{2}\right)}\int_{B(x_0,\gamma R)} a^{\frac{-r\eta_2}{q\eta_1}}(x)\,d\mu(x).
\end{aligned} \tag{4.15}$$

It remains to combine (4.11) with (4.12), and (4.15). Therefore, we obtain the estimate

$$I_2\leq CR^{\theta-\left(1+\frac{\sigma}{2}\right)\eta_1'}\left(\int_{B(x_0,\gamma R)} a^{\frac{-r\eta_2}{q\eta_1}}(x)\,d\mu(x)\right)^{\frac{\eta_1'}{\eta_2}}. \tag{4.16}$$

Notice that by the definition of the function φ and the properties of the function χ, we have

$$\frac{\partial^{i-1}\varphi}{\partial t^{i-1}}(0,x)=\begin{cases}\xi_R^{\ell}(x) & \text{if } i=1,\\ 0 & \text{if } i\in\{2,3,\cdots,k\}.\end{cases}$$

Hence, there holds

$$\sum_{i=1}^{k}(-1)^{i+1}\int_{\mathbb{M}} u_{k-i}(x)\frac{\partial^{i-1}\varphi}{\partial t^{i-1}}(0,x)\,d\mu(x)=\int_{\mathbb{M}} u_{k-1}(x)\xi_R^{\ell}(x)\,d\mu(x).$$

Since $u_{k-1}\geq 0$, we deduce that

$$\sum_{i=1}^{k}(-1)^{i+1}\int_{\mathbb{M}} u_{k-i}(x)\frac{\partial^{i-1}\varphi}{\partial t^{i-1}}(0,x)\,d\mu(x)\geq 0. \tag{4.17}$$

On the other hand, by the properties of the functions χ and ξ_R, we have

$$\int_{\Omega}|u|^p\mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r\varphi\,d\mu(x)\,dt\geq\int_0^{R^{\theta}}\int_{B(x_0,R)}|u|^p\mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r\,d\mu(x)\,dt. \tag{4.18}$$

Thus, collecting (4.6), (4.10), (4.16), (4.17), and (4.18), we get that

$$\int_0^{R^\theta}\int_{B(x_0,R)}|u|^p\mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r\,d\mu(x)\,dt$$
$$\le C\left(R^{\theta(1-k\eta_1')}+R^{\theta-\left(1+\frac{\sigma}{2}\right)\eta_1'}\right)\left(\int_{B(x_0,\gamma R)}a^{\frac{-r\eta_2}{q\eta_1}}(x)\,d\mu(x)\right)^{\frac{\eta_1'}{\eta_2}}.$$

Taking $\theta=\frac{1+\frac{\sigma}{2}}{k}>0$, after some rearranging, the above inequality reduces to

$$\int_0^{R^\theta}\int_{B(x_0,R)}|u|^p\mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r\,d\mu(x)\,dt$$
$$\le CR^{\left(1+\frac{\sigma}{2}\right)\left(\frac{1}{k}-\eta_1'\right)}\left(\int_{B(x_0,\gamma R)}a^{\frac{-r\eta_2}{q\eta_1}}(x)\,d\mu(x)\right)^{\frac{\eta_1'}{\eta_2}}.$$

Using (3.3) and (4.2), we deduce that

$$\int_0^{R^\theta}\int_{B(x_0,R)}|u|^p\mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r\,d\mu(x)\,dt\le C,$$

which yields

$$|u|^p\mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r\in L^1(\Omega). \tag{4.19}$$

