

BIG VECTOR BUNDLES ON SURFACES AND FOURFOLDS

GILBERTO BINI AND FLAMINIO FLAMINI

ABSTRACT. The aim of this note is to exhibit explicit sufficient cohomological criteria ensuring bigness of globally generated, rank- r vector bundles, $r \geq 2$, on smooth, projective varieties of even dimension $d \leq 4$. We also discuss connections of our general criteria to some recent results of other authors, as well as applications to tangent bundles of Fano varieties, to suitable Lazarsfeld-Mukai bundles on four-folds, etcetera.

1. INTRODUCTION

Let V be a smooth projective variety over the field of complex numbers, and denote by d the dimension of it. As well known, the geometry of V can be described by way of linear systems of divisors D on V . The resulting mapping from V to projective space has different characteristics according to the positivity of $\mathcal{O}_V(D)$: see [10] for a comprehensive treatment of these topics. In particular, we recall that a divisor D is *big* if and only if the *Kodaira-Iitaka dimension* of $\mathcal{O}_V(D)$ is equal to $\dim(V)$. More geometrically, the Iitaka fibration theorem implies that $\mathcal{O}_V(D)$ is big if and only if the mapping $\phi_m : V \dashrightarrow \mathbb{P}H^0(V, \mathcal{O}_V(mD))^\vee$ is birational onto its image for some $m > 0$: see, for instance, [10, p. 139]. Moreover, just to mention a few results, there are cohomological and numerical criteria for a divisor to be big. Remarkably, a globally generated line bundle - hence nef - is big if and only if the top self intersection $c_1(\mathcal{O}_V(D))^d$ is positive (cf. e.g. [10, Thm. 2.2.16]).

As recalled in the Introduction to [11], in the past 60 years there has been a considerable effort to generalize the theory of positivity of line bundles to vector bundles, in particular to extend the cohomological and topological properties of ample divisors. In this paper, we will focus on some aspects of the whole theory, which is rather articulated: the reader may find a recent exposition in [7], where the various notions of positivity for vector bundles are studied in connection with topics from Hodge Theory, Satake-Baily-Borel completion of period mappings, Iitaka conjecture, etcetera. Also, positivity of vector bundles, especially the tangent bundle T_V , is related to the classification of projective manifolds: see, for instance, [3] and, for the more general Kähler manifolds, [5].

We recall that a rank $r \geq 2$ vector bundle E on V is ample (nef) if the tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ of the projective bundle $\pi : \mathbb{P}(E) \rightarrow V$ is an ample (nef) line bundle. As for the notion of bigness, there are various definitions: see, for instance, [1] for them and their relation to base loci of vector bundles. Here we will deal with the notion of *L-bigness*, i.e., a vector bundle E is *L-big* if and only if the tautological bundle of $\mathbb{P}(E)$ is a big line bundle (cf. [1, (6.1.2) in Def. 6.1]). In what follows, we will drop the *L* and simply talk about *big vector bundles*. As in the case of line bundles, bigness of vector bundles has a geometric interpretation in terms of birational images of the ruled variety $\mathbb{P}(E)$ in suitable projective spaces.

In this paper, our aim is to investigate natural cohomological conditions for a globally generated vector bundle E to be big on V . Roughly speaking, this is our strategy. Since a globally generated vector bundle is nef (see, for the sake of completeness, Remark 2.1), [5, Theorem 2.5] implies that the nefness of E can be measured in terms of the non-negativity of $(-1)^d s_d(E)$, where $s_d(E)$ is the top Segre class of E . What's more, a nef vector bundle has a well-defined *numerical dimension* $n(E)$, which is the numerical dimension of the tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, i.e., the largest non-negative integer $n(E)$ such that $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{n(E)}$ is not numerically equivalent to 0 (cf. Def. 2.3 below).

If the d^{th} Segre class of E is positive, which restricts our investigation to d even, one can see that the numerical dimension $n(E)$ equals the dimension of $\mathbb{P}(E)$, which in turn means that the tautological bundle on V is a big line bundle, so E is big.

This research started during the visit of both authors at the *Laboratoire de Mathématiques et Applications*–Université de Poitiers; the authors thank this institution for the excellent hospitality and the friendly atmosphere. The collaboration has been partially supported by the Research Project “*Families of curves: their moduli and their related varieties*” (CUP: E81-18000100005) - Mission Sustainability - University of Rome Tor Vergata. Finally, the authors would like to thank the anonymous referee for carefully reading the manuscript, for his/her comments on the first version of this work (which have improved the exposition of the paper) and for valuable remarks inspiring Examples 3.1-(e) and 4.1-(b) of the present version.

