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On Toroidal groups in relationship with generalized Jacobians, and non-totally real number fields

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Declaration of Authorship

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Signed: Alessandro Dioguardi Burgio

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"My work is now finished. Here at last, on the shores of the sea... comes the end of our Fellowship. I will not say do not weep, for not all tears are an evil."

Gandalf the White (The Lord of the Rings: The Return of the King)

UNIVERSITY OF PALERMO

Abstract

Department of Mathematics and Computer Sciences

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by Alessandro DIOGUARDI BURGIO

This dissertation explores the connections between toroidal groups, generalized Jacobians, and non-totally real number fields, consolidating results from articles and preprints with unpublished material. After presenting the foundational theory of toroidal groups, their construction from number fields, and the essential properties of Jacobians and hyperelliptic functions in Chapter 1, the work delves into the isomorphisms between quasi-Abelian varieties and generalized Jacobians of elliptic and hyperelliptic curves, employing meromorphic periodic functions (Chapter 2). Chapter 3 investigates the relationship between toroidal groups and non-totally real number fields, starting with cubic fields and extending to any dimension. The quintic case has been also studied in relations with toroidal groups of complex rank three and real rank five. The main results of this chapter include proving that certain toroidal groups with specific ranks correspond to non-totally real number fields via their endomorphism rings. Additionally, the m-torsion subgroups of toroidal groups are parametrized in the geometric correspondence with generalized Jacobians. These results deepen our understanding of algebraic and geometric structures tied to number fields and periodic functions.

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i	imaginary unit
$\bar{z}$	complex conjugate of z
$\mathbb{C}$	complex numbers
$\mathbb{R}$	real numbers
$\mathbb{Q}$	rational numbers
$\mathbb{Z}$	integer numbers
$\mathbb{Z}[x]$	ring of integer polynomials
$\mathrm{M}_n(R)$	algebra of square matrices of coefficients in the ring R
$\mathcal{T}$	toroidal group
Λ	lattice
$\mathrm{rank}_{\mathbb{R}}$	real rank
$\mathrm{rank}_{\mathbb{C}}$	complex rank
Π	period matrix
I_n	identity matrix of dimension n
$\mathrm{Im}(A)$	imaginary part of matrix A
$\widehat{f}$	lift of f
$\mathrm{Hom}(X_1, X_2)$	group of homomorphisms from X_1 to X_2
$\mathrm{End}(X)$	group of endomorphisms of X
ϱ_r	rational representation
ϱ_a	analytic representation
$\mathrm{End}_0(X)$	algebra $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$
$\mathcal{O}_K$	ring of integers of the number field K
(r_1, r_2)	singature of a number field
Φ	type of a number field
$\mathcal{O}$	order of a number field
Δ_K, $\mathrm{disc}(K)$	discriminant of the number field K
$\mathrm{disc}(\mathcal{O})$	discriminant of the order $\mathcal{O}$
μ, μ_Φ	Minkowski map relative to the type Φ
$\mathrm{Div}(X)$	group of divisors of X
$\mathrm{Div}^0(X)$	group of degree zero divisors of X
$\mathrm{PDiv}(X)$	group of principal divisors of X
$\mathrm{Pic}(X)$	Picard group of divisors of X
$\mathrm{Pic}^0(X)$	zero part of Picard group of X
$\mathcal{C}$	elliptic curve
$\mathcal{H}$	hyperelliptic curve
Ω	point at the infinity
$\mathfrak{m}$	modulus
$\mathfrak{J}$	Jacobian
$\mathfrak{J}_{\mathfrak{m}}$	generalized Jacobian relative to the modulus $\mathfrak{m}$
c_L	factor system of modulus L
$V(a_1, \ldots, a_n)$	Vandermonde matrix in $a_1, \ldots, a_n$
$V_k(a_1, \ldots, a_n)$	Vandermonde matrix in $a_1, \ldots, a_n$ missing the k-th power

Introduction

Toroidal groups were introduced in 1964 by Kopfermann [44]. These groups are the non-compact generalizations of complex tori and play a fundamental role for the studying of any Abelian complex Lie group. Morimoto [51], [52] studied complex Lie groups without non-constant holomorphic functions. The term *(H, C)-groups*, where H=holomorphic and C=constants, can indeed be found in the literature as a reference to toroidal groups. One of the most important results of the works of Morimoto, along with that of Remmert (cf. [44, pp. 16-17]), is the decomposition named after them. In this theorem (cf. Theorem 1.8) the importance of the toroidal groups was highlighted. Actually, every connected Abelian complex Lie group can be decomposed into the product of a linear part, an additive one, and a toroidal group. It was clear then that the study of Abelian complex Lie groups could be reduced to the study of the toroidal ones. Later, the cohomology and the line bundles of toroidal groups have been studied by Kazama [40], Umeno [66], Abe [2], [3], [5], and Vogt [70], [71]. A special mention goes to the book by Abe and Kopfermann [6], the first comprehensive survey to systematically cover the theory of toroidal groups and the main reference for whoever wants to explore the topic.

On the other hand, Andreotti and Gherardelli [7] in 1973 worked on the link between number fields and *quasi-Abelian varieties* [8], associating to any quotient $\mathbb{C}^n/\Lambda$, with the real rank of Λ equal to $n+1$, a non-totally real number field [33], with a particular attention to the case of $n = 2$. This connection was studied later also by Abe [4], [1] and Vailleres [69] but its origin is very old. As early as 1722, when De Moivre published his celebrated formula, and even better when Euler rewrote it in terms of the exponential map $\vartheta \longmapsto e^{2\pi i \vartheta}$, it was clear that the fractional ideals of $\mathbb{Q}$ are related to the torsion subgroups of the unit circle $\mathbb{S}^1 \leq \mathbb{C}^*$. It was Gauss who first described in 1799 a continuous doubly periodic function on $\mathbb{C}$ with two $\mathbb{R}$-independent periods. This gives a correspondence between fractional ideals of a non-totally real quadratic number field and the torsion subgroups of a complex torus, given by the Jacobian $\mathfrak{J}(\mathcal{C})$ of an elliptic curve $\mathcal{C}$ (on the contrary, when a quadratic field is totally real any fractional ideal is dense in $\mathbb{R}$, reflecting the fact that we cannot have a continuous doubly periodic function on $\mathbb{R}$). Toroidal groups play the roles of the unit circle $\mathbb{S}^1$ firstly, and of the complex torus secondly in the higher-dimensional setting for this correspondence.

Historically, the first example of a toroidal group was given by Cousin and for this reason these groups are sometimes called in literature *Cousin-quasi-tori* or *Cousin groups*. In 1889, Poincaré won a prize, sponsored by King Oscar II of Sweden, which was offered to investigate two-variate complex functions with four real-independent periods. Nevertheless, Appell gave, as well, a solution which gained the second place, and was inspiring for further researches: the solutions left the question of a two-variate complex function with three real-independent periods unexplored, and it was Cousin, a student of Poincaré, who proved the existence of such a function, and showed its main properties [21]. With necessary restrictions on the periods, such a periodic function defines a toroidal group, given by the quotient of $\mathbb{C}^2$ by the lattice spanned by the three periods.

Periodic meromorphic functions and toroidal groups are the subject of a paper dated 1991 [20] by Capocasa and Catanese, where the authors proved that, for a lattice $\Lambda \subset \mathbb{C}^n$, there exists a non-constant Λ-periodic meromorphic function on $\mathbb{C}^n$ if and only if $\mathbb{C}^n/\Lambda$ is quasi-Abelian (in the sense of Andreotti and Gherardelli [8]).

Quasi-Abelian varieties were first introduced in 1947 by Severi [63] exactly in the context of periodic functions. However, by the theorem proved by Capocasa and Catanese, the two concepts of quasi-Abelian varieties given respectively by Severi and Andreotti-Gherardelli coincide. These varieties were later studied by Rosenlicht [59] (see also [58]) as generalizations of the classic Jacobians of a curve. Rosenlicht was awarded the Frank Nelson Cole Prize in Algebra 1960 for this construction, which gives an explicit example of a connected commutative Lie (resp. algebraic) group which is neither linear, nor compact (resp. complete), nor the direct product of a linear group and a compact (resp. complete) group. As such, generalized Jacobians are a central subject of Serre's first book [62].

This dissertation primarily focuses on the connection between toroidal groups, generalized Jacobians, and non-totally real number fields. The aim of this work was to gather articles and preprints ([27], [25], [26]) produced over the past three years and combining them with some unpublished materials.

Chapter 1 collects all the necessary preliminaries. More precisely, we recall the basic theory of toroidal groups as well as the construction of a toroidal group from a number field. We furnish the main definitions and results concerning the ordinary and generalized Jacobians, and, in passing, we introduce the theory of hyperelliptic functions.

In Chapter 2, we study the isomorphisms between a quasi-Abelian variety and the generalized Jacobian of a suitable elliptic first and hyperelliptic curve later. The content of this chapter continues the research work started in [23] and continued in [27]. The results of this chapter are obtained by the means of meromorphic periodic functions, whose existence is guaranteed by the fact that we are considering a quasi-Abelian variety. The main results of this section are Theorem 2.6 and the construction of the periodic functions (2.12).

Chapter 3 presents the results of two papers. Firstly, the results obtained in the paper of mine [27] are presented and generalized. Secondly, the derived results of the preprint [25] are presented. In particular, we study the above mentioned connection between toroidal groups and non-totally real number fields. We start from the case of cubic fields, which is the nicest case for this correspondence. Indeed, we prove in Theorem 3.4 that if one considers a non-totally real cubic number field, then it arises as the ring of endomorphisms (tensorized with $\mathbb{Q}$) of a toroidal group (having extra multiplications) with complex rank two and real rank three. Conversely, any toroidal groups $\mathbb{C}^2/\Lambda$ having extra multiplications with real rank three is isomorphic to one arising from a cubic number field. After that, we move to the quartic case, where the situation is already more delicate. Under some assumption on the field, the results of the cubic case are extended to the quartic fields (with one pair of complex embeddings), which illustrate the general case proved in the following (the main results for the quartic case are Theorems 3.13, 3.15, and 3.17, and for the general case are Theorems 3.27, 3.29). Lastly, Theorems 3.33, 3.35, and 3.36 describe the more intricate case of the relationship between quintic fields and toroidal groups with complex rank three and real rank five (the first case where the group is not necessarily a quasi-Abelian variety). In passing, we also provide in the cubic case a parametrization of the m-torsion subgroups of a toroidal group in the geometric correspondence with the generalized Jacobian of an elliptic curve (cf. Theorem 3.10).

Chapter 1

Backgrounds

1.1 Abelian complex Lie groups

Definition 1.1. A **toroidal group** is a connected complex Lie group $\mathcal{T}$ such that every holomorphic function on $\mathcal{T}$ is constant.

A **lattice** Λ of $\mathbb{C}^n$ is a discrete subgroup, therefore it is a free $\mathbb{Z}$-module. A **basis** of a lattice $\Lambda \subset \mathbb{C}^n$ is a set $\{\lambda_1, \ldots, \lambda_r\}$ of elements of Λ that are $\mathbb{R}$-independent $\mathbb{Z}$-generators. Every basis of Λ has the same cardinality r that we will call the **real rank** of Λ and it will be denoted by $\operatorname{rank}_{\mathbb{R}} \Lambda$.

Proposition 1.2. *Every connected Abelian complex Lie group is isomorphic to* $\mathbb{C}^n/\Lambda$, *where* Λ *is lattice of* $\mathbb{C}^n$.

Proof. One can consider the exponential map of the complex Lie group, which is surjective with discrete kernel. Alternatively, one can discuss about the universal covering group as in [6, Proposition 1.1.2]. □

By [51, p. 260] every toroidal group must be Abelian, then isomorphic to $\mathbb{C}^n/\Lambda$, for a lattice Λ, from Proposition 1.2.

For a lattice $\Lambda \subset \mathbb{C}^n$ with basis $\{\lambda_1, \ldots, \lambda_r\}$, we will define

$$\mathbb{R}_\Lambda := \operatorname{span}_{\mathbb{R}}\{\lambda_1, \ldots, \lambda_r\} = \{x_1\lambda_1 + \cdots + x_r\lambda_r \mid x_1, \ldots, x_r \in \mathbb{R}\}$$

and $\mathbb{C}_\Lambda := \operatorname{span}_{\mathbb{C}}\{\lambda_1, \ldots, \lambda_r\} = \mathbb{R}_\Lambda + i\mathbb{R}_\Lambda$.

Definition 1.3. The **complex rank** of Λ is the dimension of $\mathbb{C}_\Lambda$ as $\mathbb{C}$-linear space, that is, the number of $\mathbb{C}$-linearly independent elements of each basis of Λ. We will denote it by $\operatorname{rank}_{\mathbb{C}} \Lambda$.

Fixed a ordered basis $\{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$ and a ordered basis $\{\lambda_1, \ldots, \lambda_r\}$ of Λ, the matrix

$$\Pi = \begin{pmatrix} \lambda_{1,1} & \ldots & \lambda_{1,r} \\ \vdots & & \vdots \\ \lambda_{n,1} & \ldots & \lambda_{n,r} \end{pmatrix} \in \mathrm{M}_{n\times r}(\mathbb{C}),$$

where $\lambda_j = \sum_{i=1}^n \lambda_{i,j} v_i$, is called a **period matrix** of $X = \mathbb{C}^n/\Lambda$. Clearly, a period matrix Π depends on the choice of a basis of the vector space $\mathbb{C}^n$ and a basis of the lattice Λ. Notice that the real rank of Λ is equal to the number of $\mathbb{R}$-independent columns of Π, while the complex rank is just the rank of the matrix Π.

We will call complex rank, real rank and a period matrix of a toroidal group $\mathcal{T}$, denoted by $\operatorname{rank}_{\mathbb{C}} \mathcal{T}$ and $\operatorname{rank}_{\mathbb{R}} \mathcal{T}$ respectively, the complex rank, real rank and a period matrix of the lattice Λ defining $\mathcal{T}$, that is $\mathcal{T} = \mathbb{C}^n/\Lambda$.

If the complex rank of $\Lambda \subset \mathbb{C}^n$ is $m < n$, then after a linear change of coordinates we can assume that $\mathbb{C}_\Lambda$ is the $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ of the first m coordinates.

If the complex rank and real rank of $\Lambda \subset \mathbb{C}^n$ are equal to n, then after a linear change of coordinates we can assume that Λ is generated by the canonical basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$, therefore

$$\mathbb{C}^n/\Lambda \cong \mathbb{C}^n/\mathbb{Z}^n \cong (\mathbb{C}/\mathbb{Z})^n \cong (\mathbb{C}^*)^n,$$

where we recall that $\mathbb{C}/\mathbb{Z}$ is isomorphic to the multiplicative group of the complex numbers $\mathbb{C}^*$ via the exponential map $z \mapsto \exp(2\pi iz)$. The group $(\mathbb{C}^*)^n$ is not toroidal (since we can define many holomorphic functions on it, for example the exponentiation).

If the complex rank of $\Lambda \subset \mathbb{C}^n$ is n and real rank is $n+q$ with $1 \leq q \leq n$, then after a linear change of coordinates we can assume that $e_1, \dots, e_n$ are in Λ and that $\Lambda = \mathbb{Z}^n \oplus \Gamma$, where Γ is a lattice of $\mathbb{C}^n$ of real rank q.

Definition 1.4. A **complex torus** $\mathbb{T}$ is a compact connected complex Lie group.

It turns out from definition that every complex torus is an Abelian group, then isomorphic to a quotient $\mathbb{C}^n/\Lambda$, for a lattice Λ of *full real rank* $2n$. A complex torus is a toroidal group (see [51, p. 260]). In particular, complex tori are exactly compact toroidal groups, in other words: a toroidal group is a complex torus if, and only if, it is compact. Classical examples of complex tori are provided by the *elliptic curves*, that is, the set of points of the complex plane satisfying the equation $y^2 = x^3 + ax + b$ (for $a, b \in \mathbb{C}$) equipped with chord-and-tangent group law (cf. Example 1.47 below).

Proposition 1.5. *Let Λ be a lattice in $\mathbb{C}^n$. If* $\operatorname{rank}_{\mathbb{C}} \Lambda = m < n$, *then*

$$\mathbb{C}^n/\Lambda \cong \mathbb{C}^{n-m} \oplus (\mathbb{C}^m/\Lambda),$$

where Λ is considered as lattice of $\mathbb{C}^m$.

Since one can define many holomorphic functions on $\mathbb{C}^{n-m}$, a fundamental corollary of the above proposition is that every lattice $\Lambda \subset \mathbb{C}^n$ that defines a toroidal group must be of maximal rank.

Corollary 1.6. *Every toroidal group $\mathcal{T} = \mathbb{C}^n/\Lambda$ has maximal complex rank n.*

Let $\pi\colon \mathbb{C}^n \to \mathbb{C}^n/\Lambda$ be the natural projection. Let $M\mathbb{C}_\Lambda$ be the maximal complex subspace of the real space $\mathbb{R}_\Lambda$, that is $M\mathbb{C}_\Lambda = \mathbb{R}_\Lambda \cap i\mathbb{R}_\Lambda$. The subgroup $\pi(\mathbb{R}_\Lambda)$ is the **maximal real torus** of $\mathbb{C}^n/\Lambda$, note indeed that $\mathbb{R}_\Lambda/\Lambda$ is a real torus of dimension equal to the real rank of Λ and it is the maximal compact real subgroup of $\mathbb{C}^n/\Lambda$. On the other hand, $\pi(M\mathbb{C}_\Lambda)$ is a complex subgroup of $\mathbb{C}^n/\Lambda$. It is the **maximal complex subgroup** of the maximal real torus $\pi(\mathbb{R}_\Lambda)$.

complex span of Λ	$\mathbb{C}_\Lambda$	$\mathcal{T} = \mathbb{C}_\Lambda/\Lambda$	toroidal group
	|	|	
real span of Λ	$\mathbb{R}_\Lambda$	$\mathbb{R}_\Lambda/\Lambda$	maximal real torus
	|	|	
maximal complex subspace	$M\mathbb{C}_\Lambda$	$M\mathbb{C}_\Lambda/(M\mathbb{C}_\Lambda \cap \Lambda)$	maximal complex real group

Theorem 1.7. *Let Λ be a lattice in $\mathbb{C}^n$ of maximal complex rank. The following are equivalent*

- *$\mathbb{C}^n/\Lambda$ is toroidal;*

- ***Irrationality condition:*** *There exists no $l \in \mathbb{C}^n, l \neq 0$ such that the scalar product $l \cdot \lambda \in \mathbb{Z}$, for all $\lambda \in \Lambda$;*

- ***Density condition:*** *The maximal complex subgroup $\mathbb{MC}_\Lambda/(\mathbb{MC}_\Lambda \cap \Lambda)$ is dense in the maximal real torus $\mathbb{R}_\Lambda/\Lambda$.*

Theorem 1.8 (Decomposition of Remmert-Morimoto)**.** *Every connected complex Abelian group is isomorphic to a*

$$\mathbb{C}^\ell \times (\mathbb{C}^*)^m \times \mathcal{T},$$

where $\mathcal{T}$ is a toroidal group. The decomposition is unique.

Remark 1.9. The Remmert-Morimoto decomposition ([44], [51]) is a complex equivalent of the decomposition of a connected Abelian real Lie group. In fact, every connected Abelian real Lie group is isomorphic to

$$\mathbb{R}^\ell \times (\mathbb{R}/\mathbb{Z})^m,$$

where the second factor is a m-dimensional real torus. In particular, every compact connected Abelian real Lie group is isomorphic to a real torus ($\ell = 0$ in the decomposition).

We introduce now two special systems of coordinates for toroidal groups.

Proposition 1.10. *Let $\Lambda \subset \mathbb{C}^n$ be a lattice of maximal complex rank n and real rank $n + q$.*

1. ***Standard coordinates:*** *There exists a basis such that Λ has period matrix*

$$\Pi = (I_n|T) = \begin{pmatrix} I_q & 0 & \widehat{T} \\ 0 & I_{n-q} & \widetilde{T} \end{pmatrix}$$

with $\mathrm{Im}(\widehat{T}) \in \mathrm{GL}_q(\mathbb{C})$ and $\widetilde{T} \in \mathrm{M}_{(n-q)\times q}(\mathbb{C})$;

2. ***Toroidal coordinates:*** *There exists a basis such that Λ has period matrix*

$$\Pi' = \begin{pmatrix} 0 & I_q & \widehat{T} \\ I_{n-q} & R_1 & R_2 \end{pmatrix}$$

where $B = (I_q|\widehat{T})$ is the period matrix of a q-dimensional complex torus, the matrices $R_1 = -\mathrm{Im}(\widetilde{T})\,\mathrm{Im}(\widehat{T})^{-1}$ and $R_2 = \widetilde{T} + R_1\widehat{T}$ have real entries.

The $(n-q) \times 2q$ real matrix $R = (R_1|R_2)$ is called the *glueing matrix*. Standard and toroidal coordinates are not uniquely determined. The irrationality condition becomes more practical to utilize in standard and toroidal coordinates.

Proposition 1.11. *Let Λ be a lattice of maximal rank n. The following are equivalent.*

1. *$\mathbb{C}^n/\Lambda$ is toroidal;*
2. *There exists no non-zero $\mathbf{s} \in \mathbb{Z}^n$ such that $\mathbf{s}^\mathrm{T} T \in \mathbb{Z}^n$;*
3. *There exists no non-zero $\mathbf{s} \in \mathbb{Z}^{n-q}$ such that $\mathbf{s}^\mathrm{T} R \in \mathbb{Z}^{2q}$.*

Example 1.12. Let $\mathcal{T}$ be an Abelian complex Lie group of period matrix

$$\begin{pmatrix} -i & 1 & 0 & \sqrt{2}+3+i \\ -1 & 1-i & -i & 4-i(2+\sqrt{2}+\sqrt{3}) \\ -i & 1 & 1 & 3+\sqrt{2}+\sqrt{3}+i \end{pmatrix}$$

having real rank four. A period matrix in standard coordinates is given by

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & i\sqrt{2} \\ 0 & 1 & 0 & 3+i \\ 0 & 0 & 1 & \sqrt{3} \end{pmatrix}.$$

So, $\widehat{T} = i\sqrt{2} \in \mathrm{M}_1(\mathbb{C}) = \mathbb{C}$ having invertible imaginary part and $\widetilde{T} = \left(3+i, \sqrt{3}\right)^{\mathrm{T}} \in \mathrm{M}_{2\times 1}(\mathbb{C})$. A transformation from standard coordinates to toroidal coordinates is given by (see [6, p. 10], [4])

$$\begin{pmatrix} I_q & 0 \\ R_1 & I_{n-q} \end{pmatrix} \begin{pmatrix} I_q & 0 & \widehat{T} \\ 0 & I_{n-q} & \widetilde{T} \end{pmatrix} = \begin{pmatrix} I_q & 0 & \widehat{T} \\ R_1 & I_{n-q} & R_1\widehat{T} + \widetilde{T} \end{pmatrix},$$

where

$$R_1 = -\operatorname{Im}(\widetilde{T}) \operatorname{Im}(\widehat{T})^{-1} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \in \mathrm{M}_{2\times 1}(\mathbb{R}).$$

Hence, we obtain a period matrix in toroidal coordinates

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & i\sqrt{2} \\ 0 & 1 & 0 & 3+i \\ 0 & 0 & 1 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & i\sqrt{2} \\ -\frac{1}{\sqrt{2}} & 1 & 0 & 3 \\ 0 & 0 & 1 & \sqrt{3} \end{pmatrix},$$

where $R_2 = (3, \sqrt{3})^{\mathrm{T}} \in \mathrm{M}_{2\times 1}(\mathbb{R})$. We verify here that $\mathcal{T}$ is toroidal by the irrational condition in the two systems of coordinates. Let us suppose first that there exists $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{Z}^3$ such that

$$s_1 i\sqrt{2} + s_2(3+i) + s_3\sqrt{3} = S \in \mathbb{Z}.$$

It follows that $(3s_2 + s_3\sqrt{2} - S) + (s_1\sqrt{2} + s_2)i = 0$, therefore $3s_2 + s_3\sqrt{2} - S = 0$ and $s_1\sqrt{2} + s_2 = 0$. We get that $s_1 = 0 = s_2$ and $s_3 = 0 = S$, thus the irrationality condition in standard coordinates is verified.
Let us suppose that there exists $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$ such that

$$\begin{pmatrix} s_1 & s_2 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & 3 \\ 0 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$$

for some $S_1, S_2 \in \mathbb{Z}$. It follows that $-s_1\frac{1}{\sqrt{2}} = S_1$ and $3s_1 + s_2\sqrt{3} = S_2$. One get that $s_1 = 0 = s_2$. Thus, $\mathcal{T}$ is toroidal.

1.1.1 Homomorphism of complex Lie group

A **homomorphism of complex Lie groups** is a holomorphic homomorphism of groups. Let $X_1 = \mathbb{C}^{n_1}/\Lambda_1, X_2 = \mathbb{C}^{n_2}/\Lambda_2$ be two connected Abelian complex Lie groups of maximal complex rank and $f\colon X_1 \to X_2$ a homomorphism. There exists a $\mathbb{C}$-linear map $\widehat{f}\colon \mathbb{C}^{n_1} \to \mathbb{C}^{n_2}$, named the **lift** or the **linear extension** of f, such that the diagram

$$\begin{array}{ccc} \mathbb{C}^{n_1} & \xrightarrow{\widehat{f}} & \mathbb{C}^{n_2} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{C}^{n_1}/\Lambda_1 & \xrightarrow[f]{} & \mathbb{C}^{n_2}/\Lambda_2 \end{array} \tag{1.1}$$

commutes and $\widehat{f}(\Lambda_1) \subseteq \Lambda_2$, where π_1 and π_2 are the natural projections. Conversely, every $\mathbb{C}$-linear map $\widehat{f}\colon \mathbb{C}^{n_1} \to \mathbb{C}^{n_2}$ with $\widehat{f}(\Lambda_1) \subseteq \Lambda_2$ induces a complex homomorphism $f\colon X_1 \to X_2$ such that the diagram (1.1) commutes.

Let us denote by $\mathrm{Hom}(X_1, X_2)$ the group (with respect to pointwise addition) of homomorphisms $X_1 \to X_2$. If $n_1 + q_1$ and $n_2 + q_2$ are respectively the real ranks of X_1 and X_2, then we can define two homomorphisms

$$\varrho_a\colon \mathrm{Hom}(X_1, X_2) \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n_1}, \mathbb{C}^{n_2}) \cong \mathrm{M}_{n_2\times n_1}(\mathbb{C}),\ f \mapsto \varrho_a(f) = \widehat{f}$$

and

$$\varrho_r\colon \mathrm{Hom}(X_1, X_2) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda_1, \Lambda_2) \cong \mathrm{M}_{(n_2+q_2)\times(n_1+q_1)}(\mathbb{Z}),\ f \mapsto \varrho_r(f) = \widehat{f}\restriction_{\Lambda}.$$

The homomorphism ϱ_r is called the **rational representation** and ϱ_a the **analytic representation** of $\mathrm{Hom}(X_1, X_2)$. Let Π_1 be a period matrix of X_1 and Π_2 of X_2. Let A be the matrix associated to the $\mathbb{C}$-linear function $\varrho_a(f) = \widehat{f}\colon \mathbb{C}^{n_1} \to \mathbb{C}^{n_2}$ with respect the above fixed bases. The matrices Π_1, Π_2 and A satisfy

$$A\Pi_1 = \Pi_2 N, \tag{1.2}$$

where $N \in \mathrm{M}_{r_2\times r_1}(\mathbb{Z})$. The equation (1.2) is called **Hurwitz relation**. We wish to emphasize that there are many Hurwitz relations, one for each homomorphism.

Endomorphisms. Denote $\mathrm{End}(X) := \mathrm{Hom}(X, X)$ the **ring of endomorphisms** of $X = \mathbb{C}^n/\Lambda$, with maximal complex rank and $\mathrm{rank}_{\mathbb{R}}\,\Lambda = n + q$, and let Π be a period matrix of X. In this case we have

$$\varrho_a\colon \mathrm{End}(X) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) \cong \mathrm{M}_n(\mathbb{C}),\ f \mapsto \varrho_a(f) = \widehat{f}$$

and

$$\varrho_r\colon \mathrm{End}(X) \to \mathrm{End}_{\mathbb{Z}}(\Lambda) \cong \mathrm{M}_{n+q}(\mathbb{Z}),\ f \mapsto \varrho_r(f) = \widehat{f}\restriction_{\Lambda},$$

which can be extended to the algebra $\mathrm{End}_0(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ to a representation

$$\mathrm{End}_0(X) \to \mathrm{End}_{\mathbb{Q}}(\mathrm{span}_{\mathbb{Q}}\,\Lambda) \cong \mathrm{M}_{n+q}(\mathbb{Q}) \tag{1.3}$$

that we will denote again by the same symbol ϱ_r. Note that in this case the Hurwitz relations is

$$A\Pi = \Pi N \tag{1.4}$$

for $A \in \mathrm{M}_n(\mathbb{C})$ and $N \in \mathrm{M}_{n+q}(\mathbb{Z})$. The matrix A or the relation (1.4) is sometimes called a **multiplication** of Π.

Clearly, for a homomorphism $f\colon X_1 \to X_2$ being bijective one must have that X_1 and X_2 have the same complex and real ranks. In particular, a homomorphism f is bijective if, and only if, $A \in \mathrm{GL}_n(\mathbb{C})$ and $N \in \mathrm{GL}_r(\mathbb{Z})$. Finally, epimorphic image of toroidal groups are toroidal groups.

Proposition 1.13. *Let $f\colon \mathcal{T} \to X$ be a homomorphism from a toroidal group to a complex Abelian Lie group. Then $f(\mathcal{T})$ is a toroidal group.*

Isogenies. In several case, in the context of connected algebraic groups, one needs a weaker notion than isomorphism, the so-called *isogenies*.

Definition 1.14. An **isogeny** is an epimorphism $f\colon X_1 \to X_2$ with finite kernel.

For our case of Abelian connected complex Lie groups, if there exists an isogeny $X_1 \to X_2$ then there exists always a further isogeny $X_2 \to X_1$. In fact, we have the following proposition.

Proposition 1.15 ([24]). *A homomorphism $f\colon X_1 \to X_2$ is an isogeny if and only if A and N, as in (1.4), are square matrices with non-zero determinant. In this case, there exists a further isogeny $g\colon X_2 \to X_1$ such that*

$$g \circ f = l\,\mathrm{id}_{X_1} \quad \textit{and} \quad f \circ g = l\,\mathrm{id}_{X_2},$$

where $l = \det(N)$.

Corollary 1.16 ([24]). *An isogeny f is an isomorphism if and only if $\det(N) = \pm 1$, where N is a matrix associated to $\varrho_r(f)$.*

We will say that two connected Abelian complex Lie groups X_1 and X_2 are **isogenous** if there exists an isogeny $f\colon X_1 \to X_2$.

1.1.2 Extension of Abelian complex Lie groups

We want to look at closed linear subtori, i.e. isomorphic to $(\mathbb{C}^*)^m$ for some integer $m > 0$, of a connected Abelian complex Lie group $X = \mathbb{C}^n/\Lambda$ of maximal rank n. Let $\Pi = (I_n|T)$ be a period matrix in standard coordinates. Let $l_1, \dots, l_{n-m} \in \{1, \dots, n\}$ be fixed distinct integers and consider the $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ given by

$$H(l_1, \dots, l_{n-m}) = \{(z_1, \dots, z_n) \mid z_{l_k} = 0, \text{ for } k = 1, .., n-m\}.$$

The subspace $H = H(l_1, \dots, l_{n-m})$ has dimension m and let consider the matrix $C_H(\Pi)$ obtained starting from the matrix Π canceling any row with exception of those labelled by $l_1, \dots, l_{n-m}$ as well as any column of the first n with exception of those labelled by $l_1, \dots, l_{n-m}$. The matrix obtained is $C_H(\Pi) = (I_{n-m}|T')$, with $T' \in \mathrm{M}_{(n-m)\times q}(\mathbb{C})$. The following proposition holds.

Proposition 1.17 ([24]). *Let $X = \mathbb{C}^n/\Lambda$ be a connected complex Abelian Lie group of maximal rank n and let $\Pi = (I_n|T)$ be a period matrix of X in standard coordinates. If the columns of $C_H(\Pi)$ are $\mathbb{R}$-linearly independent, then $X_0 = (H + \Lambda)/\Lambda$ is a m-dimensional closed linear subtorus of X.*

In the case of the above proposition, the matrix $C_H(\Pi)$ is a period matrix of the quotient group X/X_0.

1.2 Number fields

Definition 1.18. A **number field** K is a finite field extension of rational numbers $\mathbb{Q}$. The **degree** of the number field K is the degree of the extension.

Definition 1.19. An element $\alpha \in K$ is an **algebraic integer** if it is root of a monic polynomial $f(x) \in \mathbb{Z}[X]$. The set of algebraic integers

$$\mathcal{O}_K = \{\alpha \in K \mid f(\alpha) = 0 \text{ for some monic } f(x) \in \mathbb{Z}[x]\ \}$$

is called the **ring of integers** of K.

Definition 1.20. A **fractional ideal** of a number field K is a subring

$$\frac{1}{\nu}\mathcal{J} = \left\{\frac{a}{\nu} \,:\, a \in \mathcal{J}\right\},$$

where $\mathcal{J}$ is a ideal of $\mathcal{O}_K$ and $\nu \in \mathcal{O}_K$.

Example 1.21. The trivial example $K = \mathbb{Q}$ is a number field of degree one, the ring of integers of $\mathbb{Q}$ is $\mathbb{Z}$ and every fractional ideal has form $\frac{1}{d}(m)$, where m and d are integers and (m) is the ideal of $\mathbb{Z}$ generated by m.

For every number field K there exists a primitive element ω, that is, such that $K = \mathbb{Q}(\omega)$. Let $p(x) \in \mathbb{Q}[x]$ be the minimal polynomial of ω over $\mathbb{Q}$, then the degree of $p(x)$ is equal to the degree of the number field. The polynomial $p(x)$ has r_1 real roots and, since the coefficients of $p(x)$ are all rational numbers, $2r_2$ complex roots. Let n be the degree of K and $\beta_1, \ldots, \beta_n$ the roots of $p(x)$. We can define $n = r_1 + 2r_2$ injective maps

$$\sigma_j \colon K \to \mathbb{C},\ \omega \mapsto \beta_j$$

for $j = 1, \ldots, n$. Each of σ_j is an embedding of fields, i.e. a monomorphism of fields, and we will say that σ_j is a **real embedding** if $\sigma_j(K) \subset \mathbb{R}$, otherwise we will say that σ_j is a **complex embedding**. Hence, r_1 is the number of real embeddings of a number field and r_2 is the number of complex embeddings modulo the conjugation.

Definition 1.22. A complete set of embeddings $\Phi = \{\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1} \ldots, \sigma_{r_1+r_2}\}$ of K modulo the conjugation is called a **type** of K.

This means that $\sigma_{j_1} \neq \bar{\sigma}_{j_2}$, for every $j_1, j_2 \in \{1, \ldots, r_1 + r_2\}$, $j_1 \neq j_2$. A number field is **totally real** if $r_2 = 0$, otherwise it is **non-totally real**. A non-totally real number field such that $r_1 = 0$ is sometimes called **totally complex**. The pair (r_1, r_2) is called the **signature** of the number field.

Example 1.23. $K = \mathbb{Q}$ is a totally-real number field with $r_1 = 1$ and trivial embedding $1 \mapsto 1$.

Example 1.24. Every quadratic field is some $K = \mathbb{Q}(\sqrt{d})$, with d a squarefree integer. If $d > 0$ then K is totally real, if $d < 0$ then K is totally complex. For instance, $K = \mathbb{Q}(\sqrt{2})$ is totally real: the minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}$ is $p(x) = x^2 - 2$ and its roots are $\pm\sqrt{2}$. The embeddings are $\sigma_1(\sqrt{2}) = \sqrt{2}$ and $\sigma_2(\sqrt{2}) = -\sqrt{2}$. On the other hand, $\mathbb{Q}(i)$ is a totally complex quadratic field.

Example 1.25. $K = \mathbb{Q}(\sqrt[3]{2})$ is a non-totally real number field. The minimal polynomial of $\omega = \sqrt[3]{2}$ is $p(x) = x^3 - 2$ and the roots of $p(x)$ are $\omega, \varepsilon\omega, \varepsilon^2\omega$, where ε is a primitive third root of unity. On the contrary, adjoining a root of $x^3 + x^2 - 2x - 1$ to $\mathbb{Q}$ yields a totally real number field.

Definition 1.26. An **order** $\mathcal{O}$ of a number field K is a subring $\mathcal{O} \subseteq \mathcal{O}_K$ which is also a $\mathbb{Z}$-module of rank equal to the degree of K.

The ring of integers $\mathcal{O}_K$ of K is an order. An **integral basis** of $\mathcal{O}_K$ is a basis of $\mathcal{O}_K$ as $\mathbb{Z}$-module, that is a subset $\{\alpha_0, \dots, \alpha_{n-1}\}$ such that every $x \in \mathcal{O}_K$ can be written uniquely as $x = \sum_{j=0}^{n-1} m_j \alpha_j$, where $m_0, \dots, m_{n-1} \in \mathbb{Z}$. A **power integral basis** of $\mathcal{O}_K$ is an integral basis of the form $\{\alpha^0 = 1, \alpha, \dots, \alpha^{n-1}\}$.

The ring of integers $\mathcal{O}_K$ results the unique maximal order of K.

Definition 1.27. Let K be an algebraic number field. Let $\{b_1, \dots, b_n\}$ be an integral basis of $\mathcal{O}_K$ and let $\{\sigma_1, \dots, \sigma_n\}$ be a set of embeddings of K into the complex numbers. The **discriminant** Δ_K of K is

$$\Delta_K = \det \begin{pmatrix} \sigma_1(b_1) & \sigma_1(b_2) & \cdots & \sigma_1(b_n) \\ \sigma_2(b_1) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_n(b_1) & \cdots & \cdots & \sigma_n(b_n) \end{pmatrix}^2 .$$

We will also denote the discriminant by $\operatorname{disc}(K)$.

One can proves that Δ_K is always an integer number and it does not depend on the choice of an integral basis of $\mathcal{O}_K$ and of a primitive element of K. More in general, we can define the discriminant $\operatorname{disc}(\mathcal{O})$ of an order $\mathcal{O}$ in the same way fixing an integral basis of the order $\mathcal{O}$.

Proposition 1.28. *The discriminant of the number field and the discriminant of an order satisfy* $\operatorname{disc}(\mathcal{O}) = \operatorname{disc}(\mathcal{O}_K)[\mathcal{O}_K \colon \mathcal{O}]^2$, *where* $[\mathcal{O}_K \colon \mathcal{O}]$ *is the index as* $\mathbb{Z}$*-module.*

Hence, the discriminant of any proper order is always greater than the discriminant of the unique maximal order $\mathcal{O}_K$.

Definition 1.29. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Q}[x]$ be a polynomial, with $a_n \neq 0$. If $\theta_1, \dots, \theta_n$ are the complex roots of such a polynomial, then its **discriminant** is defined as

$$\operatorname{disc}(f) := a_n^{2n-2} \prod_{i<j} (\theta_i - \theta_j)^2.$$

It is clear by the definition that $\operatorname{disc}(f) \in \mathbb{Q}$.

If $f(x)$ is a irreducible integer monic polynomial, then $\operatorname{disc}(f) = \prod_{i<j}(\theta_i - \theta_j)^2$ coincides, by definition, with the discriminant of the order $\mathbb{Z}[\theta]$ in the number field $K = \mathbb{Q}(\theta) \cong \frac{\mathbb{Q}[x]}{(f)}$, for any root θ of $f(x)$. Therefore, by Proposition 1.28,

$$\operatorname{disc}(f) = \operatorname{disc}(K)[\mathcal{O}_K \colon \mathbb{Z}[\theta]]^2.$$

Thus, we have seen that, for every integer irreducible monic polynomial $f(x)$, the number $\operatorname{disc}(f)/\operatorname{disc}(K)$, where K is the field obtained adjoining one of any root of $f(x)$ to the rational numbers, is always a square integer. The square root of this number, that is $[\mathcal{O}_K \colon \mathbb{Z}[\theta]]$, is called the **index** of $f(x)$ and denoted by $\operatorname{ind}(f)$.

More in general, given an algebraic integer θ of a number field K, the **index** of θ is defined to be the index of $\mathbb{Z}[\theta]$ as $\mathbb{Z}$-submodule of ring of integers, that is $\operatorname{ind}(\theta) := [\mathcal{O}_K : \mathbb{Z}[\theta]]$. See, e.g., [46, Chapter III], for a detailed discussion in a more general context.

1.2.1 The construction of a toroidal group from a field

Let X be an Abelian complex Lie group. Since X is an Abelian group, the integer multiplication $[k]\colon X \to X$, where $k \in \mathbb{Z}$, defined by

$$x \mapsto k \cdot x := \underbrace{x + \cdots + x}_{k \text{ times}}, \qquad \text{if } k > 0,$$
$$x \mapsto k \cdot x := -(\underbrace{x + \cdots + x}_{|k| \text{ times}}), \quad \text{if } k < 0,$$

is an endomorphism of X. The set of integer multiplications $\{[k]\colon X \to X \mid k \in \mathbb{Z}\}$ is a subring of $\mathrm{End}(X)$ isomorphic to $\mathbb{Z}$. Thus, up to isomorphism, $\mathbb{Z} \subseteq \mathrm{End}(X)$. These isomorphisms are called also **trivial**.

Definition 1.30. A toroidal group $\mathcal{T}$ **has extra multiplication** if $\mathbb{Z} \subsetneq \mathrm{End}(\mathcal{T})$.

In terms of matrix representations, a toroidal group $\mathcal{T}$ has extra multiplication if there exists some endomorphism such that its corresponding matrix is not scalar.