To this point, it is useful to return to the inequality (4.3). Thus, integrating it (with respect to t) and taking in consideration the properties of the functions φ and F, we obtain

$$\int_\Omega|u|\left|\frac{\partial^k\varphi}{\partial t^k}\right|\,d\mu(x)\,dt$$
$$=\int_{R^\theta}^{2R^\theta}\int_{\mathbb{M}}|u|\left|\frac{\partial^k\varphi}{\partial t^k}\right|\,d\mu(x)\,dt$$
$$\le\int_{R^\theta}^{2R^\theta}\left(\int_{\mathbb{M}}|u|^p\varphi\,d\mu(x)\right)^{\frac{1}{\eta_1}}\left(\int_{\mathbb{M}}a(x)|u|^q\,d\mu(x)\right)^{\frac{r}{q\eta_1}}\left(\int_{\mathbb{M}}a^{\frac{-r\eta_2}{q\eta_1}}(x)\left|\frac{\partial^k\varphi}{\partial t^k}\right|^{\eta_2}\varphi^{\frac{-\eta_2}{\eta_1}}\,d\mu(x)\right)^{\frac{1}{\eta_2}}\,dt$$
$$\le\left(\int_{(R^\theta,2R^\theta)\times M}|u|^p\mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r\varphi\,d\mu(x)\,dt\right)^{\frac{1}{\eta_1}}\left(\int_0^\infty\left(\int_{\mathbb{M}}a^{\frac{-r\eta_2}{q\eta_1}}(x)\left|\frac{\partial^k\varphi}{\partial t^k}\right|^{\eta_2}\varphi^{\frac{-\eta_2}{\eta_1}}\,d\mu(x)\right)^{\frac{\eta_1'}{\eta_2}}\,dt\right)^{\frac{1}{\eta_1'}}$$
$$\le\left(\int_{(R^\theta,2R^\theta)\times M}|u|^p\mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r\,d\mu(x)\,dt\right)^{\frac{1}{\eta_1}}\left(\int_0^\infty\left(\int_{\mathbb{M}}a^{\frac{-r\eta_2}{q\eta_1}}(x)\left|\frac{\partial^k\varphi}{\partial t^k}\right|^{\eta_2}\varphi^{\frac{-\eta_2}{\eta_1}}\,d\mu(x)\right)^{\frac{\eta_1'}{\eta_2}}\,dt\right)^{\frac{1}{\eta_1'}},$$

that is, we have the first estimate

$$\int_\Omega|u|\left|\frac{\partial^k\varphi}{\partial t^k}\right|\,d\mu(x)\,dt\le I_1^{\frac{1}{\eta_1'}}\left(\int_{(R^\theta,2R^\theta)\times M}|u|^p\mathcal{N}_q\left(a^{\frac{1}{q}}u\right)^r\,d\mu(x)\,dt\right)^{\frac{1}{\eta_1}}. \tag{4.20}$$

Similarly, we have to combine the properties of the functions φ and ξ_R, and hence we obtain the second estimate

$$\int_{\Omega} |u||\Delta\varphi|\, d\mu(x)\, dt \leq I_2^{\frac{1}{\eta_1'}} \left(\int_{(0,\infty)\times B(x_0,\gamma R)\setminus B(x_0,R)} |u|^p \mathcal{N}_q \left(a^{\frac{1}{q}} u\right)^r d\mu(x)\, dt \right)^{\frac{1}{\eta_1}}. \quad (4.21)$$

If we use (3.3) together with (4.1), (4.10), (4.16), (4.17), (4.18), (4.20), and (4.21), then we deduce the inequality

$$\begin{aligned} &C \int_0^{R^\theta} \int_{B(x_0,R)} |u|^p \mathcal{N}_q \left(a^{\frac{1}{q}} u\right)^r d\mu(x)\, dt \\ &\leq \left(\int_{(R^\theta,2R^\theta)\times M} |u|^p \mathcal{N}_q \left(a^{\frac{1}{q}} u\right)^r d\mu(x)\, dt \right)^{\frac{1}{\eta_1}} \\ &\quad + \left(\int_{(0,\infty)\times B(x_0,\gamma R)\setminus B(x_0,R)} |u|^p \mathcal{N}_q \left(a^{\frac{1}{q}} u\right)^r d\mu(x)\, dt \right)^{\frac{1}{\eta_1}}. \end{aligned}$$

To finish the proof, we need to show that $u \equiv 0$ a.e. in Ω. Passing to the limit as $R \to \infty$ in the above inequality and using (4.19), we obtain

$$\int_{\Omega} |u|^p \mathcal{N}_q \left(a^{\frac{1}{q}} u\right)^r d\mu(x)\, dt = 0,$$

which yields $u \equiv 0$ a.e. in Ω. This fact concludes the proof of the nonexistence of nontrivial global weak solutions to (1.1)-(1.2) when $\sigma \in (-2, 2]$.