MSC2000 Classification: Primary 14J60. Secondary 14J35.

With this setting, here are our results.

Theorem (cf. Theorem 3.1 below) *Let V be any smooth, irreducible projective surface. Let E be a globally-generated, rank- r vector bundle on V , $r \geq 2$, such that $h^0(E) \geq r + 2$. Assume further that $h^1((\det E)^{-1}) = 0$. Then E is a big vector bundle on V .*

As for V of dimension $d = 4$, in order to state our result, we first need to recall that global generation of E gives rise to the exact sequence:

$$0 \rightarrow M_E \rightarrow H^0(E) \otimes \mathcal{O}_V \xrightarrow{ev} E \rightarrow 0, \quad (1.1)$$

where M_E is the so called *Lazarsfeld-Mukai bundle* associated to E . Tensoring with E and passing to cohomology, one has a natural induced map

$$H^0(E)^{\otimes 2} \xrightarrow{\mu_E} H^0(E^{\otimes 2}). \quad (1.2)$$

Theorem (cf. Theorem 4.1 below) *Let V be any smooth, irreducible projective four-fold. Let E be a globally-generated, rank- r vector bundle on V , $r \geq 2$, such that $h^0(E) \geq r + 4$. Assume further that:*

$$\begin{aligned} q(V) := h^1(\mathcal{O}_V) &= 0, \\ h^i(V, (\det E)^{-1}) &= 0, \quad 1 \leq i \leq 3, \\ h^3(V, E^\vee \otimes (\det E)^{-1}) &= 0, \\ \mu_E &\text{ is injective.} \end{aligned}$$

Then E is a big vector bundle on V .

In order to write down the cohomological constraints appearing in both theorems, we assume that the d^{th} Segre class of E vanishes; so does the top Chern class of a suitable rank- d , associated vector bundle N^\vee on V , where N is the kernel of the evaluation map from $W \otimes \mathcal{O}_V$ to E , W being a general subspace of $H^0(E)$ of dimension $r + d$. This would imply the existence of a nowhere vanishing section of N^\vee . The conditions in Theorem 3.1 and 4.1 are sufficient to contradict the existence of any such section.

In principle, our results can be extended to any even dimension but the cohomological conditions are in fact more complicated to be written down. Indeed, already in the case of surfaces and four-folds, we need to investigate exterior powers of vector bundles that are defined in terms of short exact sequences. This is possible via successive short exact sequences which become more numerous, as the dimension of V increases. Nonetheless, already in dimension 2 and 4 our results give interesting applications.

As for dimension 2, Theorem 3.1 can be viewed as the "bigness"-version of the ampleness criterion given in [2, Prop. 1]. Theorem 3.1 applies to any smooth, projective surface and to any vector bundle E on it, which is globally generated and has arbitrary rank, not only two; moreover, we do not assume the Neron-Severi group to be cyclic and generated by $c_1(E)$. In Section 3.1, we explore some of the various applications of Theorem 3.1; in Example (a) and Example (b), we exhibit vector bundles that are big but not ample. In Example (c), we discuss unsplit vector bundles on Segre-Hirzebruch surfaces \mathbb{F}_e , which turn out to be very ample. In Example (d) the reader may find split vector bundles of higher rank. Possible other applications, along the lines of Example (c), give unsplit vector bundles of rank higher than two. Example (e), which has been inspired by questions of the Referee, shows that condition $h^1((\det E)^{-1}) = 0$ in Theorem 3.1 is sufficient, but not necessary, for bigness.

As for dimension 4, we present two possible applications. First, let V be a Fano manifold, i.e. a smooth projective variety such that the anti-canonical is ample. To start with, as proved, for instance, in [9, Proposition 4.1], if T_V is nef and big, then V is a Fano manifold. Conversely, as, for instance, in loc. cit., Question 4.5., one might ask

If V is Fano with nef tangent bundle T_V , is it true that T_V is big?

As explained in [9, p.1550098-8], the affirmative answer to the previous question has been proved up to dimension 3. Theorem 4.1 allows us to answer this question in dimension 4 under the assumption E is globally generated and $h^0(E) \geq 9$. Inspired by [2, Prop. 2], dealing with Lazarsfeld-Mukai bundles on a smooth surface of irregularity 0 with cyclic Neron-Severi group, we also discuss examples of suitable Lazarsfeld-Mukai bundles on four-folds which turn out to be big but which satisfy all but one of the assumptions in Theorem 4.1 below, proving that Theorem 4.1 gives sufficient but non-necessary conditions for bigness.