In [69] (see also [7]), for the case of real rank $n+1$, and, more in general, in [4] is given the costruction of a toroidal group starting from a non-totally real number field. These are examples of toroidal groups having extra multiplication. In particular, Andreotti e Gheradelli in [7] and Vaillères in [69] proved the converse for cubic fields, that is, every toroidal group of dimension two and real rank three having extra multiplication comes from a non-totally real cubic number field. This is not true for the general case of complex rank n and real rank $n+1$, but it can be proved with some technical assumptions (cfr. Chapter §3.1).

In the following, for the sake of clarity, we show how to construct a toroidal group starting from a number field. Let K be a number field of degree $n = r_1 + 2r_2$ and fix a type

$$\Phi = \{\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1} \ldots, \sigma_{r_1+r_2}\},$$

where $\sigma_1, \ldots, \sigma_{r_1}$ are real embeddings and the others r_2 are complex. We can define a injective map

$$\begin{aligned} \mu_\Phi\colon K &\to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \subset \mathbb{C}^{r_1+r_2} \\ \omega &\mapsto (\sigma_1(\omega), \ldots, \sigma_{r_1}(\omega), \sigma_{r_1+1}(\omega), \ldots, \sigma_{r_1+r_2}), \end{aligned}$$

associated to Φ. The map μ_Φ is the so-called **Minkowski map**.

Lemma 1.31. *The set $\mu_\Phi(\mathcal{O}_K)$ is lattice of $\mathbb{C}^{r_1+r_2}$ of complex rank $r_1 + r_2$ and real rank $r_1 + 2r_2$.*

Theorem 1.32 ([4])**.** *The Abelian complex Lie group $\mathcal{T} = \mathbb{C}^{r_1+r_2}/\mu_\Phi(\mathcal{O}_K)$ is toroidal of real rank $r_1 + 2r_2$ with extra multiplication.*

Conversely, Andreotti and Gherardelli in 1973 and Vaillères in 2012, for the cubic case, proved the following.

Theorem 1.33 ([7],[69])**.** *Let $\mathcal{T}$ be a toroidal group of dimension n and real rank $n+1$. Then $\mathrm{End}_0(\mathcal{T})$ is a non-totally real number field.*

In the general case, the ring $\mathrm{End}_0(\mathcal{T})$ results a $\mathbb{Q}$-algebra of finite dimension, for which we have a linear representation given by the rational representation in (1.3).

Lemma 1.34. *Let $\mathcal{T}$ be a toroidal group. If $\mathrm{End}_0(\mathcal{T})$ is a division algebra then*

$$\dim_{\mathbb{Q}} \mathrm{End}_0(\mathcal{T}) \mid \mathrm{rank}_{\mathbb{R}} \Lambda.$$

Proof. For a toroidal group, we can define the rational representation (1.3), which is linear, of the algebra $\mathrm{End}_0(\mathcal{T})$. The $\mathbb{Q}$-vector space $V = \mathrm{span}_{\mathbb{Q}} \Lambda$ has dimension equal to $\mathrm{rank}_{\mathbb{R}} \Lambda$. The space V is also a $\mathrm{End}_0(\mathcal{T})$-vector space, defining the action $x \cdot v := \varrho_r(x)(v)$ for $x \in \mathrm{End}_0(\mathcal{T})$ and $v \in V$. Therefore, we have the formula

$$\dim_{\mathbb{Q}} V = \dim_{\mathbb{Q}} \mathrm{End}_0(\mathcal{T}) \cdot \dim_{\mathrm{End}_0(\mathcal{T})} V.$$

□

Combining Theorem 1.33 and Lemma 1.34, we have the following.

Corollary 1.35. *Let $\mathcal{T}$ be a toroidal group of dimension n and real rank $n+1$. Then $\mathrm{End}_0(\mathcal{T})$ is a non-totally real number field of degree a divisor of $n+1$.*

Theorem 1.36 ([69], see also [7])**.** *Let K be a non-totally real cubic number field and Φ a type of K. Then $\mathcal{T} = \mathbb{C}^2/\mu_{\Phi}(\mathcal{O}_K)$ is a toroidal group of real rank three with extra multiplication. Moreover, $\mathrm{End}_0(\mathcal{T})$ is a non-totally real cubic number field, $\mathcal{O}_K \cong \mathrm{End}(\mathcal{T})$ and $K \cong \mathrm{End}_0(\mathcal{T})$.*

1.3 Abelian and quasi-Abelian varieties

Toroidal groups play the role that complex tori have in the definition of Abelian varieties, but in the context of quasi-Abelian varieties. We refer to the standard references [54], [55], [16], [45] for the well-known theory of Abelian varieties.

An *Abelian variety* over a field K is a connected and complete algebraic group over K. If $K = \mathbb{C}$, then the underlying variety is compact and isomorphic as Lie group to a complex torus. In particular, there exists a criterion to decide whether or not a given complex torus is an Abelian variety.

Theorem 1.37 (Riemann)**.** *Let $\mathbb{T} = V/\Lambda$ be a complex torus. The following statements are equivalent.*

- *$\mathbb{T}$ is an Abelian variety;*
- *There exists a **Riemann form** on $\mathbb{T}$, that is, a positive definite hermitian form H on V whose imaginary part takes integral values on $\Lambda \times \Lambda$;*

Theorem 1.37 is often taken as the definition of a complex Abelian variety. One may define a complex Abelian variety also as a complex torus that admits a polarisation (see, e.g., [16]).

The following definition is due to Andreotti and Gherardelli in [8].

Definition 1.38. A toroidal group $\mathcal{T} = \mathbb{C}^n/\Lambda$ is a **quasi-Abelian variety** if there exists a Hermitian form H on $\mathbb{C}^n \times \mathbb{C}^n$ such that

(QA1) $H \restriction_{M\mathbb{C}_\Lambda \times M\mathbb{C}_\Lambda}$ is positive definite;

(QA2) $E := \mathrm{Im}(H) \restriction_{\Lambda \times \Lambda}$ is a $\mathbb{Z}$-valued skew-symmetric form.

We say that the **ample Riemann form** H defines a structure of quasi-Abelian variety on $\mathcal{T}$. The conditions (QA1) and (QA2) are also called the *generalized Riemann conditions*. It is said to be of **kind** s (where $0 \leq 2s \leq n-q$) if there exists an ample Riemann form H such that $\text{rank}\,(\text{Im}(H) \restriction_{\Lambda\times\Lambda}) = 2q+2s$.

It is clear from the definitions that every Abelian variety is also a quasi-Abelian variety (indeed, if the real rank of Λ is equal to $2n$, then the conditions (QA1) and (QA2) are the classical Riemann relations [8, p. 204]). In particular, the following fibration theorem holds (for $s=0$).

Theorem 1.39 (Andreotti-Gherardelli [7], [8])**.** *Let $\mathcal{T} = \mathbb{C}^n/\Lambda$ be a toroidal group of real rank $n+q$. Then $\mathcal{T}$ is a quasi-Abelian variety of kind* 0 *if, and only if, there exists an Abelian variety $\mathbb{T}^q$ of dimension q with an exact sequence*

$$0 \longrightarrow (\mathbb{C}^*)^{n-q} \longrightarrow \mathcal{T} \longrightarrow \mathbb{T}^q \longrightarrow 0 \tag{1.5}$$

Proof. See [6] or [68]. □

Theorem 1.40 (Abe)**.** *Every quasi-Abelian variety is a quasi-projective variety.*

Proof. See [6, Theorem 3.1.18] for the proof. □

1.4 Jacobian of a curve

In this section, we recall the notion of divisors on a curve, the construction and definition of the (ordinary) Jacobian of a Riemann surface and we introduce the hyperelliptic functions. Furthermore, the generalized Jacobian is introduced and we provide notions about extensions of Abelian varities by linear algebraic groups.

1.4.1 Divisors on a curve

Let X be a curve on the complex plane.

Definition 1.41.

- A **divisor** D is a formal sum of points of X, $D = \sum_{P\in X} n_P(P)$ with $n_P \in \mathbb{Z}$. Let us denote by $\text{Div}(X)$ the Abelian group of divisors of X; (in other words, $\text{Div}(X)$ is the free Abelian group on the set of points of X)
- The **degree** of a divisor $D = \sum_{P\in X} n_P(P)$ is $\deg(D) = \sum_{P\in X} n_P$. Let $\text{Div}^0(X)$ be the subgroup of $\text{Div}(X)$ of degree zero divisors;
- Let f be a rational function on X, $f \in k(X)$. The divisor

  $$(f) = \sum_{P\in X} \text{ord}_P(f)(P)$$

  is called a **principal divisor** of X. Let PDiv(X) be the subgroup of $\text{Div}(X)$ of principal divisors of X.

Note that a principal divisor has degree zero (since the number of zeros is equal to the number of poles). Hence, PDiv(X) is a subgroup of $\text{Div}^0(X)$.

Definition 1.42. Define $\text{Pic}(X) := \frac{\text{Div(X)}}{\text{PDiv(X)}}$ the **Picard group** of X and $\text{Pic}^0(X) := \frac{\text{Div}^0(X)}{\text{PDiv(X)}}$ the **zero part of Picard group** of X.

The elements $[D] = D + \mathrm{PDiv}(X)$ of the quotient group $\mathrm{Pic}(X)$ are called **divisor classes**. We will sometimes denote $\bar{D}$ the class of the divisor D. Two divisors in the same class, that is, if they differ by a principal divisor, are said **linearly equivalent**. The degree of a divisor class $[D]$ is the degree of any representative.

1.4.2 Jacobian of a Riemann surface

We follow [48] and [34] for the following definitions.

If X is a compact Riemann surface of genus g, then the set of all holomorphic differential one-forms $\Omega^1(X)$ is a g-dimensional complex vector space. By Stoke's theorem (see, e.g., [34, p. 21]), the integral $\int_\gamma \omega$ of a one-form ω along a path γ depends only on the *holomogy class* of γ. Thus, the integral of a one-form can be naturally extended to the entire homology class

$$\int_{[\gamma]} \omega := \int_\gamma \omega. \tag{1.6}$$

The integral $\int_{[\gamma]} : \omega \mapsto \int_\gamma \omega$ is a linear functional of $\Omega^1(X)$, therefore we have a well-defined (injective) map on the *first homology group*

$$\begin{aligned} H_1(X) &\to \Omega^1(X)^* \\ [\gamma] &\mapsto \left(\int_{[\gamma]} : \Omega^1(X) \to \mathbb{C} \right), \end{aligned}$$

where $\Omega^1(X)^*$ denotes the dual space of $\Omega^1(X)$. Since $H_1(X)$ is a free Abelian group of rank $2g$, its image Λ under this map results a lattice of the g-dimensional complex vector space $\Omega^1(X)^*$ of full rank $2g$. The quotient space $\mathfrak{J}(X) := \Omega^1(X)^*/\Lambda$ is a complex torus, named the **Jacobian variety** of X.

Remark 1.43. The above construction can be done starting from the *fundamental group* $\pi(X, P)$, for a basepoint P of X. In fact, if ω is a fixed one-form on X, then the map

$$\begin{aligned} \int_{(-)} \omega : \pi(X, P) &\to \mathbb{C} \\ [\gamma]_{\text{homotopy}} &\mapsto \int_\gamma \omega \end{aligned}$$

is well-defined (independent of the choice of a path γ) and a group homomorphism, whose kernel is the commutator subgroup of $\pi(X, P)$. It is well-known that, for a path-connected space, the first homology group is isomorphic to the Abelianization of the fundamental group (see, e.g., [48, pp. 125-126]).

Theorem 1.44. *The Jacobian variety is an Abelian variety.*

Let Q be a fixed basepoint of X and $\omega_1, \ldots, \omega_g$ a basis of $\Omega^1(X)$. Choosing a basis allow us to identify $\mathfrak{J}(X)$ with a quotient $\mathbb{C}^g/\Lambda$, with Λ lattice of $\mathbb{C}^g$. We can embed the curve X into the Jacobian via the so-called **Abel-Jacobi map**

$$\mathfrak{A} \colon P \in X \mapsto \left(\int_Q^P \omega_1, \ldots, \int_Q^P \omega_g \right) \pmod{\Lambda}, \tag{1.7}$$

which can be extended linearly to the group of divisors

$$D = \sum_{P \in X} n_P(P) \mapsto \sum_{P \in X} n_P \mathfrak{A}(P). \tag{1.8}$$

Restricted to the divisors of degree zero, the map (1.8) determines canonically an isomorphism $\mathfrak{A}_0\colon \operatorname{Pic}^0(X) \to \mathbb{C}^g/\Lambda$ (this statement named *Abel-Jacobi theorem* is the union of two results usually called *Abel's theorem* and *Jacobi inversion theorem*).

Remark 1.45. We make use of this remark to point out all of the needful observations about the previous functions. The Abel-Jacobi map $\mathfrak{A}$ is well-defined because if one changes a path γ from Q to P with another path γ', the value of $\mathfrak{A}(P)$ changes by the integration around the closed chain $\gamma - \gamma'$. Moreover, while the previous maps depend on the choice of a basis for $H_1(X)$ and $\Omega^1(X)$, the fact that $\mathfrak{A}_0$ is an isomorphism results independent of these choices. Finally, it can be proven that $\mathfrak{A}_0$ results also independent of the chosen basepoint.

1.4.3 Canonical bases of differential forms and canonical cycles

Roughly speaking, the genus of a Riemann surface is the number of "handles" in the topological space (see, e.g., [34, p. 8]).

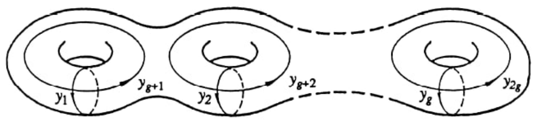

FIGURE 1.1: Riemann surface of genus g, [34, p. 106].

A (non-singular) *hyperelliptic curve* $\mathcal{H}$ of genus g is the set of points satisfying the equation $y^2 = f(x)$, for a polynomial $f(x)$ of degree $2g+2$ or $2g+1$ having no multiple roots. An hyperelliptic curve has a canonically structure of a one-dimensional Riemann surface (of genus g), see [19].[1] A basis $\{a_1, \ldots, a_g; b_1, \ldots, b_g\}$ of the first homology group $H_1(\mathcal{H})$ is called **canonical** if

$$a_i \circ a_j = 0 = b_i \circ b_j \quad \text{and} \quad a_i \circ b_j = 1 = -(b_i \circ a_j), \tag{1.9}$$

that is, if the *intersection matrix* is $J = \begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}$, where 0_g and I_g denote respectively the zero and identity matrix of dimension $g \times g$. Given a canonical basis, we define the a-**periods** and b-**periods** of a differential one-form ω as, respectively,

$$\int_{a_1} \omega, \ldots, \int_{a_g} \omega \quad a\text{-periods, and}$$
$$\int_{b_1} \omega, \ldots, \int_{b_g} \omega \quad b\text{-periods.}$$

On the other hand, traditionally, three kinds of differential one-forms are distinguished on a hyperelliptic curve. The differentials of *first kind* or *holomorphic* are the

[1] The following definitions and results hold more generally for a *hyperelliptic Riemann surface*, but we only need the case of a hyperelliptic curve.

elements of $\Omega^1(\mathcal{H})$ and they are given by the linear combinations of the so-called **canonical holomorphic differentials** ([34, Proposition 4.3, p. 141], see also [18])

$$du_j = \frac{x^{j-1}dx}{y}, \quad \text{for } j = 1, \ldots, g\,.$$

Their matrices of a-periods and b-periods are respectively

$$2\omega_1 = \left(\int_{a_i} \frac{x^{j-1}dx}{y}\right)_{ij} \quad \text{and} \quad 2\omega_2 = \left(\int_{b_i} \frac{x^{j-1}dx}{y}\right)_{ij}. \tag{1.10}$$

Concerning *differentials of second kind* or *meromorphic differentials*, there exists a canonical set of g differentials $\{dr_j\}$ such that each differential of this kind is the linear combination of the canonical ones, but we limit ourselves to introduce their matrices of a-periods and b-periods,

$$2\eta_1 = \left(-\int_{a_i} dr_j\right)_{ij} \quad \text{and} \quad 2\eta_2 = \left(-\int_{b_i} dr_j\right)_{ij}. \tag{1.11}$$

Indeed, the matrices in (1.10) and (1.11) satisfy the following **generalized Legendre relation**

$$\begin{pmatrix}\omega_1 & \omega_2\\ \eta_1 & \eta_2\end{pmatrix}\begin{pmatrix}0_g & I_g\\ -I_g & 0_g\end{pmatrix}\begin{pmatrix}\omega_1 & \omega_2\\ \eta_1 & \eta_2\end{pmatrix}^T = -\frac{\pi i}{2}\begin{pmatrix}0_g & I_g\\ -I_g & 0_g\end{pmatrix}. \tag{1.12}$$

There exist also differentials of *third kind* but we omitt them.

Remark 1.46. The identity (1.12) for $g = 1$ is the well-known relation (see, e.g., [47, p. 151])

$$\eta_1\omega_2 - \eta_2\omega_1 = -\frac{1}{2}\pi i$$

involving the constants η_1 and η_2 of the Jacobi theta function $\theta(z, \omega_1, \omega_2)$. More precisely, ω_1, ω_2 are the half-periods of θ and

$$\eta_1 = -\frac{\pi^2}{12\omega_1}\frac{\theta_1'''(0)}{\theta_1'(0)} \tag{1.13}$$

where $\theta_1(z)$ is the first theta functions.

Example 1.47. An elliptic curve on the complex plane is the subset of points satisfy the equation $y^2 = x^3 + ax + b$ (in Weierstrass form). Let E denote the elliptic curve as well as the set of its points and Ω the point at infinity. It is well-known that one can sum the points in E by the chord-tangent rule $\oplus$ and $(E, \oplus)$ is an Abelian group.

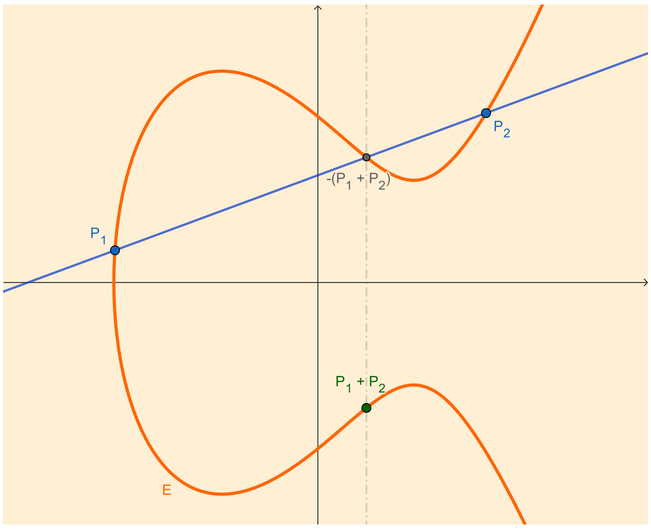

FIGURE 1.2: Sum on an elliptic curve of equation $y^2 = x^3 - 2.6x + 2.6$.

In particular, E is isomorphic to a complex torus $\mathbb{C}/\Lambda$, for a lattice Λ of $\mathbb{C}$, via the Weierstrass elliptic function

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right), \tag{1.14}$$

which is a Λ-periodic meromorphic function. In fact, the map

$$\mathrm{P}\colon \mathbb{C} \to E, \quad z \mapsto \left[\wp(z) : \frac{1}{2}\wp'(z) : 1 \right] \tag{1.15}$$

is an epimorphism with Λ as kernel. The space of holomorphic differential one-forms is a 1-dimensional complex space spanned by $\frac{dx}{y}$. The map $\widetilde{\mathrm{P}}\colon \mathbb{C}/\Lambda \to E$ is the inverse of the Abel-Jacobi map

$$\begin{aligned} \mathfrak{A}\colon E \equiv \mathrm{Pic}^0(E) &\to \mathbb{C}/\Lambda \\ P &\mapsto \int_\Omega^P \tfrac{dx}{y} \pmod{\Lambda} = \int_\Omega^P \tfrac{dx}{\sqrt{x^3+ax+b}} \pmod{\Lambda}. \end{aligned} \tag{1.16}$$

Historically, indeed, elliptic functions were introduced as formal inverse of the elliptic integrals [65], such as the one in (1.16). Their names derive from the fact that these integrals appear in the expression of the arc length of an ellipse. The Weierstrass function $\wp$ plays an important role in the theory of elliptic functions. Indeed, $\wp$ and its derivative $\wp'$ generate the field of all elliptic functions with respect to a given period lattice. For the standard theory on the elliptic curves see [64], [39].

Example 1.48 (Hyperelliptic curve of genus 2)**.** Let $\mathcal{H}$ be a (imaginary) hyperelliptic curve of genus two over the complex numbers, which is the set of points satisfying the equation $y^2 = f(x)$, where $f(x)$ is a separable polynomial of degree five (that is, $f(x)$ has five distinct roots). In this case, the curve has only one point at infinity Ω which we choose as the basepoint and the Abel-Jacobi map

$$\begin{aligned} \mathfrak{A}\colon \mathcal{H} &\to \mathbb{C}^2/\Lambda \\ P &\mapsto \left(\int_\Omega^P \tfrac{dx}{y}, \int_\Omega^P \tfrac{xdx}{y} \right) \pmod{\Lambda} \end{aligned} \tag{1.17}$$

is injective but not surjective. The isomorphism between $\mathrm{Pic}^0(\mathcal{H})$ and $\mathbb{C}^2/\Lambda$ is given by

$$[D] = [(P) + (R) - 2(\Omega)] \mapsto \mathfrak{A}(P) + \mathfrak{A}(R). \tag{1.18}$$

Indeed, for genus two, every divisor class of $\mathrm{Pic}^0(\mathcal{H})$ is determined by one or two points, as we explain in the following section.

1.4.4 Symmetric product

Let A be a set, we define the **n-th symmetric power** of A to be the set

$$A^{(n)} = \{\{a_1, \dots, a_n\} \subset A \mid a_i \in A, i = 0, \dots, n\},$$

i.e. the set of non-ordered n-tuples of elements of A. Another way to see $A^{(n)}$ is as the quotient group $A^n/\sim$, where A^n is the Cartesian product of n copies of A and $\sim$ is the equivalence relation $(a_1, \dots, a_n) \sim (a_{\sigma(1)}, \dots, a_{\sigma(n)})$, for any permutation σ of the symmetric group on n elements.

It can be proven that every divisor of degree zero on a hyperelliptic curve X of genus g is linearly equivalent to a unique **reduced** divisor, that is, a divisor of the form $\sum_{j=1}^{k}(P_i) - k(\Omega)$ with $k \le g$, such that P_i is not the symmetric point of P_j, for $i \neq j$. Hence, we can choose as a representative of each class in $\mathrm{Pic}^0(X)$ a reduced divisor and we can therefore write the group $\mathrm{Pic}^0(X)$ as

$$\left\{ \sum_{j=1}^{k}(P_i) - k(\Omega) + \mathrm{PDiv}(X) \colon k \le g,\ P_i \in X,\ P_i \neq P_j' \text{ for } i \neq j \right\},$$

where P_i' denotes the symmetric point with respect to the x-axis. It turns out that there is a (birationally) isomorphism between $\mathrm{Pic}^0(X)$ and the symmetric product $X^{(g)}$, see [35, pp. 235-237], also [34, Proposition 4.9, p. 185].

1.4.5 Klenian functions

The multidimensional sigma function was introduced by Klein in [43]. We follow the approach of Baker ([11], [10], [12]) as well as the one of the series of works of Buchstaber, Enolski et alter ([17], [18] [19]).

A canonical basis of holomorphic differentials can be always *normalized*, applying the transformation $(2\omega_1)^{-1}$, to obtain a basis $\{dv_1, \dots, dv_g\}$ satisfying

$$\int_{a_i} dv_j = \delta_{ij}, \qquad \text{for every } i, j = 1, \dots, g,$$

where δ_{ij} denotes as usual the Kronecker delta. On the other hand, it is well-known that the matrix

$$\tau = \omega_1^{-1}\omega_2 = \left(\int_{b_i} dv_j\right)_{ij}$$

belongs to the **Siegel half-upper space**

$$\mathfrak{H}_g = \{A \in M(\mathbb{C}) \colon A = A^{\mathrm{T}},\ \mathrm{Im}(A) > 0\},$$

where $\mathrm{Im}(A) > 0$ denotes that the imaginary part of the complex (symmetric) matrix A is a positive definite real matrix (see also [61]).

Now we are ready to introduce the Kleinian hyperelliptic functions. From now on, the bold symbols ($\boldsymbol{z}$, $\boldsymbol{u}$, $\boldsymbol{m}$, ...) denotes columns vector. Let $\boldsymbol{z} \in \mathbb{C}^g$ and $\tau \in \mathfrak{H}_g$, the **standard θ-function** is defined by the Fourier series

$$\theta(\boldsymbol{z};\tau) := \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp\left\{\pi i(\boldsymbol{m}^T\tau\boldsymbol{m} + 2\boldsymbol{z}^T\boldsymbol{m})\right\}. \tag{1.19}$$

Definition 1.49 (See [17]). A **fundamental hyperelliptic Kleinian σ-function** is defined in terms of the standard θ-function as follows:

$$\sigma(\boldsymbol{u}) = C\, e^{\boldsymbol{u}^{\mathrm{T}}\varkappa\boldsymbol{u}}\theta((2\omega_1)^{-1}\boldsymbol{u} - K_\Omega;\tau)\exp\left\{2\pi\, i\boldsymbol{q}_2^{\mathrm{T}}\left(-(2\omega_1)^{-1}\boldsymbol{u} + \tfrac{1}{2}\tau\boldsymbol{q}_2 - \boldsymbol{q}_1\right)\right\} \tag{1.20}$$

where $\varkappa = (2\omega_1)^{-1}\eta_1$, K_Ω is the vector of Riemann constant with basepoint Ω, C is a constant, and $\begin{bmatrix}\boldsymbol{q}_2\\ \boldsymbol{q}_1\end{bmatrix}$ is the half-integer characteristic of K_Ω.

For genus two, denoting the components of $\boldsymbol{u}$ by $(\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)})$, the constant C is chosen such that the expansion of $\sigma(\boldsymbol{u})$ near the zero vector $(0,0)$ is equal to $u^{(1)}$ + higher order terms. More in general, the constant C is chosen such that the expansion of $\sigma(\boldsymbol{u})$ near $\mathbf{0}$ is starts with a Schur-Weierstrass polynomial (see [19]).

The following quasi-periodicity property holds

$$\begin{aligned}\sigma(\boldsymbol{u} + \Omega(\boldsymbol{m}_1,\boldsymbol{m}_2)) = \exp&\left\{\mathrm{E}^{\mathrm{T}}(\boldsymbol{m}_1,\boldsymbol{m}_2)\left(\boldsymbol{u} + \frac{1}{2}\Omega(\boldsymbol{m}_1,\boldsymbol{m}_2)\right)\right\}\\ &\exp\left\{2\pi i\left(\boldsymbol{m}_1^T\boldsymbol{q}_2 - \boldsymbol{m}_2^{\mathrm{T}}\boldsymbol{q}_1\right) - \pi i\boldsymbol{m}_1^{\mathrm{T}}\boldsymbol{m}_2\right\}\sigma(\boldsymbol{u}),\end{aligned} \tag{1.21}$$

where

$$\Omega(\boldsymbol{m}_1,\boldsymbol{m}_2) = 2\omega_1\boldsymbol{m}_1 + 2\omega_2\boldsymbol{m}_2 \quad \text{and} \quad \mathrm{E}(\boldsymbol{m}_1,\boldsymbol{m}_2) = 2\eta_1\boldsymbol{m}_1 + 2\eta_2\boldsymbol{m}_2,$$

for any $\boldsymbol{u} \in \mathbb{C}^g$ and $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathbb{Z}^g$.

We define the $\wp$-functions as the logarithmic derivatives of the σ-function

$$\wp_{ij}(\boldsymbol{z}) = -\frac{\partial^2}{\partial z_i\partial z_j}\ln(\sigma(\boldsymbol{z})) \quad \text{and} \quad \wp_{ijk}(\boldsymbol{z}) = -\frac{\partial^3}{\partial z_i\partial z_j\partial z_k}\ln(\sigma(\boldsymbol{z})), \tag{1.22}$$

for $i,j,k = 1,\dots,g$. In the case of an elliptic curve, the $\wp$-function in (1.14) resolves the inverse problem to find, given an element $z + \Lambda \in \mathbb{C}/\Lambda$ of the Jacobian (which, we recall, is a complex torus), a point P of the curve such that P corresponds to z; in other words the problem to invert the Abel-Jacobi map, given in (1.16). The functions $\wp_{ij}$ play the same role in higher dimension. For the following theorem, already stated in [43], see, e.g., [17].

Theorem 1.50 (Solution of Jacobi inverse problem). *If $x = \boldsymbol{z} + \Lambda \in \mathbb{C}^g/\Lambda$ is an element of the Jacobian of a hyperelliptic curve, then $\mathfrak{A}^{-1}(x) = \{(x_k, y_k) \colon k = 1,\dots,g\}$ where x_k are the solutions of the polynomial equation*

$$X^g - \sum_{\ell=1}^{g}\wp_{\ell g}(\boldsymbol{z})X^{\ell-1} = 0$$

and y_k are given by

$$y_k = \sum_{\ell=1}^{g} \wp_{\ell gg}(z) x_k^{\ell-1}.$$

As mentioned in Example 1.47, the Jacobian variety of an elliptic curve is isomorphic to the curve itself, thus it lies in the complex plane. On the contrary, the Jacobian of a hyperelliptic curve of genus g is a variety of dimension g that lies in a complex projective space of dimension $\frac{g(g+1)}{2} + g$ as the intersection of $g(g+1)/2$ cubics (see Theorem 1.51 below). Moreover, Jacobian varieties of hyperelliptic curves are naturally equipped with an involution and if one identifies a point with the point mapped by the involution (i.e. the quotient of the Jacobian by the involution), then one obtains a so-called *Kummer surface*, a variety that lies in a $g(g+1)/2$-dimensional complex projective space (see [17]).

More precisely, let us denote by $\wp$ the $g \times g$ symmetric matrix $\wp = (\wp_{ij})$ and the following g-dimensional vectors

$$\boldsymbol{\wp} = \begin{pmatrix} \wp_{g1} \\ \vdots \\ \wp_{gg} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\wp}' = \frac{\partial}{\partial z_g} \boldsymbol{\wp} = \begin{pmatrix} \wp_{g1g} \\ \vdots \\ \wp_{ggg} \end{pmatrix}, \tag{1.23}$$

where $\boldsymbol{\wp}$ is the g-th column (or row) of the matrix $\wp$.

Theorem 1.51 ([18]). *The map*

$$\varphi \colon \mathfrak{J}(\mathcal{H}) \setminus (\sigma) \to \mathbb{C}^{\frac{g(g+1)}{2}+g}, \; \boldsymbol{z} + \Lambda \mapsto (\wp(\boldsymbol{z}), \boldsymbol{\wp}'(\boldsymbol{z})), \tag{1.24}$$

where Λ is the lattice of the Jacobian and (σ) denotes the divisor of zeros of σ, is a meromorphic embedding. The image contained in $\mathbb{C}^{\frac{g(g+1)}{2}+g}$ is the intersection of $\frac{g(g+1)}{2}$ cubics.

Example 1.52. The Jacobian of a (imaginary) hyperelliptic curve $\mathcal{H}$ of genus $g = 2$ lies in $\mathbb{CP}^5$. The embedding (1.24) is given by

$$(\wp(z_1, z_2), \boldsymbol{\wp}'(z_1, z_2)) = \left(\begin{array}{cc|c} \wp_{11}(z_1, z_2) & \wp_{12}(z_1, z_2) & \wp_{212}(z_1, z_2) \\ \wp_{12}(z_1, z_2) & \wp_{22}(z_1, z_2) & \wp_{222}(z_1, z_2) \end{array}\right)$$

The function can be extended projectively including the infinity point $[0:0:0:0:0:1]$ by

$$\begin{aligned}[\sigma(z_1,z_2)^3 :\ & \sigma(z_1,z_2)^3 \wp_{11}(z_1,z_2) : \sigma(z_1,z_2)^3 \wp_{12}(z_1,z_2) \\ & : \sigma(z_1,z_2)^3 \wp_{22}(z_1,z_2) : \sigma(z_1,z_2)^3 \wp_{212}(z_1,z_2) : \sigma(z_1,z_2)^3 \wp_{222}(z_1,z_2)] \end{aligned} \in \mathbb{CP}^5.$$

1.4.6 Generalized Jacobian

We follow [62], [59]. The idea of a *generalized Jacobian* is to refine the equivalence classes of the divisors of the ordinary Jacobian. In particular two divisors will be equivalent if they are linearly equivalent (that is, equivalent in the ordinary Jacobian) and if they satisfy a further condition that depends only on *a priori* fixed divisor on the curve.

Let X be a curve on the complex plane of genus g and S a subset of points of X.

Definition 1.53. A divisor $D = \sum_{P \in X} n_P(P)$ is **effective** if $n_P \geq 0$, for every point P.

Definition 1.54. A **modulus** $\mathfrak{m}$ on X supported on S is an effective divisor $\sum_{P \in S} n_P(P)$.

A modulus is an assignment of a integer n_P for every point P of the subset S. A divisor D is **prime** to a divisor D' if their supports are disjoint.

Definition 1.55. Let $\mathfrak{m}$ be a modulus supported on $S_\mathfrak{m}$ and $g \in k(X)$ be a rational function. The principal divisor (g) is **prime** to $S_\mathfrak{m}$ if $v_P(1-g) \geq n_P$ for all $P \in S_\mathfrak{m}$. We will denote this by $g \equiv 1 \mod \mathfrak{m}$.

Definition 1.56. Two divisors $D, D' \in \mathrm{Div}(X)$ prime to $S_\mathfrak{m}$ are $\mathfrak{m}$-**equivalent** if $D - D' = (g)$ for some rational function $g \equiv 1 \mod \mathfrak{m}$. We will denote this by $D \sim_\mathfrak{m} D'$.

For $\mathfrak{m} = 0$, we obtain the classical linear equivalence. Let $\mathrm{Div}_\mathfrak{m}(X)$ be the set of divisors prime to $S_\mathfrak{m}$ and $\mathrm{Div}^0_\mathfrak{m}(X)$ the subset of divisors of $\mathrm{Div}_\mathfrak{m}(X)$ of degree zero. Let $\mathrm{PDiv}_\mathfrak{m}(X)$ be the subgroup of principal divisors of $\mathrm{Div}^0_\mathfrak{m}(X)$. The quotient group $\mathrm{Pic}^0_\mathfrak{m}(X) := \frac{\mathrm{Div}^0_\mathfrak{m}(X)}{\mathrm{PDiv}_\mathfrak{m}(X)}$ is the **zero part of the $\mathfrak{m}$-Picard group**. We denote the $\mathfrak{m}$-divisor class by $[D]_\mathfrak{m}$.

Theorem 1.57 (Roselincht). *There exists a commutative algebraic group $\mathfrak{J}_\mathfrak{m}$ isomorphic to $\mathrm{Pic}^0_\mathfrak{m}(X)$. The dimension π of $\mathfrak{J}_\mathfrak{m}$ is*

$$\pi = \begin{cases} |S_\mathfrak{m}| & \text{if } \mathfrak{m} = 0, \\ g + |S_\mathfrak{m}| - 1 & \text{otherwise.} \end{cases}$$

Definition 1.58. The commutative algebraic group $\mathfrak{J}_\mathfrak{m}$ is the **generalized Jacobian** of X.

The following theorem holds (see [62, Chapter V, Section 3]).

Theorem 1.59. *For any curve X with modulus $\mathfrak{m}$, the generalized Jacobian $\mathfrak{J}_\mathfrak{m}(X)$ is a group extension of an algebraic group $\mathfrak{L}_\mathfrak{m}(X)$, isomorphic to $\mathbb{G}_m^{|S_\mathfrak{m}|-1} \times \prod_{P \in S} \mathbb{G}_a^{n_P - 1}$, by the ordinary Jacobian $\mathfrak{J}(X)$. In other words, the generalized Jacobian fits in the following exact sequence*

$$0 \longrightarrow \mathfrak{L}_\mathfrak{m} \longrightarrow \mathfrak{J}_\mathfrak{m} \longrightarrow \mathfrak{J} \longrightarrow 0. \tag{1.25}$$

For genus $g \geq 1$ and $|S_\mathfrak{m}| \geq 2$, the sequence (1.25) is non-split. In particular, for a modulus $\mathfrak{m} = (T_1) + \cdots + (T_n)$ with $n \geq 2$ distinct points $T_1, \ldots, T_n \in X$, the sequence is non-split and $\mathfrak{L}_\mathfrak{m} \cong \mathbb{G}_m^{n-1}$ is a linear torus. In this work, we will see only generalized Jacobian for a modulus with pairwise distinct points, thus without the additive part.

1.4.7 Factor systems

In this section we provide notions about extensions of commutative algebraic groups.

Definition 1.60. Let X be an algebraic variety. A **Weil divisor** is an integer linear combination of irreducible subvarieties of X of codimension one.

We follow the approach of Serre's book [62]. Let $f\colon A \to B$ a homomorphism of Abelian groups. A **factor system** on A with values in B is a function $f\colon A \times A \to B$ such that

$$f(x+y, z) = f(y, z) - f(x, y+z) - f(x, z) \tag{1.26}$$

for every $x, y, z \in A$. If g is any map $A \to B$, we define $\delta(g)\colon A \times A \to B$ by the formula

$$\delta(g)(x, y) = g(x+y) - g(x) - g(y) \tag{1.27}$$

for $x, y \in A$. The function $\delta(g)$ results a factor system. We will call these factor systems **trivial** or 2-**coboundary**. The set of factor systems forms a group. Denote $H^2(A, B)$ the set of factor systems modulo the trivial ones. A factor system f is **symmetric** if $f(x, y) = f(y, x)$ for every $x, y \in A$. The set of symmetric factor systems forms a subgroup of $H^2(A, B)$ that we will denote by $H^2(A, B)_s$. The elements of $H^2(A, B)$ corresponds to the class of central extensions of A by B, while the elements of $H^2(A, B)_s$ corresponds to the class of commutative extensions of A by B.

Let A, B be two connected commutative algebraic groups. A rational map $A \times A \to B$ satisfying (1.26) is a **rational factor system**; if $f = \delta(g)$ for some rational map $g\colon A \to B$, then it is a **trivial rational factor system**. We will denote $H^2_{rat}(A, B)$ the group of rational factor systems modulo the trivial rational factor systems and $H^2_{rat}(A, B)_s$ the subgroup of classes of symmetric ones. Considering regular maps instead of rational ones, we can define in the same manner **regular factor systems** and the group $H^2_{reg}(A, B)$ with subgroup $H^2_{reg}(A, B)_s$.

Proposition 1.61 (Chapter VII, Section 1.4, [62])**.**

- *The group $H^2_{reg}(A, B)_s$ is isomorphic to the subgroup of* $\mathrm{Ext}(A, B)$ *given by extensions which admit a regular section.*
- *The group $H^2_{rat}(A, B)_s$ is isomorphic to the subgroup of* $\mathrm{Ext}(A, B)$ *given by extensions which admit a rational section.*
- *There exits a canonical homomorphism $H^2_{reg}(A, B)_s \to H^2_{rat}(A, B)_s$, whose is injective.*

Proposition 1.62. *If B is a linear algebraic group, then $H^2_{rat}(A, B)_s = \mathrm{Ext}(A, B)$. If A and B are linear algebraic groups, then $H^2_{reg}(A, B)_s = \mathrm{Ext}(A, B)$.*

Weil, in the published notes [72], was the first who studied extensions of an Abelian variety by the multiplicative group of the ground field. Barsotti [15] also studied this problem from the point of view of factors systems.

Theorem 1.63. *If A is an Abelian variety, the group $\mathrm{Ext}(A, \mathbb{G}_m)$ is canonically isomorphic, as abstract group, to the group underlying $\widehat{A}$, the dual variety of A.*

Proof. For the proof see [62], Chapter VII, sections 3.14, 3.15 and 3.16. □

In particular, denoting by $s_A\colon A \times A \to A$ the sum in the Abelian variety and p_1 and p_2 the two projections $p_j\colon A \times A \to A$ $(j = 1, 2)$, every divisor X such that the two divisors $s_A^{-1}(X)$ and $p_1^{-1}(X) + p_2^{-1}(X)$ are linearly equivalent in $A \times A$ (this is equivalent to the fact that $X \equiv 0$ by [45, Theorem 2, p. 90] and to the fact that X is *algebraically equivalent* to 0 by [13], see also [14]) corresponds to a class of rational factor systems. Explicitly, there exists a rational function $f\colon A \times A \to \mathbb{G}_m$ such that

$$(f) = s_A^{-1}(X) - p_1^{-1}(X) - p_2^{-1}(X) = s_A^{-1}(X) - X \times A - A \times X.$$

The function f is the factor system corresponding to the divisor X in the isomorphism of the above theorem.

Since the Jacobian of a curve is a principally polarized Abelian variety, it is self-dual, then the group $\mathrm{Ext}(\mathfrak{J}, \mathbb{G}_m)$ is canonically isomorphic to the commutative group underlying the Jacobian $\mathcal{J}$ itself [62], see also [49] and [50].

Linear equivalence in the generalized Jacobian of an elliptic curve. The following direct computations can be found, for instance, in [29, Section 5.3.1] and also cited in [28]. It is worthwhile to mention Section 1 of [23], where the difference between ordinary Jacobians and generalized Jacobians has been highlighted. The results of this section has been utilized in Section 2.1 and illustrate the case of next dimension (cf. Section 2.2).

The first goal is to describe explicitly the equivalence relation in the generalized Jacobian of an elliptic curve. Let $D_1 = (P_1) - (\Omega) + (f_1)$ and $D_2 = (P_2) - (\Omega) + (f_2)$ be two divisors and $L = (M) + (N)$ the considered modulus. The divisors D_1, D_2 are linearly equivalent if $D_1 - D_2$ is a principal divisor, that is, if $D_1 - D_2 = (h)$ for some rational function h on the curve $\mathcal{C}$. To be linearly equivalent one must have that $P_1 = P_2$. The condition relative to L consists in the fact that if $D_1 - D_2 = (h)$, then $v_M(1-h) \geq 1$ and $v_N(1-h) \geq 1$. Therefore, we have that $h(M) = 1 = h(N)$, that is, $\frac{f_1}{f_2}(M) = \frac{f_1}{f_2}(N)$. We can formulate this last condition as $\frac{f_1(M)}{f_1(N)} = \frac{f_2(M)}{f_2(N)}$. Thus, the equivalence class of a divisor D in $\mathfrak{J}_L(\mathcal{C})$ is determined by a (unique) point P and a non-zero complex number k such that $D = (P) - (\Omega) + (f)$ with $k = \frac{f(M)}{f(N)}$ (cf. [62, Chapter V, Section 3]).