It remains to consider now the case $\sigma > 2$. In this case, we have

$$-C_0(N-1)\left(1+\rho^2(x)\right)^{-\frac{\sigma}{2}} \geq -C_0(N-1)\left(1+\rho^2(x)\right)^{-1}.$$

Hence, (3.2) is also satisfied for $\sigma = 2$. Then the conclusion follows from the case $\sigma = 2$ already studied above. The proof of Theorem 3.1 is completed. □

5 Conclusions

This manuscript focused on extension and refinement of certain results about the nonexistence of nontrivial solutions to (higher order) evolution inequalities. The framework setting is a Riemannian manifold satisfying the condition:

$$\mathrm{Ric} \geq -C_0(N-1)\left(1+\rho^2(x)\right)^{-\frac{\sigma}{2}},$$

for some $C_0 > 0$ and $\sigma \geq -2$, on the Ricci curvature. This lower bound condition is a key assumption in our analysis, concerning the geometry of the manifold. Thus, without adopting additional topological assumptions and according to the classical test function method, we were able to obtain technical estimates of various integral terms along the manuscript. It is worth mentioning that these estimates depend on the choice of point x_0. Additionally, we remark the importance in controlling the gradient of the test functions and assuming an explicit decay for the Laplacian term, as stated in Lemma 2.1 (see also the related discussion in [25]). Indeed, these control bounds together with the above negative bound to the Ricci curvature made our approach well-suited to the study.

Acknowledgements
The authors wish to thank the knowledgeable referees for the constructive remarks.
Disclosure statement
No potential conflict of interest was reported by the authors.

References

[1] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J Fac Sci Univ Tokyo Sect. I 13 (1966) 109–124.

[2] D.G. Aronson, H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics. Adv Math. 30 (1978) 33–76.

[3] K. Hayakawa, On nonexistence of global solutions of some semilinear parabolic differential equations. Proc Japan Acad. 49 (1973) 503–505.

[4] K. Kobayashi, T. Sirao, H. Tanaka, On the growing up problem for semilinear heat equations. J Math Soc Japan. 29 (1977) 407–424.

[5] E. Mitidieri, S.I. Pohozaev, A priori estimates and blow-up of solutions of nonlinear partial differential equations and inequalities. Proc Steklov Inst Math. 234 (2001) 1–362.

[6] T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations. Comm Pure Appl Math. 33 (1980) 501–505.

[7] V. Georgiev, H. Lindblad, C.D. Sogge, Weighted Strichartz estimates and global existence for semilinear wave equations. Am J Math. 119 (1997) 1291–1319.

[8] R.T. Glassey, Existence in the large for $\Box u = f(u)$ in two space dimensions. Math Z. 178 (1981) 233–261.

[9] N.-A. Lai, Y. Zhou, An elementary proof of Strauss conjecture. J Funct Anal. 267 (2014) 1364–1381.

[10] J. Schaeffer, The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p. Proc R Soc Edinb. 101A (1985) 31–44.

[11] T.C. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions. J. Differential Equations. 52 (1984) 378–406.

[12] B. Yordanov, Q.S. Zhang, Finite time blow up for critical wave equations in high dimensions. J Funct Anal. 231 (2006) 361–374.

[13] Y. Zhou, Blow up of solutions to semilinear wave equations with critical exponent in high dimensions. Chin Ann Math Ser B. 28 (2007) 205–212.

[14] G.G. Laptev, Some nonexistence results for higher order evolution inequalities in cone-like domains. Electron Res Announc Amer Math Soc. 7 (2001) 87–93.

[15] C.S. Chen, J.C. Huang, Some nonexistence results for degenerate parabolic inequalities with local and nonlocal nonlinear terms. J Nanjing Univ Math Biq. 21 (2004) 12–20.