As for the plan of the paper, in Section 2.1 we recall some preliminary results, in particular on Chern and Segre classes, as well as on positivity on vector bundles. In Section 3, we prove Theorem 3.1 and some possible applications of it. Finally, in Section 4 we pass to dimension 4.

In what follows, we work over the complex field \mathbb{C} . For any smooth, projective variety V , $A_n(V)$ will denote the group of n -cycles modulo rational equivalence on V , where $0 \leq n \leq \dim(V)$ (cf. [6, § 1]). Unless otherwise stated, from now on we will set $d := \dim(V)$ and E a vector bundle of rank r on V . The *dual bundle* of E will be denoted by E^\vee , unless $E = L$ is a line bundle whose dual will be simply denoted by L^{-1} . For not reminded terminology and notation, we refer the reader to [8].

2. PRELIMINARIES

We briefly recall some results which are frequently used in the paper.

2.1. Chern and Segre classes. For V and E as above, we set $\mathbb{P}(E) := \text{Proj}(\text{Sym}(E))$ (i.e. $\mathbb{P}(E)$ is the projective-bundle parametrizing 1-dimensional quotients of the fibres of E), $\mathcal{O}_{\mathbb{P}(E)}(1)$ the *tautological line-bundle* on $\mathbb{P}(E)$ and $\mathbb{P}(E) \xrightarrow{\pi} V$ the canonical projection (cf. e.g. [8]). By [6, § 1–3], there are homomorphisms

$$A_n(V) \rightarrow A_{n-k}(V), \quad \alpha \rightarrow s_k(E) \cap \alpha,$$

which are defined by the formula

$$s_k(E) \cap \alpha := \pi_* (c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{r-1+k} \cap \pi^*(\alpha)), \quad (2.1)$$

where $\pi^* : A_n(V) \rightarrow A_{n+r-1}(\mathbb{P}(E))$ is the flat pull-back (cf. [6, § 1.7]), $\pi_* : A_{n-k}(\mathbb{P}(E)) \rightarrow A_{n-k}(V)$ the push-forward (cf. [6, § 1.4]) whereas $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{r-1+k} \cap - : A_{r-1+k}(\mathbb{P}(E)) \rightarrow A_{n-k}(\mathbb{P}(E))$ the iterated first Chern class homomorphism (cf. [6, § 2.5]).

$s_k(E)$ in (2.1) is called the *k^{th} -Segre class of E* whereas $s(E) := 1 + s_1(E) + s_2(E) + \dots$ the *total Segre class* of E . $s_k(E)$ is a polynomial in the Chern classes $c_1(E), \dots, c_r(E)$ of E ; indeed given the Chern polynomial of E ,

$$c_E(t) := \sum_{k=0}^r c_k(E)t^k = 1 + c_1(E)t + c_2(E)t^2 + \dots + c_r(E)t^r,$$

the Segre classes defined in (2.1) turn out to be coefficients of the formal power series

$$s_E(t) := \sum_{k=0}^{+\infty} s_k(E)t^k = 1 + s_1(E)t + s_2(E)t^2 + \dots$$

defined to be the inverse power series of $c_E(t)$, i.e. $s_E(t) = c_E(t)^{-1}$ (cf. e.g. [6, § 3.2]). Explicitly, one has (cf. also [11, Examples 8.3.3–8.3.5]):

$$c_1(E) = -s_1(E), \quad c_2(E) = s_1(E)^2 - s_2(E), \dots, \quad c_k(E) = -s_1(E)c_{k-1}(E) - s_2(E)c_{k-2}(E) - \dots - s_k(E), \quad \forall k \geq 3. \quad (2.2)$$

If L is any line bundle on V , then one has (cf. [6, Rem. 3.2.3 (a), Ex. 3.2.2, Ex. 3.1.1]):

$$c_k(E^\vee) = (-1)^k c_k(E) \quad \text{and} \quad c_k(E \otimes L) = \sum_{j=0}^k \binom{r-j}{k-j} c_j(E) c_1(L)^{k-j}, \quad 1 \leq k \leq r, \quad (2.3)$$

$$s_k(E^\vee) = (-1)^k s_k(E) \quad \text{and} \quad s_k(E \otimes L) = \sum_{j=0}^k (-1)^{k-j} \binom{r-1+k}{r-1+j} s_j(E) c_1(L)^{k-j}, \quad (2.4)$$

where $c_0(E) = s_0(E) = 1$ and where $c_1(L)^{k-j}$ denotes the $(k-j)^{\text{th}}$ self-intersection of $c_1(L)$.