Now, we are able to determine a factor system for the generalized Jacobian of an elliptic curve, identified as the cartesian product $\mathcal{C} \times \mathbb{C}^*$. Here, the Jacobian of $\mathcal{C}$ is naturally identified with the curve $\mathcal{C}$. Let $(P_1, k_1), (P_2, k_2), (P_3, k_3)$ be three pairs, where P_j are points on $\mathcal{C}$ and k_j are non-zero complex numbers, such that

$$(P_1, k_1)(P_2, k_2) = (P_3, k_3).$$

Let $D_1 = (P_1) - (\Omega) + (f_1)$ and $D_2 = (P_2) - (\Omega) + (f_2)$, where f_1, f_2 are rational functions such that $k_j = \frac{f_j(M)}{f_j(N)}$, for $j = 1, 2$. The divisor given by the sum $D_1 + D_2 = (P_1) + (P_2) - 2(\Omega) + (f_1 f_2)$ must be linearly equivalent to a divisor $D_3 = (P_3) - (\Omega) + (f_3)$. So, the goal is to determine the point $P_3 \in \mathcal{C}$ and the rational function f_3 associated to the (class of) divisor D_3. We know, by the group law of the points in an elliptic curve (cf. Example 1.47), that

$$(P_1) + (P_2) + (-(P_1 + P_2)) - 3(\Omega) = (\ell_{P_1,P_2}), \tag{1.28}$$

where ℓ_{P_1,P_2} denotes the straight line through the points P_1 and P_2. Since

$$(P_1 + P_2) + (-(P_1 + P_2)) - 2(\Omega) = (\ell_{P_1+P_2,\Omega}), \tag{1.29}$$

where $\ell_{P_1+P_2,\Omega}$ is the vertical line through the point $P_1 + P_2$, substituting (1.29) in (1.28) we have that

$$(P_1) + (P_2) - (P_1 + P_2) - (\Omega) = (\ell_{P_1,P_2}) - (\ell_{P_1+P_2,\Omega})$$

Thus, the divisor $D_1 + D_2$ is linearly equivalent to $(P_1 + P_2) - (\Omega)$ and their difference is the principal divisor $\left(f_1 f_2 \frac{\ell_{P_1,P_2}}{\ell_{P_1+P_2,\Omega}}\right)$. It follows that $P_3 = P_1 + P_2$, $f_3 = f_1 f_2 \frac{\ell_{P_1,P_2}}{\ell_{P_1+P_2,\Omega}}$ and the complex number is given by

$$k_3 = \frac{f_3(M)}{f_3(N)} = \frac{f_1(M)}{f_1(N)} \frac{f_2(M)}{f_2(N)} \frac{\ell_{P_1,P_2}(M)}{\ell_{P_1+P_2,\Omega}(M)} \frac{\ell_{P_1+P_2,\Omega}(N)}{\ell_{P_1,P_2}(N)},$$

which is, recalling what k_1 and k_2 are,

$$k_3 = k_1 k_2 \frac{\ell_{P_1,P_2}(M)}{\ell_{P_1+P_2,\Omega}(M)} \frac{\ell_{P_1+P_2,\Omega}(N)}{\ell_{P_1,P_2}(N)}.$$

As mentioned above, we will talk again about the identification of the generalized Jacobian by pair (or, more in general, by a n-tuple) in Section 2.2 for the hyperelliptic case. As pointed out in Section 1.4.3, for genus two or greater the situation is more intricate. In fact, the Jacobian is no longer the curve itself but a variety that contains properly the curve.

Chapter 2

Toroidal groups, generalized Jacobians, and periodic functions

In this chapter we study the relationship between toroidal groups and generalized Jacobians. More precisely, in Section 2.1, we consider a toroidal group with complex rank n and real rank $n+1$, which can be always seen as the group extension of an elliptic curve. In particular, we start from the case of dimension $n = 2$ and generalize the results to an arbitrary dimension n. In section 2.2, we study a toroidal group of dimension three and real rank five as the group extension of the Abelian variety given by the Jacobian of a hyperelliptic curve of genus two. Each of the two above mentioned classes of toroidal groups are included in the case of so-called *quasi-Abelian variety* (cf. Definition 1.38).

2.1 Toroidal groups as group extension of an elliptic curve

In [23] the authors considered a toroidal group $\mathcal{T} = \mathbb{C}^2/\Lambda$ of real rank $\mathrm{rk}_{\mathbb{R}}\,\Lambda = 3$ and they proved that it is birationally isomorphic to a generalized Jacobian of a suitable elliptic curve. In particular, let $\Lambda = \langle (1,0),(0,1),(\tau,s)\rangle$ be the lattice of a toroidal group in standard coordinates, where we can assume that τ is not real. In this case, the irrationality condition consists in the fact that there no exists a non-trivial pair of integers (a,b) such that $a\tau + bs \in \mathbb{Z}$. It is immediate that $\mathcal{T}$ is the group extension of a 1-dimensional linear subtorus, isomorphic to $\mathbb{C}^*$, by an elliptic curve of periods 1 and τ.

Theorem 2.1. *Let $\mathcal{T} = \mathbb{C}^2/\Lambda$ be a toroidal group of period matrix in standard coordinates given by*

$$\left(\begin{array}{cc|c} 1 & 0 & \tau \\ 0 & 1 & s \end{array}\right)$$

with $\tau \notin \mathbb{R}$. Then the group $\mathcal{T}$ is extension of a 1-dimensional closed linear subtorus $\mathcal{L}$, isomorphic to $\mathbb{C}^*$, by the Jacobian of an elliptic curve of lattice $\langle 1,\tau\rangle_{\mathbb{Z}}$.

Proof. Define the 1-dimensional subspace of $\mathbb{C}^2$, $H = \{(z_1,z_2) \in \mathbb{C}^2 \colon z_2 = 0\}$, then the quotient $\mathcal{L} = (H+\Lambda)/\Lambda$ is a closed linear subgroup of $\mathcal{T}$ and the factor group $\mathcal{T}/\mathcal{L}$ is a 1-dimensional complex torus with period matrix $(1|\tau)$, by Proposition 1.17. □

As explained in Backgrounds, the Jacobian of an elliptic curve concides with the curve itself by the Abel-Jacobi map (cf. Example 1.47). We give now the following definition to fix the notation of function c_L already introduced at the end of Section 1.4.7.

Definition 2.2. Let the elliptic curve $\mathcal{C}$ be defined by the lattice $\langle 1, \tau \rangle_{\mathbb{Z}}$, and let Ω be its point at infinity. Let $L = (M) + (N)$, with distinct $M, N \in \mathcal{C}$, be a modulus (that is, an effective divisor), and let $\ell_{P,Q}(X) = 0$ be (up to an uninfluent scalar) the equation of the straight line through the points P and Q, or the tangent line in P to $\mathcal{C}$ in the case where $Q = P$. We define the map

$$c_L(P_1, P_2) = \frac{\ell_{P_1,P_2}(M)}{\ell_{P_1+P_2,\Omega}(M)} \frac{\ell_{P_1+P_2,\Omega}(N)}{\ell_{P_1,P_2}(N)}, \tag{2.1}$$

for any two points P_1, P_2 such that $(P_1) + (P_2) - 2(\Omega)$ is not linearly equivalent to either $-(N) + (\Omega)$ or $-(M) + (\Omega)$.

The function c_L is a (non-regular) factor system of the non-split sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathfrak{J}_L \longrightarrow \mathfrak{J} \longrightarrow \Omega, \tag{2.2}$$

as well as one can compute it directly, see Section 1.4.7 (also [23], [28]).

Remark 2.3. It must be noticed that the partial operation

$$(P_1, k_1)(P_2, k_2) = (P_1 + P_2, k_1 k_2\, c_L(P_1, P_2))$$

cannot be extended continuously to $\mathcal{C} \times \mathbb{C}^*$. Thus, the factor system c_L is rational but not regular, a not frequent example of the fact that the group of extensions $\mathrm{Ext}(A, B)$ coincides with the class $H^2(A, B)_{reg}$ of regular factor systems, only if both A and B are linear, whereas it coincides with the class $H^2(A, B)_{rat}$ of rational factor systems, if B is linear, as in the present case (cf. Proposition 1.62).

The elliptic curve is parametrized by complex numbers through the surjective map $\mathrm{P}\colon z \mapsto (\wp(z), \wp'(z))$ defined in (1.15), given by the Weierstrass elliptic function $\wp$. Thus, $\mathcal{C}$ is equal to $\mathrm{P}(\mathbb{C})$. The factor system $c_L\colon \mathfrak{J} \times \mathfrak{J} \to \mathbb{C}^*$ induces a further factor system

$$c_L\mathrm{P}\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}^*, \quad c_L\mathrm{P}(z_1, z_2) = c_L(\mathrm{P}(z_1), \mathrm{P}(z_2)). \tag{2.3}$$

Since every commutative extension of $\mathbb{C}^*$ by $\mathbb{C}$ is splitting, we have that $c_L\mathrm{P}$ is a trivial factor system, that is, a coboundary $\delta^1(\varsigma)$ for some function $\varsigma\colon \mathbb{C} \to \mathbb{C}^*$. The following theorem furnish the explicit expression of ς, among other things.

Theorem 2.4. *Let $M = \mathrm{P}(t_M)$ and $N = \mathrm{P}(t_N)$ be two different point of $\mathcal{C}$ such that $(\mathrm{P}(s)) - (\Omega)$ is linearly equivalent to the divisor $(M) - (N)$. The function $G(x, y) = \exp(2\pi i y) g(x)$ with*

$$g(x) = \exp(2\eta_1(t_N - t_M)x) \frac{\sigma(t_M)}{\sigma(t_N)} \frac{\sigma(x - t_N)}{\sigma(x - t_M)}, \tag{2.4}$$

where σ is the sigma Weierstrass function of lattice generated by $\{1, \tau\}$ and η_1 is the constant (1.13), is triply periodic with period lattice Λ.
The generalized Jacobian $\mathfrak{J}_L$ of $\mathcal{C}$ with modulus $L = (M) + (N)$ is isomorphic to the toroidal group $\mathcal{T} = \mathbb{C}^2/\Lambda$. In particular, $\delta^1(g) = c_L\mathrm{P}$ and the map

$$\mathbb{C}^2 \to \Xi_L, \quad (x, y) \mapsto (\mathrm{P}(x), G(x, y)), \tag{2.5}$$

defines an isomorphism from $\mathcal{T}$ to a birationally isomorphic image Ξ_L of $\mathfrak{J}_L$.

The first part of the claim could be verified *a posteriori* (as one can check in Section 2.1.1), but the authors in [23] furnished a constructive proof of Theorem 2.4, explaining better the link between the toroidal group $\mathcal{T} = \mathbb{C}^2/\Lambda$ and the generalized Jacobian $\mathfrak{J}_L$. Indeed, they determined explicitly the function $g(x)$ imposing the condition $c_L(\mathrm{P}(x_1), \mathrm{P}(x_2))g(x_1)g(x_2) = g(x_1 + x_2)$, by utilizing the explicit form of the epimorphism $\mathrm{P}\colon \mathbb{C} \to \mathcal{C}$ given by the Weierstrass elliptic function $\wp$ of periods 1 and τ, the equations of the straight lines through two points in the projective complex plane, and the quasi-periodicity of the σ-function.

Remark 2.5. We already know that the group $\mathrm{Ext}(\mathfrak{J}(\mathcal{C}), \mathbb{C}^*)$ is canonically isomorphic to $\mathfrak{J}(\mathcal{C}) \equiv \mathcal{C}$ (cf. Section 1.4.7). This isomorphism can be seen, in the light of the above theorem, in terms of the toroidal group. The points M and N of the modulus L in Theorem 2.4 are arbitrarily chosen under the only hypothesis that the divisor $(\mathrm{P}(s)) - (\Omega)$ is linearly equivalent to $(M) - (N)$. In terms of the complex torus $\mathbb{C}/\langle 1, \tau\rangle_{\mathbb{Z}}$, this hypothesis means that the corresponding complex numbers must satisfy the condition $s - t_M + t_N \in \langle 1, \tau\rangle_{\mathbb{Z}}$. This allows us to fix one of the two points, suppose M, and then choosing a suitable point $N \equiv \mathrm{P}(t_N)$ such that $s = t_M - t_N + a + b\tau$, for some $a, b \in \mathbb{Z}$. Therefore, from the point of view of the toroidal group, every points N corresponds a factor system c_L, where L satisfies the hypothesis of the above theorem.

In [27] the authors extended the isomorphism (2.5) to one between a toroidal group of arbitrary dimension n and real rank $n+1$ and a generalized Jacobian of an elliptic curve. Furthermore, this correspondence has been used to determine a parametrization of the torsion subgroups of a toroidal group and to establish a "canonical" method to construct a periodic meromorphic function of a given period lattice.

Let $\mathcal{T}$ be a toroidal group of complex rank n and real rank $n+1$. Up to a change of basis, we can assume that $\mathcal{T} = \mathbb{C}^n/\Lambda$ and the lattice to be in standard coordinates

$$\Lambda = \langle (1,0,\dots,0), (0,1,\dots,0), \dots, (0,\dots,0,1), (\tau, s_1, \dots, s_{n-1})\rangle_{\mathbb{Z}}$$

with $\tau \notin \mathbb{R}$. We recall that for such a lattice Λ the irrationality condition must hold, that is, the only n-tuple $(a_0, \dots, a_{n-1}) \in \mathbb{Z}^n$ such that

$$a_0\tau + a_1 s_1 + \cdots + a_{n-1}s_{n-1} \in \mathbb{Z}$$

is $(0, \dots, 0)$. In this case, we are rolling out elementary holomorphic periodic functions such as

$$G(z_0, z_1, \dots, z_{n-1}) = \exp\{2\pi i(a_0 z_0 + a_1 z_1 + \cdots + a_{n-1}z_{n-1})\}$$

with $a_0\tau + a_1 s_1 + \cdots + a_{n-1}s_{n-1} \in \mathbb{Z}$, having manifestly Λ as period lattice. In the following theorem, we denote the first variable by ζ and the other ones by $z_1, \dots, z_{n-1}$ because the first one plays a special role, as one can see in (2.11).

Theorem 2.6. *Let $\Lambda = \langle (1,0,\dots,0), \dots, (0,\dots,0,1), (\tau, s_1, \dots, s_{n-1})\rangle_{\mathbb{Z}}$, with $\tau \notin \mathbb{R}$, be a lattice defining the toroidal group $\mathcal{T} = \mathbb{C}^n/\Lambda$. If $\mathcal{V}(\zeta) = \{(\zeta, z_1, \dots, z_{n-1}) \in \mathbb{C}^n : \zeta = 0\}$, then the subgroup $\mathcal{L} := \big(\mathcal{V}(\zeta) + \Lambda\big)/\Lambda$ of $\mathcal{T}$ is a linear subtorus, isomorphic to $(\mathbb{C}^*)^{n-1}$, with the quotient group $\mathcal{T}/\mathcal{L}$ isomorphic to the Jacobian $\mathfrak{J}(\mathcal{C})$ of the elliptic curve $\mathcal{C}$ with fundamental parallelogram generated by $\{1, \tau\}$. Moreover, in the corresponding* *non-splitting extension*

$$1 \longrightarrow \mathcal{L} \longrightarrow \mathcal{T} \longrightarrow \mathfrak{J}(\mathcal{C}) \longrightarrow \Omega, \tag{2.6}$$

we can represent an element of $\mathcal{T}$ with a n-tuple

$$(P, k_1, \dots, k_{n-1}) \in \mathcal{C} \times (\mathbb{C}^*)^{n-1}, \tag{2.7}$$

in such a way that $(\Omega, 1, \dots, 1)$ is the zero element, and

$$\begin{aligned}&(P_1, k_1, \dots, k_{n-1})(P_2, h_1, \dots, h_{n-1}) \\ &= (P_1 + P_2, k_1 h_1\, c_{L_1}(P_1, P_2), \dots, k_{n-1} h_{n-1}\, c_{L_{n-1}}(P_1, P_2)),\end{aligned} \tag{2.8}$$

where c_{L_j} is defined as in (2.1) for $j = 1, \dots, n-1$, by $n-1$ given moduli $L_j = (T_j) + (T_{j+1})$, such that $(T_j) - (T_{j+1})$ is linearly equivalent to $(P_j) - (\Omega)$, where $P_j = \mathrm{P}(s_j)$, and P is the map defined in (1.15).

Proof. Since $\mathcal{V}(\zeta) \cap \Lambda = \langle (0,1,0,\dots,0), \dots, (0,\dots,0,1) \rangle_{\mathbb{Z}}$, the subgroup $\mathcal{L}$ is a linear subtorus and, by Proposition 1.17, the quotient group $\mathcal{T}/\mathcal{L}$ is isomorphic to $\mathfrak{J}(\mathcal{C})$. Since the functor Ext is additive, i.e. $\mathrm{Ext}((\mathbb{C}^*)^{n-1}, \mathbb{C}) \cong \oplus^{n-1} \mathrm{Ext}(\mathbb{C}^*, \mathbb{C})$, we can reduce the argument to the case where $n = 2$, for which the situation is described in Theorem 2.4. □

Remark 2.7. Indeed, one can choose the moduli L_j to be defined by any two points M_j, N_j such that $(M_j) - (N_j)$ is linearly equivalent to $(P_j) - (\Omega)$, however we choose $L_j = (T_j) + (T_{j+1})$ just in order to simplify the function G we will exhibit in (2.12).

As remarked above for the case $n = 2$, the non-trivial factor system c_{L_j} induces a further factor system

$$c_{L_j}\mathrm{P} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^*, \quad (\zeta_1, \zeta_2) \mapsto c_{L_j}\big(\mathrm{P}(\zeta_1), \mathrm{P}(\zeta_2)\big).$$

Again, as every commutative Lie group extension of $\mathbb{C}^*$ by $\mathbb{C}$ is splitting, $c_{L_j}\mathrm{P}$ turns out indeed to be a trivial factor system, that is, a coboundary $\delta^1(g_j) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^*$, where $g_j : \mathbb{C} \to \mathbb{C}^*$ is a section (cf. [23, Th., p. 128]) defined by

$$g_j(\zeta) = \exp(-2\eta_1 s_j \zeta) \frac{\sigma(t_j)}{\sigma(t_{j+1})} \frac{\sigma(\zeta - t_{j+1})}{\sigma(\zeta - t_j)}, \tag{2.9}$$

where $t_j \in \mathbb{C}$ is such that $\mathrm{P}(t_j) = T_j$ and η_1 is the constant in (1.13). This unveils the cocycle $c_{L_j}\big(\mathrm{P}(\zeta_1), \mathrm{P}(\zeta_2)\big)$ as the coboundary $(c_{L_j}\mathrm{P})(\zeta_1, \zeta_2) = \delta^1(g_j)(\zeta_1, \zeta_2)$.

The map

$$\begin{aligned}\mathbb{C}^n &\longrightarrow \mathcal{C} \times (\mathbb{C}^*)^{n-1}, \\ (\zeta, z_1, \dots, z_n) &\mapsto \big(\mathrm{P}(\zeta), G_1(\zeta, z_1), \dots, G_{n-1}(\zeta, z_{n-1})\big),\end{aligned} \tag{2.10}$$

with

$$G_j(\zeta, z_j) := g_j(\zeta) \exp(2\pi i z_j) = \exp(-2\eta_1 s_j \zeta) \frac{\sigma(t_j)}{\sigma(t_{j+1})} \frac{\sigma(\zeta - t_{j+1})}{\sigma(\zeta - t_j)} \exp\left(2\pi i z_j\right),$$

determines an isomorphism between the toroidal group $\mathcal{T} = \mathbb{C}^n/\Lambda$ and a birational image of the generalized Jacobian (with modulus L) of the elliptic curve $\mathcal{C}$ with fundamental paralleogram spanned by $\{1, \tau\}$. In particular, as in the two-dimensional case that we have considered above, each G_j is a two-variate meromorphic function with period lattice given by $\Lambda_j = \langle (1,0), (0,1), (\tau, s_j) \rangle_{\mathbb{Z}}$.

Turning back to n-variate meromorphic functions with $n+1$ $\mathbb{R}$-independent periods, we point out that while by definition of a toroidal group $\mathcal{T}$ there no exist holomorphic functions on $\mathcal{T}$, on the contrary, the existence of meromophic functions on $\mathcal{T}$ is guaranteed by the fact that such toroidal groups are quasi-Abelian varieties (in the sense of Severi [63]). Indeed, Capocasa and Catanese [20, Main Theorem] in 1991 proved that there exists a non-degenerate Λ-periodic meromorphic function if and only if $\mathbb{C}^n/\Lambda$ is a quasi-Abelian variety, as defined by Andreotti and Gherardelli [8].

In order to find out how such a meromorphic function $G(\zeta, z_1, \dots, z_{n-1})$ is built, we stress its geometric meaning. Note in passing that, whereas such a function determines a unique period lattice, a lattice $\Lambda \subset \mathbb{C}^n$ of real rank $n+1$ determines a whole family of periodic functions. Being Λ-periodic, any such a function G defines a further function

$$\hat{G}\big((\zeta, z_1, \dots, z_{n-1}) + \Lambda\big) := G(\zeta, z_1, \dots, z_{n-1}),$$

from $\mathcal{T} = \mathbb{C}^n/\Lambda$ to $\mathbb{C}$, giving in turn the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & (\mathbb{C}^*)^{n-1} & \longrightarrow & \mathcal{T} & \xrightarrow{\pi} & \mathfrak{J}(\mathcal{C}) & \longrightarrow & 0 \\
 & & & & \varpi\uparrow & \searrow^{\hat{G}} & & & \\
 & & & & \mathbb{C}^n & \xrightarrow[G]{} & \mathbb{C} & &
\end{array}$$

where ϖ denotes the canonical projection. Whilst ϖ is independent of the elliptic curve $\mathcal{C}$, the role played by the choice of $\mathcal{C}$ comes in here: by the Theorem 2.4 and the above Theorem 2.6, if we let again $t_j \in \mathbb{C}$ be such that $\mathrm{P}(t_j) = T_j$, then

$$\varpi(\zeta, z_1, \dots, z_{n-1}) \equiv \big(\mathrm{P}(\zeta), G_1(\zeta, z_1), \dots, G_{n-1}(\zeta, z_{n-1})\big), \tag{2.11}$$

by the isomorphism with the generalized Jacobian. The map ϖ can be seen as an epimorphism from the additive group of $\mathbb{C}^n$ to (a representation of) the toroidal group $\mathcal{T}$ with Λ as the kernel. As mentioned above in passing, the period lattice Λ_j of any function of $G_1, \dots, G_{n-1}$ is contained in Λ. As a prototype of such a n-variate meromorphic function with period lattice Λ, we simply take here their product,

$$G(\zeta, z_1, \dots, z_{n-1}) := G_1(\zeta, z_1) \cdots G_{n-1}(\zeta, z_{n-1}) \tag{2.12}$$

which, with the above choice of $L_i = (T_i) + (T_{i+1})$, reduces to

$$\begin{aligned}
G(\zeta, z_1, \dots, z_{n-1}) &= \prod_{j=1}^{n} \exp(-2\eta_1 s_j \zeta) \frac{\sigma(t_j)}{\sigma(t_{j+1})} \frac{\sigma(\zeta - t_{j+1})}{\sigma(\zeta - t_j)} \exp\left(2\pi, i z_j\right) \\
&= \exp(-2\eta_1 s \zeta) \frac{\sigma(t_1)}{\sigma(t_n)} \frac{\sigma(\zeta - t_n)}{\sigma(\zeta - t_1)} \exp(2\pi i (z_1 + \cdots + z_{n-1})),
\end{aligned}$$

with $s = s_1 + s_2 + \cdots + s_{n-1}$.

We will make use of this geometric relationship between toroidal groups and generalized Jacobians in Section 3.1.2, where we will provide a parametrization of the torsion subgroups of a toroidal group.

2.1.1 A *new* relation between Weierstrass functions

The function c_L is a factor system of the non-split exact sequence (1.25). Moreover, in the previous section we showed that this map, composed with the function $\mathrm{P}\colon \mathbb{C} \to \mathcal{C}$, turns out to be a 2-coboundary. In particular, the equality $\delta^1(g) = c_L\mathrm{P}$, where g is the function in (2.4), holds. Substituting the explicit expressions of g and c_L, we can deduce the further equality

$$\frac{\sigma(t_N)\sigma(x_1+x_2-t_N)\sigma(x_1-t_M)\sigma(x_2-t_M)}{\sigma(t_M)\sigma(x_1+x_2-t_M)\sigma(x_1-t_N)\sigma(x_2-t_N)} = \frac{\ell_{\mathrm{P}(x_1),\mathrm{P}(x_2)}(M)}{\ell_{\mathrm{P}(x_1+x_2),\Omega}(M)}\frac{\ell_{\mathrm{P}(x_1+x_2),\Omega}(N)}{\ell_{\mathrm{P}(x_1),\mathrm{P}(x_2)}(N)}, \tag{2.13}$$

first appeared in [23], which is apparently a new relationship between the two Weierstrass functions σ and $\wp$ (where we recall that the elliptic function $\wp$ occurs in (2.13) in the definition of the map P).

As mentioned in the previous section, the function g in (2.4) comes out from an ad hoc construction imposing the condition $\delta^1(g) = c_L\mathrm{P}$ (see [23, proof of Theorem] for details). As a further proof of the fact, we want to show *a posteriori* this equality proving the relation (2.13). In order to do that, we use the well-know relations occurring between the Weierstrass functions σ and $\wp$. Recall the following two addition theorems

$$-\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\,\sigma(v)^2} = \begin{vmatrix} 1 & \wp(u) \\ 1 & \wp(v) \end{vmatrix}, \quad \text{for every } u, v \in \mathbb{C}, \tag{2.14}$$

(see, e.g., [10, p. 1] or [47, p. 158]) and the more general *Frobenius-Stickelberger formula*, appeared in [30],

$$\begin{vmatrix} 1 & \wp(u_1) & \wp'(u_1) & \dots & \wp^{(n-2)}(u_1) \\ 1 & \wp(u_2) & \wp'(u_2) & \dots & \wp^{(n-2)}(u_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u_n) & \wp'(u_n) & \dots & \wp^{(n-2)}(u_n) \end{vmatrix} = (-1)^{n-1}\prod_{i=1}^{n-1} i!\,\frac{\sigma\left(\sum_{i=1}^{n} u_i\right)\prod_{1\le i<j\le n}\sigma(u_j-u_i)}{\prod_{i=1}^{n}\sigma(u_i)^n} \tag{2.15}$$

for every $u_1, \dots, u_n \in \mathbb{C}$.

First note that, writing the equations of the straight lines as determinants, we have the following equality

$$\frac{\ell_{\mathrm{P}(x_1),\mathrm{P}(x_2)}(N)}{\ell_{\mathrm{P}(x_1+x_2),\Omega}(N)} = \frac{\begin{vmatrix} 1 & \wp(x_1) & \wp'(x_1) \\ 1 & \wp(x_2) & \wp'(x_2) \\ 1 & \wp(t_N) & \wp'(t_N) \end{vmatrix}}{\begin{vmatrix} 1 & \wp(x_1+x_2) \\ 1 & \wp(t_N) \end{vmatrix}}.$$

Now, combining the formula (2.15) for $u_1 = x_1$, $u_2 = x_2$, and $u_3 = t_N$,

$$\begin{vmatrix} 1 & \wp(x_1) & \wp'(x_1) \\ 1 & \wp(x_2) & \wp'(x_2) \\ 1 & \wp(t_N) & \wp'(t_N) \end{vmatrix} = -2\frac{\sigma(x_1+x_2+t_N)\sigma(x_1-x_2)\sigma(x_1-t_N)\sigma(x_2-t_N)}{\sigma(x_1)^3\sigma(x_2)^3\sigma(t_N)^3},$$

with the formula (2.14) for $u = x_1 + x_2$, and $v = t_N$,

$$-\frac{\sigma(x_1+x_2+t_N)\,\sigma(x_1+x_2-t_N)}{\sigma(x_1+x_2)^2\,\sigma(t_N)^2} = \begin{vmatrix} 1 & \wp(x_1+x_2) \\ 1 & \wp(t_N) \end{vmatrix},$$

we obtain

$$\frac{\begin{vmatrix} 1 & \wp(x_1) & \wp'(x_1) \\ 1 & \wp(x_2) & \wp'(x_2) \\ 1 & \wp(t_N) & \wp'(t_N) \end{vmatrix}}{\begin{vmatrix} 1 & \wp(x_1+x_2) \\ 1 & \wp(t_N) \end{vmatrix}} = 2\frac{\sigma(x_1-x_2)\sigma(x_1-t_N)\sigma(x_2-t_N)\sigma(x_1+x_2)^2}{\sigma(x_1)^3\sigma(x_2)^3\sigma(t_N)\sigma(x_1+x_2-t_N)}.$$

The same relation holds for t_M and dividing the two relations each other we have that

$$c_L\mathrm{P}(x_1,x_2) = \frac{\begin{vmatrix} 1 & \wp(x_1) & \wp'(x_1) \\ 1 & \wp(x_2) & \wp'(x_2) \\ 1 & \wp(t_M) & \wp'(t_M) \end{vmatrix}}{\begin{vmatrix} 1 & \wp(x_1+x_2) \\ 1 & \wp(t_M) \end{vmatrix}} \frac{\begin{vmatrix} 1 & \wp(x_1+x_2) \\ 1 & \wp(t_N) \end{vmatrix}}{\begin{vmatrix} 1 & \wp(x_1) & \wp'(x_1) \\ 1 & \wp(x_2) & \wp'(x_2) \\ 1 & \wp(t_N) & \wp'(t_N) \end{vmatrix}}$$

$$= \frac{\sigma(t_N)\sigma(x_1+x_2-t_N)\sigma(x_1-t_M)\sigma(x_2-t_M)}{\sigma(t_M)\sigma(x_1+x_2-t_M)\sigma(x_1-t_N)\sigma(x_2-t_N)} = \delta^1(g)(x_1,x_2).$$

2.2 Toroidal groups as group extension of a Jacobian of a hyperelliptic curve of genus 2

In this section, we investigate the connection between toroidal groups and hyperelliptic curves. We focus our attention to imaginary hyperelliptic curves of genus two, which have only one point at infinity Ω. From now on in this section, σ denotes the two-variate quasi-periodic sigma function defined in (1.20), whose properties have been described in Section 1.4.5, and the bold symbols ($\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{m}, \ldots$) denote the 2-tuples considered as column vectors.

By Theorem 1.59, for a modulus $L = (T_1) + \cdots + (T_n)$ with distinct points $T_1, \ldots, T_n$ of a (imaginary) hyperelliptic curve $\mathcal{H}$, the generalized Jacobian $\mathfrak{J}_L(\mathcal{H})$ is the group extension of the ordinary Jacobian variety $\mathfrak{J}(\mathcal{H})$ by an algebraic group isomorphic to $(\mathbb{C}^*)^{n-1}$, that is, $\mathfrak{J}_L(\mathcal{H})$ fits in the following non-split exact sequence

$$1 \longrightarrow (\mathbb{C}^*)^{n-1} \longrightarrow \mathfrak{J}_L(\mathcal{H}) \longrightarrow \mathfrak{J}(\mathcal{H}) \longrightarrow \Omega \tag{2.16}$$

of commutative algebraic groups. Let us restrict ourselves, for the moment, to the case of $n = 2$. Thus, we are considering the generalized Jacobian of modulus $L = (M) + (N)$ of $\mathcal{H}$, with $M \neq N$, which is the group extension of a one-dimensional linear torus, isomorphic to $\mathbb{C}^*$, and we write the corresponding non-split exact sequence in order to fix the notation

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathfrak{J}_L(\mathcal{H}) \longrightarrow \mathfrak{J}(\mathcal{H}) \longrightarrow \Omega. \tag{2.17}$$

Again, as in the elliptic case, by the means of the exact sequence (2.17), we want to represent $\mathfrak{J}_L(\mathcal{H})$ by pairs.

Linear equivalence in the generalized Jacobian of a hyperelliptic curve. We proceed as we have done in the case of an elliptic curve in Section 1.4.7. The first step is to explicit the elements of a class of a divisor (of degree zero) in $\mathfrak{J}_L(\mathcal{H})$. Two divisors D_1 and D_2 are equivalent in the generalized Jacobian $\mathfrak{J}_L(\mathcal{H})$ if they differ by a principal divisor (h) of some rational function h on $\mathcal{H}$ such that $v_P(1-h) \geq n_P$, for $P \in \{M, N\}$. Since every divisor of degree zero is linearly equivalent to a unique reduced divisor (cf. Section 1.4.4), for D_1 and D_2 being linearly equivalent they must be linearly equivalent to the same reduced divisor (hence, they must be determined by the same points of the curve). More explicitly, if $D_j = (P_j) + (R_j) - 2(\Omega) + (f_j)$ for $j = 1, 2$, then D_1 and D_2 are linearly equivalent if and only if $\{P_1, R_1\} = \{P_2, R_2\}$ as sets. Therefore, their difference is given by $D_1 - D_2 = \left(\frac{f_1}{f_2}\right)$ and, thus, D_1 and D_2 are equivalent in the generalized Jacobian if and only if $\frac{f_1(M)}{f_2(M)} = 1 = \frac{f_1(N)}{f_2(N)}$, that we can rewrite as in the elliptic case as the condition $\frac{f_1(M)}{f_1(N)} = \frac{f_2(M)}{f_2(N)}$. In other words, a class in the generalized Jacobian is uniquely determined by two points (or one point for the divisors coming from the embedding of the curve into the Jacobian) and a constant number. More precisely, for the exact sequence (1.25), a divisor class in the generalized Jacobian is identified by a pair $(\bar{D}, k) \in \mathfrak{J}(\mathcal{H}) \times \mathbb{C}^*$, where $D = (P) + (R) - 2(\Omega) + (f)$ and $k = \frac{f(M)}{f(N)}$, for a rational function f on $\mathcal{H}$ (see [62, Chapter V, Section 3], also [23, Section 1]).

The second step is to determine a factor system. Let $(\bar{D}_1, k_1), (\bar{D}_2, k_2), (\bar{D}_3, k_3)$ be three pairs such that

$$(\bar{D}_1, k_1)(\bar{D}_2, k_2) = (\bar{D}_3, k_3).$$

The pair $(\bar{D}_3, k_3)$ is associated to a divisor of degree zero $D_3 = (S_1) + (S_2) - 2(\Omega) + (f_3)$ such that $k_3 = \frac{f_3(M)}{f_3(N)}$. The goal is to find D_3 and k_3. We know how to sum two divisors in the ordinary Jacobian variety of a hyperelliptic curve of genus two: the sum of two divisors $D_1 + D_2 = (P_1) + (R_1) + (P_2) + (R_2) - 4(\Omega) + (f_1 f_2)$ is equivalent to a (unique) divisor $(U) + (V) - 2(\Omega) + (f_3)$. Let us consider the (unique) cubic curve κ through the points P_1, R_1, P_2, R_2. The intersection of κ and $\mathcal{H}$ consists of six points: the starting four P_1, R_1, P_2, R_2 and two further T_1, T_2. Let S_i be the symmetric point of T_i, hence $(S_i) + (T_i) - 2(\Omega) = (\ell_i)$, where ℓ_i is the vertical line through S_i and T_i. We have

$$(\kappa) = (P_1) + (R_1) + (P_2) + (R_2) + (T_1) + (T_2) - 6(\Omega)$$

and

$$\begin{aligned} D_1 + D_2 &= (P_1) + (R_1) + (P_2) + (R_2) - 4(\Omega) + (f_1 f_2) \\ &= (\kappa) - (T_1) - (T_2) + 2(\Omega) + (f_1 f_2) \\ &= (\kappa) + (S_1) + (S_2) - 2(\Omega) - (\ell_1) - (\ell_2) + (f_1 f_2) \\ &= (S_1) + (S_2) - 2(\Omega) + \left(f_1 f_2 \frac{\kappa}{\ell_1 \ell_2} \right). \end{aligned}$$

We conclude that $U = S_1$, $V = S_2$ and $f_3 = f_1 f_2 \frac{\kappa}{\ell_1 \ell_2}$. Thus, the corresponding non-zero complex number is given by

$$k_3 = \frac{f_3(M)}{f_3(N)} = k_1 k_2 \frac{\kappa(M)}{\ell_1(M)\ell_2(M)} \frac{\ell_1(N)\ell_2(N)}{\kappa(N)}$$

(of course, if $D_1 = (P_1) - (\Omega)$ one has to take the parabola through P_1, P_2, R_2 instead of the above cubic).

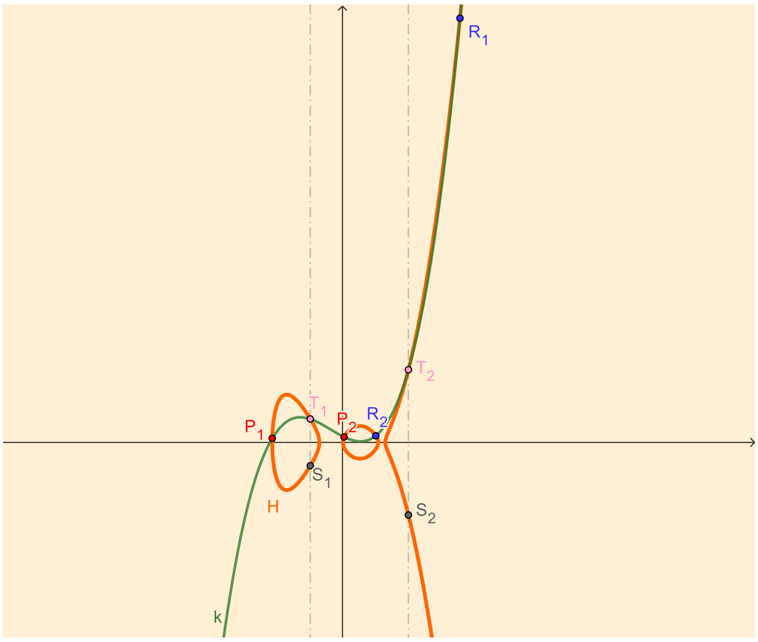

FIGURE 2.1: Sum of two divisors in a hyperelliptic curve.

This allows us to give the following definition.

Definition 2.8. We define the function

$$c_L(\bar{D}_1, \bar{D}_2) = \frac{\kappa(M)}{\ell_1(M)\ell_2(M)} \frac{\ell_1(N)\ell_2(N)}{\kappa(N)} \tag{2.18}$$

for any pair of divisors (D_1, D_2) with $D_j = (P_j) + (R_j) - 2(\Omega) + (f_j)$, where

- κ is the equation (up to scalar) of the cubic through the four points (counting the multiplicity) of the considered two divisors;
- ℓ_1 and ℓ_2 are respectively the equation (up to scalar) of the vertical line through the fifth and sixth point of intersection between κ and $\mathcal{H}$.

By the above arguments, c_L is a factor system of (1.25), in the sense that

$$(\bar{D}_1, k_1)(\bar{D}_2, k_2) = (\bar{D}_1 + \bar{D}_2, k_1 k_2 c_L(\bar{D}_1, \bar{D}_2)).$$

It is worthwhile to remark once more that c_L is not everywhere defined.

We recall that, if $\mathcal{H}$ is a (imaginary) hyperelliptic curve of genus two, the quotient group $\mathrm{Pic}^0(\mathcal{H}) = \mathrm{Div}^0(\mathcal{H}) / \mathrm{PDiv}(\mathcal{H})$ is isomorphic to a complex torus $\mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathbb{C}^2$ is a lattice of (full) rank four. The isomorphism $\mathrm{Pic}^0(\mathcal{H}) \to \mathbb{C}^2/\Gamma$ is determined by the Abel-Jacobi map (1.18). The problem to invert the Abel-Jacobi map, resolved in the case of genus one by the Weierstrass elliptic function $\wp$, is now resolved by the functions $\wp_{ij}$, defined in (1.22). The ordinary Jacobian $\mathfrak{J}(\mathcal{H})$ is no more a curve on the complex projective plane: it is a surface of dimension equal to the genus, which is two, that lies in a five-dimensional complex projective space. In particular, to each point

$$(\wp_{11}(z_1, z_2), \wp_{12}(z_1, z_2), \wp_{22}(z_1, z_2), \wp_{212}(z_1, z_2), \wp_{222}(z_1, z_2))$$

in the surface corresponds the divisor $(P_1) + (P_2) - 2(\Omega)$ with $P_i = (x_i, y_i) \in \mathcal{H}$, where

$$x_i^2 - \wp_{22}(\boldsymbol{z})x_i - \wp_{12}(\boldsymbol{z}) = 0 \quad \text{and} \quad y_i = \wp_{122}(\boldsymbol{z}) + \wp_{222}(\boldsymbol{z})x_i. \tag{2.19}$$

Thus, the epimorphism $\mathrm{P}\colon \mathbb{C}^2 \to \mathfrak{J}(\mathcal{H})$ with Γ as kernel is given by $\boldsymbol{z} = (z_1, z_2) \mapsto \bar{D}$, where $D = (P_1) + (P_2) - 2(\Omega)$ with $P_j = (x_j, y_j)$ satisfying (2.19).

The non-trivial (non-regular) factor system $c_L\colon \mathfrak{J} \times \mathfrak{J} \to \mathbb{C}^*$ induces a further factor system

$$c_L\mathrm{P}\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^*,\ (\boldsymbol{u}_1, \boldsymbol{u}_2) \mapsto c_L(\mathrm{P}(\boldsymbol{u}_1), \mathrm{P}(\boldsymbol{u}_2)).$$

Since every commutative Lie extension of $\mathbb{C}^*$ by $\mathbb{C}^2$ is trivial, we have that the factor system $c_L\mathrm{P}$ must be a coboundary, that is, $c_L\mathrm{P} = \delta^1(\varsigma)$, for some function $\varsigma\colon \mathbb{C}^2 \to \mathbb{C}^*$. As in the one-dimensional case, we want to give the explicit expression of such ς.