[16] S. Xiao, Z.B. Fang, Nonexistence of solutions for the quasilinear parabolic differential inequalities with singular potential term and nonlocal source. J Inequal Appl. 2020:64 (2020).

[17] A. Nabti, A. Alsaedi, M. Kirane, B. Ahmad, Nonexistence of global solutions of fractional diffusion equation with time-space nonlocal source. Adv Differ Equ. 2020:625 (2020).

[18] S. Xiao, Z.B. Fang, Nonexistence of solutions for quasilinear hyperbolic inequalities. J Inequal Appl. 2021:151 (2021).

[19] Q. Ru, A nonexistence result for a nonlinear wave equation with damping on a Riemannian manifold. Bound Value Probl. 2016:198 (2016).

[20] A. Grigor'yan, V. A. Kondratiev, On the existence of positive solutions of semilinear elliptic inequalities on Riemannian manifolds, In Around the research of Vladimir Maz'ya. II, volume 12 of Int. Math. Ser. (N. Y.), pages 203–218, Springer, New York, 2010.

[21] A. Grigor'yan, Y. Sun, On non-negative solutions of the inequality $\Delta u + u^\sigma \leq 0$ on Riemannian manifolds. Comm Pure Appl Math. 67 (2014) 1336–1352.

[22] P. Mastrolia, D. Monticelli, F. Punzo, Nonexistence results for elliptic differential inequalities with a potential on Riemannian manifolds. Calc Var Part Diff. Eq. 54 (2015) 1345–1372.

[23] P. Mastrolia, D. Monticelli, F. Punzo, Nonexistence of solutions to parabolic differential inequalities with a potential on Riemannian manifolds. Math Ann. 367 (2017) 929–963.

[24] D.D. Monticelli, F. Punzo, M. Squassina, Nonexistence for hyperbolic problems on Riemannian manifolds. Asymptot Anal. 120 (2020) 87–101.

[25] D. Bianchi, A. Setti, Laplacian cut-offs, porous and fast diffusion on manifolds and other applications. Calc Var Part Diff Eq. 57:4 (2018) 1–33.

[26] I. Azman, M. Jleli, B. Samet, Blow-up of solutions to parabolic inequalities in the Heisenberg group. Electron J Differ Equ. 2015:167 (2015).

[27] C. Bandle, M.A. Pozio, A. Tesei, The Fujita exponent for the Cauchy problem in the hyperbolic space. J Differential Equations. 251 (2011) 2143–2163.

[28] S. Bordoni, R. Filippucci, P. Pucci, Nonlinear elliptic inequalities with gradient terms on the Heisenberg group. Nonlinear Anal. 121 (2015) 262–279.

[29] M. Boutefnouchet, M. Kirane, Nonexistence of solutions of some non-linear non-local evolution systems on the Heisenberg group. Fract Calc Appl Anal. 18 (2015) 1336–1349.

[30] L. D'Ambrosio, Critical degenerate inequalities on the Heisenberg group. Manuscripta Math. 106 (2001) 519–536.

[31] M. Jleli, M. Kirane, B. Samet, Nonexistence results for pseudo-parabolic equations in the Heisenberg group. Monatsh Math. 180 (2016) 255–270.

[32] M. Ruzhansky, N. Tokmagambetov, Nonlinear damped wave equations for the sub-Laplacian on the Heisenberg group and for Rockland operators on graded Lie groups. J Differential Equations. 265 (2018) 5212–5236.

[33] B. Torebek A. Kassymov, N. Tokmagambetov, Nonexistence results for the hyperbolic-type equations on graded Lie groups. Bull Malays Math Sci Soc. 43 (2020) 4223–4243.

[34] L. J. Alias, P. Mastrolia, M. Rigoli, Maximum principles and geometric applications, Springer, Cham, 2016.

[35] P. Petersen, Riemannian geometry. Graduate Texts in Mathematics, Springer, New York, 2006.

[36] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. Bull Amer Math Soc. 36 (1999) 135–249.