2.2. Positivity of vector bundles. We remind some definitions concerning certain *dimension* and *positivity* notions related to vector bundles over a smooth, projective variety V from [5], [7, § II] and [11, § 6–8]. These concepts will be first reminded for line bundles L on V and then, for vector bundles E of rank $r \geq 2$, the definitions being related via the canonical association $E \rightarrow V \rightsquigarrow \mathcal{O}_{\mathbb{P}(E)}(1) \rightarrow \mathbb{P}(E)$.

• **Kodaira–Itaka dimension, bigness, nefness.** Take L any line bundle on V ; its *Kodaira–Itaka dimension*, denoted by $k(L)$, is defined as follows:

$$k(L) := \begin{cases} -\infty & \text{if } h^0(L^{\otimes m}) = 0, \forall m \in \mathbb{N} \\ \max_{m \in \mathbb{N}} \dim(\varphi_{L^{\otimes m}}(V)), & \text{otherwise} \end{cases}$$

where $V \xrightarrow{\varphi_{L^{\otimes m}}} \mathbb{P}(H^0(L^{\otimes m})^\vee)$ denotes the rational map given by the linear system $|L^{\otimes m}|$ (cf. e.g. [7, § II.A]). Then,

$$L \text{ is said to be } \textit{big} \text{ if } k(L) = \dim(V) \quad (2.5)$$

(cf. [11, Def. 2.2.1]). Finally, L is said to be *nef* if $L \cdot C \geq 0$ for any effective curve $C \subset V$.

Let now E be any rank- r vector bundle on V , with $r \geq 2$. Similary as above, its *Kodaira–Iitaka dimension* $k(E)$ is defined to be $k(E) := k(\mathcal{O}_{\mathbb{P}(E)}(1))$. E is said to be a *big* vector bundle if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a big line bundle on $\mathbb{P}(E)$ (cf. e.g. [11, Ex. 6.1.23]). From (2.5), we have therefore

$$E \text{ is a big vector bundle if and only if } k(E) = \dim(V) + r - 1. \quad (2.6)$$

Finally, E is said to be *nef* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle on $\mathbb{P}(E)$ (cf. e.g. [5, Definition 1.9]).

Remark 2.1. Assume that E is globally generated, then E is nef. Indeed, taking $\mathbb{P}(E) \xrightarrow{\pi} V$ the natural projection, global generation of E ensures that π^*E is globally generated. Since $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a quotient of π^*E , the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is globally generated too. Hence, since $|\mathcal{O}_{\mathbb{P}(E)}(1)|$ defines a morphism to a suitable projective space \mathbb{P} , then $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef because it is the pull-back via this morphism of the very-ample line bundle $\mathcal{O}_{\mathbb{P}}(1)$, proving the assertion.

• **Numerical dimension.** As above, we start with the line bundle case.

Definition 2.2. (cf. [7, II.E, p. 24]) Let L be any nef line bundle. The *numerical dimension* of L is defined to be the largest integer $n(L)$ such that $c_1(L)^{n(L)} \neq 0$.

Relating the Kodaira–Iitaka and the numerical dimensions of a nef line bundle L , from [4] one has (cf. also [7, (II.E.1), p.24]):

$$k(L) \leq n(L), \text{ and equality holds if } n(L) = 0, d. \quad (2.7)$$

Let now E be a globally generated vector bundle, of rank $r \geq 2$. From Remark 2.1 E is nef, i.e. $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle on $\mathbb{P}(E)$. Therefore, it makes sense to consider the numerical dimension of such a nef line bundle. Indeed, in accordance with [7, § II.E, p.25], we set

Definition 2.3. Let E be a globally generated vector bundle of rank r on V . The *numerical dimension* of E is $n(E) := n(\mathcal{O}_{\mathbb{P}(E)}(1))$.