On the other hand, looking at the toroidal groups, the following theorem holds.

Theorem 2.9. *Let $\tau \in \mathfrak{H}_2$ and $s_1, s_2 \in \mathbb{C}$. The toroidal group $\mathcal{T} = \mathbb{C}^3/\Lambda$ with $\mathrm{rk}_{\mathbb{R}}\,\Lambda = 5$ and period matrix in standard coordinates given by*

$$\Pi = \left(\begin{array}{cc|c|cc} \multicolumn{2}{c|}{\multirow{2}{*}{I_2}} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{τ}} \\ & & 0 & & \\ \hline 0 & 0 & 1 & s_1 & s_2 \end{array}\right), \tag{2.20}$$

is an extension of a linear subtorus $\mathcal{L}$, isomorphic to $\mathbb{C}^*$, by the Jacobian $\mathfrak{J}(\mathcal{H})$ of the hyperelliptic curve of period matrix $(I_2|\tau)$.

Proof. Let $H = \{(x, y, z) \in \mathbb{C}^3\colon x = 0 = y\}$ be a 2-dimensional linear subspace of $\mathbb{C}^3$. Since $H \cap \Lambda = \langle(0, 0, 1)\rangle_{\mathbb{Z}}$, the subgroup $\mathcal{L} := (H + \Lambda)/\Lambda$ is a one-dimensional linear subtorus and, by Proposition 1.17, $\mathcal{T}/\mathcal{L}$ is isomorphic to an Abelian complex Lie group with period matrix $(I_2|\tau)$, which is the period matrix of the Jacobian variety $\mathfrak{J}(\mathcal{H})$ of a hyperelliptic curve (cf. [61]). □

Remark 2.10. We point out that we are naturally considering again quasi-Abelian varieties. Indeed, $\mathcal{T}$ is the extension of $\mathbb{C}^*$ by the Abelian variety $\mathfrak{J}(\mathcal{H})$, thus, by Theorem 1.39, the toroidal group $\mathcal{T}$ is a quasi-Abelian variety (of kind 0).

We are looking for an epimorphism $\mathbb{C}^3 \to \mathfrak{J}(\mathcal{H}) \times \mathbb{C}^*$ with the lattice Λ as kernel. Let $\mathcal{H}$ be the hyperelliptic curve of genus two having matrices of a-periods and b-periods given by

$$2\omega_1 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad 2\omega_2 = \tau \in \mathfrak{H}_2,$$

respectively (thus, the Jacobian variety of $\mathcal{H}$ is a complex torus of lattice exactly spanned by the columns of $(I_2|\tau)$). By (1.12), the matrices η_1 and η_2, defined in (1.11), satisfy the following relations

$$\eta_1\eta_2^{\mathrm{T}} = \eta_2\eta_1^{\mathrm{T}} \quad \text{and} \quad \eta_2^{\mathrm{T}} - \tau\eta_1^{\mathrm{T}} = -\pi i I_2. \tag{2.21}$$

Let $\boldsymbol{t}_M, \boldsymbol{t}_N \in \mathbb{C}^2$ and let us define the function

$$g(\boldsymbol{u}) = \exp\left\{(2\eta_1\boldsymbol{u})^{\mathrm{T}}(\boldsymbol{t}_N - \boldsymbol{t}_M)\right\} \frac{\sigma(\boldsymbol{t}_M)}{\sigma\boldsymbol{t}_N)} \frac{\sigma(\boldsymbol{u} - \boldsymbol{t}_N)}{\sigma(\boldsymbol{u} - \boldsymbol{t}_M)}, \quad \boldsymbol{u} \in \mathbb{C}^2, \tag{2.22}$$

where σ is the 2-variate sigma function corresponding to the above curve, that is, having the quasi-periodicity property

$$\begin{aligned}\sigma(\boldsymbol{u}+\Omega(\boldsymbol{m}_1,\boldsymbol{m}_2)) = &\exp\left\{\mathrm{E}^{\mathrm{T}}(\boldsymbol{m}_1,\boldsymbol{m}_2)\left(\boldsymbol{u}+\frac{1}{2}\Omega(\boldsymbol{m}_1,\boldsymbol{m}_2)\right)\right\}\\ &\times\exp\left\{-\pi i\boldsymbol{m}_1^{\mathrm{T}}\boldsymbol{m}_1+2\pi i(\boldsymbol{m}_1^{\mathrm{T}}\boldsymbol{q}_1-\boldsymbol{m}_2^{\mathrm{T}}\boldsymbol{q}_2)\right\}\sigma(\boldsymbol{u}),\end{aligned} \tag{2.23}$$

where $\Omega(\boldsymbol{m}_1,\boldsymbol{m}_2) = \boldsymbol{m}_1+\tau\boldsymbol{m}_2$ and $\mathrm{E}(\boldsymbol{m}_1,\boldsymbol{m}_2) = 2\eta_1\boldsymbol{m}_1+2\eta_2\boldsymbol{m}_2$. Let us denote $\boldsymbol{e}_1 = (1,0)^{\mathrm{T}}$ and $\boldsymbol{e}_2 = (0,1)^{\mathrm{T}}$ the canonical basis of $\mathbb{C}^2$ and $\boldsymbol{\tau}_j$ the vector given by j-th column (or row) of the (symmetric) matrix τ, for $j = 1,2$. Thus, with the above symbols, we have that $(I_2|\tau) = (\boldsymbol{e}_1|\boldsymbol{e}_2|\boldsymbol{\tau}_1|\boldsymbol{\tau}_2)$ and note that $\boldsymbol{e}_j = \Omega(\boldsymbol{e}_j,\boldsymbol{0})$ and $\boldsymbol{\tau}_j = \Omega(\boldsymbol{0},\boldsymbol{e}_j)$. The property (2.23) implies that the function g satisfies the following lemma.

Lemma 2.11. *For all $\boldsymbol{u}\in\mathbb{C}^2$ and $j\in\{1,2\}$,*

(1) $g(\boldsymbol{u}+\boldsymbol{e}_j) = g(\boldsymbol{u})$;

(2) $g(\boldsymbol{u}+\boldsymbol{\tau}_j) = \exp\left(2\pi i\boldsymbol{e}_j^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right)g(\boldsymbol{u})$.

Proof. 1. For $j\in\{1,2\}$, by definition

$$\begin{aligned}g(\boldsymbol{u}+\boldsymbol{e}_j) &= \exp\left\{(2\eta_1(\boldsymbol{u}+\boldsymbol{e}_j))^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\frac{\sigma(\boldsymbol{t}_M)}{\sigma(\boldsymbol{t}_N)}\frac{\sigma(\boldsymbol{u}-\boldsymbol{t}_N+\boldsymbol{e}_j)}{\sigma(\boldsymbol{u}-\boldsymbol{t}_M+\boldsymbol{e}_j)}\\ &= \exp\left\{(2\eta_1\boldsymbol{u})^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\exp\left\{(2\eta_1\boldsymbol{e}_j)^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\\ &\qquad\times\frac{\sigma(\boldsymbol{t}_M)}{\sigma(\boldsymbol{t}_N)}\frac{\sigma(\boldsymbol{u}-\boldsymbol{t}_N+\boldsymbol{e}_j)}{\sigma(\boldsymbol{u}-\boldsymbol{t}_M+\boldsymbol{e}_j)}.\end{aligned}$$

By the quasi-periodicity,

$$\begin{aligned}&\exp\left\{(2\eta_1\boldsymbol{u})^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\exp\left\{(2\eta_1\boldsymbol{e}_j)^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\frac{\sigma(\boldsymbol{t}_M)}{\sigma(\boldsymbol{t}_N)}\frac{\sigma(\boldsymbol{u}-\boldsymbol{t}_N)}{\sigma(\boldsymbol{u}-\boldsymbol{t}_M)}\\ &\quad\times\frac{\exp\left\{\mathrm{E}^{\mathrm{T}}(\boldsymbol{e}_j,\boldsymbol{0})\left(\boldsymbol{u}-\boldsymbol{t}_N+\frac{1}{2}\Omega(\boldsymbol{e}_j,\boldsymbol{0})\right)-\pi i\boldsymbol{e}_j^{\mathrm{T}}\boldsymbol{e}_j+2\pi i(\boldsymbol{e}_j^{\mathrm{T}}\boldsymbol{q}_1)\right\}}{\exp\left\{\mathrm{E}^{\mathrm{T}}(\boldsymbol{e}_j,\boldsymbol{0})\left(\boldsymbol{u}-\boldsymbol{t}_M+\frac{1}{2}\Omega(\boldsymbol{e}_j,\boldsymbol{0})\right)-\pi i\boldsymbol{e}_j^{\mathrm{T}}\boldsymbol{e}_j+2\pi i(\boldsymbol{e}_j^{\mathrm{T}}\boldsymbol{q}_1)\right\}}\\ &= g(\boldsymbol{u})\cdot\exp\left\{(2\eta_1\boldsymbol{e}_j)^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\exp\left\{(2\eta_1\boldsymbol{e}_j)^{\mathrm{T}}(\boldsymbol{t}_M-\boldsymbol{t}_N)\right\}\\ &= g(\boldsymbol{u})\end{aligned}$$

2. For $j\in\{1,2\}$, by definition

$$\begin{aligned}g(\boldsymbol{u}+\boldsymbol{\tau}_j) &= \exp\left\{(2\eta_1(\boldsymbol{u}+\boldsymbol{\tau}_j))^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\frac{\sigma(\boldsymbol{t}_M)}{\sigma(\boldsymbol{t}_N)}\frac{\sigma(\boldsymbol{u}-\boldsymbol{t}_N+\boldsymbol{\tau}_j)}{\sigma(\boldsymbol{u}-\boldsymbol{t}_M+\boldsymbol{\tau}_j)}\\ &= \exp\left\{(2\eta_1\boldsymbol{u})^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\exp\left\{(2\eta_1\boldsymbol{\tau}_j)^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\\ &\qquad\times\frac{\sigma(\boldsymbol{t}_M)}{\sigma(\boldsymbol{t}_N)}\frac{\sigma(\boldsymbol{u}-\boldsymbol{t}_N+\boldsymbol{\tau}_j)}{\sigma(\boldsymbol{u}-\boldsymbol{t}_M+\boldsymbol{\tau}_j)}.\end{aligned}$$

By the quasi-periodicity,

$$\begin{aligned}
&\exp\left\{(2\eta_1 \boldsymbol{u})^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\exp\left\{(2\eta_1 \boldsymbol{\tau}_j)^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\frac{\sigma(\boldsymbol{t}_M)}{\sigma(\boldsymbol{t}_N)}\frac{\sigma(\boldsymbol{u}-\boldsymbol{t}_N)}{\sigma(\boldsymbol{u}-\boldsymbol{t}_M)}\\
&\times\frac{\exp\left\{\mathrm{E}^{\mathrm{T}}(\boldsymbol{0},\boldsymbol{e}_j)\left(\boldsymbol{u}-\boldsymbol{t}_N+\frac{1}{2}\Omega(\boldsymbol{0},\boldsymbol{e}_j)\right)-2\pi i(\boldsymbol{e}_j^{\mathrm{T}}\boldsymbol{q}_2)\right\}}{\exp\left\{\mathrm{E}^{\mathrm{T}}(\boldsymbol{0},\boldsymbol{e}_j)\left(\boldsymbol{u}-\boldsymbol{t}_M+\frac{1}{2}\Omega(\boldsymbol{0},\boldsymbol{e}_j)\right)-2\pi i(\boldsymbol{e}_j^{\mathrm{T}}\boldsymbol{q}_2)\right\}}\\
&=g(\boldsymbol{u})\exp\left\{(2\eta_1\boldsymbol{\tau}_j)^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}\exp\left\{(2\eta_2\boldsymbol{e}_j)^{\mathrm{T}}(\boldsymbol{t}_M-\boldsymbol{t}_N)\right\}
\end{aligned}$$

Since $\boldsymbol{\tau}_j=\tau\boldsymbol{e}_j$, we have that

$$g(\boldsymbol{u})\exp\left\{2((\eta_1\tau-\eta_2)\boldsymbol{e}_j)^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}$$

which, applying the second relation $\eta_1\tau-\eta_2=\pi i I_2$ in (2.21), is equal to

$$g(\boldsymbol{u})\exp\left\{2\pi i\boldsymbol{e}_j^{\mathrm{T}}(\boldsymbol{t}_N-\boldsymbol{t}_M)\right\}.$$

□

Theorem 2.12. *With the above notation, let M and N be two points of $\mathcal{H}$ such that the divisor class of $(M)-(N)$ is given by $\mathrm{P}(s_1,s_2)$. If $\boldsymbol{t}_M,\boldsymbol{t}_N\in\mathbb{C}^2$ are such that $\mathrm{P}(\boldsymbol{t}_M)$ is the divisor class of $(M)-(\Omega)$ and $\mathrm{P}(\boldsymbol{t}_N)$ of $(N)-(\Omega)$, then the function $G(x,y,z)=\exp(2\pi iz)g(x,y)$, where g is defined in (2.22), is a 3-variate periodic meromorphic function with period lattice Λ spanned by the columns of the matrix (2.20).*

Proof. We have that $G(x+1,y,z)=G(x,y+1,z)=G(x,y,z)$ from (1) of Lemma 2.11 and, clearly, $G(x,y,z+1)=G(x,y,z)$ from the periodicity of expotential function.
The last fact to prove is that $G((x,y)^{\mathrm{T}}+\boldsymbol{\tau}_1,z+s_1)=G(x,y,z)$. From (2) of Lemma 2.11 , we have that

$$G((x,y)^{\mathrm{T}}+\boldsymbol{\tau}_1,z+s_1)=\exp(2\pi iz)\exp(2\pi is_1)\exp(2\pi i\boldsymbol{m}_1{}^T(\boldsymbol{t}_N-\boldsymbol{t}_M))g(x,y).$$

Since $(M)-(N)$ is linearly equivalent to the divisor identified by the pair (s_1,s_2), we have that $\boldsymbol{t}_M-\boldsymbol{t}_N\equiv(s_1,s_2)^{\mathrm{T}}\mod\Gamma$, where Γ is the lattice spanned by the columns of the matrix $(I_2|\tau)$. Hence,

$$\boldsymbol{t}_M-\boldsymbol{t}_N=(s_1,s_2)^T+\ell_1\boldsymbol{m}_1+\ell_2\boldsymbol{m}_2+\ell_3\boldsymbol{\tau}_1+\ell_4\boldsymbol{\tau}_2$$

with $\ell_j\in\mathbb{Z}$. But the kernel of P is Γ, then we can choose the representative $\boldsymbol{t}_M$ and $\boldsymbol{t}_N$ such that ℓ_3 and ℓ_4 do not occur, that is,

$$\boldsymbol{t}_M-\boldsymbol{t}_N=(s_1,s_2)^T+\ell_1\boldsymbol{m}_1+\ell_2\boldsymbol{m}_2=\begin{pmatrix}s_1+\ell_1\\ s_2+\ell_2\end{pmatrix}.$$

Therefore,

$$\exp(2\pi i\boldsymbol{m}_1{}^T(\boldsymbol{t}_N-\boldsymbol{t}_M))=\exp(-2\pi is_1)\exp(-2\pi i\ell_1)=\exp(-2\pi is_1).$$

In the same way, one can prove that $G((x,y)^T+\boldsymbol{\tau}_2,z+s_2)=G(x,y,z)$. □

Conjecture 2.13. $\delta^1(g)=c_L\mathrm{P}$.

If the equality of Conjecture 2.13 holds, then we have that the map

$$\mathbb{C}^3 \to \Xi_L, \quad (x, y, z) \mapsto (\mathrm{P}(x, y), G(x, y, z)) \tag{2.24}$$

gives in turn the epimorphism that we were looking for.

More in general, lattices $\Lambda \subset \mathbb{C}^n$ of real rank $n+2$ arise naturally when one considers n-variate meromorphic functions with $n+2$ $\mathbb{R}$-independent periods. In the following we generalize the previous results to toroidal groups of an arbitrary complex rank n and real rank $n+2$. As in the elliptic case, we denote the first two coordinates of the n-tuples by different symbols ζ_1 and ζ_2, while the remaining coordinates are denoted by $z_1, \ldots, z_{n-2}$. This distinction is made because the roles of ζ_1 and ζ_2 are closely related to the Jacobian variety of $\mathcal{H}$.

Proposition 2.14. *Let $\mathcal{T} = \mathbb{C}^n/\Lambda$ be a toroidal group of real rank $n+2$ and period matrix in standard coordinates given by*

$$\Pi = \left(\begin{array}{cc|ccc|cc} \multicolumn{2}{c|}{I_2} & 0 & \ldots & 0 & \multicolumn{2}{c}{\tau} \\ & & 0 & \ldots & 0 & & \\ \hline 0 & 0 & & & & s_{1,1} & s_{1,2} \\ \vdots & \vdots & & I_{n-2} & & \vdots & \vdots \\ 0 & 0 & & & & s_{n-2,1} & s_{n-2,2} \end{array}\right),$$

where $\tau \in \mathfrak{H}_2$ and $s_{i,j} \in \mathbb{C}$. If $V(\zeta_1, \zeta_2) = \{(\zeta_1, \zeta_2, z_1, \ldots, z_{n-2}) \in \mathbb{C}^n \colon \zeta_1 = 0 = \zeta_2\}$, then $\mathcal{T}$ is extension of a linear subtorus $\mathcal{L} = (V(\zeta_1, \zeta_2) + \Lambda)/\Lambda$, isomorphic to $(\mathbb{C}^*)^{n-2}$, by the Jacobian $\mathfrak{J}(\mathcal{H})$ of the hyperelliptic curve of period matrix $(I_2|\tau)$.

Proof. We proceed as in the proof of Theorem 2.9. Since

$$\mathcal{V}(\zeta_1, \zeta_2) \cap \Lambda = \langle (0, 0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1) \rangle_{\mathbb{Z}},$$

the subgroup is a linear subtorus of dimension $n-1$. Thus, by Proposition 1.17, the quotient group $\mathcal{T}/\mathcal{L}$ is isomorphic to an Abelian complex Lie group having period matrix given by $(I_2|\tau)$, which is the period matrix of the Jacobian variety of a hyperelliptic curve (cf. [61]). □

Theorem 2.15. *Assume that Conjecture 2.13 holds and let $\mathcal{T}$ be the toroidal group of the previous proposition. In the corresponding non-splitting extension*

$$1 \longrightarrow \mathcal{L} \longrightarrow \mathcal{T} \longrightarrow \mathfrak{J} \longrightarrow \Omega,$$

we can represent an element of $\mathcal{T}$ with an $(n-1)$-tuple $(\bar{D}, k_1, \ldots, k_{n-2}) \in \mathfrak{J}(\mathcal{H}) \times (\mathbb{C}^*)^{n-2}$, in such a way $(\bar{\Omega}, 1, \ldots, 1)$ is the zero element, and

$$(\bar{D}_1, k_1, \ldots, k_{n-2})(\bar{D}_2, h_1, \ldots, h_{n-2}) = \\ (\bar{D}_1 + \bar{D}_2, k_1 h_1 c_{L_1}(\bar{D}_1, \bar{D}_2), \ldots, k_{n-2} h_{n-2} c_{L_{n-2}}(\bar{D}_1, \bar{D}_2))$$

where c_{L_j} is the factor system (2.18), for $j = 1, \ldots, n-2$, by $n-2$ given moduli $L_j = (T_j) + (T_{j+1})$, such that the class of divisor $(T_j) - (T_{j+1})$ is given by $\mathrm{P}(s_{j,1}, s_{j,2})$.

Proof. Since the functor Ext is additive, we can reduce the argument to the case $n = 3$, which has been described above. More precisely, we have the isomorphism (2.24). □

Each of c_{L_j} determines a further factor system $c_{L_j}\mathrm{P}$, which results a coboundary. In particular, $c_{L_j}\mathrm{P} = \delta^1(g_j)$ with g_j defined as in (2.22). We have the epimorphism

$$(\zeta_1, \zeta_2, z_1, \dots, z_{n-2}) \mapsto (\mathrm{P}(\zeta_1, \zeta_2), G_1(\zeta_1, \zeta_2, z_1), \dots, G_{n-2}(\zeta_1, \zeta_2, z_{n-2}))$$

where $G_j(\zeta_1, \zeta_2, z_j) = g(\zeta_1, \zeta_2)\exp(2\pi i z_j)$, from the additive group $\mathbb{C}^n$ to (a representation of) the generalized Jacobian $\mathfrak{J}_L(\mathcal{H})$ with Λ as kernel. Furthermore, the product

$$G(\zeta_1, \zeta_2, z_1, \dots, z_{n-2}) := G_1(\zeta_1, \zeta_2, z_1) \cdots G_{n-2}(\zeta_1, \zeta_2, z_{n-2}),$$

is a n-variate meromorphic function with period lattice given by Λ (thus, $n+2$ $\mathbb{R}$-independent periods).

Remark 2.16. As pointed out in Remark 2.5 for the elliptic case, the isomorphism between $\mathrm{Ext}(\mathfrak{J}(\mathcal{H}), \mathbb{C}^*)$ and $\mathfrak{J}(\mathcal{H})$ can be seen in terms of toroidal groups in light of the results of this section. To each modulus $L = (M) + (N)$ with $M \neq N$, such that $(M) - (N)$ is the divisor class $\mathrm{P}(s_1, s_2)$, corresponds a (non-regular) factor system c_L defined in (2.18). One can fix one of the two points M and N, suppose M, and choosing a suitable point N such that the above condition holds.

Equality. In order to prove the relation stated in Conjecture 2.13 between the factor system c_L and the function $g\colon \mathbb{C}^2 \to \mathbb{C}^*$, we would make use of the following generalized version of the Frobenius-Stickerberg formula (2.15) for the case of genus two, provided by Ônishi in 2002. Let us denote by σ_2 the partial derivative of σ with respect to the second variable.

Proposition 2.17 ([56]). *Let n be a positive integer and $\boldsymbol{u}_0, \dots, \boldsymbol{u}_n$ be such that $\mathrm{P}(u_j)$ is the divisor class of $(P_j) - (\Omega)$ in the Jacobian variety of $\mathcal{H}$. Then*

$$-\frac{\sigma(\boldsymbol{u}_0 + \cdots + \boldsymbol{u}_n)\prod_{i<j}\sigma(\boldsymbol{u}_j - \boldsymbol{u}_i)}{\sigma_2(\boldsymbol{u}_0)^n \cdots \sigma_2(\boldsymbol{u}_n)^n}$$

is equal to

$$\begin{vmatrix} 1 & x(\boldsymbol{u}_0) & x^2(\boldsymbol{u}_0) & y(\boldsymbol{u}_0) & x^3(\boldsymbol{u}_0) & yx(\boldsymbol{u}_0) & \dots & yx^{\frac{n-4}{2}}(\boldsymbol{u}_0) & x^{\frac{n+2}{2}}(\boldsymbol{u}_0) \\ 1 & x(\boldsymbol{u}_1) & x^2(\boldsymbol{u}_1) & y(\boldsymbol{u}_1) & x^3(\boldsymbol{u}_1) & yx(\boldsymbol{u}_1) & \dots & yx^{\frac{n-4}{2}}(\boldsymbol{u}_1) & x^{\frac{n+2}{2}}(\boldsymbol{u}_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x(\boldsymbol{u}_n) & x^2(\boldsymbol{u}_n) & y(\boldsymbol{u}_n) & x^3(\boldsymbol{u}_n) & yx(\boldsymbol{u}_n) & \dots & yx^{\frac{n-4}{2}}(\boldsymbol{u}_n) & x^{\frac{n+2}{2}}(\boldsymbol{u}_n) \end{vmatrix}$$

if n is even, or

$$\begin{vmatrix} 1 & x(\boldsymbol{u}_0) & x^2(\boldsymbol{u}_0) & y(\boldsymbol{u}_0) & x^3(\boldsymbol{u}_0) & yx(\boldsymbol{u}_0) & \dots & yx^{\frac{n-4}{2}}(\boldsymbol{u}_0) & x^{\frac{n+2}{2}}(\boldsymbol{u}_0) \\ 1 & x(\boldsymbol{u}_1) & x^2(\boldsymbol{u}_1) & y(\boldsymbol{u}_1) & x^3(\boldsymbol{u}_1) & yx(\boldsymbol{u}_1) & \dots & yx^{\frac{n-4}{2}}(\boldsymbol{u}_1) & x^{\frac{n+2}{2}}(\boldsymbol{u}_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x(\boldsymbol{u}_n) & x^2(\boldsymbol{u}_n) & y(\boldsymbol{u}_n) & x^3(\boldsymbol{u}_n) & yx(\boldsymbol{u}_n) & \dots & yx^{\frac{n+1}{2}}(\boldsymbol{u}_n) & x^{\frac{n-3}{2}}(\boldsymbol{u}_n) \end{vmatrix}$$

otherwise.

More precisely, we want to prove that $c_L\mathrm{P}(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is equal to non-zero complex number

$$\delta^1(g)(\boldsymbol{u}_1, \boldsymbol{u}_2) = \frac{g(\boldsymbol{u}_1 + \boldsymbol{u}_2)}{g(\boldsymbol{u}_1)g(\boldsymbol{u}_2)}, \tag{2.25}$$

for any $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{C}^2$. This is equal to

$$\frac{\sigma(\boldsymbol{t}_N)\sigma(\boldsymbol{u}_1+\boldsymbol{u}_2-\boldsymbol{t}_N)\sigma(\boldsymbol{u}_1-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_2-\boldsymbol{t}_M)}{\sigma(\boldsymbol{t}_M)\sigma(\boldsymbol{u}_1+\boldsymbol{u}_2-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_1-\boldsymbol{t}_N)\sigma(\boldsymbol{u}_2-\boldsymbol{t}_N)}. \tag{2.26}$$

On the other hand, if $D_j = (P_j) + (R_j) - 2(\Omega) + (f_j)$ with $P_j = (x_{P_j}, y_{P_j})$ and $R_j = (x_{R_j}, y_{R_j})$, for $j = 1, 2$, then one can write the quantity $\frac{\kappa(M)}{\ell_1(M)\ell_2(M)}$ in terms of determinants as

$$\frac{\begin{vmatrix} 1 & x_{P_1} & x_{P_1}^2 & x_{P_1}^3 & y_{P_1} \\ 1 & x_{R_1} & x_{R_1}^2 & x_{R_1}^3 & y_{R_1} \\ 1 & x_{P_2} & x_{P_2}^2 & x_{P_2}^3 & y_{P_2} \\ 1 & x_{R_2} & x_{R_2}^2 & x_{R_2}^3 & y_{R_2} \\ 1 & x_M & x_M^2 & x_M^3 & y_M \end{vmatrix}}{\begin{vmatrix} 1 & x_{S_1} \\ 1 & x_M \end{vmatrix} \begin{vmatrix} 1 & x_{S_2} \\ 1 & x_M \end{vmatrix}}, \tag{2.27}$$

where S_1 and S_2 are the symmetric points of the fifth and sixth point of intersection of the cubic curve κ and the hyperelliptic curve $\mathcal{H}$ (cf. Figure 2.1). Moreover, if $\mathrm{P}(\boldsymbol{u}_{P_j})$ and $\mathrm{P}(\boldsymbol{u}_{R_j})$ are the divisor class determined by $(P_j) - (\Omega)$ and $(R_j) - (\Omega)$ respectively ($j = 1, 2$), then the numerator of (2.27) is equal, by Proposition 2.17, to

$$\begin{aligned} &\frac{\sigma(\boldsymbol{u}_{P_1}+\boldsymbol{u}_{R_1}+\boldsymbol{u}_{P_2}+\boldsymbol{u}_{R_2}+\boldsymbol{t}_M)}{\sigma(\boldsymbol{u}_{P_1})^5\sigma_2(\boldsymbol{u}_{P_2})^5\sigma_2(\boldsymbol{u}_{R_1})^5\sigma_2(\boldsymbol{u}_{R_2})^5\sigma_2(\boldsymbol{t}_M)^5} \\ &\times \sigma(\boldsymbol{u}_{P_1}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{P_2}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{R_1}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{R_2}-\boldsymbol{t}_M) \\ &\times \sigma(\boldsymbol{u}_{P_1}-\boldsymbol{u}_{R_1})(\boldsymbol{u}_{P_1}-\boldsymbol{u}_{P_2})(\boldsymbol{u}_{P_1}-\boldsymbol{u}_{R_2})\sigma(\boldsymbol{u}_{R_1}-\boldsymbol{u}_{P_2})\sigma(\boldsymbol{u}_{R_1}-\boldsymbol{u}_{R_2})\sigma(\boldsymbol{u}_{P_2}-\boldsymbol{u}_{R_2}). \end{aligned} \tag{2.28}$$

The denominator is equal to $x_M^2 - (x_{S_1} + x_{S_2})x_M + x_{S_1}x_{S_2}$, which is also equal to

$$-\frac{\sigma(\boldsymbol{u}_{S_j}+\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{S_j}-\boldsymbol{t}_M)}{\sigma_2(\boldsymbol{u}_{S_j})^2\sigma_2(\boldsymbol{t}_M)^2}. \tag{2.29}$$

So, dividing (2.28) by (2.29), for $j = 1, 2$, we obtain

$$C\frac{\sigma(\boldsymbol{u}_{P_1}+\boldsymbol{u}_{R_1}+\boldsymbol{u}_{P_2}+\boldsymbol{u}_{R_2}+\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{P_1}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{R_1}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{P_2}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{R_2}-\boldsymbol{t}_M)}{\sigma_2(\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{S_1}+\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{S_2}+\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{S_1}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{S_2}-\boldsymbol{t}_M)} \tag{2.30}$$

where

$$C = \frac{\sigma_2(\boldsymbol{u}_{S_1})^2\sigma_2(\boldsymbol{u}_{S_2})^2\sigma(\boldsymbol{u}_{P_1}-\boldsymbol{u}_{R_1})\sigma(\boldsymbol{u}_{P_1}-\boldsymbol{u}_{P_2})\sigma(\boldsymbol{u}_{P_1}-\boldsymbol{u}_{R_2})\sigma(\boldsymbol{u}_{R_1}-\boldsymbol{u}_{P_2})\sigma(\boldsymbol{u}_{R_1}-\boldsymbol{u}_{R_2})\sigma(\boldsymbol{u}_{P_2}-\boldsymbol{u}_{R_2})}{\sigma_2(\boldsymbol{u}_{P_1})^5\sigma_2(\boldsymbol{u}_{R_1})^5\sigma_2(\boldsymbol{u}_{P_2})^5\sigma_2(\boldsymbol{u}_{R_2})^5}$$

is a number independent of the point M. The same relation (2.30) holds substituting M with N, therefore dividing the one relative to M by the one relative to N we obtain

$$\begin{aligned} &\frac{\sigma_2(\boldsymbol{t}_N)}{\sigma_2(\boldsymbol{t}_M)}\frac{\sigma(\boldsymbol{u}_{P_1}+\boldsymbol{u}_{R_1}+\boldsymbol{u}_{P_2}+\boldsymbol{u}_{R_2}+\boldsymbol{t}_M)}{\sigma(\boldsymbol{u}_{P_1}+\boldsymbol{u}_{R_1}+\boldsymbol{u}_{P_2}+\boldsymbol{u}_{R_2}+\boldsymbol{t}_N)} \\ &\times \frac{\sigma(\boldsymbol{u}_{P_1}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{R_1}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{P_2}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{R_2}-\boldsymbol{t}_M)}{\sigma(\boldsymbol{u}_{P_1}-\boldsymbol{t}_N)\sigma(\boldsymbol{u}_{R_1}-\boldsymbol{t}_N)\sigma(\boldsymbol{u}_{P_2}-\boldsymbol{t}_N)\sigma(\boldsymbol{u}_{R_2}-\boldsymbol{t}_N)} \\ &\times \frac{\sigma(\boldsymbol{u}_{S_1}+\boldsymbol{t}_N)\sigma(\boldsymbol{u}_{S_2}+\boldsymbol{t}_N)\sigma(\boldsymbol{u}_{S_1}-\boldsymbol{t}_N)\sigma(\boldsymbol{u}_{S_2}-\boldsymbol{t}_N)}{\sigma(\boldsymbol{u}_{S_1}+\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{S_2}+\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{S_1}-\boldsymbol{t}_M)\sigma(\boldsymbol{u}_{S_2}-\boldsymbol{t}_M)}. \end{aligned} \tag{2.31}$$

Future directions. If Conjecture 2.13 (or, possibly, a reformulation thereof) were to be proven, then the general case can be addressed using the same technique, constructing periodic meromorphic functions by leveraging, for instance, the generalization of the relation of Proposition 2.17 to a Frobenius–Stickelberger-type formula for the multi-dimensional sigma function of any genus provided by Ônishi in [57].

Chapter 3

Toroidal groups and non-totally real number fields

In this chapter we study the relationship between toroidal groups and non-totally real number fields.

3.1 Dimension n and real rank $n+1$

In this section we study the relationship between non-totally real number fields of degree $d = r_1 + 2r_2$ with $r_2 = 1$ and isogeny classes of toroidal groups of dimension $n = d - 1$ and real rank d with extra multiplications, introduced in 1973 by Andreotti and Gherardelli in [7] and investigated in 2012 by Vallières in [69], and in 2013 by Abe in [4]. In this section, we consider non-totally real number fields with precisely one pair of complex embeddings.

As introduced in Section 1.2.1, let K be a non-totally real number field, $\sigma_1, \dots, \sigma_{r_1}$ the real embeddings $K \longrightarrow \mathbb{R} \subset \mathbb{C}$, and σ_n one of the two conjugate complex embeddings $K \longrightarrow \mathbb{C}$. For any fractional ideal $\frac{1}{\nu}\mathcal{J}$ of K, where $\nu \in \mathcal{O}_K$ and $\mathcal{J}$ is an ideal of $\mathcal{O}_K$, the Minkowski embedding $\mu : K \longrightarrow \mathbb{C}^n$, defined by the map

$$z \mapsto \big(\sigma_1(z), \dots, \sigma_{r_1}(z), \sigma_n(z)\big),$$

gives in turn the lattice $\mu(\frac{1}{\nu}\mathcal{J})$ of $\mathbb{C}^n$, which has real rank equal to the rank of $\frac{1}{\nu}\mathcal{J}$, that is, $d = n + 1$ (cf. Lemma 1.31).

Now we note that replacing the lattice $\mu(\frac{1}{\nu}\mathcal{J})$ with $\mu(\mathcal{O}_K)$ corresponds to applying an isogeny.

Theorem 3.1. *Let K be a number field, $\mu\colon K \to \mathbb{C}^n$ a Minkowski map, $\mathcal{J}$ an ideal of $\mathcal{O}_K$ and $\nu \in \mathcal{O}_K$. The quotient groups $\mathbb{C}^n/\mu(\frac{1}{\nu}\mathcal{J})$, $\mathbb{C}^n/\mu(\mathcal{J})$ and $\mathbb{C}^n/\mu(\mathcal{O}_K)$ are isogeneous.*

Proof. Let us consider the maps

$$\mathbb{C}^n/\mu(\tfrac{1}{\nu}\mathcal{J}) \stackrel{\iota}{\longrightarrow} \mathbb{C}^n/\mu(\mathcal{J}) \stackrel{\pi}{\longrightarrow} \mathbb{C}^n/\mu(\mathcal{O}_K),$$

where π is the canonical projection and ι is induced by the following linear isomorphism $\mathbb{C}^n \to \mathbb{C}^n$:

$$(x_1, \dots, x_n) \longmapsto \big(\sigma_1(\nu)x_1, \dots, \sigma_n(\nu)x_n\big).$$

The kernel of π is $\mu(\mathcal{O}_K)/\mu(\mathcal{J})$, that is finite because $\mu(\mathcal{J})$ is a free $\mathbb{Z}$-submodule of $\mu(\mathcal{O}_K)$ of the same rank. The map ι is an isogeny by construction. □

The quotient group $\mathcal{T} = \mathbb{C}^n/\mu(\mathcal{O}_K)$ is proved to be a toroidal group with extra multiplications (cf. Theorem 1.32), and, in this situation, $\mathrm{End}_0(\mathcal{T}) = \mathrm{End}(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$

is a non-totally real number field which is isomorphic to K and End$(\mathcal{T})$ turns out to be isomorphic, not only to an order of K, but to the whole ring of integers $\mathcal{O}_K$.

Using Theorem 3.1, we can generalize Theorem 1.32.

Corollary 3.2. *The group $\mathbb{C}^n/\mu(\mathfrak{a})$ is toroidal, for every fractional ideal $\mathfrak{a} = \frac{1}{v}\mathcal{J}$ of K.*

Proof. According to the proof of Theorem 3.1, given a fractional ideal $\mathfrak{a}$ of K, we know that there exists an isogeny from the group $\mathcal{T}_1 = \mathbb{C}^n/\mu(\mathfrak{a})$ to $\mathcal{T}_2 = \mathbb{C}^n/\mu(\mathcal{O}_K)$, hence we have a further isogeny $\mathcal{T}_2 \to \mathcal{T}_1$ (see Proposition 1.15). The claim follows from the well-known fact that every epimorphic image of a toroidal group is again toroidal (see Proposition 1.13). □

On the other hand, for those non-totally real number fields K admitting an **essential polynomial**, that is, an irreducible polynomial

$$f(x) = a_1x^{n+1} + \cdots + a_nx + a_{n+1} + \in \mathbb{Z}[x]$$

such that $K = \mathbb{Q}(\omega)$ with $f(\omega) = 0$ and the discriminant of $f(x)$ is equal to the discriminant of the field K, by [36] an integral basis of the ring of integers $\mathcal{O}_K$ is given by $\{\rho_0, \rho_1, \ldots, \rho_n\}$ where

$$\rho_0 = 1 \text{ and } \rho_j = \sum_{k=1}^{j} a_k\omega^{j+1-k}, \text{ for } j = 1, \ldots, n. \tag{3.1}$$

In this case, let $\alpha_1, \ldots, \alpha_{r_1}$ be the real roots of $f(x)$ and let $\gamma, \bar{\gamma}$ be the two non-real roots. Let $\Phi = \{\sigma_1, \ldots, \sigma_{r_1}, \sigma_n\}$ be the type defined by

$$\sigma_1 \colon \omega \mapsto \alpha_1, \quad \ldots \quad \sigma_{r_1} \colon \omega \mapsto \alpha_{r_1}, \quad \sigma_n \colon \omega \mapsto \gamma.$$

The lattice $\mu_\Phi(\mathcal{O}_K)$ is generated by the vectors

$$\begin{aligned}
\mu_\Phi(1) &= (1, \ldots, 1, 1),\\
\mu_\Phi(\rho_1) &= (a_1\alpha_1, \ldots, a_1\alpha_{r_1}, a_1\gamma),\\
\mu_\Phi(\rho_2) &= (a_1\alpha_1^2 + a_2\alpha_1, \ldots, a_1\alpha_{r_1}^2 + a_2\alpha_{r_1}, a_1\gamma^2 + a_2\gamma),\\
&\vdots\\
\mu_\Phi(\rho_n) &= (a_1\alpha_1^n + \cdots + a_n\alpha_1, \ldots,\\
&\qquad a_1\alpha_{r_1}^n + \cdots + a_n\alpha_{r_1}, a_1\gamma^n + \cdots + a_n\gamma),
\end{aligned}$$

therefore its period lattice Λ is generated by the columns of the matrix

$$\begin{pmatrix}
1 & a_1\alpha_1 & a_1\alpha_1^2 + a_2\alpha_1 & \ldots & a_1\alpha_1^n + \cdots + a_n\alpha_1\\
\vdots & \vdots & \vdots & \ldots & \vdots\\
1 & a_1\alpha_{r_1} & a_1\alpha_{r_1}^2 + a_2\alpha_{r_1} & \ldots & a_1\alpha_{r_1}^n + \cdots + a_n\alpha_{r_1}\\
1 & a_1\gamma & a_1\gamma^2 + a_2\gamma & \ldots & a_1\gamma^n + \cdots + a_n\gamma
\end{pmatrix}. \tag{3.2}$$

Being isomorphic to $\mathcal{O}_K$, End$(\mathcal{T})$ consists of the endomorphisms induced by the multiplications by algebraic integers.

In the sections 3.1.1, 3.1.3, and 3.1.4, we write down the relations between the essential polynomial and the parameters in the last column of a period matrix of $\mathcal{T}$ in standard form, and we give the analytic and rational representations of End$(\mathcal{T})$.

The cases of low dimension (that is, $n = 2, 3$) of Sections 3.1.1, 3.1.3, 3.1.4 illustrate the general case, which has been studied in Section 3.1.6.

The name *essential* is given by Ash, Brakenhoff, and Zarrabi in [9], to refer to *monic* irreducible polynomials having integer coefficients such that the discriminant of the polynomial is equal to the discriminant of the field obtained adjoining one of its roots to the rational numbers. Since the discriminant of a number field is, by definition, the discriminant (as order) of the ring of integers, if $\mathrm{disc}(K) = \mathrm{disc}(f)$ for such a monic polynomial $f(x) \in \mathbb{Z}[x]$, then we have that $\mathcal{O}_K = \mathbb{Z}[\theta]$, where $f(\theta) = 0$ and $K = \mathbb{Q}(\theta)$, that is, K is monogenic. The authors studied the probability that a random monic irreducible integer polynomial (with a certain height) is essential and they estimated that it approaches $\frac{6}{\pi^2}$, as the height to infinity. In this thesis, we refer to *essential* also to the non-monic ones. These polynomials come from [36], where the author defines an *essential form* as an irreducible binary form $\mathcal{B}(x, y)$ of degree n such that the discriminant of $\mathcal{B}$ is equal to the discriminant of a number field of degree n and, more in general, an *essential pair* as $[a_0, \mathcal{B}]$, where $\mathcal{B}(x, y) = a_1 x^n + a_2 x^{n-1} y + \cdots + a_n y^n$ is an essential form and a_0 an integer, such that the discriminant of $\mathcal{B}$ is equal to $a_0^2 \,\mathrm{disc}(K)$, for a number field K of degree n, with a_0^2 dividing a_1 and a_0 dividing a_2 (we have back the essential forms when $a_0 = 1$). For the cubic case, there always exist essential forms, the so-called *index form* (see, e.g, [32], also [37, Section 1.4]), while for $n > 3$ they may not exist.