Notice that, since $\mathcal{O}_{\mathbb{P}(E)}(1)$ is very-ample on the fibres of the projection $\mathbb{P}(E) \xrightarrow{\pi} V$, one has

$$r - 1 \leq n(E) \leq \dim(\mathbb{P}(E)) = \dim(V) + r - 1 = d + r - 1. \quad (2.8)$$

On the other hand, since E is nef, by (2.1) and Definition 2.2 we have

$$n(E) \text{ is the largest integer with } s_{n(E)-r+1}(E) \neq 0. \quad (2.9)$$

Notice that (2.9) coincides with (II.E.2) in [7], where the authors consider a wider class of vector bundles.

Taking into account the definition of Kodaira–Iitaka dimension $k(E)$ above, (2.7) and Definition 2.3, one has therefore

$$k(E) \leq n(E), \text{ where the equality holds when } n(E) = \dim(V) + r - 1 = d + r - 1. \quad (2.10)$$

Notice that $n(E) = d + r - 1$ (and so $k(E) = n(E) = \dim(\mathbb{P}(E))$) implies that E is big. Moreover, by the global generation (and so nefness) of E , $s_{n(E)-r+1}(E) = s_d(E) \neq 0$ is equivalent to $s_d(E) > 0$, as it follows from [5, Theorem 2.5], with $d = k$ and $Y = V$, which applies to the Segre class $s_d(E)$ considered as a suitable Schur polynomial (cf. the "second interesting example" after [5, Theorem 2.5]).

To sum up, for a globally generated rank- r vector bundle E on a d -dimensional smooth projective variety V , the bigness of E is encoded by the positivity of the d^{th} Segre class of E . In the sequel, we will be concerned in finding sufficient cohomological conditions on a globally generated vector bundle E ensuring the positivity of $s_d(E)$.

3. THE SURFACE CASE

In this section, $d = 2$. Inspired by [2, Lemma], one can prove the following

Theorem 3.1. *Let V be any smooth, irreducible projective surface. Let E be a globally-generated, rank- r vector bundle on V , $r \geq 2$, such that $h^0(E) \geq r + 2$. Assume further that $h^1((\det E)^{-1}) = 0$. Then E is a big vector bundle on V .*

Proof. Since E is globally generated, one has

$$H^0(E) \otimes \mathcal{O}_V \xrightarrow{ev} E \rightarrow 0. \quad (3.1)$$

When $h^0(E) = r + 2$, we set $W := H^0(E)$. When otherwise $h^0(E) > r + 2$, we take $W \subset H^0(E)$ corresponding to the general point of the Grassmannian $\mathbb{G}(r + 2, H^0(E))$ parametrizing $(r + 2)$ -dimensional sub-vector spaces of $H^0(E)$. As in [11, Ex. 6.1.5, p.9], (3.1) defines a morphism

$$\mathbb{P}(E) \xrightarrow{\phi} \mathbb{P}(H^0(E)) = \text{Proj}(\text{Sym}(H^0(E)))$$

(i.e. the projective space of one-dimensional quotients of $H^0(E)$, equivalently of one-dimensional sub-vector spaces of $H^0(E)^\vee$). Then $x := \dim(\text{Im}(\phi)) \leq \dim(\mathbb{P}(E)) = r + 1$. From the definition of W , the surjection $H^0(E)^\vee \rightarrow W^\vee \rightarrow 0$ gives rise to the linear projection

$$\mathbb{P}(H^0(E)) \xrightarrow{\pi_\Lambda} \mathbb{P}(W),$$

whose center Λ is a linear subspace of $\mathbb{P}(H^0(E))$ of dimension $h^0(E) - r - 3$. The generality of W implies that Λ is a general linear subspace of $\mathbb{P}(H^0(E))$. Thus, since $x \leq r + 1$, the subvariety $\Lambda \cap \text{Im}(\phi)$ is empty, which implies that $\pi_\Lambda \circ \phi : \mathbb{P}(E) \rightarrow \mathbb{P}(W)$ is a morphism.

To sum up, in any case one has the exact sequence:

$$0 \rightarrow N \rightarrow W \otimes \mathcal{O}_V \xrightarrow{ev_W} E \rightarrow 0, \quad (3.2)$$

where $N := \ker(ev_W)$ is a rank-2 vector bundle on V . Dualizing (3.2) shows that N^\vee is globally generated. Let $\sigma \in H^0(N^\vee)$ be a general section; then the zero-locus $V(\sigma) \subset V$ is a zero-dimensional scheme of length $c_2(N^\vee) \geq 0$. From (2.3), one has also $c_2(N) \geq 0$.