Remark 3.3. We point out that, if a field is monogenic then it admits trivially an essential *monic* polynomial (indeed, the minimal polynomial of an algebraic integer belongs always to $\mathbb{Z}[x]$). On the converse, a field $K = \mathbb{Q}(\theta)$ admitting an essential polynomial $f(x)$, with $f(\theta) = 0$, is not necessarily monogenic, because if $f(x)$ is not monic then its discriminant $\mathrm{disc}(f)$ could be different from the discriminant of $\mathbb{Z}[\theta]$. In fact, for instance, every cubic number field admits an essential polynomial but there exist non-monogenic cubic number fields (cfr. Example 3.8).

3.1.1 Toroidal groups and non-totally real cubic number fields

Let now K be a non-totally real cubic number field, $\sigma_1 \colon K \longrightarrow \mathbb{R} \subset \mathbb{C}$ the real embedding, and $\sigma_2 : K \longrightarrow \mathbb{C}$ one of the two conjugate complex embeddings of K. The Minkowski embedding $\mu : K \longrightarrow \mathbb{C}^2$, defined by the map

$$z \mapsto \big(\sigma_1(z), \sigma_2(z)\big),$$

gives in turn the lattice $\mu(\mathcal{O}_K)$ of $\mathbb{C}^2$, which has real rank equal to the rank of $\mathcal{O}_K$, that is three, and the quotient group $\mathcal{T} = \mathbb{C}^2/\mu(\mathcal{O}_K)$ is a toroidal group with extra multiplications.

On the other hand, for any non-totally real cubic number field K, it is always possible to choose an element $\omega \in K$ such that $K = \mathbb{Q}(\omega)$, with $a\omega^3 + b\omega^2 + c\omega + d = 0$ (with $a, b, c, d \in \mathbb{Z}$) and

$$\mathcal{O}_K = \langle 1, \rho_1, \rho_2 \rangle,$$

where $\rho_1 = a\omega$, $\rho_2 = a\omega^2 + b\omega$ (see, e.g., [37, §1.3]). Note that $a\rho_2 = \rho_1^2 + b\rho_1$.

In this case, the embeddings σ_1 and σ_2 are defined by $\sigma_1(\omega) = \alpha$, and $\sigma_2(\omega) = \beta$, where

$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \bar{\beta}).$$

As claimed above, we write down the period matrix in standard form of $\mathcal{T}$, and we give the analytic and rational representations of $\mathrm{End}(\mathcal{T})$.

Theorem 3.4. *Let $K = \mathbb{Q}(\omega)$ be any non-totally real cubic number field, with $a\omega^3 + b\omega^2 + c\omega + d = 0$, and let $\mu : K \longrightarrow \mathbb{C}^2$ be a Minkowski embedding, as above. If $\mathcal{T} = \mathbb{C}^2/\mu(\mathcal{O}_K)$ is the toroidal group associated to $\mathcal{O}_K$, then the parameters defining the lattice in standard form are $\tau = -a\alpha\beta$ and $s_1 = -\bar{\beta}$ (hence $\tau s_1 = -d$).*

Moreover, $\mathcal{T}$ has extra multiplications, and the analytic and rational representations of End$(\mathcal{T})$ *yield the free $\mathbb{Z}$-modules generated, respectively, by the matrices $I_2, (\rho_1)_a, (\rho_2)_a$ and $I_3, (\rho_1)_r, (\rho_2)_r$,[1] where*

$$(\rho_1)_a = \begin{pmatrix} 0 & a\tau \\ 1 & -b + as_1 \end{pmatrix}, \quad (\rho_2)_a = \begin{pmatrix} \tau & -ad \\ s_1 & -c \end{pmatrix}, \tag{3.3}$$

and

$$(\rho_1)_r = \begin{pmatrix} 0 & 0 & -ad \\ 1 & -b & -c \\ 0 & a & 0 \end{pmatrix}, \quad (\rho_2)_r = \begin{pmatrix} 0 & -ad & -bd \\ 0 & -c & -d \\ 1 & 0 & -c \end{pmatrix}. \tag{3.4}$$

Proof. We can assume that the Minkowski embedding $\mu : K \longrightarrow \mathbb{C}^2$ is defined, as above, by $\mu(z) = (\sigma_1(z), \sigma_2(z))$, with $\sigma_1(\omega) = \alpha$, and $\sigma_2(\omega) = \beta$. Since we have chosen the primitive element $\omega \in K$ such that $\mathcal{O}_K = \langle 1, \rho_1, \rho_2 \rangle$, with $\rho_1 = a\omega$, $\rho_2 = a\omega^2 + b\omega$, the Minkowski embedding μ produces the lattice

$$\mu(\mathcal{O}_K) = \langle (1,1), (a\alpha, a\beta), (a\alpha^2 + b\alpha, a\beta^2 + b\beta) \rangle_{\mathbb{Z}}.$$

It is clear that the set $\{\mu(1) = (1,1), \mu(\rho_1) = (a\alpha, a\beta)\}$ is a basis of $\mathbb{C}^2$ because $\alpha \neq \beta$. Replacing this basis with the canonical basis of $\mathbb{C}^2$ we obtain that the lattice can be rewritten as

$$\langle (1,0), (0,1), (\tau, s_1) \rangle_{\mathbb{Z}},$$

with

$$\begin{aligned}
\begin{pmatrix} \tau \\ s_1 \end{pmatrix} &= \begin{pmatrix} 1 & a\alpha \\ 1 & a\beta \end{pmatrix}^{-1} \cdot \begin{pmatrix} a\alpha^2 + b\alpha \\ a\beta^2 + b\beta \end{pmatrix} \\
&= -\frac{1}{a(\alpha - \beta)} \begin{pmatrix} a\beta & -a\alpha \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a\alpha^2 + b\alpha \\ a\beta^2 + b\beta \end{pmatrix} \\
&= -\frac{1}{a(\alpha - \beta)} \begin{pmatrix} a\beta(a\alpha^2 + b\alpha) - a\alpha(a\beta^2 + b\beta) \\ -(a\alpha^2 + b\alpha) + a\beta^2 + b\beta \end{pmatrix} \\
&= -\frac{1}{a(\alpha - \beta)} \begin{pmatrix} a\left(a\alpha^2\beta - a\alpha\beta^2\right) \\ -a(\alpha^2 - \beta^2) - b(\alpha - \beta) \end{pmatrix} \\
&= -\frac{1}{a(\alpha - \beta)} \begin{pmatrix} a^2\alpha\beta(\alpha - \beta) \\ -(\alpha - \beta)\left(a(\alpha + \beta) + b\right) \end{pmatrix} = \begin{pmatrix} -a\alpha\beta \\ -\bar{\beta} \end{pmatrix},
\end{aligned}$$

and the first assertion follows, since $b = -a(\alpha + \beta + \bar{\beta})$.

By Theorem 1.36, we know that the toroidal group $\mathcal{T}$ has extra multiplications, which in our setting are induced by multiplications by ρ_1 and ρ_2. Since $\{\mu(1), \mu(\rho_1)\}$ is a basis of $\mathbb{C}^2$, we have to check how the multiplications by ρ_1 and ρ_2 induce endomorphisms of $\mathbb{C}^2$. Seeing that

$$\mu(\rho_1^2) = \mu(-b\rho_1 + a\rho_2) = a\tau\mu(1) + (-b + as_1)\mu(\rho_1),$$

[1] Here, we used the simpler notation $(\rho_j)_a$ and $(\rho_j)_r$ instead of $\varrho_a(\rho_j)$ and $\varrho_r(\rho_j)$, respectively.

and that

$$\mu(\rho_2) = \tau\mu(1) + s_1\mu(\rho_1),$$
$$\mu(\rho_2\rho_1) = \mu(a^2\omega^3 + ab\omega^2) = \mu(-ad - ac\omega) = -ad\mu(1) - c\mu(\rho_1),$$

the analytic representation of ρ_1 and ρ_2 is given by the matrices in (3.3).

Since $\{\mu(1), \mu(\rho_1), \mu(\rho_2)\}$ is an integral basis of the lattice $\mu(\mathcal{O}_K)$, the same relations, together with

$$\begin{aligned}\mu(\rho_2^2) &= \mu(a^2\omega^4 + 2ab\omega^3 + b^2\omega^2)\\ &= \mu\big((-ab\omega^3 - ac\omega^2 - ad\omega) + 2ab\omega^3 + b^2\omega^2\big)\\ &= \mu\big((-b^2\omega^2 - bc\omega - bd) - ac\omega^2 - ad\omega + b^2\omega^2\big)\\ &= \mu(-bc\omega - bd - ac\omega^2 - ad\omega)\\ &= -bd\mu(1) - d\mu(\rho_1) - c\mu(\rho_2)\end{aligned}$$

(where we applied twice the equality $a\omega^3 + b\omega^2 + c\omega + d = 0$), prove that the rational representations of ρ_1 and ρ_2 are the ones given in (3.4). □

Remark 3.5. As a matter of fact, these matrix representations satisfy the following Hurwitz relations:

$$(\rho_j)_a \begin{pmatrix} 1 & 0 & \tau \\ 0 & 1 & s_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau \\ 0 & 1 & s_1 \end{pmatrix} (\rho_j)_r, \quad j = 1, 2.$$

Under the necessary conditions seen before, the following theorem of Vaillères characterizes, conversely, a general toroidal group $\mathcal{T}$ as determined by the lattice $\mu(\frac{1}{\nu}\mathcal{J}) \subset \mathbb{C}^2$, for some fractional ideal $\frac{1}{\nu}\mathcal{J}$ of the non-totally real cubic number field $K = \mathrm{End}_0(\mathcal{T})$.

Theorem 3.6 ([69])**.** *Given a toroidal group $\mathcal{T}$ of complex rank $\mathrm{rk}_{\mathbb{C}}\,\mathcal{T} = 2$ and real rank $\mathrm{rk}_{\mathbb{R}}\,\mathcal{T} = 3$ with extra multiplications, the field $K = \mathrm{End}_0(\mathcal{T})$ is a non-totally real cubic number field, and if the ring $\mathrm{End}(\mathcal{T})$ coincides with the ring of algebraic integers $\mathcal{O}_K$, then there exists a fractional ideal $\frac{1}{\nu}\mathcal{J}$, with $\nu \in \mathcal{O}_K$ and $\mathcal{J}$ an ideal of $\mathcal{O}_K$, such that $\mathcal{T}$ is isomorphic to the quotient*

$$\mathbb{C}^2/\mu(\tfrac{1}{\nu}\mathcal{J}),$$

for a suitable Minkowski embedding $\mu : K \longrightarrow \mathbb{C}^2$.

Example 3.7 (Monogenic)**.** Let $K = \mathbb{Q}(\omega)$ with $\omega^3 = 2$. The minimal polynomial of ω over $\mathbb{Q}$ is $f(x) = x^3 - 2$ having roots $\alpha, \varepsilon\alpha, \varepsilon^2\alpha$, where $\alpha = \sqrt[3]{2}$ and $\varepsilon = \exp\left(\frac{2\pi i}{3}\right)$ is a primitive third root of unity (hence, defined by the cyclotomic polynomial $\varepsilon^2 + \varepsilon + 1 = 0$). The cubic field K is non-totally real and monogenic, that is, the ring of integers is $\mathbb{Z}[\omega] = \langle 1, \omega, \omega^2 \rangle_{\mathbb{Z}}$. The discriminant is $\mathrm{disc}(K) = \mathrm{disc}(\mathbb{Z}[\omega]) = -108$. Let $\Phi = \{\sigma_1 \colon \omega \mapsto \alpha,\ \sigma_2 \colon \omega \mapsto \varepsilon\alpha\}$ be a type and $\mathcal{T} := \mathbb{C}^2/\mu_\Phi(\mathcal{O}_K)$. The matrix associated to $\mu_\Phi(\mathcal{O}_K)$ is

$$\begin{pmatrix} 1 & \alpha & \alpha^2 \\ 1 & \varepsilon\alpha & (\varepsilon\alpha)^2 \end{pmatrix},$$

that, in standard coordinates, is given by

$$\begin{pmatrix} 1 & \alpha \\ 1 & \varepsilon\alpha \end{pmatrix}^{-1} \begin{pmatrix} 1 & \alpha & \alpha^2 \\ 1 & \varepsilon\alpha & (\varepsilon\alpha)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau \\ 0 & 1 & s_1 \end{pmatrix},$$

where, applying Cramer's rules,

$$\tau = \frac{\begin{vmatrix} \alpha^2 & \alpha \\ (\varepsilon\alpha)^2 & \varepsilon\alpha \end{vmatrix}}{\begin{vmatrix} 1 & \alpha \\ 1 & \varepsilon\alpha \end{vmatrix}} = \frac{(\varepsilon - \varepsilon^2)\alpha^3}{(\varepsilon - 1)\alpha} = -(\varepsilon+1)\alpha^2 = -\varepsilon\alpha^2$$

and

$$s_1 = \frac{\begin{vmatrix} 1 & \alpha^2 \\ 1 & (\varepsilon\alpha)^2 \end{vmatrix}}{\begin{vmatrix} 1 & \alpha \\ 1 & \varepsilon\alpha \end{vmatrix}} = \frac{(\varepsilon^2 - 1)\alpha^2}{(\varepsilon - 1)\alpha} = (\varepsilon+1)\alpha = -\varepsilon^2\alpha.$$

Example 3.8 (Non-monogenic)**.** The first example of a non-monogenic field is given by Dedekind ([22]). Let θ be a root of the irreducible polynomial $h(x) = x^3 - x^2 - 2x - 8$ and $K = \mathbb{Q}(\theta)$. The field K is non-totally real with discriminant $\Delta_K = -503$ and ring of integers

$$\mathcal{O}_K = \langle 1, \theta, \theta' \rangle_{\mathbb{Z}},$$

where $\theta' = 4/\theta$. Following [37, Section 1.3], we can construct a new integer basis $\{1, \rho_1, \rho_2\}$ such that

$$\rho_1 = a\omega \quad \text{and} \quad \rho_2 = a\omega^2 + b\omega,$$

where $K = \mathbb{Q}(\omega)$, ω is a root of the irreducible integer polynomial $f(x) = ax^3 + bx^2 + cx + d$ and $\operatorname{disc}(f) = \Delta_K$. In particular, $\omega = \theta/2$ is a root of

$$f(x) = 2x^3 - x^2 - x - 2,$$

of discriminant -503 and the integral basis is given by

$$\rho_1 = 2\omega = \theta \quad \text{and} \quad \rho_2 = 2\omega^2 - \omega = \frac{\theta^2}{2} - \frac{\theta}{2}.$$

Let Φ be the type defined by $\sigma_1 \colon \omega \mapsto \alpha \in \mathbb{R}$ and $\sigma_2 \colon \omega \mapsto \beta \in \mathbb{C} \setminus \mathbb{R}$, where $\alpha, \beta, \bar{\beta}$ are the roots of $f(x)$. The associated toroidal group $\mathbb{C}^2/\mu_\Phi(\mathcal{O}_K)$ has period matrix

$$\begin{pmatrix} 1 & 2\alpha & 2\alpha^2 - \alpha \\ 1 & 2\beta & 2\beta^2 - \beta \end{pmatrix},$$

that, in standard coordinates, is given by

$$\begin{pmatrix} 1 & 2\alpha \\ 1 & 2\beta \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2\alpha & 2\alpha^2 - \alpha \\ 1 & 2\beta & 2\beta^2 - \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau \\ 0 & 1 & s_1 \end{pmatrix},$$

where, applying Cramer's rule,

$$\begin{aligned} \tau = \frac{\begin{vmatrix} 2\alpha^2 - \alpha & 2\alpha \\ 2\beta^2 - \beta & 2\beta \end{vmatrix}}{\begin{vmatrix} 1 & 2\alpha \\ 1 & 2\beta \end{vmatrix}} &= \frac{2\beta(2\alpha^2 - \alpha) - 2\alpha(2\beta^2 - \beta)}{2(\beta - \alpha)} \\ &= \frac{2\alpha^2\beta - 2\alpha\beta^2}{\beta - \alpha} = \frac{2\alpha\beta(\alpha - \beta)}{\beta - \alpha} = -2\alpha\beta \end{aligned}$$

and

$$s_1 = \frac{\begin{vmatrix} 1 & 2\alpha^2 - \alpha \\ 1 & 2\beta^2 - \beta \end{vmatrix}}{\begin{vmatrix} 1 & 2\alpha \\ 1 & 2\beta \end{vmatrix}} = \frac{2\beta^2 - \beta - 2\alpha^2 + \alpha}{2(\beta - \alpha)} = \frac{(\beta - \alpha)\,(2(\beta + \alpha) - 1))}{2(\beta - \alpha)} = \alpha + \beta - \frac{1}{2}.$$

Since $f(x) = 2x^3 - x^2 - x - 2$ is equal to $2(x-\alpha)(x-\beta)(x-\bar{\beta})$, we obtain the relations between roots and coefficients of $f(x)$, that is, $-2(\alpha + \beta + \bar{\beta}) = -1$ and $-2\alpha\beta\bar{\beta} = -2$. Hence, we have that $\tau = -2/\bar{\beta}$ and $s_1 = -\bar{\beta}$, as claimed in Theorem 3.4. By (3.3) and (3.4), we write the analytic and rational representation

$$(\rho_1)_a = \begin{pmatrix} 0 & -4/\bar{\beta} \\ 1 & -1-2\bar{\beta} \end{pmatrix}, \quad (\rho_2)_a = \begin{pmatrix} -2/\bar{\beta} & 4 \\ -\bar{\beta} & 1 \end{pmatrix},$$

and

$$(\rho_1)_r = \begin{pmatrix} 0 & 0 & 4 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \quad (\rho_2)_r = \begin{pmatrix} 0 & 4 & -2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

3.1.2 Fractional ideals and torsion points, in the geometric correspondence with a generalized Jacobian

In this section we represent the torsion points of a toroidal group $\mathbb{C}^2/\Lambda$ in the geometric correspondence with a generalized Jacobian with modulus $L = (M) + (N)$, with $M = \mathrm{P}(t_M)$ and $N = \mathrm{P}(t_N)$. Before that, we briefly resume the case of quadratic fields.

Fractional ideals of quadratic fields. Let us consider a totally complex quadratic field $K = \mathbb{Q}(\omega)$, with $\omega^2 = d$, for a square-free negative integer d, and its ring of integers $\mathcal{O}_K = \langle 1, \rho \rangle_{\mathbb{Z}}$. Recall indeed, in passing, that a basis of $\mathcal{O}_K$ is given by $\{1, \rho\}$ and ρ depend on the primitive element in the following way (see e.g. [60, p. 35])

$$\rho = \begin{cases} \omega, & \text{if } d \equiv 2,3 \pmod 4, \\ \frac{1+\omega}{2}, & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

A fixed complex embedding $K \to \mathbb{C}$ coincides with the corresponding Minkowski map μ and we have that $\mu(\mathcal{O}_K) = \langle 1, \tau \rangle_{\mathbb{Z}}$ (with τ the embedding of ρ) is a lattice of $\mathbb{C}$ of real rank 2, with the resulting quotient group $\mathbb{C}/\mu(\mathcal{O}_K)$ being a 1-dimensional complex torus, hence isomorphic to the Jacobian of an elliptic curve $\mathcal{C}$ by the map $\mathrm{P}\colon z \longmapsto (\wp(z), \wp'(z))$, defined in (1.15), where $\wp$ and $\wp'$ are the periodic Weierstrass functions with period lattice $\Lambda := \mu(\mathcal{O}_K)$ (cf. Example 1.47). Every element of a given fractional ideal $\frac{1}{a+\rho b}\mathcal{J}$, where $\mathcal{J}$ is an ideal of $\mathcal{O}_K$ and $a, b \in \mathbb{Z}$ not both zero, is mapped onto a point $P = \mathrm{P}\left(\frac{1}{a+\tau b}c\right)$ (with $c \in \mu(\mathcal{J})$), such that, if we put $m = \|a + \tau b\|$, then

$$mP = \|a+\tau b\|\,\mathrm{P}\left(\frac{1}{a+\tau b}c\right) = \|a+\tau b\|\,\mathrm{P}\left(\frac{\overline{a+\tau b}}{\|a+\tau b\|}c\right) = \mathrm{P}\left((\overline{a+\tau b})c\right) = \Omega,$$

because $(\overline{a+\tau b})c \in \mu(\mathcal{O}_K)$, mirroring the fact that evaluating the map P on the elements

$$\frac{1}{m}(d_1 + \tau d_2), \quad d_1, d_2 = 0, \ldots, m-1,$$

with $\frac{d_1}{m}, \frac{d_2}{m}$ dotting the fundamental parallelogram, we get points of order dividing m.

Coming back to cubic fields, recall that any fractional ideal $\frac{1}{\nu}\mathcal{J}$ of a cubic number field K defines the isogeny $\pi\iota$ as in Theorem 3.1. This allows us to consider just toroidal groups of the form $\mathcal{T} = \mathbb{C}^2/\mu(\mathcal{O}_K)$. With the notation of Section 2.1, in this case, $\mathcal{T}$ is birationally isomorphic to the generalized Jacobian $\mathfrak{J}_L(\mathcal{C})$ (cf. Theorem 2.4) of the elliptic curve $\mathcal{C}$ having periods 1 and $\tau = -a\alpha\beta$, with $L = (M) + (N)$ such that $(M) - (N) = (\mathrm{P}(s)) - (\Omega)$, $s = -\bar{\beta}$, where a, α and β are given as in Section 3.1.1.

Remark 3.9. The fact that, in the case where the toroidal group comes from the ring $\mathcal{O}_K$, the entry s is not a real number distinguishes this from the general case of an arbitrary toroidal group. Note moreover that, in such a given case, we could make the same construction, starting from the elliptic curve defined by the lattice $\langle 1, s\rangle_{\mathbb{Z}}$, obtaining a different meromorphic function $G(\zeta, z)$ with the same lattice Λ.

It is worthwhile to point out that, even if the m-torsion subgroup $\mathcal{T}[m]$ of $\mathcal{T}$ is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^3$, its elements are parametrized only by the complex variable ζ of the pair $(\zeta, z) \in \mathbb{C}^2$, the complex variable z playing, virtually, no role (cf. Remark 3.11 below).

Theorem 3.10. *In the above notation, if $m > 0$ is a fixed integer, then the torsion subgroup*

$$\mathcal{T}[m] = \mu(\tfrac{1}{m}\mathcal{O}_K)/\mu(\mathcal{O}_K)$$

is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^3$. Furthermore, an element of $\mathfrak{J}_L(\mathcal{C})$, represented as in (2.7), by the pair $(P, k) \in \mathcal{C} \times \mathbb{C}^*$, with $P = \mathrm{P}(t)$, belongs to the m-torsion subgroup if and only if there exists $\zeta \in \langle 1, \tau\rangle_{\mathbb{Z}}$ such that

$$\begin{cases} t = \frac{1}{m}\zeta, \\ k = \dfrac{\sigma\left(\frac{1}{m}\zeta - t_N\right)}{\sigma\left(\frac{1}{m}\zeta - t_M\right)} \sqrt[m]{\dfrac{\sigma(\zeta - t_M)}{\sigma(\zeta - t_N)} \dfrac{\sigma(t_M)^{m-1}}{\sigma(t_N)^{m-1}}}. \end{cases}$$

Proof. The first assertion follows from the trivial fact that

$$m\big((\zeta, z) + \mu(\mathcal{O}_K)\big) = \mu(\mathcal{O}_K) \iff (\zeta, z) \in \tfrac{1}{m}\mu(\mathcal{O}_K) = \mu\left(\tfrac{1}{m}\mathcal{O}_K\right),$$

so

$$\mathcal{T}[m] = \left\{\tfrac{d_1}{m}(1,0) + \tfrac{d_2}{m}(0,1) + \tfrac{d_3}{m}(\tau, s_1) + \mu(\mathcal{O}_K) \,:\, 0 \le d_j \le m-1\right\} \cong (\mathbb{Z}/m\mathbb{Z})^3.$$

As the sum in the generalized Jacobian is given by

$$\begin{aligned} &\big(\mathrm{P}(\zeta_1), G(\zeta_1, z_1)\big) + \big(\mathrm{P}(\zeta_2), G(\zeta_2, z_2)\big) \\ &\quad = \Big(\mathrm{P}(\zeta_1 + \zeta_2), G(\zeta_1, z_1)G(\zeta_2, z_2)(c_L\mathrm{P})(\zeta_1, \zeta_2)\Big), \end{aligned}$$

where

$$G(\zeta, z) = \exp(-2\eta_1 s_1 \zeta)\frac{\sigma(t_M)}{\sigma(t_N)}\frac{\sigma(\zeta - t_N)}{\sigma(\zeta - t_M)} \exp(2\pi i z), \tag{3.5}$$

recursively we find

$$G(m\zeta, mz) = G(\zeta, z)^m \prod_{j=1}^{m-1} (c_L\mathrm{P})(\zeta, j\zeta).$$

Therefore, if $(\zeta, z) \in \Lambda$, then

$$1 = G(\zeta, z) = G(\tfrac{1}{m}\zeta, \tfrac{1}{m}z)^m \prod_{j=1}^{m-1} (c_L\mathrm{P})(\tfrac{1}{m}\zeta, j\tfrac{1}{m}\zeta), \tag{3.6}$$

whence we obtain

$$G(\tfrac{1}{m}\zeta, \tfrac{1}{m}z)^m = \prod_{j=1}^{m-1} \frac{1}{(c_L\mathrm{P})(\frac{1}{m}\zeta, j\frac{1}{m}\zeta)}.$$

By equations (3.5) and (3.6), we can write now

$$\frac{\sigma(t_M)}{\sigma(t_N)} \frac{\sigma(\zeta - t_N)}{\sigma(\zeta - t_M)} = \frac{\sigma(t_M)^m}{\sigma(t_N)^m} \frac{\sigma(\frac{1}{m}\zeta - t_N)^m}{\sigma(\frac{1}{m}\zeta - t_M)^m} \prod_{j=1}^{m-1} (c_L\mathrm{P})(\tfrac{1}{m}\zeta, j\tfrac{1}{m}\zeta),$$

and the second assertion follows. □

Remark 3.11. Apparently, z plays no role in the parametrization of the elements of $\mathcal{T}[m]$, but if $(\zeta, z) \in \Lambda$, then

$$\left(\frac{1}{m}\zeta, \frac{1}{m}z\right) = \left(\frac{d_1 + d_3\tau}{m}, \frac{d_2 + d_3 s_1}{m}\right),$$

so the role of the integer parameter d_2 in z is linked to that of the m-th root.

3.1.3 Toroidal groups arising from quartic fields

Now we want to exhibit the construction of a toroidal group arising from a non-totally real quartic number field with one pair of complex embeddings. Let K be such a number field, $\Phi = \{\sigma_1, \sigma_2, \sigma_3\}$ a type of K and μ_Φ the corresponding Minkowski map. Let us consider the connected Abelian complex Lie group $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$. If there exists an essential polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e \in \mathbb{Z}[x]$ for the quartic number field K, then $\mathcal{O}_K$ admits an integral basis of the form

$$\{\rho_0 = 1,\ \rho_1 = a\omega,\ \rho_2 = a\omega^2 + b\omega,\ \rho_3 = a\omega^3 + b\omega^2 + c\omega\}, \tag{3.7}$$

where ω is a primitive element of K with $f(\omega) = 0$.

The quartic fields form the first case where a field could have more than one pair of complex embeddings. Nevertheless, we consider fields with exactly one pair in this section, as pointed out in the following remark. The first example of the link between toroidal groups and number fields with more than one pair of complex embeddings is treated in Section 3.2.

Remark 3.12. We note that the fact that K has precisely one pair of complex embeddings is necessary in order to have a toroidal group of complex dimension 3. Indeed, if we suppose that it admits 2 pairs of complex embeddings then, for every type Φ and fractional ideal $\mathfrak{a}$, we would have a lattice $\Lambda = \mu_\Phi(\mathfrak{a}) \subset \mathbb{C}^2$.

Theorem 3.13. *Let K be a quartic field admitting an essential polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ with real roots α, β and complex roots $\gamma, \bar{\gamma}$, and let us fix the type Φ corresponding to the set of roots $\{\alpha, \beta, \gamma\}$. The group $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal and its period matrix in standard coordinates*

$$\Pi = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \tau \\ 0 & 1 & 0 & s_1 \\ 0 & 0 & 1 & s_2 \end{array}\right)$$

is given by

$$\tau = \frac{e}{\bar{\gamma}}, \quad s_1 = -\bar{\gamma}^2, \quad s_2 = -\bar{\gamma}. \tag{3.8}$$

Proof. The proof proceeds as in the case where $n = 3$. Let $K = \mathbb{Q}(\omega)$ be a quartic field admitting an essential polynomial $f(x)$ with real roots α, β and complex roots $\gamma, \bar{\gamma}$. Let $\Phi = \{\sigma_1, \sigma_2, \sigma_3\}$ be the type defined by

$$\sigma_1 \colon \omega \mapsto \alpha, \quad \sigma_2 \colon \omega \mapsto \beta, \quad \sigma_3 \colon \omega \mapsto \gamma.$$

The lattice $\mu_\Phi(\mathcal{O}_K)$ is generated by the vectors

$$\begin{aligned}
\mu_\Phi(1) &= (1, 1, 1), \\
\mu_\Phi(\rho_1) &= (a\alpha, a\beta, a\gamma), \\
\mu_\Phi(\rho_2) &= (a\alpha^2 + b\alpha, a\beta^2 + b\beta, a\gamma^2 + b\gamma), \\
\mu_\Phi(\rho_3) &= (a\alpha^3 + b\alpha^2 + c\alpha, a\beta^3 + b\beta^2 + c\beta, a\gamma^3 + b\gamma^2 + c\gamma),
\end{aligned}$$

therefore its period matrix is given by

$$\begin{pmatrix} 1 & a\alpha & a\alpha^2 + b\alpha & a\alpha^3 + b\alpha^2 + c\alpha \\ 1 & a\beta & a\beta^2 + b\beta & a\beta^3 + b\beta^2 + c\beta \\ 1 & a\gamma & a\gamma^2 + b\gamma & a\gamma^3 + b\gamma^2 + c\gamma \end{pmatrix},$$

which, changing basis, gives in turn the period matrix in standard coordinates $\Pi = (I_3|T)$ where T is the column

$$T = \begin{pmatrix} a\alpha\beta\gamma \\ -\frac{1}{a^2}\left(b^2 - ac + a^2(\alpha\beta + \alpha\gamma + \beta\gamma) + ab(\alpha + \beta + \gamma)\right) \\ \frac{b}{a} + \alpha + \beta + \gamma \end{pmatrix}.$$

Using the following relations occuring among the roots and the coefficients of a polynomial,

$$\begin{aligned}
\alpha\beta\gamma\bar{\gamma} &= \frac{e}{a}, \\
\alpha + \beta + \gamma + \bar{\gamma} &= -\frac{b}{a}, \\
\alpha\beta + \alpha\gamma + \alpha\bar{\gamma} + \beta\gamma + \beta\bar{\gamma} + \gamma\bar{\gamma} &= \frac{c}{a},
\end{aligned}$$

we obtain that

$$\tau = a\alpha\beta\gamma = \frac{a\alpha\beta\gamma\bar{\gamma}}{\bar{\gamma}} = \frac{e}{\bar{\gamma}},$$

$$\begin{aligned} s_1 &= -\frac{b^2 - ac + a^2(\alpha\beta + \alpha\gamma + \beta\gamma) + ab(\alpha+\beta+\gamma)}{a^2} \\ &= -\frac{b^2 - ac + ac + a\bar{\gamma}(b + a\bar{\gamma}) + b(-b - a\bar{\gamma})}{a^2} = -\frac{a^2\bar{\gamma}^2}{a^2} = -\bar{\gamma}^2, \end{aligned}$$

$$s_2 = \frac{b}{a} + \alpha + \beta + \gamma = \frac{b}{a} - \frac{b + a\bar{\gamma}}{a} = -\bar{\gamma}.$$

Now we show that this group is toroidal. The irrationality condition in standard coordinates reduces to the fact that the only vector $(l_1, l_2, l_3) \in \mathbb{Z}^3$ that satisfies $(l_1, l_2, l_3) \cdot (\tau, s_1, s_2) \in \mathbb{Z}$ is the zero vector. Let us suppose that $(l_1, l_2, l_3) \cdot (\tau, s_1, s_2) = l \in \mathbb{Z}$. Therefore we have

$$l_1 \frac{e}{\bar{\gamma}} - l_2\bar{\gamma}^2 - l_3\bar{\gamma} = l,$$

that implies

$$l_2\bar{\gamma}^3 + l_3\bar{\gamma}^2 + l\bar{\gamma} - l_1 e = 0.$$

This gives $(l_1, l_2, l_3) = (0, 0, 0)$, otherwise we would have a contradiction with the fact that $f(x)$ is a times the minimal polynomial of $\bar{\gamma}$ over $\mathbb{Q}$. □

Remark 3.14. The fact that $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal is already in Theorem 1.32 proved in [4], but the proof given here shows that, in the case where K admits an essential polynomial, it can be proved more directly. Finally, using Corollary 3.2, $\mathbb{C}^3/\mu_\Phi(\mathfrak{a})$ is, as well, toroidal for every fractional ideal $\mathfrak{a}$ of K.

Now we can generalize Theorem 1.36, the proof of which is similar.

Theorem 3.15. *If $\Phi = \{\sigma_1, \sigma_2, \sigma_3\}$ is a type of a quartic field K admitting an essential polynomial, and if $\mathfrak{a}$ is a fractional ideal of K, then the toroidal group $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathfrak{a})$ has extra multiplications. Moreover, the ring $\mathcal{O}_K$ is isomorphic to* $\mathrm{End}(\mathcal{T})$ *and the field K is isomorphic to* $\mathrm{End}_0(\mathcal{T})$.

Proof. For every $\nu \in \mathcal{O}_K$, the matrix

$$D(\nu) = \begin{pmatrix} \sigma_1(\nu) & 0 & 0 \\ 0 & \sigma_2(\nu) & 0 \\ 0 & 0 & \sigma_3(\nu) \end{pmatrix} \tag{3.9}$$

induces an endomorphism of $\mathcal{T}$, that we keep denoting with the same symbol. In fact, if $(\sigma_1(\varepsilon), \sigma_2(\varepsilon), \sigma_3(\varepsilon)) \in \mu_\Phi(\mathfrak{a})$ then

$$D(\nu)\begin{pmatrix} \sigma_1(\varepsilon) \\ \sigma_2(\varepsilon) \\ \sigma_3(\varepsilon) \end{pmatrix} = \begin{pmatrix} \sigma_1(\nu) & 0 & 0 \\ 0 & \sigma_2(\nu) & 0 \\ 0 & 0 & \sigma_3(\nu) \end{pmatrix}\begin{pmatrix} \sigma_1(\varepsilon) \\ \sigma_2(\varepsilon) \\ \sigma_3(\varepsilon) \end{pmatrix} = \begin{pmatrix} \sigma_1(\nu\varepsilon) \\ \sigma_2(\nu\varepsilon) \\ \sigma_3(\nu\varepsilon) \end{pmatrix},$$

and $(\sigma_1(\nu\varepsilon), \sigma_2(\nu\varepsilon), \sigma_3(\nu\varepsilon)) \in \mu_\Phi(\mathfrak{a})$. Since $\mathbb{Z}$ is strictly contained in $\mathcal{O}_K$, we have that $\mathcal{T}$ has extra multiplications. Tensoring with $\mathbb{Q}$, the injective map

$$\begin{aligned} D\colon \mathcal{O}_K &\longrightarrow \mathrm{End}(\mathcal{T}) \\ \nu &\longmapsto D(\nu) \end{aligned}$$

induces a further map

$$K \longrightarrow \mathrm{End}_0(\mathcal{T}). \tag{3.10}$$

Since by Theorem 1.33 we know that $\mathrm{End}_0(\mathcal{T})$ is a non-totally real number field, and its dimension must divide 4, the map in (3.10) turns out to be an isomorphism. We also conclude that $\mathrm{End}(\mathcal{T})$ is an order in $\mathrm{End}_0(\mathcal{T}) \cong K$, and since it contains (up to isomorphism) the maximal order $\mathcal{O}_K$, we obtain that the map D is an isomorphism. □

Remark 3.16. While by Theorem 1.33 we know that for all toroidal groups $\mathbb{C}^3/\Lambda$ of real rank four, the ring $\mathrm{End}_0(\mathbb{C}^3/\Lambda)$ is actually a non-totally real number field, Theorem 3.13 shows that the period lattice of a toroidal group must fulfill very strict conditions in order to be a group arising from a non-totally real quartic field admitting an essential polynomial (cf. Section 3.1.5).

This can be seen as a further evidence that the class of fields with essential polynomial is very small compared to all the non-totally real number fields.

3.1.4 Representations of the endomorphisms

Coming back to the 3-dimensional general case, let $\mathcal{T} = \mathbb{C}^3/\Lambda$ be a toroidal group of complex dimension $\mathrm{rk}_\mathbb{C}\, \mathcal{T} = 3$ and real rank $\mathrm{rk}_\mathbb{R}\, \mathcal{T} = 4$ having extra multiplications. We recall again that $\mathrm{End}_0(\mathcal{T})$ is a non-totally real number field. Since

$$\dim_\mathbb{Q} \mathrm{End}_0(\mathcal{T}) \mid \dim_\mathbb{Q} \mathrm{End}_\mathbb{Q}(\mathrm{span}_\mathbb{Q}(\Lambda)) = \mathrm{rk}_\mathbb{R}\, \Lambda = 4,$$

we have that $\dim_\mathbb{Q} \mathrm{End}_0(\mathcal{T})$ can be equal to 2 or 4, that is, $\mathrm{End}_0(\mathcal{T})$ can be respectively a quadratic or a quartic number field. The natural map

$$\mathrm{End}(\mathcal{T}) \to \mathrm{End}(\mathcal{T}) \otimes_\mathbb{Z} \mathbb{Q} = \mathrm{End}_0(\mathcal{T})$$

is injective because $\mathrm{End}(\mathcal{T})$ is torsion-free. We conclude that $\mathrm{End}(\mathcal{T})$ is an order of the number field $\mathrm{End}_0(\mathcal{T})$ since $\mathrm{rk}_\mathbb{R}\, \mathrm{End}(\mathcal{T}) = [\mathrm{End}_0(\mathcal{T}) : \mathbb{Q}]$. Let us suppose from now on that $\mathrm{End}_0(\mathcal{T})$ is a quartic number field K admitting an essential polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e \in \mathbb{Z}[x]$ and that the order $\mathrm{End}(\mathcal{T})$ is maximal, so it coincides with the ring of integers $\mathcal{O}_K$. Again by [36], we have an integral basis of the form (3.7) and hence a matrix representation of $\mathcal{O}_K$ into $\mathrm{M}_4(\mathbb{Z})$ given by

$$\begin{aligned} &\rho_0 = 1 \mapsto I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \rho_1 \mapsto N_1 = \begin{pmatrix} 0 & 0 & 0 & -ae \\ 1 & -b & -c & -d \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \end{pmatrix}, \\ &\rho_2 \mapsto N_2 = \begin{pmatrix} 0 & 0 & -ae & -be \\ 0 & -c & -d & -e \\ 1 & 0 & -c & -d \\ 0 & a & b & 0 \end{pmatrix}, \ \rho_3 \mapsto N_3 = \begin{pmatrix} 0 & -ae & -be & -ce \\ 0 & -d & -e & 0 \\ 0 & 0 & -d & -e \\ 1 & 0 & 0 & -d \end{pmatrix}. \end{aligned} \tag{3.11}$$

Theorem 3.17. *Let $\mathcal{T}$ be a toroidal group of complex dimension 3 and real rank 4 having extra multiplications and period matrix in standard coordinates given by*

$$\Pi = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \tau \\ 0 & 1 & 0 & s_1 \\ 0 & 0 & 1 & s_2 \end{array}\right). \tag{3.12}$$

Suppose that the non-totally real number field $K = \mathrm{End}_0(\mathcal{T})$ is a quartic field admitting an essential polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ and that $\mathrm{End}(\mathcal{T}) = \mathcal{O}_K$. Suppose (3.11) defines the rational representation of $\mathrm{End}(\mathcal{T})$. Then $\mathcal{T}$ is isomorphic to $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$, for a suitable Minkowski map μ_Φ. Furthermore, the analytic representation in standard coordinates is given by

$$\rho_0 = 1 \mapsto I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_1 \mapsto A_1 = \begin{pmatrix} 0 & 0 & a\tau \\ 1 & -b & -c+as_1 \\ 0 & a & as_2 \end{pmatrix}, \tag{3.13}$$

$$\rho_2 \mapsto A_2 = \begin{pmatrix} 0 & a\tau & -ae+b\tau \\ 0 & -c+as_1 & -d+bs_1 \\ 1 & as_2 & -c+bs_2 \end{pmatrix}, \quad \rho_3 \mapsto A_3 = \begin{pmatrix} \tau & -ae & -be \\ s_1 & -d & -e \\ s_2 & 0 & -d \end{pmatrix}.$$

Proof. Suppose that the matrix Π in (3.12) is the period matrix of $\mathcal{T}$ in standard coordinates. With the embedding in (3.11) as the rational representation of the ring $\mathrm{End}(\mathcal{T})$, the Hurwitz relations

$$A_j\Pi = \Pi N_j, \quad j = 1, 2, 3, \tag{3.14}$$

must hold, where $A_j \in \mathrm{M}_3(\mathbb{C})$ is the analytic representation of ρ_j. The direct computations

$$\begin{aligned}
(A_1|A_1T) &= \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & 1 & 0 & s_1 \\ 0 & 0 & 1 & s_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -ae \\ 1 & -b & -c & -d \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & a\tau & -ae \\ 1 & -b & -c+as_1 & -d \\ 0 & a & as_2 & 0 \end{pmatrix}, \\
(A_2|A_2T) &= \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & 1 & 0 & s_1 \\ 0 & 0 & 1 & s_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & -ae & -be \\ 0 & -c & -d & -e \\ 1 & 0 & -c & -d \\ 0 & a & b & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & a\tau & b\tau-ae & -be \\ 0 & -c+as_1 & -d+bs_1 & -e \\ 1 & as_2 & -c+bs_2 & -d \end{pmatrix}, \\
(A_3|A_3T) &= \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & 1 & 0 & s_1 \\ 0 & 0 & 1 & s_2 \end{pmatrix} \begin{pmatrix} 0 & -ae & -be & -ce \\ 0 & -d & -e & 0 \\ 0 & 0 & -d & -e \\ 1 & 0 & 0 & -d \end{pmatrix} \\
&= \begin{pmatrix} \tau & -ae & -be & -d\tau-ce \\ s_1 & -d & -e & -ds_1 \\ s_2 & 0 & -d & -ds_2-e \end{pmatrix},
\end{aligned} \tag{3.15}$$

give in turn the matrices A_j, that are

$$A_1 = \begin{pmatrix} 0 & 0 & a\tau \\ 1 & -b & -c+as_1 \\ 0 & a & as_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & a\tau & -ae+b\tau \\ 0 & -c+as_1 & -d+bs_1 \\ 1 & as_2 & -c+bs_2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \tau & -ae & -be \\ s_1 & -d & -e \\ s_2 & 0 & -d \end{pmatrix}.$$

Comparing the last columns of Hurwitz relation (3.15), that is,

$$\begin{pmatrix} 0 & 0 & a\tau \\ 1 & -b & -c+as_1 \\ 0 & a & as_2 \end{pmatrix} \begin{pmatrix} \tau \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} -ae \\ -d \\ 0 \end{pmatrix},$$

we obtain that τ, s_1 and s_2 must satisfy the following conditions:

$$\begin{cases} \tau = -\frac{e}{s_2}, \\ s_1 = -s_2^2, \\ f(-s_2) = 0. \end{cases}$$

Equation (3.14) with $j = 2, 3$ does not give any further information. Hence we have that $-s_2$ is a root of the polynomial $f(x)$ which cannot be real (otherwise the complex rank of Λ would be 3).
Let α and β be the real roots of $f(x)$. If we consider the type $\Phi = \{\sigma_1, \sigma_2, \sigma_3\}$ defined by

$$\sigma_1 \colon \omega \mapsto \alpha, \quad \sigma_2 \colon \omega \mapsto \beta, \quad \sigma_3 \colon \omega \mapsto -\bar{s}_2,$$

we have that Λ coincides with the lattice $\mu_\Phi(\mathcal{O}_K)$ described in Theorem 3.13. □

Remark 3.18. The above representation in $\mathrm{M}_4(\mathbb{Z})$ of the ring of integers $\mathcal{O}_K$ of a non-totally real quartic field K with essential polynomial is already in [36]. We remark here that the representation of $\mathrm{End}(\mathcal{T})$ in (3.13) gives in turn, as well, a representation of smaller degree into $\mathrm{M}_3(\mathbb{C})$ of $\mathcal{O}_K$.

3.1.5 Constructing fields admitting an essential polynomial

Now we want to investigate which toroidal groups satisfy the hypotheses of Theorem 3.17. The following explicit computations allow us to furnish some examples of toroidal groups such that the ring of endomorphisms determines a non-totally real admitting an essential polynomial and, conversely, some toroidal groups arising from such a field.

If $\mathcal{T} = \mathbb{C}^3/\Lambda$ is a toroidal group with $\mathrm{rk}_\mathbb{R}\,\Lambda = 4$ having extra multiplications, then $K := \mathrm{End}_0(\mathcal{T})$ is a non-totally real number field of degree at most four. If K satisfy the hypotheses of Theorem 3.17 proved in the previous section, then $\tau = -\frac{a_5}{s_2}$, $s_1 = -s_2^2$ and $f(-s_2) = 0$, where $(\tau, s_1, s_2)^\mathrm{T}$ is the last column of a period matrix in standard coordinates and $f(x) = a_1x^4 + \cdots + a_4x + a_5 \in \mathbb{Z}[x]$ is an essential polynomial for K. Let us assume therefore that $\mathcal{T} = \mathbb{C}^3/\Lambda$ is a toroidal group having extra multiplications with period matrix in standard coordinates given by

$$\Pi = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -e/s_2 \\ 0 & 1 & 0 & -s_2^2 \\ 0 & 0 & 1 & s_2 \end{array}\right), \tag{3.16}$$

with $e \in \mathbb{Z}$, $e \neq 0$ and $s_2 \in \mathbb{C} \setminus \mathbb{R}$. We directly compute the ring $\mathrm{End}(\mathcal{T})$ by using the Hurwitzs relations $A\Pi = \Pi N$, where $A \in \mathrm{M}_3(\mathbb{C})$ and $N \in \mathrm{M}_4(\mathbb{Z})$. From the condition $(A|AT) = \Pi N$, we obtain the following three relations

$$n_{12}s_2^4 - (n_{13} + n_{42}e)s_2^3 + (n_{14} + n_{43}e)s_2^2 + (n_{11} - n_{44})es_2 - n_{41}e^2 = 0, \tag{3.17}$$
$$n_{32}s_2^4 + (n_{22} - n_{33})s_2^3 + (n_{34} - n_{23})s_2^2 + (n_{24} + n_{31}e)s_2 + n_{21}e = 0, \text{ and} \tag{3.18}$$
$$n_{42}s_2^4 + (n_{32} - n_{43})s_2^3 + (n_{44} - n_{33})s_2^2 + (n_{34} + n_{41}e)s_2 + n_{31}e = 0. \tag{3.19}$$

If s_2 is algebraic over $\mathbb{Q}$ of degree greater than five or trascendental, then manifestly $\mathrm{End}(\mathcal{T}) \cong \mathbb{Z}$. On the other hand, from the irrationality condition in standard coordinates for the period matrix Π in (3.16), there no exists a non-zero triple of integers (l_1, l_2, l_3) such that

$$-l_1\frac{e}{s_2} - l_2s_2^2 + l_3s_2 = L, \text{ for some } L \in \mathbb{Z},$$

and this fact implies that s_2 cannot be algebraic over $\mathbb{Q}$ of degree smaller than four (otherwise, the powers $1, s_2, s_2^2, s_2^3$ would be linearly dependent over $\mathbb{Z}$ and one could find such integers l_1, l_2, l_3). Therefore, s_2 is an algebraic complex number of degree four, thus $s_2^4 = q_0 + q_1s_2 + q_2s_2^2 + q_3s_2^3$, with $q_j \in \mathbb{Q}$, $j = 0, 1, 2, 3$. The conditions (3.17)-(3.19) become

$$(-n_{13} - n_{42}e + n_{12}q_3)s_2^3 + (n_{14} + n_{43}e + n_{12}q_2)s_2^2 + ((n_{11} - n_{44})e + n_{12}q_1)s_2 + (n_{12}q_0 - n_{41}e^2) = 0,$$

$$(n_{22} - n_{33} + n_{32}q_3)s_2^3 + (n_{34} - n_{23} + n_{32}q_2)s_2^2 + (n_{24} + n_{31}e + n_{32}q_1)s_2 + (n_{32}q_0 + n_{21}e) = 0,$$

and

$$(n_{32} - n_{43} + n_{42}q_3)s_2^3 + (n_{44} - n_{33} + n_{42}q_2)s_2^2 + (n_{34} + n_{41}e + n_{42}q_1)s_2 + (n_{42}q_0 + n_{31}e) = 0.$$

Since $1, s_2, s_2^2, s_2^3$ are linearly independent on $\mathbb{Q}$, we obtain the twelve relations

$$\begin{array}{ccc} n_{13} + n_{42}e - n_{12}q_3 = 0, & n_{22} - n_{33} + n_{32}q_3 = 0, & n_{32} - n_{43} + n_{42}q_3 = 0, \\ n_{14} + n_{43}e + n_{12}q_2 = 0, & n_{34} - n_{23} + n_{32}q_2 = 0, & n_{44} - n_{33} + n_{42}q_2 = 0, \\ (n_{11} - n_{44})e + n_{12}q_1 = 0, & n_{31}e + n_{24} + n_{32}q_1 = 0, & n_{41}e + n_{34} + n_{42}q_1 = 0, \\ n_{12}q_0 - n_{41}e^2 = 0; & n_{32}q_0 + n_{21}e = 0; & n_{42}q_0 + n_{31}e = 0. \end{array} \tag{3.20}$$

Example 3.19 (Monogenic)**.** The complex number $s_2 = i\sqrt[4]{2} - \sqrt{2}$ is such that $s_2^4 = 4s_2^2 + 8s_2 - 2$. Let us consider the toroidal group of period matrix in standard coordinates given by (3.16), that is

$$\Pi = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{e}{i\sqrt[4]{2}-\sqrt{2}} \\ 0 & 1 & 0 & -2 + \sqrt{2} + 2i\sqrt[4]{8} \\ 0 & 0 & 1 & i\sqrt[4]{2} - \sqrt{2} \end{array}\right),$$

and compute End$(\mathcal{T})$. For $c_0 = -2, c_1 = 8, c_2 = 4, c_3 = 0$ in the relations (3.20), the rational representation of End$(\mathcal{T})$ is given by the matrices

$$N = \begin{pmatrix} n_{11} & -\frac{e^2}{2}n_{41} & -en_{42} & 2e^2n_{41} - en_{43} \\ \frac{2}{e}n_{43} & n_{11} - 4en_{41} + 4n_{42} & -en_{41} - 8n_{42} + 4n_{43} & -2n_{42} - 8n_{43} \\ \frac{2}{e}n_{42} & n_{43} & n_{11} - 4en_{41} + 4n_{42} & -en_{41} - 8n_{42} \\ n_{41} & n_{42} & n_{43} & n_{11} - 4en_{41} \end{pmatrix},$$

such that $n_{42}, n_{43} \in \frac{e}{(e,2)}\mathbb{Z}$ and $n_{41} \in \frac{2}{(e^2,2)}\mathbb{Z}$, where (x, y) denotes the greatest common divisor of the integers x and y. Thus, End$(\mathcal{T})$ can be identified with the order of the matrices of the form

$$A = \begin{pmatrix} n_{11} - n_{41}\frac{e}{s_2} & -\frac{e^2}{2}n_{41} - n_{42}\frac{e}{s_2} & -en_{42} - n_{43}\frac{e}{s_2} \\ \frac{2}{e}n_{43} - n_{41}s_2^2 & n_{11} - 4en_{41} + 4n_{42} - n_{42}s_2^2 & -en_{41} - 8n_{42} + 4n_{43} - n_{43}s_2^2 \\ \frac{2}{e}n_{42} + n_{41}s_2 & n_{43} + n_{42}s_2 & n_{11} - 4en_{41} + 4n_{42} + n_{43}s_2 \end{pmatrix}$$

of the number field End$_0(\mathcal{T})$. A basis of End$(\mathcal{T}) \subset \mathrm{M}_3(\mathbb{C})$ is given by I_3,

$$A_1 = \frac{2}{(e^2,2)} \begin{pmatrix} -\frac{e}{s_2} & -\frac{e^2}{2} & 0 \\ -s_2^2 & -4e & -e \\ s_2 & 0 & -4e \end{pmatrix},$$

$$A_2 = \frac{e}{(e,2)} \begin{pmatrix} 0 & -\frac{e}{s_2} & -e \\ 0 & 4 - s_2^2 & -8 \\ \frac{2}{e} & s_2 & 4 \end{pmatrix},$$

$$A_3 = \frac{e}{(e,2)} \begin{pmatrix} 0 & 0 & -\frac{e}{s_2} \\ \frac{2}{e} & 0 & 4 - s_2^2 \\ 0 & 1 & s_2 \end{pmatrix}.$$

Let us suppose, for instance, that $e = 2$, then $e/(e,2) = 1 = 2/(e^2,2)$, so we have the following matrices

$$A_1 = \begin{pmatrix} -\frac{2}{s_2} & -2 & 0 \\ -s_2^2 & -8 & -2 \\ s_2 & 0 & -8 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -\frac{2}{s_2} & -2 \\ 0 & 4 - s_2^2 & -8 \\ 1 & s_2 & 4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & -\frac{2}{s_2} \\ 1 & 0 & 4 - s_2^2 \\ 0 & 1 & s_2 \end{pmatrix}.$$

Computing the powers of A_3,

$$A_3^2 = \begin{pmatrix} 0 & \frac{-2}{s_2} & -2 \\ 0 & 4 - s_2^2 & \frac{-s_2^4 + 4s_2^2 - 2}{s_2} \\ 1 & s_2 & 4 \end{pmatrix}$$

$$A_3^3 = \begin{pmatrix} \frac{-2}{s_2} & -2 & \frac{-8}{s_2} \\ 4 - s_2^2 & \frac{-s_2^4 + 4s_2^2 - 2}{s_2} & 14 - 4s_2^2 \\ s_2 & 4 & \frac{-s_2^4 + 8s_2^2 - 2}{s_2} \end{pmatrix}$$

one can note that $A_2 = A_3^2$, $A_1 = A_3^3 - 4A_3$ and

$$A_3^4 - 4A_3^2 + 8A_3 + 2I_3 = 0.$$

One can change the basis $\{1, A_1, A_2, A_3\}$ to $\{1, A_3, A_3^2, A_3^3\}$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

which has determinant equal to -1. Thus, the order $\mathrm{End}(\mathcal{T})$ has a power integer basis $\{1, A_3, A_3^2, A_3^3\}$ and the field $\mathrm{End}_0(\mathcal{T})$ is monogenic.

On the other hand, the field $\mathrm{End}_0(\mathcal{T})$ is isomorphic to the monogenic field $K = \mathbb{Q}(-s_2)$, for which the defining polynomial is $f(x) = x^4 - 4x^2 + 8x + 2$. The real roots of $f(x)$ are $\alpha = \sqrt[4]{2} - 2$ and $\beta = -\sqrt[4]{2} - 2$, and complex ones $-s_2, -\bar{s}_2$. Fixing the type Φ corresponding to the set of roots $\{\alpha, \beta, -\bar{s}_2\}$, the toroidal group $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ coincides with the starting one.

Pure case. From now on, we limit ourselves to the pure case. Let us suppose that $s_2^4 = c$, where $c = \frac{c_1}{c_2} \in \mathbb{Q}$ for c_1 and $c_2 > 0$ fourth power free non-zero integers and relative prime. The relations are now

$$\begin{aligned} &-(n_{13} + n_{42}e)s_2^3 + (n_{14} + n_{43}e)s_2^2 + (n_{11} - n_{44})es_2 + (n_{12}c - n_{41}e^2) = 0, \\ &(n_{22} - n_{33})s_2^3 + (n_{34} - n_{23})s_2^2 + (n_{24} + n_{31}e)s_2 + (n_{32}c + n_{21}e) = 0, \\ &(n_{32} - n_{43})s_2^3 + (n_{44} - n_{33})s_2^2 + (n_{34} + n_{41}e)s_2 + (n_{42}c + n_{31}e) = 0. \end{aligned}$$

which, since the powers $s_2^0 = 1, s_2, s_2^2, s_2^3$ are $\mathbb{Q}$-independent, imply the following twelve relations

$$\begin{array}{ccc} n_{42}e + n_{13} = 0, & n_{22} = n_{33}, & n_{32} = n_{43}, \\ n_{43}e + n_{14} = 0, & n_{34} = n_{23}, & n_{44} = n_{33}, \\ n_{11} = n_{44}, & n_{31}e + n_{24} = 0, & n_{41}e + n_{34} = 0, \\ n_{12}c - n_{41}e = 0; & n_{32}c + n_{21}e = 0; & n_{42}c + n_{31}e = 0. \end{array} \tag{3.21}$$

The rational representation of the ring $\mathrm{End}(\mathcal{T})$ is given by the matrices

$$N = \begin{pmatrix} n_{11} & \frac{e^2}{c}n_{41} & -en_{42} & -en_{43} \\ -\frac{c}{e}n_{43} & n_{11} & -en_{41} & cn_{42} \\ -\frac{c}{e}n_{42} & n_{43} & n_{11} & -en_{41} \\ n_{41} & n_{42} & n_{43} & n_{11} \end{pmatrix},$$

which must have integer entries, thus $n_{41} \in \frac{c_1}{(c_1,e^2)}\mathbb{Z}$ and $n_{42}, n_{43} \in \frac{c_2 e}{(c_1,e)} \in \mathbb{Z}$, where (x, y) denotes the greatest common divisor of the integers x and y. The Hurwitz relations provide the analytic representation, which is given by the matrices $A(n_{11}, n_{41}, n_{42}, n_{43})$ defined by

$$\begin{pmatrix} n_{11} - \frac{c_1}{(c_1,e^2)}\frac{e}{s_2}n_{41} & \frac{c_2e^2}{(c_1,e^2)}n_{41} - \frac{c_2e}{(c_1,e)}\frac{e}{s_2}n_{42} & -\frac{c_2e^2}{(c_1,e)}n_{42} - \frac{c_2e}{(c_1,e)}\frac{e}{s_2}n_{43} \\ -\frac{c_1}{(c_1,e)}n_{43} - \frac{c_1}{(c_1,e^2)}s_2^2n_{41} & n_{11} - \frac{c_2e}{(c_1,e)}s_2^2n_{42} & -\frac{c_1e}{(c_1,e^2)}n_{41} - \frac{c_2e}{(c_1,e)}s_2^2n_{43} \\ -\frac{c_1}{(c_1,e)}n_{42} + \frac{c_1}{(c_1,e^2)}s_2n_{41} & \frac{c_2e}{(c_1,e)}n_{43} + \frac{c_2e}{(c_1,e)}s_2n_{42} & n_{11} + \frac{c_2e}{(c_1,e)}s_2n_{43} \end{pmatrix},$$

with $n_{11}, n_{41}, n_{42}, n_{43} \in \mathbb{Z}$.

The ring $\mathcal{O} := \mathrm{End}(\mathcal{T})$ is generated over $\mathbb{Z}$ by the matrices $I_3 = A(1,0,0,0)$,

$$A_1 = A(0,1,0,0) = \frac{1}{(c_1,e^2)}\begin{pmatrix} -c_1\frac{e}{s_2} & c_2e^2 & 0 \\ -c_1s_2^2 & 0 & -c_1e \\ c_1s_2 & 0 & 0 \end{pmatrix}, \tag{3.22}$$

$$A_2 = A(0,0,1,0) = \frac{1}{(c_1,e)}\begin{pmatrix} 0 & -c_2e\frac{e}{s_2} & -c_2e^2 \\ 0 & -c_2es_2^2 & 0 \\ -c_1 & c_2es_2 & 0 \end{pmatrix} \tag{3.23}$$

$$\text{and} \quad A_3 = A(0,0,0,1) = \frac{1}{(c_1,e)}\begin{pmatrix} 0 & 0 & -c_2e\frac{e}{s_2} \\ -c_1 & 0 & -c_2es_2^2 \\ 0 & c_2e & c_2es_2 \end{pmatrix}. \tag{3.24}$$

Hence, $\mathcal{O}$ is a free $\mathbb{Z}$-module of rank four and K must be necessarily a quartic field. The complex numbers z such that $z^4 = c \in \mathbb{Q}$ have the form $\pm i\sqrt[4]{c}$ if $c > 0$ and $\frac{\sqrt[4]{-c}}{\sqrt{2}}(\pm 1 \pm i)$ if $c < 0$. These cases are respectively $r_2 = 1$ and $r_2 = 2$, therefore, in order to have exactly one pair of complex embeddings ($r_2 = 1$), we are considering the case $s_2 = \pm i\sqrt[4]{c}$. Computing the powers of A_1,

$$A_1^2 = \frac{c_1e}{(c_1,e^2)^2}\begin{pmatrix} 0 & -c_2e\frac{e}{s_2} & -c_2e^2 \\ 0 & -c_2es_2^2 & 0 \\ -c_1 & c_2es_2 & 0 \end{pmatrix} = \frac{c_1e(c_1,e)}{(c_1,e^2)^2}A_2,$$

$$A_1^3 = -\frac{c_1^2e^2}{(c_1,e^2)^3}\begin{pmatrix} 0 & 0 & -c_2e\frac{e}{s_2} \\ -c_1 & 0 & -c_2es_2^2 \\ 0 & c_2e & c_2es_2 \end{pmatrix} = -\frac{c_1^2e^2(c_1,e)}{(c_1,e^2)^3}A_3,$$

$$A_1^4 = \frac{c_1^3c_2e^4}{(c_1,e^2)^4}I_3 = \left(\frac{c_1e}{(c_1,e^2)}\right)^4\frac{1}{c}I_3,$$

we conclude that $\omega := \frac{(c_1,e^2)}{c_1e}A_1$ is a primitive element of the field K (indeed, in terms of ω, we have that $A_2 = \omega^2$, $A_3 = \omega^3$ and $I_3 = c\omega^4$). The minimal polynomial of ω on $\mathbb{Q}$ is $p(x) = x^4 - \frac{1}{c}$, which has roots $\pm\frac{1}{\sqrt[4]{c}}, \pm\frac{1}{i\sqrt[4]{c}}$. Let us fix the embeddings of K

$$\left\{\sigma_1\colon \omega \mapsto \frac{1}{\sqrt[4]{c}},\ \sigma_2\colon \omega \mapsto -\frac{1}{\sqrt[4]{c}},\ \sigma_3\colon \omega \mapsto \frac{1}{i\sqrt[4]{c}},\ \sigma_4\colon \omega \mapsto -\frac{1}{i\sqrt[4]{c}}\right\}.$$

The field K is therefore manifestly isomorphic to $\mathbb{Q}(s_2) = \mathbb{Q}\left(\pm i\sqrt[4]{c}\right) = \mathbb{Q}\left(\pm i\sqrt[4]{c_1c_2^3}\right)$, a pure quartic (non-totally real) field and the discriminants of such fields have been completely computed by Fukanura (see Appendix A).

On the other hand, we can compute by definition the discriminant of $\mathcal{O}$ as

$$\begin{vmatrix} 1 & \frac{h}{\sqrt[4]{c}} & \frac{k}{\sqrt[4]{c^2}} & -\frac{k}{\sqrt[4]{c^3}} \\ 1 & -\frac{h}{\sqrt[4]{c}} & \frac{k}{\sqrt[4]{c^2}} & \frac{k}{\sqrt[4]{c^3}} \\ 1 & \frac{h}{i\sqrt[4]{c}} & -\frac{k}{\sqrt[4]{c^2}} & \frac{k}{i\sqrt[4]{c^3}} \\ 1 & -\frac{h}{i\sqrt[4]{c}} & -\frac{k}{\sqrt[4]{c^2}} & -\frac{k}{i\sqrt[4]{c^3}} \end{vmatrix}^2 = \left(16i\frac{hk^2}{\sqrt[4]{c^6}}\right)^2 = -256\frac{h^2k^4}{c^3},$$

where $h = \frac{c_1e}{(c_1,e^2)}$ and $k = \frac{c_1e}{(c_1,e)}$. Hence, $\mathrm{disc}(\mathcal{O}) = -256\frac{c_1^3c_2^3e^6}{(c_1,e^2)^2(c_1,e)^4}$.
If $\mathcal{O} = \mathcal{O}_K$ (that is, $\mathcal{O}$ is a maximal order of K) and c_1 divides e (this is not a restrictive condition, indeed this is exactly the case of Theorem 3.17), then $f(x) = -\frac{e}{c}x^4 + e \in$

$\mathbb{Z}[x]$ has root the primitive element ω^{-1} of K and discriminant $\mathrm{disc}(f) = -256\frac{e^6}{c^3}$, hence it is essential. Thus, the type Φ of K defined by

$$\sigma_1\colon \omega \mapsto \sqrt[4]{c}, \quad \sigma_2\colon \omega \mapsto -\sqrt[4]{c}, \quad \sigma_3\colon \omega \mapsto i\sqrt[4]{c},$$

determines the toroidal group $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$, which is equal to the starting one $\mathcal{T}$. We summarize in the following lemma.

Lemma 3.20. *Let $\mathcal{T} = \mathbb{C}^3/\Lambda$ be a toroidal group having extra multiplications with period matrix in standard coordinate given by*

$$\Pi = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -e/s_2 \\ 0 & 1 & 0 & -s_2^2 \\ 0 & 0 & 1 & s_2 \end{array}\right),$$

with $s_2 = \pm i\sqrt[4]{c}$, where $c = \frac{c_1}{c_2} \in \mathbb{Q}$, c_1, c_2 are positive relative prime and fourth power free integers, and $e \in c_1\mathbb{Z}$. Then $K := \mathrm{End}_0(\mathcal{T})$ is a non-totally real quartic pure number field with only one pair of complex embeddings.
Furthermore, if the order $\mathrm{End}(\mathcal{T})$ is maximal, that is $\mathrm{End}(\mathcal{T}) = \mathcal{O}_K$, then the number field K admits an essential polynomial $f(x) = -\frac{e}{c}x^4 + e$. In this case, $\mathcal{T}$ is the toroidal group $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$, for a suitable Minkowski map μ_Φ.

Remark 3.21. Under the hypothesis of Lemma 3.20, looking at the corresponding analytic representation given by (3.13) of $\mathrm{End}(\mathcal{T})$ and the matrices A_1, A_2, A_3 of (3.22), (3.23), (3.24), we point out that

$$A_1 = (\rho_1)_r, \quad A_2 = -(\rho_2)_r, \quad \text{and} \quad A_3 = -(\rho_3)_r.$$

The previous lemma allow us to make the following examples.

Example 3.22 (Monogenic)**.** The toroidal group $\mathcal{T}$ of period matrix in standard coordinates

$$\Pi = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2i\sqrt[4]{\frac{3}{2}} \\ 0 & 1 & 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 & 1 & i\sqrt[4]{\frac{2}{3}} \end{array}\right), \tag{3.25}$$

is an example of a group of the form given in Lemma 3.20 (for $e = 2 = c_1$ and $c_2 = 3$). The toroidal group $\mathcal{T}$ has extra multiplications and

$$\mathrm{disc}(\mathrm{End}(\mathcal{T})) = -256 \cdot 2^3 \cdot 3^3 = \mathrm{disc}(K)$$

(cf. Theorem A.1 in Appendix A, applied for $a = c_1c_2^2$ and $b = c_2$), where the field K is $\mathrm{End}_0(\mathcal{T}) = \mathrm{End}(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence, $\mathrm{End}(\mathcal{T}) = \mathcal{O}_K$ and it can be identified in $\mathrm{M}_5(\mathbb{Z})$ with the subring

$$N = \begin{pmatrix} n_{11} & 6n_{41} & -6n_{42} & -6n_{43} \\ -n_{43} & n_{11} & -2n_{41} & 2n_{42} \\ -n_{42} & 3n_{43} & n_{11} & -2n_{41} \\ n_{41} & 3n_{42} & 3n_{43} & n_{11} \end{pmatrix}$$

and in $M_3(\mathbb{C})$ with the order

$$A = \begin{pmatrix} n_{11} + 2i\sqrt[4]{\frac{3}{2}}n_{41} & 6n_{41} + 6i\sqrt[4]{\frac{3}{2}}n_{42} & -6n_{42} + 6i\sqrt[4]{\frac{3}{2}}n_{43} \\ -n_{43} + \sqrt{\frac{2}{3}}n_{41} & n_{11} + 3\sqrt{\frac{2}{3}}n_{42} & -2n_{41} + 6\sqrt{\frac{2}{3}}n_{43} \\ -n_{42} + i\sqrt[4]{\frac{2}{3}}n_{41} & 3n_{43} + 3i\sqrt[4]{\frac{2}{3}}n_{42} & n_{11} + 3i\sqrt[4]{\frac{2}{3}}n_{43} \end{pmatrix}.$$

The field K admits an essential polynomial $f(x) = -3x^4 + 2$, where $K = \mathbb{Q}(\omega)$, $f(\omega) = 0$ with

$$\omega = 2\begin{pmatrix} 2i\sqrt[4]{\frac{3}{2}} & 6 & 0 \\ \sqrt{\frac{2}{3}} & 0 & -2 \\ i\sqrt[4]{\frac{2}{3}} & 0 & 0 \end{pmatrix}^{-1} = -\frac{1}{3}A_3.$$

The toroidal group arises from the non-totally real number field K, which admits essential polynomial. In particular, $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ for the type Φ corresponding to the set of roots $\left\{\alpha = -\sqrt[4]{\frac{2}{3}},\ \beta = \sqrt[4]{\frac{2}{3}},\ \gamma = i\sqrt[4]{\frac{2}{3}}\right\}$.

On the other hand, the field K is monogenic by [31, Theorem 7-a, p. 37], hence a further toroidal group can be constructed considering the power integral basis provided by Fukanura.

Example 3.23. The toroidal group of period matrix in standard coordinates

$$\Pi = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3i\sqrt[4]{\frac{5}{4}} \\ 0 & 1 & 0 & 2\sqrt{5} \\ 0 & 0 & 1 & \sqrt[4]{\frac{4}{5}} \end{array}\right), \tag{3.26}$$

is an example of a group having period matrix of the form given in Lemma 3.20 (for $e = 3$, $c_1 = 4$ and $c_2 = 5$) but which does not satisfy the hypothesis that $\mathrm{End}(\mathcal{T})$ is maximal and c_1 does not divide the integer e.

Indeed, the discriminant of the number field is $-16 \cdot 2^2 \cdot 5^3$ by Theorem A.1, while the discriminant of $\mathrm{End}(\mathcal{T})$ is, by the above computations, $-256 \cdot 2^6 \cdot 3^3 \cdot 5^3$.

Example 3.24. The toroidal group of period matrix in standard coordinates

$$\Pi = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 10i\sqrt[4]{\frac{17}{5}} \\ 0 & 1 & 0 & \sqrt{\frac{5}{17}} \\ 0 & 0 & 1 & \sqrt[4]{\frac{5}{17}} \end{array}\right), \tag{3.27}$$

is an example of a group having period matrix of the form given in Lemma 3.20 (for $e = 10$, $c_1 = 5$, and $c_2 = 17$) but which does not satisfy the hypothesis that $\mathrm{End}(\mathcal{T})$ is maximal.

Indeed, the discriminant of the number field $\mathbb{Q}(i\sqrt[4]{4 \cdot 17^3}) = \mathbb{Q}(i\sqrt[4]{24565})$ is $-16 \cdot 5^3 \cdot 17^3$ by Theorem A.1, while the discriminant of $\mathrm{End}(\mathcal{T})$ is, by the above computations, $-256 \cdot 5^3 \cdot 17^3$. This show that the condition $e \in c_1\mathbb{Z}$ is necessary but not sufficient in order to be a toroidal group inducing a quartic field admitting essential polynomial.

Remark 3.25. Theorem 3.17 can be seen as a tool to construct number field admitting essential polynomial. Changing point of view, Theorem 3.13 can be used as test if a given number field can admit an essential polynomial, looking at the corresponding toroidal group.

3.1.6 General case of dimension n

In the following, we write down, in the general case of $r_2 = 1$, the relations between the essential polynomial and the parameters in the last column of the period matrix of $\mathcal{T}$ in standard form, and we give the analytic and rational representations of $\mathrm{End}(\mathcal{T})$.

Remark 3.26. Recall that the Vandermonde matrix in $a_1, \dots, a_n$, that we will denote by $V(a_1, \dots, a_n)$, is the $n \times n$ matrix

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{pmatrix}.$$

We will denote the Vandermonde matrix in $a_1, \dots, a_n$ missing the k-th power by $V_k(a_1, \dots, a_n)$, for $k = 0, \dots, n-1$, and its determinant is given by

$$\begin{vmatrix} 1 & a_1 & \dots & a_1^{k-1} & a_1^{k+1} & \dots & a_1^n \\ 1 & a_2 & \dots & a_2^{k-1} & a_2^{k+1} & \dots & a_2^n \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & a_n & \dots & a_n^{k-1} & a_n^{k+1} & \dots & a_n^n \end{vmatrix} = e_{n-k}(a_1, a_2, \dots, a_n) \det(V),$$

where e_j is the j-th fundamental symmetric polynomial.

Theorem 3.27. *Let K be a non-totally real number field of degree $n+1$ with only a pair of complex embeddings. Suppose that the field K admits an essential polynomial $f(x) = a_1 x^{n+1} + \dots + a_{n+1} x + a_{n+2}$ and let $\alpha_1, \dots, \alpha_{n-1}$ be the real roots of the polynomial f and $\gamma, \bar{\gamma}$ the complex ones. Fix Φ be the type corresponding to the set of roots $\{\alpha_1, \dots, \alpha_{n-1}, \gamma\}$. The Abelian complex Lie group $\mathcal{T} = \mathbb{C}^n / \mu_\Phi(\mathcal{O}_K)$ is toroidal with extra multiplications and, if*

$$\Pi = \left(\begin{array}{cccc|c} 1 & 0 & \dots & 0 & \tau \\ 0 & 1 & \dots & 0 & s_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & s_{n-1} \end{array}\right)$$

is a period matrix in standard coordinates, then

$$\tau = \frac{a_{n+2}}{\bar{\gamma}}, \quad s_i = -\bar{\gamma}^{n-i}, \tag{3.28}$$

for $i = 1, \dots, n-1$. Moreover, $\mathrm{End}_0(\mathcal{T}) \cong K$ and $\mathrm{End}(\mathcal{T}) \cong \mathcal{O}_K$.

Proof. By [36], if $K = \mathbb{Q}(\omega)$ such that $f(\omega) = 0$ we have a basis of the ring of integers as in (3.1). The lattice $\mu_\Phi(\mathcal{O}_K)$ is generated by the vectors $\mu_\Phi(1), \mu_\Phi(\rho_1), \dots, \mu_\Phi(\rho_n)$ and let $A = (A_0|A_1|\dots|A_{n-1}|A_n)$ be the period matrix corresponding to this basis, which is given by (3.2). Let $\Pi = (I_n|T)$ be a period matrix in standard coordinates, then $\Pi = P^{-1}A$, where P is the square matrix $P = (A_0|A_1|\dots|A_{n-1})$ given by the first n columns of A. Therefore $T = P^{-1}A_n$, that is, T is the solution of the linear system $PX = A_n$. Applying Cramer's rule, we find that

$$\tau = \frac{\det(A_n|A_1|\dots|A_{n-1})}{\det(A_0|A_1|\dots|A_{n-1})}$$

and

$$s_i = \frac{\det(A_0|A_1|\dots|A_{i-1}|A_n|A_{i+1}|A_{n-1})}{\det(A_0|A_1|\dots|A_{n-1})},$$

for $i = 1, \dots, n-1$. Denoting

$$\phi^{(j)} = \begin{pmatrix} \alpha_1^j \\ \vdots \\ \alpha_{n-1}^j \\ \gamma^j \end{pmatrix}, \text{ for } j = 0, \dots, n,$$

we have that $A_i = \sum_{j=1}^{i} a_j \phi^{(i-j+1)}$. Notice that

$$\det(P) = \det(A_0|A_1|\dots|A_{n-1}) = a_1^{n-1} \det(\phi^{(0)}|\phi^{(1)}|\dots|\phi^{(n-1)})$$

and that $(\phi^{(0)}|\phi^{(1)}|\dots|\phi^{(n-1)})$ is the Vandermonde matrix $V = V(\alpha_1, \dots, \alpha_{n-1}, \gamma)$.

Computing τ, after the permutation of the columns and by the linearity of the determinant, we have that

$$\tau = (-1)^{n-1} \frac{\det(A_1|\dots|A_{n-1}|A_n)}{\det(A_0|A_1|\dots|A_{n-1})} = (-1)^{n-1} \frac{a_1^n \det(\phi^{(1)}|\dots|\phi^{(n-1)}|\phi^{(n)})}{a_1^{n-1} \det(\phi^{(0)}|\phi^{(1)}|\dots|\phi^{(n-1)})}.$$

The matrix $(\phi^{(1)}|\dots|\phi^{(n-1)}|\phi^{(n)})$ is the Vandermonde matrix missing the zero powers $V_0(\alpha_1, \dots, \alpha_{n-1}, \gamma)$ and by Remark 3.26 its determinant is $\det(V) e_n(\alpha_1, \dots, \alpha_{n-1}, \gamma)$, then $\tau = (-1)^{n-1} a_1 \alpha_1 \cdots \alpha_{n-1} \gamma = \frac{a_{n+2}}{\bar{\gamma}}$.

For the parameters s_i, we prove (3.28) by induction on $j = n - i$. For $j = 1$ (that is, $i = n-1$), permuting the columns and applying the linearity of the determinant

$$\begin{aligned}
s_{n-1} &= \frac{\det(A_0|A_1|\dots|A_{n-2}|A_n)}{\det(A_0|A_1|\dots|A_{n-1})} = \frac{a_1^{n-2} \det(\phi^{(0)}|\phi^{(1)}|\dots|\phi^{(n-2)}|a_1\phi^{(n)} + a_2\phi^{(n-1)})}{a_1^{n-1} \det(\phi^{(0)}|\phi^{(1)}|\dots|\phi^{(n-1)})} \\
&= \frac{a_1 \det(\phi^{(0)}|\phi^{(1)}|\dots|\phi^{(n-2)}|\phi^{(n)}) + a_2 \det(\phi^{(0)}|\phi^{(1)}|\dots|\phi^{(n-2)}|\phi^{(n-1)})}{a_1 \det(\phi^{(0)}|\phi^{(1)}|\dots|\phi^{(n-1)})} \\
&= \frac{a_1 \det(V_{n-1}) + a_2 \det(V)}{a_1 \det(V)} = \frac{a_1 \det(V) e_1(\alpha_1, \dots, \alpha_{n-1}, \gamma) + a_2 \det(V)}{a_1 \det(V)} \\
&= \frac{a_1 e_1(\alpha_1, \dots, \alpha_{n-1}, \gamma) + a_2}{a_1} = \frac{-a_1 \bar{\gamma}}{a_1},
\end{aligned}$$

where we applied the fact that $\det(V_{n-1}) = \det(V) e_1(\alpha_1, \dots, \alpha_{n-1}, \gamma)$ and the relation

$$a_2 = -a_1 e_2(\alpha_1, \dots, \alpha_{n-1}, \gamma, \bar{\gamma}) = -a_1 e_1(\alpha_1, \dots, \alpha_{n-1}, \gamma) - a_1 \bar{\gamma}.$$

Suppose now that $s_{n-d} = -\bar{\gamma}^d$, for $d = 2, \dots, j-1$, and compute s_{n-j}. We first prove the following claim.