By the exact sequence (3.2), the total Chern classes of E and N satisfy $c(E)c(N) = 1$, thus $c(N) = s(E)$, where $s(E)$ is the total Segre class of E as in § 2.1. From (2.2), one gets therefore $0 \leq c_2(N) = s_2(E) = c_1(E)^2 - c_2(E)$.

If $0 < s_2(E)$, from (2.9) and the nefness of E (cf. Rem. 2.1), it follows that $n(E) - r + 1 \geq 2$, where $n(E)$ is as in Definition 2.3. In such a case one has $n(E) \geq r + 1$. By (2.8), one therefore concludes that $n(E) = r + 1 = \dim(V) + (r - 1)$ which, by (2.10), implies $n(E) = k(E) = r + 1 = \dim(\mathbb{P}(E))$. This gives that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a big line bundle, as it follows from (2.5), so that E is a big vector bundle.

We want to show that, under our assumptions, the case $s_2(E) = 0$ cannot occur. To do this, we use the same argument as in [2, Proof of the Lemma]. Assume by contradiction that $s_2(E) = 0$, so also $c_2(N) = c_2(N^\vee) = 0$. This implies that $\sigma \in H^0(N^\vee)$ general as above is no-where vanishing on V giving rise to the exact sequence

$$0 \rightarrow \mathcal{O}_V \xrightarrow{\cdot\sigma} N^\vee \rightarrow \det N^\vee \cong \det E \rightarrow 0,$$

the isomorphism on the right-side following from (3.2). The previous exact sequence shows that

$$N^\vee \in \text{Ext}^1(\det E, \mathcal{O}_V) \cong H^1((\det E)^{-1}) = (0),$$

the latter equality following from assumptions. Therefore $N^\vee = \mathcal{O}_V \oplus \det E$, i.e. $N = \mathcal{O}_V \oplus (\det E)^{-1}$. Plugging into (3.2) gives

$$0 \rightarrow \mathcal{O}_V \oplus (\det E)^{-1} \rightarrow W \otimes \mathcal{O}_V \cong \mathcal{O}_V^{\oplus(r+2)} \rightarrow E \rightarrow 0$$

from which one deduces the exact sequence

$$0 \rightarrow (\det E)^{-1} \rightarrow \mathcal{O}_V^{\oplus(r+1)} \rightarrow E \rightarrow 0.$$

Since $h^1((\det E)^{-1}) = 0$, the previous exact sequence implies $h^0(E) \leq r + 1$, which contradicts assumptions. \square

3.1. Examples. We discuss some examples which satisfy assumptions in Theorem 3.1.

(a) Let $V = \mathbb{P}^2$ and consider the rank-2 vector bundle $E := \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$. The vector bundle E is globally generated, with $h^0(E) = 7$ and $h^1((\det E)^{-1}) = h^1(\mathcal{O}_{\mathbb{P}^2}(-2)) = 0$. From Theorem 3.1, E is big. Indeed, $|\mathcal{O}_{\mathbb{P}(E)}(1)|$ maps $\mathbb{P}(E)$ in \mathbb{P}^6 onto the cone over the Veronese surface in \mathbb{P}^5 ; in particular, E is big but not ample. Another example in the same vein is e.g. $E := \mathcal{O}_{\mathbb{P}^2} \oplus T_{\mathbb{P}^2}$, where $T_{\mathbb{P}^2}$ is the tangent bundle on \mathbb{P}^2 , as it follows from the Euler sequence for $T_{\mathbb{P}^2}$ and $c_1(T_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2}(3)$.

(b) The previous example can be easily extended to any smooth, projective irreducible surface V and any rank- r vector bundle on V of the form $E = \mathcal{O}_V \oplus F$, with F any ample, rank- $(r-1)$ vector bundle such that $h^0(F) \geq r+1$ and $h^1((\det F)^{-1}) = 0$. For example consider $\mathbb{F}_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ the Hirzebruch surface, for some integer $e \geq 0$. We let $\pi_e : \mathbb{F}_e \rightarrow \mathbb{P}^1$ denote the natural projection. Thus $\text{Num}(\mathbb{F}_e) = \mathbb{Z}[C_e] \oplus \mathbb{Z}[f]$, where C_e is a section of \mathbb{F}_e corresponding to the surjection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-e)$ on \mathbb{P}^1 (C_e is unique when $e > 0$), and $f = \pi_e^*(p)$, for any $p \in \mathbb{P}^1$, the class of a fibre; in particular $C_e^2 = -e$, $f^2 = 0$, $C_e f = 1$. Let b be an integer and assume $b > e \geq 0$; consider the vector bundle $E = \mathcal{O}_{\mathbb{F}_e} \oplus \mathcal{O}_{\mathbb{F}_e}(C_e + bf)$. Since $b > e$, by [8, V.Thm. 2.17,(b)], $\mathcal{O}_{\mathbb{F}_e}(C_e + bf)$ is very-ample; thus E is globally generated but not ample. Moreover