Claim: For $j \in \{1, \ldots, n-1\}$,

$$s_{n-j} = \frac{a_{j+1}}{a_1} - \sum_{k=1}^{j-1} \frac{a_{k+1}}{a_1} s_{n-j+k} + (-1)^{j-1} e_j(\alpha, \ldots, \gamma).$$

Proof of the claim. By Cramer's rule,

$$s_{n-j} = \frac{\det(A_0|A_1|\ldots|A_{n-j-1}|A_n|A_{n-j+1}|\ldots|A_{n-1})}{\det(A_0|A_1|\ldots|A_{n-1})},$$

and, permuting the columns such that A_n is the last one and applying linearity of determinant in each column, it is equal to

$$\frac{(-1)^{j-1} a_1^{n-j-1}}{a_1^{n-1} \det(\phi^{(0)}|\phi^{(1)}|\ldots|\phi^{(n-1)})} \det(\phi^{(0)}|\ldots|\phi^{(n-j-1)}|A_{n-j+1}|\ldots|A_{n-1}|A_n).$$

If we denote

$$\mathrm{I}^{(n-g)} = a_1\phi^{(n-g)} + a_2\phi^{(n-g-1)} + \cdots + a_{j-g}\phi^{(n-j+1)}$$

and

$$\mathrm{II}^{(n-g)} = a_{j-g+1}\phi^{(n-j)},$$

then

$$\begin{aligned} s_{n-j} = \frac{(-1)^{j-1} a_1^{-j}}{\det(V)} \det\Big(&\phi^{(0)}|\ldots|\phi^{(n-j-1)}|\, \mathrm{I}^{(n-j+1)} + \mathrm{II}^{(n-j+1)}\,| \\ &\ldots|\, \mathrm{I}^{(n-1)} + \mathrm{II}^{(n-1)}\,|\, \mathrm{I}^{(n)} + \mathrm{II}^{(n)}\Big), \end{aligned} \tag{3.29}$$

which, splitting by linearity in the last column, is equal to

$$\frac{(-1)^{j-1} a_1^{-j}}{\det(V)} \cdot \left\{ \begin{array}{l} \det\left(\phi^{(0)}|\ldots|\phi^{(n-j-1)}|\,\mathrm{I}^{(n-j+1)} + \mathrm{II}^{(n-j+1)}\,|\ldots|\,\mathrm{I}^{(n-1)} + \mathrm{II}^{(n-1)}\,|\,\mathrm{II}^{(n)}\right) + \\ \det\left(\phi^{(0)}|\ldots|\phi^{(n-j-1)}|\,\mathrm{I}^{(n-j+1)} + \mathrm{II}^{(n-j+1)}\,|\ldots|\,\mathrm{I}^{(n-1)} + \mathrm{II}^{(n-1)}\,|\,\mathrm{I}^{(n)}\right) \end{array} \right\}.$$

Let us calculate the two terms of the sum separately.
First term. The first addend is easy to compute,

$$\begin{aligned} &\frac{(-1)^{j-1} a_1^{-j}}{\det(V)} \det(\phi^{(0)}|\ldots|\phi^{(n-j-1)}|\,\mathrm{I}^{(n-j+1)} + \mathrm{II}^{(n-j+1)}\,|\ldots|\,\mathrm{I}^{(n-1)} + \mathrm{II}^{(n-1)}\,|\,\mathrm{II}^{(n)}) \\ &= \frac{(-1)^{j-1} a_1^{-j}}{\det(V)} \det(\phi^{(0)}|\ldots|\phi^{(n-j-1)}|\,\mathrm{I}^{(n-j+1)}\,|\ldots|\,\mathrm{I}^{(n-1)}\,|\,\mathrm{II}^{(n)}) \\ &= \frac{(-1)^{j-1} a_1^{-j}}{\det(V)} \det(\phi^{(0)}|\ldots|\phi^{(n-j-1)}|a_1\phi^{(n-j+1)}|\ldots|a_1\phi^{(n-1)}|a_1\phi^{(n-j)}) \\ &= \frac{(-1)^{j-1} a_1^{-j}}{\det(V)} (-1)^{(n-1)-(n-j+1)+1} a_1^{(n-1)-(n-j+1)+1} a_{j+1} \det(V) \\ &= (-1)^{2(j-1)} a_1^{-1} a_{j+1} = \frac{a_{j+1}}{a_1}. \end{aligned}$$

Second term. The computation of the second addend

$$\det\left(\phi^{(0)}|\dots|\phi^{(n-j-1)}|\,\mathrm{I}^{(n-j+1)}+\mathrm{II}^{(n-j+1)}\,|\dots|\,\mathrm{I}^{(n-1)}+\mathrm{II}^{(n-1)}\,|\,\mathrm{I}^{(n)}\right)$$

is more intricate. In each column one can split the determinant by linearity choosing $\mathrm{I}^{(n-j+k)}$ or $\mathrm{II}^{(n-j+k)}$. The turning point in the proof is following: as one can see in the above computation of the first term, if one splits the determinant choosing in the $(n-j+k)$-th column the addend $\mathrm{II}^{(n-j+k)}$, then by this choice the other columns must be $\mathrm{I}^{(g)}$, for $g=n-j+1,\dots,n$ and $g\neq n-j+k$. More precisely, we have the sum of $j-1$ terms of the form

$$\begin{aligned}\frac{(-1)^{j-1}a_1^{-j}}{\det(V)}\det\Big(\phi^{(0)}|\dots|\phi^{(n-j-1)}|\,\mathrm{I}^{(n-j+1)}\,|\dots\\ |\,\mathrm{I}^{(n-j+k-1)}\,|\,\mathrm{II}^{(n-j+k)}\,|\,\mathrm{I}^{(n-j+k+1)}\,|\dots|\,\mathrm{I}^{(n-1)}\,|\,\mathrm{I}^{(n)}\Big)\end{aligned} \tag{3.30}$$

for $k=1,\dots,j-1$. Thus, let us compute the quantity in (3.30) for a fixed $k\in\{1,\dots,j-1\}$,

$$\frac{(-1)^{j-1}a_1{}^{-j}}{\det(V)}\cdot\left\{\begin{aligned}&\det(\phi^{(0)}|\dots|\phi^{(n-j-1)}|a_1\phi^{(n-j+1)}+a_2\phi^{(n-j)}|\dots\\ &\qquad|a_1\phi^{(n-j+k-1)}+\dots+a_{k-1}\phi^{(n-j+1)}+a_k\phi^{(n-j)}\\ &\qquad|a_{k+1}\phi^{(n-j)}|\,a_1\phi^{(n-j+k+1)}+\dots+a_{k+2}\phi^{(n-j+1)}|\dots\\ &\qquad|a_1\phi^{(n)}+\dots+a_j\phi^{(n-j+1)})\end{aligned}\right\}.$$

After some trivial splits and $(n-j+k-1)-(n-j+1)+1=k-1$ permutations, this is equal to

$$\frac{(-1)^{j-1}a_1^{-j}}{\det(V)}a_1^{k-1}(-1)^{k-1}a_{k+1}\cdot\left\{\begin{aligned}&\det(\phi^{(0)}|\dots|\phi^{(n-j-1)}|\phi^{(n-j)}|\phi^{(n-j+1)}|\dots\\ &\qquad|\phi^{(n-j+k-1)}|a_1\phi^{(n-j+k+1)}+a_2\phi^{(n-j+k)}|\dots\\ &\qquad|a_1\phi^{(n)}+\dots+a_{j-k}\phi^{(n-j+k)})\end{aligned}\right\}$$

$$=\frac{(-1)^{k+j-2}a_1^{k-j-1}a_{k+1}}{\det(V)}\cdot\left\{\begin{aligned}&\det(\phi^{(0)}|\dots|\phi^{(n-j-1)}|\phi^{(n-j)}|\phi^{(n-j+1)}|\dots|\phi^{(n-j+k-1)}\\ &\qquad|a_1\phi^{(n-j+k+1)}+a_2\phi^{(n-j+k)}|\dots\\ &\qquad|a_1\phi^{(n)}+\dots+a_{j-k}\phi^{(n-j+k)})\end{aligned}\right\}.$$

Finally, if one compares to (3.29), then this is equal to $-\frac{a_{k+1}}{a_1}s_{n-j+k}$.

On the other hand, we have also one term which comes out when we never choose the column $\mathrm{II}^{(n-j+k)}$ (in other words, there are only columns of the kind $\mathrm{I}^{(n-j+k)}$) given by

$$\det\left(\phi^{(0)}|\dots|\phi^{(n-j-1)}|\,\mathrm{I}^{(n-j+1)}\,|\dots|\,\mathrm{I}^{(n-1)}\,|\,\mathrm{I}^{(n)}\right),$$

which is easy to compute. Indeed, it is equal to

$$\begin{aligned}&\frac{(-1)^{j-1}a_1^{-j}}{\det(V)}\det(\phi^{(0)}|\dots|\phi^{(n-j-1)}|\,\mathrm{I}^{(n-j+1)}\,|\dots|\,\mathrm{I}^{(n-1)}\,|\,\mathrm{I}^{(n)})\\ &\text{applying linearity of determinant in each column}\\ &=\frac{(-1)^{j-1}a_1^{-j}}{\det(V)}a_1^{n-(n-j+1)+1}\det(V_{n-j})\\ &=(-1)^{j-1}e_j(\alpha,\dots,\gamma),\end{aligned}$$

since $\det(V_{n-j}) = e_j(\alpha,\dots,\gamma)\det(V)$.
Thus, the claim has been proved. ■

It is easy to prove by induction that

$$a_{j+1} = (-1)^j a_1 e_j(\alpha,\dots,\gamma) - \sum_{k=1}^{j} a_k\bar{\gamma}^{j+1-k}. \tag{3.31}$$

Indeed, if $a_j = (-1)^{j-1}a_1e_{j-1}(\alpha,\dots,\gamma) - \sum_{k=1}^{j-1} a_k\bar{\gamma}^{j-k}$, then

$$\begin{aligned}a_{j+1} &= (-1)^j a_1 e_j(\alpha,\dots,\gamma,\bar{\gamma})\\ &= (-1)^j a_1 e_{j-1}(\alpha,\dots,\gamma)\bar{\gamma} + (-1)^j a_1 e_j(\alpha,\dots,\gamma)\\ &= -a_j\bar{\gamma} - \bar{\gamma}\sum_{k=1}^{j-1} a_k\bar{\gamma}^{j-k} + (-1)^j a_1 e_j(\alpha,\dots,\gamma)\\ &= (-1)^j a_1 e_j(\alpha,\dots,\gamma) - \sum_{k=1}^{j} a_k\bar{\gamma}^{j+1-k}.\end{aligned}$$

Finally, we compute s_{n-j} using the claim

$$\begin{aligned}s_{n-j} &= \frac{a_{j+1}}{a_1} - \sum_{k=1}^{j-1}\frac{a_{k+1}}{a_1}s_{n-j+k} + (-1)^{j-1}e_j(\alpha,\dots,\gamma)\\ &\text{since } s_{n-j+k} = -\bar{\gamma}^{j-k} \text{ by induction, thus}\\ &= \frac{a_{j+1}}{a_1} + \sum_{k=1}^{j-1}\frac{a_{k+1}}{a_1}\bar{\gamma}^{j-k} + (-1)^{j-1}e_j(\alpha,\dots,\gamma)\\ &\text{and by the relation (3.31)}\\ &= \frac{-a_1\bar{\gamma}^j}{a_1} = -\bar{\gamma}^j.\end{aligned}$$

Now we show that $\mathcal{T}$ is toroidal. Suppose that $l = (l_0, l_1, \dots, l_{n-1}) \in \mathbb{Z}^n$ is such that $l \cdot T = L \in \mathbb{Z}$, that is

$$l_0\tau + l_1 s_1 + \cdots + l_{n-1}s_{n-1} = L.$$

Hence,

$$l_0\frac{a_{n+2}}{\bar{\gamma}} - l_1\bar{\gamma}^{n-1} + \cdots - l_{n-1}\bar{\gamma} = L,$$

and we get

$$l_1\bar{\gamma}^n + \cdots + l_{n-1}\bar{\gamma}^2 + L\bar{\gamma} - l_0 a_{n+2} = 0.$$

This gives $l = (l_0, l_1, \dots, l_{n-1}) = (0, 0, \dots, 0)$, otherwise we would have a contradiction with the fact that $f(x)$ is a_1 times the minimal polynomial of $\bar{\gamma}$ over $\mathbb{Q}$.

Let $\Phi = \{\sigma_1, \dots, \sigma_{r_1}, \sigma_n\}$ be the type corresponding to the set of roots $\{\alpha_1, \dots, \alpha_{n-1}, \gamma\}$, and consider the injective map

$$\begin{aligned} D\colon \mathcal{O}_K &\to \mathrm{End}(\mathbb{C}^n/\mu_\Phi(\mathcal{O}_K)), \\ \nu \mapsto D(\nu) &\equiv \begin{pmatrix} \sigma_1(\nu) & & & \\ & \ddots & & \\ & & \sigma_{r_1}(\nu) & \\ & & & \sigma_n(\nu) \end{pmatrix}. \end{aligned}$$

Since $\mathbb{Z}$ is strictly contained in $\mathcal{O}_K$, the toroidal group $\mathcal{T}$ has extra multiplications. Tensoring the map D with $\mathbb{Q}$ we obtain an isomorphism because K has dimension $n+1$ and the dimension of the non-totally real number field $\mathrm{End}_0(\mathcal{T})$ is a divisor of $n+1$. We have also that, up to isomorphism, $\mathrm{End}(\mathcal{T})$ is an order of K that contains the maximal order $\mathcal{O}_K$ and then D is an isomorphism. □

Remark 3.28. It is worthwhile to remark the corresponding observations of the low dimension case. Firstly, the condition that K has precisely one pair of complex embeddings ($r_2 = 1$) is necessary in order to have a toroidal group of complex dimension n and real rank $n+1$. Secondly, again, the fact that such a complex Lie group is toroidal is already in [4], but in the case where K admits an essential polynomial, it can be proved more directly. Moreover, by Theorem 3.2, every complex Lie group $\mathbb{C}^n/\mu_\Phi(\mathfrak{a})$ is toroidal, being isogenous to one of the kind of Theorem 3.27. Lastly, while by Theorem 1.33 we know that for all toroidal groups $\mathbb{C}^n/\Lambda$ of real rank $n+1$, the ring $\mathrm{End}_0(\mathbb{C}^n/\Lambda)$ is a non-totally real number field, the above Theorem shows that period lattice of a toroidal groups must fulfill very strict conditions to be a group arising from a non-totally number field with one pair of complex embeddings admitting an essential polynomial.

On the other hand, when a number field of degree $n+1$ admits an essential polynomial $f(x) = a_1 x^{n+1} + \cdots + a_{n+1} x + a_{n+2} \in \mathbb{Z}[x]$ we have an integral basis of the form (3.1) and hence a matrix representation of $\mathcal{O}_K$ given by

$$x = x_0\rho_0 + x_1\rho_1 + \cdots + x_n\rho_n \mapsto N_x \in \mathrm{M}_{n+1}(\mathbb{Z}), \tag{3.32}$$

where N_x is the integral matrix defined by the coefficients

$$n_{11} = x_0, \tag{3.33}$$

$$n_{i1} = x_{i-1}, \quad \text{for } n+1 \geq j > 1, \tag{3.34}$$

$$n_{1j} = -a_{n+2} \sum_{k=1}^{j-1} a_k x_{k+n+1-j}, \quad \text{for } n+1 \geq j > 1, \tag{3.35}$$

$$n_{ij} = \sum_{k=1}^{j-1} a_k x_{k+i-j-1}, \quad \text{for } n+1 \geq i > j > 1, \tag{3.36}$$

$$n_{ij} = -\sum_{k=j}^{n+2} a_k x_{k+i-j-1}, \quad \text{for } n+1 \geq j > i > 1, \tag{3.37}$$

$$n_{jj} = x_0 - \sum_{k=j}^{n+1} a_k x_{k-1}, \quad \text{for } n+1 \geq j > 1. \tag{3.38}$$

Theorem 3.29. *Let $\mathcal{T} = \mathbb{C}^n/\Lambda$ be a toroidal group of dimension n and real rank $n+1$ having extra multiplications. Suppose that the non-totally real number field $K = \mathrm{End}_0(\mathcal{T})$ has degree $n+1$ with only a pair of complex embeddings admitting an essential polynomial $f(x) = a_1x^{n+1} + \cdots + a_{n+1}x^n + a_{n+2}$ and that $\mathrm{End}(\mathcal{T}) = \mathcal{O}_K$. Suppose that (3.32) defines the rational representations of $\mathrm{End}(\mathcal{T})$. Then $\mathcal{T}$ is isomorphic to $\mathbb{C}^n/\mu_\Phi(\mathcal{O}_K)$, for a suitable Minkowski map μ_Φ.*

Proof. The proof proceeds as in [27]. With the embedding (3.32) as the rational representation of the ring $\mathrm{End}(\mathcal{T})$, the Hurwitz relation $A_x\Pi = \Pi N_x$, with N_x defined by the relations (3.34)-(3.38), must hold for each $x \in \mathcal{O}_K$, where $A_x \in \mathrm{M}_n(\mathbb{C})$ is the analytic representation of x. In particular, for $x = \rho_1$ we have

$$N_{\rho_1} = \left(\begin{array}{cccccc|c} 0 & 0 & 0 & \dots & 0 & 0 & -a_{n+2}a_1 \\ 1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n & -a_{n+1} \\ 0 & a_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 & \vdots \\ 0 & 0 & \dots & 0 & a_1 & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & a_1 & 0 \end{array}\right) = \left(\begin{array}{c|c} N_{\rho_1}^{(1)} & N_{\rho_1}^{(2)} \\ \hline N_{\rho_1}^{(3)} & N_{\rho_1}^{(4)} \end{array}\right)$$

and $A_{\rho_1} = N_{\rho_1}^{(1)} + TN_{\rho_1}^{(3)}$, which is the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \tau a_1 \\ 1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n + s_1a_1 \\ 0 & a_1 & 0 & \dots & 0 & s_2a_1 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 0 & a_1 & s_{n-1}a_1 \end{pmatrix}.$$

Comparing the last column of Hurwitz relation for ρ_1, we obtain the equation $A_{\rho_1}T = N_{\rho_1}^{(2)}$, that is,

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \tau a_1 \\ 1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n + s_1a_1 \\ 0 & a_1 & 0 & \dots & 0 & s_2a_1 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 0 & a_1 & s_{n-1}a_1 \end{pmatrix} \begin{pmatrix} \tau \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} = \begin{pmatrix} -a_{n+2}a_1 \\ -a_{n+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which implies the n equations

$$\begin{cases} s_{n-1}\tau = -a_{n+2} \\ s_1 = (-1)^{n-1}s_{n-1}^{n-1} \\ \vdots \\ s_{n-3} = s_{n-1}^3 \\ s_{n-2} = -s_{n-1}^2 \\ f(-s_{n-1}) = 0 \end{cases},$$

then $-s_{n-1}$ is a root of the essential polynomial $f(x)$ which cannot be real (otherwise the real rank of Λ would be n). The Hurwitz relation for the others ρ_j, $j = 2, \dots, n$, does not give any further information. This is because $\rho_j = \frac{1}{a_1}\rho_1(\rho_{j-1} + a_j)$, hence $N_{\rho_j} = \frac{1}{a_1}N_{\rho_1}(N_{\rho_{j-1}} + a_j I_n)$.
Let $\alpha_1, \dots, \alpha_{n-1}$ be the real roots of $f(x)$ and $K = \mathbb{Q}(\omega)$ with $f(\omega) = 0$. If we consider the type $\Phi = \{\sigma_1, \dots, \sigma_{n-1}, \sigma_n\}$ defined by

$$\sigma_1 \colon \omega \mapsto \alpha_1, \quad \dots \quad \sigma_{n-1} \colon \omega \mapsto \alpha_{n-1}, \quad \sigma_n \colon \omega \mapsto -\overline{s_{n-1}},$$

we have that Λ coincides with lattice $\mu_\Phi(\mathcal{O}_K)$ described in Theorem 3.27. □

Remark 3.30. The previous Theorem 3.27 and Theorem 3.29 generalize to an arbitrary dimension n the theorems proved in the sections 3.1.1, 3.1.3 and 3.1.4 .

We end this section with the generalization of Theorem 3.10.

Theorem 3.31. *Let $\mathcal{T}$ be the toroidal group $\mathbb{C}^n/\mu(\mathcal{O}_K)$ arising from a non-totally real number field with only one pair of complex embeddings. If $m > 0$ is a fixed integer, then the torsion subgroup*

$$\mathcal{T}[m] = \mu(\tfrac{1}{m}\mathcal{O}_K)/\mu(\mathcal{O}_K)$$

is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^{n+1}$. Furthermore, an element of $\mathfrak{J}_L(\mathcal{C})$, represented as in (2.7) by the pair $(P, k_1, \dots, k_{n-1}) \in \mathcal{C} \times (\mathbb{C}^*)^{n-1}$, with $P = \mathrm{P}(t)$, belongs to the m-torsion subgroup if and only if there exists $\zeta \in \langle 1, \tau\rangle$ such that

$$\begin{cases} t = \frac{1}{m}\zeta, \\ k_1 = \dfrac{\sigma\left(\frac{1}{m}\zeta - t_2\right)}{\sigma\left(\frac{1}{m}\zeta - t_1\right)} \sqrt[m]{\dfrac{\sigma(\zeta - t_1)}{\sigma(\zeta - t_2)} \dfrac{\sigma(t_1)^{m-1}}{\sigma(t_2)^{m-1}}}, \\ \vdots \\ k_{n-1} = \dfrac{\sigma\left(\frac{1}{m}\zeta - t_{n-1}\right)}{\sigma\left(\frac{1}{m}\zeta - t_{n-2}\right)} \sqrt[m]{\dfrac{\sigma(\zeta - t_{n-2})}{\sigma(\zeta - t_{n-1})} \dfrac{\sigma(t_{n-2})^{m-1}}{\sigma(t_{n-1})^{m-1}}} \end{cases}.$$

Proof. Proceeding as in proof of Theorem 3.10, the first assertion follows from the trivial fact that

$$m\big((\zeta, z_1, \dots, z_{n-1}) + \mu(\mathcal{O}_K)\big) = \mu(\mathcal{O}_K) \iff (\zeta, z_1, \dots, z_{n-1}) \in \tfrac{1}{m}\mu(\mathcal{O}_K) = \mu\left(\tfrac{1}{m}\mathcal{O}_K\right),$$

so

$$\mathcal{T}[m] = \Big\{ \tfrac{d_0}{m}(1, 0, \dots, 0) + \dots + \tfrac{d_{n-1}}{m}(0, \dots, 0, 1) + \tfrac{d_n}{m}(\tau, s_1, \dots, s_{n-1}) + \mu(\mathcal{O}_K) : d_j = 0, \dots, m-1 \Big\},$$

manifestly isomorphic to $(\mathbb{Z}/m\mathbb{Z})^{n+1}$.
As the sum in the generalized Jacobian is given by

$$\big(\mathrm{P}(\zeta), G_1(\zeta, z_1), \dots, G_{n-1}(\zeta, z_{n-1})\big) + \big(\mathrm{P}(\zeta'), G_1(\zeta', z_1), \dots, G_{n-1}(\zeta, z_{n-1})\big) = \Big(\mathrm{P}(\zeta + \zeta'), G_1(\zeta_1, z_1) c_{L_1} \mathrm{P}(\zeta, \zeta'), \dots, G_{n-1}(\zeta_2, z_2) c_{L_{n-1}} \mathrm{P}(\zeta, \zeta')\Big),$$

where

$$G_j(\zeta, z_j) = \exp(-2\eta_1 s_j \zeta) \frac{\sigma(t_j)}{\sigma(t_{j+1})} \frac{\sigma(\zeta - t_{j+1})}{\sigma(\zeta - t_j)} \exp(2\pi i z_j),$$

we can reduce the argument to the case $n = 2$, for which the situation is described in Theorem 3.10. □

Furthermore, if K admits an essential polynomial then the last column of the period matrix in standard coordinates is given by Theorem 3.27.

Remark 3.32. As mentioned in [27] and already pointed out for the case $n = 2$ in Remark 3.11, apparently, $z_1, \dots, z_{n-1}$ play no role in the parametrization of the elements of $\mathcal{T}[m]$, but if $(\zeta, z_1, \dots, z_{n-1}) \in \Lambda$, then

$$\left(\frac{1}{m}\zeta, \frac{1}{m}z_1, \dots, \frac{1}{m}z_{n-1}\right) = \left(\frac{d_0 + d_n\tau}{m}, \frac{d_1 + d_n s_1}{m}, \dots, \frac{d_{n-1} + d_n s_{n-1}}{m}\right),$$

so the role of the integer parameters $d_1, \dots, d_{n-1}$ in, respectively, $z_1, \dots, z_{n-1}$ is linked to that of the m-th root.

3.2 Toroidal groups arising from quintic field with two pair of complex embeddings

We study now the case of a toroidal group $\mathcal{T}$ with $\mathrm{rk}_{\mathbb{C}}\,\mathcal{T} = 3$ and $\mathrm{rk}_{\mathbb{R}}\,\mathcal{T} = 5$, which is isomorphic to $\mathbb{C}^3/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}^3$ of real rank five. Such a toroidal group arises naturally from a quintic number field of signature $(r_1, r_2) = (1, 2)$, where r_1 is the number of real embeddings and r_2 is, up to conjugation, the number of complex emdeddings of K. This connection has been studied by the author in [25].

If $r_2 = 1$, then the algebra $\mathrm{End}_0(\mathbb{C}^{r_1+1}/\mu(\mathcal{O}_K))$ is always a non-totally real number field isomorphic to K. On the contrary, if $r_2 > 1$ then $\mathrm{End}_0(\mathbb{C}^{r_1+1}/\mu(\mathcal{O}_K))$ can even fail to be a division algebra, thus in the following Theorems 3.35 and 3.36, which together with Theorem 3.33 illustrate the connection between toroidal groups and quintic number fields, we must assume it in the hypotheses.

Theorem 3.33. *Let K be a quintic field admitting an essential polynomial $f(x) = a_1x^5 + a_2x^4 + \cdots + a_5x + a_6$ with real root α and complex roots $\beta, \bar{\beta}, \gamma, \bar{\gamma}$, and let us fix the type Φ corresponding to the set of roots $\{\alpha, \beta, \gamma\}$. The group $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal and its period matrix in standard coordinates*

$$\Pi = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & \tau_{11} & \tau_{12} \\ 0 & 1 & 0 & \tau_{21} & \tau_{22} \\ 0 & 0 & 1 & s_1 & s_2 \end{array}\right)$$

is given by

$$\tau_{11} = -\frac{a_6}{\bar{\beta}\bar{\gamma}}, \quad \tau_{12} = a_6\frac{\bar{\beta}+\bar{\gamma}}{\bar{\beta}\bar{\gamma}}, \quad \tau_{21} = \bar{\beta}\bar{\gamma} - (\bar{\beta}+\bar{\gamma})^2, \quad \tau_{22} = \bar{\beta}\bar{\gamma}(\bar{\beta}+\bar{\gamma}) \tag{3.39}$$

and

$$s_1 = -(\bar{\beta}+\bar{\gamma}), \quad s_2 = \bar{\beta}\bar{\gamma}. \tag{3.40}$$

Proof. The proof proceeds as in [27] as well as in the proof of Theorems 3.13 and 3.27. Let $K = \mathbb{Q}(\omega)$ be a quintic field admitting an essential polynomal $f(x)$ with real root α and complex roots $\beta, \bar{\beta}, \gamma, \bar{\gamma}$. Let $\Phi = \{\sigma_1, \sigma_2, \sigma_3\}$ be the type defined by

$$\sigma_1 \colon \omega \mapsto \alpha, \quad \sigma_2 \colon \omega \mapsto \beta, \quad \sigma_3 \colon \omega \mapsto \gamma.$$

The lattice $\mu_\Phi(\mathcal{O}_K)$ is generated by the vectors

$$\begin{aligned}
\mu_\Phi(1) &= (1,1,1),\\
\mu_\Phi(\rho_1) &= (a_1\alpha, a_1\beta, a_1\gamma),\\
\mu_\Phi(\rho_2) &= (a_1\alpha^2 + a_2\alpha, a_1\beta^2 + a_2\beta, a_1\gamma^2 + a_2\gamma),\\
\mu_\Phi(\rho_3) &= (a_1\alpha^3 + a_2\alpha^2 + a_3\alpha,\ a_1\beta^3 + a_2\beta^2 + a_3\beta,\ a_1\gamma^3 + a_2\gamma^2 + a_3\gamma),\\
\mu_\Phi(\rho_4) &= (a_1\alpha^4 + a_2\alpha^3 + a_3\alpha^2 + a_3\alpha, a_1\beta^4 + a_2\beta^3 + a_3\beta^2 + a_4\beta, a_1\gamma^4 + a_2\gamma^3 + a_3\gamma^2 + a_4\gamma),
\end{aligned}$$

therefore its period matrix is given by

$$\begin{pmatrix} 1 & a_1\alpha & a_1\alpha^2 + a_2\alpha & a_1\alpha^3 + a_2\alpha^2 + a_3\alpha & a_1\alpha^4 + a_2\alpha^3 + a_3\alpha^2 + a_3\alpha \\ 1 & a_1\beta & a_1\beta^2 + a_2\beta & a_1\beta^3 + a_2\beta^2 + a_3\beta & a_1\beta^4 + a_2\beta^3 + a_3\beta^2 + a_4\beta \\ 1 & a_1\gamma & a_1\gamma^2 + a_2\gamma & a_1\gamma^3 + a_2\gamma^2 + a_3\gamma & a_1\gamma^4 + a_2\gamma^3 + a_3\gamma^2 + a_4\gamma \end{pmatrix}, \tag{3.41}$$

which, changing basis, gives in turn the period matrix in standard coordinates $\Pi = (I_3|T_1|T_2)$ with

$$T_1 = \begin{pmatrix} a_1\alpha\beta\gamma \\ \frac{-a_2{}^2+a_1a_3-a_1a_2(\alpha+\beta+\gamma)-a_1{}^2(\alpha\beta+\alpha\gamma+\beta\gamma)}{a_1{}^2} \\ \frac{b+a_1(\alpha+\beta+\gamma)}{a_1} \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} a_1\alpha\beta\gamma\big(a_2 + (\alpha+\beta+\gamma)a_1\big) \\ \frac{-a_2a_3+a_1a_4-a_1a_2(\alpha+\beta+\gamma)^2-a_2{}^2(\alpha+\beta+\gamma)-a_1{}^2(\alpha^2\beta+\alpha\beta^2+\alpha^2\gamma+\alpha\gamma^2+\beta^2\gamma+\beta\gamma^2+2\alpha\beta\gamma)}{a_1{}^2} \\ \frac{a_3+a_2(\alpha+\beta+\gamma)+a_1(\alpha^2+\beta^2+\gamma^2)+a_1(\alpha\beta+\alpha\gamma+\beta\gamma)}{a_1} \end{pmatrix}.$$

Using the relations occuring among the coefficients of a polynomial and its roots,

$$a_i = (-1)^{i-1}a_1e_{i-1}(\alpha,\beta,\bar{\beta},\gamma,\bar{\gamma}), \quad i = 2,\dots,6,$$

where $e_j(X_1,\dots,X_d)$ is the j-th elementary symmetric polynomial in the d indeterminates $X_1,\dots,X_d$, we obtain the relations (3.39) and (3.40).

We now prove that $\mathcal{T}$ is toroidal by using the irrationality condition in standard coordinates. Suppose that $l = (l_1,l_2,l_3) \in \mathbb{Z}^3$ is such that $l \cdot T_1 = L_1 \in \mathbb{Z}$ and $l \cdot T_2 = L_2\mathbb{Z}$. Therefore, we have

$$\begin{cases} l_1\tau_{11} + l_2\tau_{21} + l_3s_1 = L_1, \\ l_1\tau_{12} + l_2\tau_{22} + l_3s_2 = L_2, \end{cases}$$

that, written in s_1 and s_2, are the relations

$$\begin{cases} G(s_1,s_2) = l_2s_2(s_2 - s_1^2) + l_3s_1s_2 - L_1s_2 - a_6l_1 = 0, \\ H(s_1,s_2) = l_2s_1s_2^2 - l_3s_2 + L_2s_2 + a_6l_1s_1 = 0. \end{cases}$$

We have that $\bar{\beta}$ is a root of the polynomial $p(x) = l_2x^4 + l_3x^3 + L_1x^2 + L_2x - a_6l_1 \in \mathbb{Z}[X]$. Indeed, using recursively the relation $\bar{\beta}^2 = -(s_1\bar{\beta} + s_2)$,

$$\begin{aligned} p(\bar{\beta}) =& l_2\bar{\beta}^4 + l_3\bar{\beta}^3 + L_1\bar{\beta}^2 + L_2\bar{\beta} - a_6l_1 \\ =& l_2(s_1\bar{\beta} + s_2)^2 - l_3\bar{\beta}(s_1\bar{\beta} + s_2) - L_1(s_1\bar{\beta} + s_2) + L_2\bar{\beta} - a_6l_1 \\ =& \bar{\beta}\left(\frac{s_1}{s_2}G(s_1, s_2) - \frac{1}{s_2}H(s_1, s_2)\right) + G(s_1, s_2) = 0 \end{aligned}$$

This gives $l = (l_1, l_2, l_3) = (0, 0, 0)$, otherwise we have a contradiction with the fact that $f(x)$ is a_1 times the minimal polynomial of $\bar{\beta}$ over $\mathbb{Q}$. □

Remark 3.34. As mentioned above, $\mathrm{End}_0(\mathbb{C}^{r_1+r_2}/\mu_\Phi(\mathcal{O}_K))$ is not necessarily a division algebra but it results a $\mathbb{Q}$-algebra, whose dimension is between $r_1 + 2r_2$ and $(r_1 + 2r_2)^2$. Moreover, it is K-vector space, then by the fact that

$$\dim_{\mathbb{Q}} \mathrm{End}_0(\mathbb{C}^{r_1+r_2}/\mu_\Phi(\mathcal{O}_K)) = \dim_{\mathbb{Q}} K \cdot \dim_K \mathrm{End}_0(\mathbb{C}^{r_1+r_2}/\mu_\Phi(\mathcal{O}_K))$$

its dimension must be a multiple of $r_1 + 2r_2$.

Theorem 3.35. *Let K be a quintic number field as in Theorem 3.33 and $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ the corresponding toroidal group. Then $\mathcal{T}$ has extra multiplications and, if $\mathrm{End}_0(\mathcal{T})$ results a division algebra, we have the following isomorphisms $K \cong \mathrm{End}_0(\mathcal{T})$ and $\mathcal{O}_K \cong \mathrm{End}(\mathcal{T})$.*

Proof. The toroidal group $\mathcal{T}$ has dimension 3 and real rank 5. Let us consider the injective map

$$\begin{aligned} D\colon \mathcal{O}_K &\to \mathrm{End}(\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)) \\ \nu &\mapsto D(\nu), \end{aligned}$$

where $D(\nu)$ is the endomorphism given by the diagonal matrix

$$\begin{pmatrix} \sigma_1(\nu) & & \\ & \sigma_2(\nu) & \\ & & \sigma_3(\nu) \end{pmatrix}.$$

Since $\mathbb{Z}$ is strictly contained in $\mathcal{O}_K$, the toroidal group $\mathcal{T}$ has extra multiplications. Now, as pointed out in Remark 3.34, the ring $\mathrm{End}_0(\mathcal{T})$ is $\mathbb{Q}$-algebra of dimension $5, 10, 15, 20$ or 25. If $\mathrm{End}_0(\mathcal{T})$ is a division algebra then, by Lemma 1.34, its dimension must divide $\mathrm{rank}_{\mathbb{R}}(\mu_\Phi(\mathcal{O}_K)) = 5$, hence it is a quintic number field. Tensoring the map D with $\mathbb{Q}$ we obtain an isomorphism because K has the same dimension of $\mathrm{End}_0(\mathcal{T})$. We have also that, up to isomorphism, $\mathrm{End}(\mathcal{T})$ is an order of K that contains the maximal order $\mathcal{O}_K$ and then D is an isomorphism. □

Theorem 3.36. *Let $\mathcal{T}$ be a toroidal group of dimension 3 and real rank 5 having extra multiplications and period matrix given by*

$$\Pi = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & \tau_{11} & \tau_{12} \\ 0 & 1 & 0 & \tau_{21} & \tau_{22} \\ 0 & 0 & 1 & s_1 & s_2 \end{array}\right).$$

If $K = \mathrm{End}_0(\mathcal{T})$ is a division algebra, then K is a quintic number field. Suppose K admits an essential polynomial $f(x) = a_1x^5 + \cdots + a_5x + a_6$ and that $\mathrm{End}(\mathcal{T}) = \mathcal{O}_K$. Suppose that (3.32) defines the rational representation of $\mathrm{End}(\mathcal{T})$. Then $\mathcal{T}$ is isomorphic to $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$, *for a suitable Minkowski map. Futhermore the analytic representation in standard coordinates is given by*

$$\rho_0 = 1 \mapsto I_3,$$

$$\rho_1 \mapsto A_1 = \begin{pmatrix} 0 & 0 & a_1\tau_{11} \\ 1 & -a_2 & -a_3 + a_1\tau_{21} \\ 0 & a_1 & a_1 s_1 \end{pmatrix}, \tag{3.42}$$

$$\rho_2 \mapsto A_2 = \begin{pmatrix} 0 & a_1\tau_{11} & a_2\tau_{11} + a_1\tau_{12} \\ 0 & -a_3 + a_1\tau_{21} & -a_4 + a_2\tau_{21} + a_1\tau_{22} \\ 1 & a_1 s_1 & -a_3 + a_2 s_1 + a_1 s_2 \end{pmatrix}, \tag{3.43}$$

$$\rho_3 \mapsto A_3 = \begin{pmatrix} \tau_{11} & a_1\tau_{12} & -a_1 a_6 + a_2\tau_{12} \\ \tau_{21} & -a_4 + a_1\tau_{22} & a_2\tau_{22} - a_5 \\ s_1 & a_1 s_2 & -a_4 + a_2 s_2 \end{pmatrix}, \quad \textit{and} \tag{3.44}$$

$$\rho_4 \mapsto A_4 = \begin{pmatrix} \tau_{12} & -a_1 a_6 & -a_2 a_6 \\ \tau_{22} & -a_5 & -a_6 \\ s_2 & 0 & -a_5 \end{pmatrix}. \tag{3.45}$$

Proof. The Hurwitz relations $A_j\Pi = \Pi N_j$ must hold, where $N_j \in \mathrm{M}_5(\mathbb{Z})$ is the matrix defined by (3.34)-(3.38) and $A_j \in \mathrm{M}_3(\mathbb{C})$ is the analytic representation of ρ_j, for $j = 0, \dots, 4$. In particular, the matrices (3.32) for a quintic field admitting essential polynomial $f(x) = a_1x^5 + \cdots + a_5x + a_6$ are $N_0 = I_5$,

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -a_6a_1 \\ 1 & -a_2 & -a_3 & -a_4 & -a_5 \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 \end{pmatrix}, \; N_2 = \begin{pmatrix} 0 & 0 & 0 & -a_6a_1 & -a_6a_2 \\ 0 & -a_3 & -a_4 & -a_5 & -a_6 \\ 1 & 0 & -a_3 & -a_4 & -a_5 \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 \end{pmatrix},$$

$$N_3 = \begin{pmatrix} 0 & 0 & -a_6a_1 & -a_6a_2 & -a_6a_3 \\ 0 & -a_4 & -a_5 & -a_6 & 0 \\ 0 & 0 & -a_4 & -a_5 & -a_6 \\ 1 & 0 & 0 & -a_4 & -a_5 \\ 0 & a_1 & a_2 & a_3 & 0 \end{pmatrix}, \; N_4 = \begin{pmatrix} 0 & -a_6a_1 & -a_6a_2 & -a_6a_3 & -a_6a_4 \\ 0 & -a_5 & -a_6 & 0 & 0 \\ 0 & 0 & -a_5 & -a_6 & 0 \\ 0 & 0 & 0 & -a_5 & -a_6 \\ 1 & 0 & 0 & 0 & -a_5 \end{pmatrix},$$

Direct computations give in turn the matrices A_j in (3.42)-(3.45). Comparing the last columns of Hurwitz relations, we obtain that

$$\tau_{11} = -\tfrac{a_6}{s_2}, \quad \tau_{12} = -a_6\tfrac{s_1}{s_2},$$
$$\tau_{21} = s_2 - s_1^2, \quad \tau_{22} = -s_1 s_2$$

and that s_1 and s_2 (which must be either non-zero) satisfy the following relations

$$G(s_1, s_2) = a_1s_1s_2^2 + a_1s_1s_2(s_2 - s_1^2) - a_2s_2(s_2 - s_1^2) - a_3s_1s_2 + a_4s_2 - a_6 = 0, \tag{3.46}$$

$$H(s_1, s_2) = a_1s_2^2(s_2 - s_1^2) + a_2s_1s_2^2 - a_3s_2^2 + a_5s_2 - a_6s_1 = 0. \tag{3.47}$$

At least one of them must be a complex not real number (otherwise the real rank of Λ would be less than five). Consider the roots u and v of the polynomial $x^2 + s_1x + s_2$. Hence, we have that $s_1 = -(u + v)$ and $s_2 = uv$. Since at least one of s_1 and s_2 is not real, at least one of u and v is not real and they are not conjugate each other.

Claim: u and v are roots of the essential polynomial $f(x)$.
We prove the claim for u. Let us compute $f(u) = a_1u^5 + a_2u^4 + a_3u^3 + a_4u^2 + a_5u + a_6$

by using the fact that $u^2 = -(us_1 + s_2)$ and the relations (3.46), (3.47),

$$\begin{aligned} f(u) =& a_1(us_1^4 + s_1^3 s_2 - us_1^2 s_2 + us_2^2 - 2us_1^2 s_2 - 2s_1 s_2^2) \\ &+ a_2(-us_1^3 - s_1^2 s_2 + s_2^2 + 2us_1 s_2) \\ &+ a_3(us_1^2 + s_1 s_2 - us_2) \\ &+ a_4(-us_1 - s_2) + a_5 u + a_6 \\ =& u\left(-\frac{s_1}{s_2} G(s_1, s_2) + \frac{1}{s_2} H(s_1, s_2)\right) - G(s_1, s_2) = 0 \end{aligned}$$

The proof that $f(v) = 0$ is the same and the claim is proved. ■

Moreover, at least one of u and v is not real, suppose $u \in \mathbb{C} \setminus \mathbb{R}$. Nevertheless, we prove that also v must be not real. Indeed, let us suppose that $v \in \mathbb{R}$. We have that $s_1 = -(\frac{s_2}{v} + v)$ and $\tau_{21} = s_2 - s_1^2 = -s_2 - \frac{s_2^2}{v^2} - v^2$, hence

$$\begin{aligned} \tau_{12} &= -a_6 \frac{s_1}{s_2} = \frac{a_6}{s_2} v + \frac{a_6}{v} = -v\tau_{11} + \frac{a_6}{v}, \\ \tau_{22} &= -s_1 s_2 = \frac{s_2^2}{v} + s_2 v = -v\tau_{21} - v^3, \\ s_2 &= -v s_1 - v^2, \end{aligned}$$

that is

$$\begin{pmatrix} \tau_{12} \\ \tau_{22} \\ s_2 \end{pmatrix} = \frac{a_6}{v} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - v^3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - v^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - v \begin{pmatrix} \tau_{11} \\ \tau_{21} \\ s_1 \end{pmatrix}$$

and this is a contradiction to the fact that $\operatorname{rk}_{\mathbb{R}} \Lambda = 5$.

Therefore, u and v are complex and not real roots of the essential polynomial $f(x)$ and they are not conjugate each other. If we consider the type $\Phi = \{\sigma_1, \sigma_2, \sigma_3\}$ defined by

$$\sigma_1 \colon \omega \mapsto \alpha, \quad \sigma_2 \colon \omega \mapsto \bar{u}, \quad \sigma_3 \colon \omega \mapsto \bar{v}.$$

we have that Λ concides with the lattice $\mu_\Phi(\mathcal{O}_K)$ described in Theorem 3.33. □

Remark 3.37. We already pointed out in this section that in the present case of a toroidal group $\mathcal{T}$ of complex rank and real rank differ by two, $\operatorname{rk}_{\mathbb{R}} \mathcal{T} = \operatorname{rk}_{\mathbb{C}} \mathcal{T} + 2$, the ring $\operatorname{End}_0(\mathcal{T})$ is not necessarily a division algebra. However, for example, Abe provided in [4], and generalized in [1], a sufficient condition for a toroidal group to be such that the ring of endomorphisms determines a division algebra. More precisely, Abe proved that for any toroidal group $\mathcal{T}$ which does not contains toroidal subgroups, the ring $\operatorname{End}_0(\mathcal{T})$ is a division algebra. This is perfectly consistent with the fact that, for a *simple* Abelian variety $\mathbb{T}$ (that is, having no Abelian subvarieties), $\operatorname{End}_0(\mathbb{T})$ is a division algebra whose center is either a totally real number field or an imaginary quadratic extension of such a field (see, e.g., [16, Chapter 5.5]).

3.2.1 Quasi-abelian varieties

In the case where the difference between the real and complex rank is one, a toroidal group is a quasi-Abelian variety. Indeed, such a toroidal group is fibred over infinitely many elliptic curves (cf. Chapter 2, see [8], [7]). In particular, every toroidal group $\mathbb{C}^2/\Lambda$ of real rank three is a quasi-Abelian variety as well as it arises from a non-totally real cubic number field (cf. Section 3.1.1 and Theorem 1.36).

On the contrary, a toroidal group arising from a non-totally real quintic number field with two pair of complex embeddings is not necessarily a quasi-Abelian variety. In this section, we give an example of such a toroidal group that is also a quasi-Abelian variety and an example where this is not the case. We also furnish, by a simple computation, a condition on the last column of a period matrix in standard coordinates.

We first recall a characterization of the period matrices of quasi-Abelian varieties provided by Umeno, that generalizes the Riemann relations for Abelian varieties.

Theorem 3.38. *Let $\mathcal{T} = \mathbb{C}^n/\Lambda$ be a toroidal group with $\operatorname{rk}_{\mathbb{R}} \Lambda = n + q$ and $\Pi = (I_n|T)$ a period matrix in standard coordinates.*

- *If $\mathcal{T}$ is a quasi-Abelian variety with ample Riemann form H then $E := \operatorname{Im}(H) \restriction_{\Lambda\times\Lambda}$ satisfies the relations:*

$$T^{\mathrm{T}}E_1T + E_2T - T^{\mathrm{T}}E_2 + E_3 = 0 \quad \textit{and} \tag{3.48}$$

$$\frac{i}{2}\left(\overline{T}^{\mathrm{T}}E_1T + E_2T - \overline{T}^{\mathrm{T}}E_2 + E_3\right) > 0, \tag{3.49}$$

  where

$$E = \begin{pmatrix} E_1 & E_2 \\ -E_2^{\mathrm{T}} & E_3 \end{pmatrix}$$

  with $E_1 \in \mathrm{M}_n(\mathbb{Z})$, $E_2 \in \mathrm{M}_{n\times q}(\mathbb{Z})$ and $E_3 \in \mathrm{M}_q(\mathbb{Z})$.

- *Conversly, if E is a $\mathbb{Z}$-valued skew-symmetric $(n+q)\times(n+q)$ matrix satisfying (3.48) and (3.49), then $\mathcal{T}$ is a quasi-Abelian variety with an ample Riemann form H such that $\operatorname{Im}(H) \restriction_{\Lambda\times\Lambda} = E$.*

Proof. See [68, Theorem 3.1]. □

For the next examples, let $\mathcal{T} = \mathbb{C}^3/\Lambda$ be a toroidal group with $\operatorname{rk}_{\mathbb{R}} \Lambda = 5$ and

$$E = \begin{pmatrix} E_1 & E_2 \\ -E_2^{\mathrm{T}} & E_3 \end{pmatrix} \in \mathrm{M}_5(\mathbb{Z}) \tag{3.50}$$

a skew-symmetric matrix, where

$$E_1 = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} d & e \\ f & g \\ h & k \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}. \tag{3.51}$$

Example 3.39. This example is due to Umeno [67]. Let $K = \mathbb{Q}(\omega)$ be the field defined by $\omega^5 = 2$. The pure field K is a monogenic quintic number field, that is, $\{1, \omega, \omega^2, \omega^3, \omega^4\}$ is a basis of the ring of integers. Let $\alpha = \sqrt[5]{2}$ and $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ be the toroidal group corresponding to the type

$$\Phi = \{\sigma_1\colon \omega \mapsto \alpha,\ \sigma_2\colon \omega \mapsto \alpha\varepsilon,\ \sigma_3\colon \omega \mapsto \alpha\varepsilon^2\},$$

where $\varepsilon = \exp\left(\frac{2\pi i}{5}\right)$ primitive 5-th root of unity. A period matrix of $\mu_\Phi(\mathcal{O}_K)$ in standard coordinates is given by

$$(I_3|T) = \begin{pmatrix} 1 & 0 & 0 & \alpha^3\varepsilon^3 & \alpha^4\left(\varepsilon^4+\varepsilon^3+1\right) \\ 0 & 1 & 0 & \alpha^2\left(\varepsilon^4+1\right) & \alpha^3\left(1+\varepsilon\right) \\ 0 & 0 & 1 & \alpha\left(\varepsilon^2+\varepsilon+1\right) & \alpha^2\varepsilon^2 \end{pmatrix},$$

Let E be given by (3.50) and (3.51), the relation (3.48) implies that

$$\left(-c\alpha^4 - e\alpha^3 + g\alpha^2 + 2b\right)\varepsilon^3 + \left((-d-c)\alpha^4 + (h+g)\alpha^2 - k\alpha + 2b\right)\varepsilon^2 + \left((-d-c)\alpha^4 + f\alpha^3 + (g+2a)\alpha^2 - k\alpha\right)\varepsilon + \left(-c\alpha^4 + f\alpha^3 - k\alpha + m\right) = 0.$$

Since the cyclotomic polynomial $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$ is irreducible on K, the field $K(\varepsilon)$ is a extension of K of degree 4 and $\{1, \varepsilon, \varepsilon^2, \varepsilon^3\}$ is a basis of such extension, therefore the powers $1, \varepsilon, \varepsilon^2$ and ε^3 are linearly independent over K and we have that

$$\begin{aligned} -c\alpha^4 - e\alpha^3 + g\alpha^2 + 2b &= 0, \\ (-d-c)\alpha^4 + (h+g)\alpha^2 - k\alpha + 2b &= 0, \\ (-d-c)\alpha^4 + f\alpha^3 + (g+2a) &= 0, \\ -c\alpha^4 + f\alpha^3 - k\alpha + m &= 0. \end{aligned}$$

Now, the powers $1, \alpha, \alpha^2, \alpha^3$ and α^4 are linearly independent over $\mathbb{Q}$, then we obtain that $a = b = c = d = e = f = g = h = k = m = 0$. Hence, $\mathcal{T}$ is not a quasi-Abelian variety.

Example 3.40. This example is due to Kazama and Umeno in [41]. They proved that the toroidal group of period matrix

$$\begin{pmatrix} 1 & 0 & 0 & i\sqrt{2} & i\sqrt{7} \\ 0 & 1 & 0 & i\sqrt{3} & i\sqrt{11} \\ 0 & 0 & 1 & i\sqrt{5} & i\sqrt{13} \end{pmatrix}$$

has no non-constant meromorphic functions, showing that its Néron–Severi group is zero. Thus, this group cannot be a quasi-Abelian variety.

Example 3.41. This example appeared in [6, p. 103] erroneously as a toroidal group which admits no non-constant meromorphic functions (that is, a toroidal group which is not a quasi-Abelian variety by the result of Capocasa and Catanese [20]). On the contrary, Kim proved in [42] that there may exist non-constant meromorphic functions by using the above mentioned result of Umeno ([68]).

Let $\mathcal{T}$ be the toroidal group with period matrix in standard coordinates given by

$$\begin{pmatrix} 1 & 0 & 0 & i & i\sqrt{2} \\ 0 & 1 & 0 & i\sqrt{3} & i\sqrt{5} \\ 0 & 0 & 1 & i\sqrt{7} & i \end{pmatrix}.$$

We show that $\mathcal{T}$ is a quasi-Abelian variety by using Theorem 3.38. Let E be given by (3.50) and (3.51), the relation (3.48) and (3.49) implies that

$$\begin{aligned} C_1 = m - (\sqrt{3} - \sqrt{35})c - (1 - \sqrt{14})b - (\sqrt{5} - \sqrt{6})a \\ + i\left(h - \sqrt{3}g + \sqrt{5}f - e + \sqrt{2}d - \sqrt{7}k\right) = 0 \end{aligned}$$

and

$$M = \begin{pmatrix} -d - \sqrt{3}f - \sqrt{7}h & \frac{C_2}{2} \\ -\frac{C_2}{2} & -k - \sqrt{2}e - \sqrt{5}g \end{pmatrix} > 0$$

with

$$C_2 = -h - \sqrt{3}g + \sqrt{5}f - e + \sqrt{2}d - \sqrt{7}k \\ +i\left(m + (\sqrt{3} - \sqrt{35})c + (1 - \sqrt{14})b + (\sqrt{5} - \sqrt{6})a\right).$$

Choosing $m = a = b = c = 0 = d = f = g = k$ and $h = e < 0$, we have that $C_1 = 0$ and

$$M = \begin{pmatrix} -\sqrt{7}e & -e \\ e & -\sqrt{2}e \end{pmatrix}$$

is positive definite. By Theorem 3.38, we conclude that $\mathcal{T}$ is a quasi-Abelian variety (of kind 2). Furthermore, it is easy to prove that $\mathrm{End}(\mathcal{T}) = \mathbb{Z}$.

Fields with essential polynomial. Now, we focus our attention to toroidal groups arising from quintic fields admitting an essential polynomial, as in Section 3.2, and we provide an example of such a group that can not be a quasi-Abelian variety.

Let $\mathcal{T}$ be a toroidal group arising from a non-totally real quintic number field with two pairs of complex embeddings admitting an essential polynomial $f(x) = a_1x^5 + \cdots + a_5x + a_6$, which has been described in Theorem 3.33. Let E be the skew-symmetric matrix given by (3.50) and (3.51), and

$$T = \begin{pmatrix} -\frac{a_6}{s_2} & -a_6\frac{s_1}{s_2} \\ s_2 - s_1^2 & -s_1s_2 \\ s_1 & s_2 \end{pmatrix}$$

the last two columns of the period matrix in standard coordinates. The relation (3.48) implies that

$$cs_2^3 + (-fs_1 + h - g)\,s_2^2 + \left(gs_1^2 + (2aa_6 - k)\,s_1 + m - a_6b\right)s_2 \\ -a_6(as_1^3 - bs_1^2 + ds_1 - e) = 0. \tag{3.52}$$

We recall that for the periods of such a toroidal group the relations (3.46) and (3.47) hold, from which we get the further one

$$a_1s_2^2 + (-3a_1s_1^2 + 2a_2 - a_3)s_2 + (a_1s_1^4 - a_2s_1^3 + a_3s_1^2 - a_4s_1 + a_5) = 0. \tag{3.53}$$

This implies that

$$s_2^2 = -\frac{1}{a_1}(-3a_1s_1^2 + 2a_2s_1 - a_3)s_2 - \frac{1}{a_1}(a_1s_1^4 - a_2s_1^3 + a_3s_1^2 - a_4s_1 + a_5). \tag{3.54}$$

We point out that s_2 is algebraic on the field $\mathbb{Q}(s_1)$ of degree two, so (3.54) is the unique way to write s_2^2 as linear combination of 1 and s_2 in the field $\mathbb{Q}(s_1)$.

On the other hand, by (3.52) and (3.46) we find the relation

$$
\begin{aligned}
&s_2^2\left(4aa_1s_1^2+(-2ba_1-2aa_2)s_1+(ba_2+c)\right)\\
&+s_2\Big(-4aa_1s_1^4+(3ba_1+3aa_2)s_1^3+(-2ba_2-2aa_3-2da_1)s_1^2\\
&\qquad+(-f+2aa_4ba_3+2ea_1+da_2)s_1+(h-g-ba_4-ea_2)\Big)\\
&+\Big(-aa_1s_1^6+(ba_1+aa_2)s_1^5+(-da_1-ba_2-aa_3)s_1^4\\
&\qquad+(-ea_1+da_2+ba_3+aa_4)s_1^3+(g-ea_2-da_3-ba_4)s_1^2\\
&\qquad+(-k+ea_3+da_4)s_1+(m-ea_4)\Big)=0
\end{aligned}
\tag{3.55}
$$

Thus, comparing (3.54) and (3.55), we find that

$$
\begin{aligned}
&a_1\Big(-4aa_1s_1^4+(3ba_1+3aa_2)s_1^3+(-2ba_2-2aa_3-2da_1)s_1^2\\
&\quad+(-f+2aa_4ba_3+2ea_1+da_2)s_1+(h-g-ba_4-ea_2)\Big)\\
&\quad=(-3a_1s_1^2+2a_2s_1-a_3)\left(4aa_1s_1^2+(-2ba_1-2aa_2)s_1+(ba_2+c)\right)
\end{aligned}
\tag{3.56}
$$

and

$$
\begin{aligned}
&a_1\Big(-aa_1s_1^6+(ba_1+aa_2)s_1^5+(-da_1-ba_2-aa_3)s_1^4+(-ea_1+da_2+ba_3+aa_4)s_1^3\\
&\quad+(g-ea_2-da_3-ba_4)s_1^2+(-k+ea_3+da_4)s_1+(m-ea_4)\Big)\\
&\quad=(a_1s_1^4-a_2s_1^3+a_3s_1^2-a_4s_1+a_5)\left(4aa_1s_1^2+(-2ba_1-2aa_2)s_1+(ba_2+c)\right)
\end{aligned}
\tag{3.57}
$$

Let us suppose that the powers $1, s_1, s_1^2, s_1^3, s_1^4$ are independent over $\mathbb{Q}$. Equation (3.56) implies that $-4aa_1^2=-12aa_1^2$, from which we conclude that $a=0$. Moreover, $3ba_1^2=6ba_1^2$ implies that also $b=0$. The others relations that we get from (3.56) are

$$2da_1=3c, \tag{3.58}$$

$$a_1(-f+2ea_1+da_2)=2ca_2,\ \text{and} \tag{3.59}$$

$$h-g-ea_2=-a_3c. \tag{3.60}$$

Equation (3.57) for $a=0=b$ is now

$$
\begin{aligned}
&a_1\Big(-da_1s_1^4+(-ea_1+da_2)s_1^3+(g-ea_2-da_3)s_1^2\\
&\qquad+(-k+ea_3+da_4)s_1+(m-ea_4)\Big)\\
&\qquad=(a_1s_1^4-a_2s_1^3+a_3s_1^2-a_4s_1+a_5)c.
\end{aligned}
\tag{3.61}
$$

We therefore find the further relations $-da_1=c$, which together $2da_1=3c$ implies $c=0=d$. By (3.61), we obtain

$$
\begin{aligned}
a_1(ea_1+da_2)=0 &\quad \text{then} \quad e=0,\\
a_1(g-ea_2-da_3)=0 &\quad \text{then} \quad g=0,\\
a_1(-k+ea_3+da_4)=0 &\quad \text{then} \quad k=0,\\
a_1(m-ea_4)=0 &\quad \text{then} \quad m=0.
\end{aligned}
$$

Lastly, by (3.59) and (3.60), one has that $f=0=h$.

One must have then that $a = b = c = d = e = f = g = h = k = m = 0$. Hence, in this case $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ cannot be a quasi-Abelian variety. This is indeed the case of Example 3.39, where $s_1 = -\alpha(\varepsilon^3 + \varepsilon^4)$ has minimal polynomial $x^{10} - 22x + 4$ on $\mathbb{Q}$. More in general, this is true for any pure monogenic quintic field. Let K be defined by $\omega^5 = a$ and monogenic, i.e. $\mathcal{O}_K = \mathbb{Z}[\omega]$.

If $a > 0$, then one can proceed exactly as in Example 3.39 for $\alpha = \sqrt[5]{a}$. The roots of $x^5 - a$ are $\varepsilon^j\sqrt[5]{a}$, for $j = 0, 1, 2, 3, 4$ and $\varepsilon = \exp(\frac{2\pi i}{5})$, and let us fix the type $\Phi = \{\omega \mapsto \sqrt[5]{a},\ \omega \mapsto \varepsilon\sqrt[5]{a},\ \omega \mapsto \varepsilon^2\sqrt[5]{a}\}$. Hence, the minimal polynomial on $\mathbb{Q}$ of $s_1 = -\sqrt[5]{a}(\varepsilon^3 + \varepsilon^4)$ is $x^{10} - 11ax^5 - a^2$.

If $a < 0$, the roots of $x^5 - a$ are $-\varepsilon^j\sqrt[5]{a}$ and let us fix the type $\Phi = \{\omega \mapsto \sqrt[5]{a},\ \omega \mapsto -\varepsilon\sqrt[5]{a},\ \omega \mapsto -\varepsilon^2\sqrt[5]{a}\}$. The minimal polynomial on $\mathbb{Q}$ of $s_1 = \sqrt[5]{a}(\varepsilon^3 + \varepsilon^4)$ is $x^{10} + 11ax^5 - a^2$.

3.2.2 Future directions

We conclude this chapter by briefly presenting some potential future research directions regarding the connection between toroidal groups and number fields. It is evident from the discussion in this thesis that there exist privileged cases in which this connection is either immediate or at least more direct with number fields, such as the condition of simplicity and the primality of the real rank (cf. Remark 3.37). One possible further investigation could be to deepen the study in the simple case, or more generally in the geometrically simple case, for a toroidal group whose real and complex ranks do not differ by one, in order to determine whether exhaustive and complete results can be obtained, possibly in combination with some technical assumptions. In such cases, the study of quasi-Abelianity can be conducted by using Theorem 3.38. In the case where complex and real ranks differ by one (cf. Section 3.1), one could attempt to further generalize the study presented in this thesis, which already provides fairly general results, by relaxing certain technical assumptions that have facilitated working with the ring of integers.

Appendix A

Period matrices of toroidal groups arising from pure quartic fields

In this appendix, we write down the period matrices of toroidal groups arising from pure quartic fields with one pair of complex embeddings. More precisely, we furnish the last column of a period matrix in standard coordinates, whose computations are summarized in Table A.1. The results of this appendix were obtained to be used in Section 3.1.5; nevertheless, they are interesting in their own right.

Theorem A.1 ([31], see also [53], [38]). *Let $K = \mathbb{Q}(\omega)$ with $\omega^4 - ab^2$ (a, b squarefree integer, $b > 0$, $a \neq 0, 1$), be a pure quartic field and $d = (a, b)$. Then a basis of the ring of integers and discriminant are given by the followings;*

(1) in the case $a \equiv 2 \pmod 4$,

$$\mathcal{O}_K = \mathbb{Z}\left[1, \omega, \frac{1}{b}\omega^2, \frac{1}{bd}\omega^3\right], \quad \Delta_K = -256\frac{a^3b^2}{d^2},$$

(2) in the case $a \equiv 3 \pmod 4$

(2.1) if $b \equiv 1 \pmod 2$, *then*

$$\mathcal{O}_K = \mathbb{Z}\left[1, \omega, \frac{1}{b}\omega^2, \frac{1}{bd}\omega^3\right], \quad \Delta_K = -256\frac{a^3b^2}{d^2},$$

(2.2) if $a \equiv 3 \pmod 8$, $b \equiv 2 \pmod 4$, *then*

$$\mathcal{O}_K = \mathbb{Z}\left[1, \frac{1+\omega+\frac{1}{b}\omega^2}{2}, \frac{1}{b}\omega^2, \frac{\omega+\frac{1}{bd}\omega^3}{2}\right], \quad \Delta_K = -16\frac{a^3b^2}{d^2},$$

(2.3) if $a \equiv 7 \pmod 8$, $b \equiv 2 \pmod 4$ *then*

$$\mathcal{O}_K = \mathbb{Z}\left[1, \frac{1+\omega+\frac{1}{b}\omega^2}{2}, \frac{1}{b}\omega^2, \frac{\frac{b}{2d}\omega+2\omega^2+\frac{1}{bd}\omega^3}{4}\right], \quad \Delta_K = -16\frac{a^3b^2}{d^2},$$

(3) in the case $a \equiv 1 \pmod 4$

(3.1) if $b \equiv 2 \pmod 4$

or

(3.2) if $a \equiv 5 \pmod 8$, $b \equiv 1 \pmod 2$,

then

$$\mathcal{O}_K = \mathbb{Z}\left[1, \omega, \frac{1+\frac{1}{b}\omega^2}{2}, \frac{\omega+\frac{1}{bd}\omega^3}{2}\right], \quad \Delta_K = -16\frac{a^3b^2}{d^2},$$

(3.3) *if* $a \equiv 3 \pmod 8$, $b \equiv 2 \pmod 4$, *then*

$$\mathcal{O}_K = \mathbb{Z}\left[1, \omega, \frac{1+\frac{1}{b}\omega^2}{2}, \frac{\frac{ab}{d^2}+b\omega+\frac{1}{b}\omega^2+\frac{1}{bd}\omega^3}{4}\right], \quad \Delta_K = -4\frac{a^3b^2}{d^2}.$$

Let $K = \mathbb{Q}(\omega)$ be a pure quartic field, which we can assume to be such that $\omega^4 = ab^2$, for $a, b \in \mathbb{Z}$ square-free, $a \neq 0, 1$ and $b > 0$.

Let us suppose that $a > 0$, otherwise the field K has two pairs of complex embeddings, hence it determines a lattice $\mu(\mathcal{O}_K) \subset \mathbb{C}^2$ of real rank four.

Let $m(x) = x^4 - ab^2$ be the minimal polynomial of ω over $\mathbb{Q}$, which has roots $\pm\sqrt[4]{ab^2} \in \mathbb{R}$ and $\pm i\sqrt[4]{ab^2} \in \mathbb{C} \setminus \mathbb{R}$. Hence, the set of embeddings

$$\Phi = \{\sigma_1 \colon \omega \mapsto \sqrt[4]{ab^2},\ \sigma_2 \colon \omega \mapsto -\sqrt[4]{ab^2},\ \sigma_3 \colon \omega \mapsto i\sqrt[4]{ab^2}\}$$

defines a type of K. The Abelian complex Lie group $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal with extra multiplications (see Theorem 1.32) of real rank four. Theorem A.1 describes completely the pure quartic fields providing an integer basis of $\mathcal{O}_K$ and the value of the discriminant. Let us study the corresponding toroidal groups for each case. Let us denote the greatest common divisor of a and b by $d = (a, b)$, $h = \sqrt{a}$, and $k = \sqrt[4]{ab^2}$ (hence, $k^2 = hb$ and $k^3 = bhk$).

(1) in the case $a \equiv 2 \pmod 4$, a integer basis of $\mathcal{O}_K$ is given by $\{1, \omega, \frac{1}{b}\omega^2, \frac{1}{bd}\omega^3\}$, so a period matrix of the lattice $\mu_\Phi(\mathcal{O}_K)$ is

$$\begin{pmatrix} 1 & k & h & \frac{hk}{d} \\ 1 & -k & h & -\frac{hk}{d} \\ 1 & ik & -h & -i\frac{hk}{d} \end{pmatrix}.$$

If $\Pi = (I_3|T)$ is a period matrix in standard coordinates, then the last column $T = (t_1, t_2, t_3)^{\mathrm{T}}$ has been computed in the following by Cramer's rule

$$t_1 = \frac{\begin{vmatrix} \frac{hk}{d} & k & h \\ -\frac{hk}{d} & -k & h \\ -i\frac{hk}{d} & ik & -h \end{vmatrix}}{\begin{vmatrix} 1 & k & h \\ 1 & -k & h \\ 1 & ik & -h \end{vmatrix}} = \frac{\begin{vmatrix} \frac{hk}{d} & k & h \\ -\frac{hk}{d} & -k & h \\ -i\frac{hk}{d} & ik & -h \end{vmatrix}}{4hk} = -i\frac{hk}{d},$$

$$t_2 = \frac{\begin{vmatrix} 1 & \frac{hk}{d} & h \\ 1 & -\frac{hk}{d} & h \\ 1 & -i\frac{hk}{d} & -h \end{vmatrix}}{4hk} = \frac{h}{d},$$

$$t_3 = \frac{\begin{vmatrix} 1 & k & \frac{hk}{d} \\ 1 & -k & -\frac{hk}{d} \\ 1 & ik & -i\frac{hk}{d} \end{vmatrix}}{4hk} = i\frac{k}{d}.$$

Note that, for such a lattice, proving that $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal is more directly by the irrationality condition in standard coordinates. Indeed, if $l_1, l_2, l_3 \in \mathbb{Z}$ are such that

$$l_1\left(-i\frac{hk}{d}\right) + l_2\left(\frac{h}{d}\right) + l_3\left(i\frac{k}{d}\right) = L \in \mathbb{Z},$$

then

$$i\left(-l_1hk + l_3k\right) + \left(l_2h - Ld\right) = 0.$$

The above complex number is zero if and only if $-l_1hk + l_3k = 0$ and $l_2h - Ld = 0$. Since $h = \sqrt{a}$ and $k = \sqrt[4]{ab^2}$ are independent over $\mathbb{Z}$, one must have that $l_1 = l_2 = l_3 = 0$. The discriminant is $\Delta_K = -256\frac{a^3b^2}{d^2}$.

(2) in the case $a \equiv 3 \pmod 4$

(2.1) if $b \equiv 1 \pmod 2$, then a integral basis is given again by $\{1, \omega, \frac{1}{b}\omega^2, \frac{1}{bd}\omega^3\}$, so we have the same period matrices of the case (1). The discriminant is again $\Delta_K = -256\frac{a^3b^2}{d^2}$.

(2.2) if $a \equiv 3 \pmod 8$, $b \equiv 2 \pmod 4$, then a integral basis of $\mathcal{O}_K$ is given by

$$\left\{1, \frac{1+\omega+\frac{1}{b}\omega^2}{2}, \frac{1}{b}\omega^2, \frac{\omega+\frac{1}{bd}\omega^3}{2}\right\},$$

so, the corresponding period matrix is

$$\begin{pmatrix} 1 & \frac{1+k+h}{2} & h & \frac{1}{2}\left(k+\frac{hk}{d}\right) \\ 1 & \frac{1-k+h}{2} & h & \frac{1}{2}\left(-k-\frac{hk}{d}\right) \\ 1 & \frac{1+ik-h}{2} & -h & \frac{1}{2}\left(ik-i\frac{hk}{d}\right) \end{pmatrix}.$$

If $\Pi = (I_3|T)$ is a period matrix in standard coordinates, then the last column $T = (t_1, t_2, t_3)^{\mathrm{T}}$ has been computed in the following by Cramer's rule

$$t_1 = \frac{\begin{vmatrix} \frac{1}{2}\left(k+\frac{hk}{d}\right) & \frac{1+k+h}{2} & h \\ \frac{1}{2}\left(-k-\frac{hk}{d}\right) & \frac{1-k+h}{2} & h \\ \frac{1}{2}\left(ik-i\frac{hk}{d}\right) & \frac{1+ik-h}{2} & -h \end{vmatrix}}{\begin{vmatrix} 1 & \frac{1+k+h}{2} & h \\ 1 & \frac{1-k+h}{2} & h \\ 1 & \frac{1+ik-h}{2} & -h \end{vmatrix}} = \frac{\begin{vmatrix} \frac{1}{2}\left(k+\frac{hk}{d}\right) & \frac{1+k+h}{2} & h \\ \frac{1}{2}\left(-k-\frac{hk}{d}\right) & \frac{1-k+h}{2} & h \\ \frac{1}{2}\left(ik-i\frac{hk}{d}\right) & \frac{1+ik-h}{2} & -h \end{vmatrix}}{2hk}$$

$$= \frac{-d-h-ihk}{2d},$$

$$t_2 = \frac{\begin{vmatrix} 1 & \frac{1}{2}\left(k+\frac{hk}{d}\right) & h \\ 1 & \frac{1}{2}\left(-k-\frac{hk}{d}\right) & h \\ 1 & \frac{1}{2}\left(ik-i\frac{hk}{d}\right) & -h \end{vmatrix}}{2hk} = \frac{d+h}{d},$$

$$t_3 = \frac{\begin{vmatrix} 1 & \frac{1+k+h}{2} & \frac{1}{2}\left(k+\frac{hk}{d}\right) \\ 1 & \frac{1-k+h}{2} & \frac{1}{2}\left(-k-\frac{hk}{d}\right) \\ 1 & \frac{1+ik-h}{2} & \frac{1}{2}\left(ik-i\frac{hk}{d}\right) \end{vmatrix}}{2hk} = \frac{-d-h+ik}{2d}.$$

Note that, for such a lattice, proving that $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal is more directly by the irrationality condition in standard coordinates. Indeed, if

$l_1, l_2, l_3 \in \mathbb{Z}$ are such that

$$l_1\left(\frac{-d-h-ihk}{2d}\right)+l_2\left(\frac{d+h}{d}\right)+l_3\left(\frac{-d-h+ik}{2d}\right)=L\in\mathbb{Z},$$

then

$$i\left(-l_1hk+l_3k\right)+\left(-l_1(d+h)+2l_2(d+h)-l_3(d+h)-2Ld\right)=0.$$

The above complex number is zero if and only if $-l_1hk+l_3k=0$ and $-l_1(d+h)+2l_2(d+h)-l_3(d+h)-2Ld=0$. Since $h=\sqrt{a}$ and $k=\sqrt[4]{ab^2}$ are independent over $\mathbb{Z}$, one must have that $l_1=l_2=l_3=0$. The discriminant is $\Delta_K=-16\frac{a^3b^2}{d^2}$.

(2.3) if $a\equiv 7 \pmod 8$, $b\equiv 2 \pmod 4$, then a integer basis is given by

$$\left\{1,\frac{1+\omega+\frac{1}{b}\omega^2}{2},\frac{1}{b}\omega^2,\frac{\frac{b}{2d}\omega+2\omega^2+\frac{1}{bd}\omega^3}{4}\right\},$$

so, the corresponding period matrix is

$$\begin{pmatrix} 1 & \frac{1+k+h}{2} & h & \frac{1}{4}\left(\frac{b}{2d}k+2bh+\frac{hk}{d}\right)\\ 1 & \frac{1-k+h}{2} & h & \frac{1}{4}\left(-\frac{b}{2d}k+2bh-\frac{hk}{d}\right)\\ 1 & \frac{1+ik-h}{2} & -h & \frac{1}{4}\left(-\frac{b}{2d}ik-2bh-i\frac{hk}{d}\right)\end{pmatrix}.$$

If $\Pi=(I_3|T)$ is a period matrix in standard coordinates, then the last column $T=(t_1,t_2,t_3)^{\mathrm{T}}$ has been computed in the following by Cramer's rule,

$$t_1=\frac{\begin{vmatrix}\frac{1}{4}\left(\frac{b}{2d}k+2bh+\frac{hk}{d}\right) & \frac{1+k+h}{2} & h\\ \frac{1}{4}\left(-\frac{b}{2d}k+2bh-\frac{hk}{d}\right) & \frac{1-k+h}{2} & h\\ \frac{1}{4}\left(-\frac{b}{2d}ik-2bh-i\frac{hk}{d}\right) & \frac{1+ik-h}{2} & -h\end{vmatrix}}{\begin{vmatrix}1 & \frac{1+k+h}{2} & h\\ 1 & \frac{1-k+h}{2} & h\\ 1 & \frac{1+ik-h}{2} & -h\end{vmatrix}}$$

$$=\frac{\begin{vmatrix}\frac{1}{2}\left(ik-\frac{hk}{d}\right) & \frac{1+k+h}{2} & h\\ \frac{1}{2}\left(ik-\frac{hk}{d}\right) & \frac{1-k+h}{2} & h\\ \frac{1}{2}\left(ik-\frac{hk}{d}\right) & \frac{1+ik-h}{2} & -h\end{vmatrix}}{2hk}=\frac{-b-2h-2ihk}{8d},$$

$$t_2=\frac{\begin{vmatrix}1 & \frac{1}{2}\left(k+\frac{hk}{d}\right) & h\\ 1 & \frac{1}{2}\left(-k-\frac{hk}{d}\right) & h\\ 1 & \frac{1}{2}\left(ik-\frac{hk}{d}\right) & -h\end{vmatrix}}{2hk}=\frac{b+2h}{4d},$$

$$t_3 = \frac{\begin{vmatrix} 1 & \frac{1+k+h}{2} & \frac{1}{2}\left(k+\frac{hk}{d}\right) \\ 1 & \frac{1-k+h}{2} & \frac{1}{2}\left(-k-\frac{hk}{d}\right) \\ 1 & \frac{1+ik-h}{2} & \frac{1}{2}\left(ik-\frac{hk}{d}\right) \end{vmatrix}}{2hk} = \frac{-b+4bd-2h+2ik}{8d}.$$

Note that, for such a lattice, proving that $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal is more directly by the irrationality condition in standard coordinates. Indeed, if $l_1, l_2, l_3 \in \mathbb{Z}$ are such that

$$l_1\left(\frac{-b-2h-2ihk}{8d}\right)+l_2\left(\frac{b+2h}{4d}\right)+l_3\left(\frac{-b+4bd-2h+2ik}{8d}\right)=L\in\mathbb{Z},$$

then

$$i\left(-2l_1hk+2l_3k\right)+\left(-l_1(b+2h)+2l_2(b+2h)+l_3(-b+4bd-2h)-8Ld\right)=0.$$

The above complex number is zero if and only if $-2l_1hk+2l_3k=0$ and $-l_1(b+2h)+2l_2(b+2h)+l_3(-b+4bd-2h)-8Ld=0$. Since $h=\sqrt{a}$ and $k=\sqrt[4]{ab^2}$ are independent over $\mathbb{Z}$, one must have that $l_1=l_2=l_3=0$. The discriminant is $\Delta_K=-16\frac{a^3b^2}{d^2}$.

(3) in the case $a \equiv 1 \pmod 4$

(3.1) if $b \equiv 2 \pmod 4$

or

(3.2) if $a \equiv 5 \pmod 8$, $b \equiv 1 \pmod 2$,
a integral basis of $\mathcal{O}_K$ is given by

$$\left\{1, \omega, \frac{1+\frac{1}{b}\omega^2}{2}, \frac{\omega+\frac{1}{bd}\omega^3}{2}\right\},$$

so, the corresponding period matrix is

$$\begin{pmatrix} 1 & k & \frac{1+h}{2} & \frac{1}{2}\left(k+\frac{hk}{c}\right) \\ 1 & -k & \frac{1+h}{2} & \frac{1}{2}\left(-k-\frac{hk}{c}\right) \\ 1 & ik & \frac{1-h}{2} & \frac{1}{2}\left(ik-i\frac{hk}{c}\right) \end{pmatrix}.$$

If $\Pi = (I_3|T)$ is a period matrix in standard coordinates, then the last column $T = (t_1, t_2, t_3)^{\mathrm{T}}$ has been computed in the following by Cramer's rule

$$t_1 = \frac{\begin{vmatrix} \frac{1}{2}\left(k + \frac{hk}{c}\right) & k & \frac{1+h}{2} \\ \frac{1}{2}\left(-k - \frac{hk}{c}\right) & -k & \frac{1+h}{2} \\ \frac{1}{2}\left(ik - i\frac{hk}{c}\right) & ik & \frac{1-h}{2} \end{vmatrix}}{\begin{vmatrix} 1 & k & \frac{1+h}{2} \\ 1 & -k & \frac{1+h}{2} \\ 1 & ik & \frac{1-h}{2} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{1}{2}\left(k + \frac{hk}{c}\right) & k & \frac{1+h}{2} \\ \frac{1}{2}\left(-k - \frac{hk}{c}\right) & -k & \frac{1+h}{2} \\ \frac{1}{2}\left(ik - i\frac{hk}{c}\right) & ik & \frac{1-h}{2} \end{vmatrix}}{2hk} = -i\frac{1+h}{2d},$$

$$t_2 = \frac{\begin{vmatrix} 1 & \frac{1}{2}\left(k + \frac{hk}{c}\right) & \frac{1+h}{2} \\ 1 & \frac{1}{2}\left(-k - \frac{hk}{c}\right) & \frac{1+h}{2} \\ 1 & \frac{1}{2}\left(ik - i\frac{hk}{c}\right) & \frac{1-h}{2} \end{vmatrix}}{2hk} = \frac{d+h}{2d},$$

$$t_3 = \frac{\begin{vmatrix} 1 & k & \frac{1}{2}\left(k + \frac{hk}{c}\right) \\ 1 & -k & \frac{1}{2}\left(-k - \frac{hk}{c}\right) \\ 1 & ik & \frac{1}{2}\left(ik - i\frac{hk}{c}\right) \end{vmatrix}}{2hk} = i\frac{k}{d}.$$

Note that, for such a lattice, proving that $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal is more directly by the irrationality condition in standard coordinates. Indeed, if $l_1, l_2, l_3 \in \mathbb{Z}$ are such that

$$l_1\left(-i\frac{1+h}{2d}\right) + l_2\left(\frac{d+h}{2d}\right) + l_3\left(i\frac{k}{d}\right) = L \in \mathbb{Z},$$

then

$$i\left(-l_1(1+h) + 2l_3k\right) + \left(l_2(d+h) - 2Ld\right) = 0.$$

The above complex number is zero if and only if $-l_1(1+h) + 2l_3k = 0$ and $l_2(d+h) - 2Ld = 0$. Since $h = \sqrt{a}$ and $k = \sqrt[4]{ab^2}$ are independent over $\mathbb{Z}$, one must have that $l_1 = l_2 = l_3 = 0$. The discriminant is $\Delta_K = -16\frac{a^3b^2}{d^2}$.

(3.3) $a \equiv 1 \pmod 8$, $b \equiv 1 \pmod 2$, a integral basis of $\mathcal{O}_K$ is given by

$$\left\{1, \omega, \frac{1+\frac{1}{b}\omega^2}{2}, \frac{\frac{ab}{d^2} + b\omega + \frac{1}{b}\omega^2 + \frac{1}{bd}\omega^3}{4}\right\}$$

so, the corresponding period matrix is

$$\begin{pmatrix} 1 & k & \frac{1}{2}(1+h) & \frac{1}{4}\left(\frac{ab}{c^2}+bk+h+\frac{hk}{d}\right) \\ 1 & -k & \frac{1}{2}(1-h) & \frac{1}{4}\left(\frac{ab}{c^2}-bk+h-\frac{hk}{d}\right) \\ 1 & ik & \frac{1}{2}(1+ih) & \frac{1}{4}\left(\frac{ab}{c^2}+bik-h-i\frac{hk}{d}\right) \end{pmatrix}.$$

If $\Pi = (I_3|T)$ is a period matrix in standard coordinates, then the last column $T = (t_1, t_2, t_3)^{\mathrm{T}}$ has been computed in the following by Cramer's rule

$$t_1 = \frac{\begin{vmatrix} \frac{1}{4}\left(\frac{ab}{c^2}+bk+h+\frac{hk}{d}\right) & k & \frac{1+h}{2} \\ \frac{1}{4}\left(\frac{ab}{c^2}-bk+h-\frac{hk}{d}\right) & -k & \frac{1+h}{2} \\ \frac{1}{4}\left(\frac{ab}{c^2}+bik-h-i\frac{hk}{d}\right) & ik & \frac{1-h}{2} \end{vmatrix}}{\begin{vmatrix} 1 & k & \frac{1+h}{2} \\ 1 & -k & \frac{1+h}{2} \\ 1 & ik & \frac{1-h}{2} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \frac{1}{2}\left(k+\frac{hk}{c}\right) & k & \frac{1+h}{2} \\ \frac{1}{2}\left(-k-\frac{hk}{c}\right) & -k & \frac{1+h}{2} \\ \frac{1}{2}\left(ik-i\frac{hk}{c}\right) & ik & \frac{1-h}{2} \end{vmatrix}}{2hk} = \frac{ab-d^2-idk-idhk}{d^2},$$

$$t_2 = \frac{\begin{vmatrix} 1 & \frac{1}{2}\left(k+\frac{hk}{c}\right) & \frac{1+h}{2} \\ 1 & \frac{1}{2}\left(-k-\frac{hk}{c}\right) & \frac{1+h}{2} \\ 1 & \frac{1}{2}\left(ik-i\frac{hk}{c}\right) & \frac{1-h}{2} \end{vmatrix}}{2hk} = \frac{bd+h}{d},$$

$$t_3 = \frac{\begin{vmatrix} 1 & k & \frac{1}{2}\left(k+\frac{hk}{c}\right) \\ 1 & -k & \frac{1}{2}\left(-k-\frac{hk}{c}\right) \\ 1 & ik & \frac{1}{2}\left(ik-i\frac{hk}{c}\right) \end{vmatrix}}{2hk} = \frac{2d+2ik}{d}.$$

Note that, for such a lattice, proving that $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal is more directly by the irrationality condition in standard coordinates. Indeed, if $l_1, l_2, l_3 \in \mathbb{Z}$ are such that

$$l_1\left(\frac{ab-d^2-idk-idhk}{d^2}\right) + l_2\left(\frac{bd+h}{d}\right) + l_3\left(\frac{2d+2ik}{d}\right) = L \in \mathbb{Z},$$

then

$$i\left(-l_1(dk+dhk)+2l_3dk\right) + \left(l_1(ab-d^2)+l_2d(bd+h)+2l_3d^2-Ld^2\right) = 0.$$

The above complex number is zero if and only if $-l_1(dk+dhk)+2l_3dk = 0$ and $l_1(ab-d^2)+l_2d(bd+h)+2l_3d^2-Ld^2 = 0$. Since $h = \sqrt{a}$ and $k = \sqrt[4]{ab^2}$ are independent over $\mathbb{Z}$, one must have that $l_1 = l_2 = l_3 = 0$. The discriminant is $\Delta_K = -4\frac{a^3b^2}{d^2}$.

The following Table A.1 summarizes the above results.

Case	Δ_K	$T = (t_1, t_2, t_3)^{\mathrm{T}}$
$a \equiv 2 \pmod 4$ or $a \equiv 3 \pmod 4$, $b \equiv 1 \pmod 2$	$-256\frac{a^3b^2}{d^2}$	$\left(-i\frac{hk}{d}, \frac{h}{d}, i\frac{k}{d}\right)$
$a \equiv 3 \pmod 8$, $b \equiv 2 \pmod 4$	$-16\frac{a^3b^2}{d^2}$	$\left(\frac{-d-h-ihk}{2d}, \frac{d+h}{d}, \frac{-d-h+ik}{2d}\right)$
$a \equiv 7 \pmod 8$, $b \equiv 2 \pmod 4$	$-16\frac{a^3b^2}{d^2}$	$\left(\frac{-b-2h-2ihk}{8d}, \frac{b+2h}{4d}, \frac{-b+4bd-2h+2ik}{8d}\right)$
$b \equiv 2 \pmod 4$, or $a \equiv 5 \pmod 8$, $b \equiv 1 \pmod 2$	$-16\frac{a^3b^2}{d^2}$	$\left(-i\frac{1+h}{2d}, \frac{d+h}{2d}, i\frac{k}{d}\right)$
$a \equiv 1 \pmod 8$, $b \equiv 1 \pmod 2$	$-4\frac{a^3b^2}{d^2}$	$\left(\frac{ab-d^2-idk-idhk}{d^2}, \frac{bd+h}{d}, \frac{2d+2ik}{d}\right)$

TABLE A.1: Last columns of a period matrix in standard coordinates

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