$$h^0(E) = 1 + h^0(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(C_e + bf)) = 1 + h^0(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(b))$$

the second equality following from Leray isomorphism. Since $b > e$, then

$$h^0(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(b)) = h^0(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(b-e))) = 2b + 2 - e.$$

Thus $h^0(E) = 2b + 3 - e > 4$, as it follows by $b > e \geq 0$. Finally

$$h^1((\det E)^{-1}) = h^1(\mathcal{O}_{\mathbb{F}_e}(-C_e - bf)) = h^1(\omega_{\mathbb{F}_e} \otimes \mathcal{O}_{\mathbb{F}_e}(C_e + bf)) = 0$$

where the second equality follows from Serre duality whereas the last equality from Kodaira vanishing. By Theorem 3.1, it follows that E is big.

(c) To discuss examples of big (or even very-ample) unsplitting vector bundles, we use same notation as in Example (b) above and consider integers $e \geq 1$, $b \geq 4e + 3$ and $\frac{3b+2-4e}{2} \leq k < 2b - 4e$. Let $A := \mathcal{O}_{\mathbb{F}_e}(2C_e + (2b - k - 2e)f)$ and $B := \mathcal{O}_{\mathbb{F}_e}(C_e + (k - b + 2e)f)$ be line bundles on \mathbb{F}_e . Any $u \in \text{Ext}^1(B, A)$ gives rise to a rank-two vector bundle E_u fitting in the exact sequence

$$0 \rightarrow A \rightarrow E_u \rightarrow B \rightarrow 0, \tag{3.3}$$

with $\det(E_u) = \mathcal{O}_{\mathbb{F}_e}(3C_e + bf)$. Notice that

$$\dim(\text{Ext}^1(B, A)) = 9e + 4k - 6b - 2 \geq e + 2 \geq 3,$$

where the inequalities follow from numerical assumptions $k \geq \frac{3b+2-4e}{2}$ and $e \geq 1$. To show the equality, consider

$$\dim \text{Ext}^1(B, A) = h^1(\mathbb{F}_e, A \otimes B^{-1}) = h^1(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(C_e + (3b - 2k - 4e)f)) = h^1(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(3b - 2k - 4e)).$$

Since $k \geq \frac{3b+2-4e}{2}$ both $h^1(\mathcal{O}_{\mathbb{P}^1}(3b - 2k - 4e))$ and $h^1(\mathcal{O}_{\mathbb{P}^1}(3b - 2k - 5e))$ are positive and they add-up to $9e + 4k - 6b - 2$.

We claim that the general $u \in \text{Ext}^1(B, A)$ gives rise to an unsplitting vector bundle. To prove this, we use that E_u is of rank-two and that it fits in the exact sequence (3.3), thus $E_u^\vee \cong E_u \otimes A^\vee \otimes B^\vee$, since $\det E_u = A \otimes B$. Tensoring (3.3) respectively by $E_u^\vee \cong E_u \otimes A^\vee \otimes B^\vee$, B^\vee , A^\vee , we get the following exact diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A \otimes B^\vee & \rightarrow & E_u \otimes B^\vee & \rightarrow & \mathcal{O}_{\mathbb{F}_e} & \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \rightarrow & E_u \otimes B^\vee & \rightarrow & E_u \otimes E_u^\vee & \rightarrow & E_u \otimes A^\vee & \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \rightarrow & \mathcal{O}_{\mathbb{F}_e} & \rightarrow & E_u \otimes A^\vee & \rightarrow & B \otimes A^\vee & \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ & & 0 & & 0 & & 0 & . \end{array} \tag{3.4}$$

One needs to compute $h^0(E_u \otimes B^\vee)$ and $h^0(E_u \otimes A^\vee)$. From the cohomology sequence associated to the first row of diagram (3.4) we get

$$0 \rightarrow H^0(A \otimes B^\vee) \rightarrow H^0(E_u \otimes B^\vee) \rightarrow H^0(\mathcal{O}_{\mathbb{F}_e}) \xrightarrow{\hat{\delta}} H^1(A \otimes B^\vee).$$

The coboundary map

$$H^0(\mathcal{O}_{\mathbb{F}_e}) \xrightarrow{\hat{\partial}} H^1(A \otimes B^\vee) \cong \text{Ext}^1(B, A),$$

has to be injective since it corresponds to the choice of the non-trivial general extension class $u \in \text{Ext}^1(B, A)$ associated to E_u . Thus one gets

$$h^0(E_u \otimes B^\vee) = h^0(A \otimes B^\vee) = h^0(\mathcal{O}_{\mathbb{P}^1}(3b - 2k - 4e)) + h^0(\mathcal{O}_{\mathbb{P}^1}(3b - 2k - 5e)) = 0, \quad (3.5)$$

the last equality following from the assumption $k \geq \frac{3b+2-4e}{2}$, which gives $3b - 2k - 4e \leq -2$ and $3b - 2k - 5e \leq -2 - e$.

From the third row of diagram (3.4), since $B \otimes A^\vee = \mathcal{O}_{\mathbb{F}_e}(-C_e + (2k - 3b + 4e)f)$ is not effective, it follows that $h^0(E_u \otimes A^\vee) = h^0(\mathcal{O}_{\mathbb{F}_e}) = 1$, thus $H^0(E_u \otimes A^\vee) \cong \mathbb{C}$.

From the second column of diagram (3.4), we have

$$0 \rightarrow H^0(E_u \otimes B^\vee) \rightarrow H^0(E_u \otimes E_u^\vee) \xrightarrow{\psi} H^0(E_u \otimes A^\vee) \cong \mathbb{C} \rightarrow H^1(E_u \otimes B^\vee) \rightarrow \dots$$

We claim that the map ψ is surjective. To prove this, notice that from the first two columns of diagram (3.4) and the fact that the coboundary map $\hat{\partial}$ is injective (as remarked above) we have

$$\begin{array}{ccccc} & 0 & & H^0(E_u \otimes E_u^\vee) & \\ & \downarrow & & \downarrow \psi & \\ 0 \rightarrow & H^0(\mathcal{O}_{\mathbb{F}_e}) & \xrightarrow{\cong} & H^0(E_u \otimes A^\vee) & \rightarrow 0 \\ & \downarrow \hat{\partial} & & \downarrow \hat{\partial} & \\ & H^1(A \otimes B^\vee) & \longrightarrow & H^1(E_u \otimes B^\vee) & . \end{array}$$

Since $H^0(E_u \otimes A^\vee) \cong \mathbb{C}$, ψ is not surjective if and only if $\psi = 0$, which is equivalent to $\hat{\partial}$ to be injective. The latter is impossible since, from the first column of diagram (3.4), we have

$$H^0(\mathcal{O}_{\mathbb{F}_e}) \xrightarrow{\hat{\partial}} H^1(A \otimes B^\vee) \rightarrow H^1(E_u \otimes B^\vee)$$

and the composition of the above two maps is $\tilde{\partial}$. From the surjectivity of ψ , we conclude that

$$h^0(E_u \otimes E_u^\vee) = h^0(E_u \otimes B^\vee) + 1. \quad (3.6)$$

Combining (3.5) and (3.6) we determine $h^0(E_u \otimes E_u^\vee) = 1$ when $u \in \text{Ext}^1(B, A)$ is general. Since E_u is simple, we deduce that E_u must be unsplitting.

Once we have produced E_u unsplitting with arguments above, we want to show that it satisfies assumptions as in Theorem 3.1, so E_u will be big. Notice indeed that

$$h^0(E_u) \geq 4b - 6e - k + 5 > 6e + 11 > rk(E_u) + 2;$$

the second inequality above follows from the assumptions $k < 2b - 4e$ and $b \geq 4e + 3$, whereas the first inequality is a consequence of: (1) standard computations which show that $h^2(A) = h^j(B) = 0$, for $j \geq 1$, so $h^2(E_u) = 0$; (2) the exact sequence (3.3), from which one gets

$$h^0(E_u) = h^0(A) + h^0(B) - h^1(A) + h^1(E_u) = \chi(A) + \chi(B) + h^1(E_u) \geq \chi(A) + \chi(B)$$

where the latter can be easily computed via Riemann-Roch (left to the reader). Furthermore,

$$h^1((\det E_u)^\vee) = h^1(\mathcal{O}_{\mathbb{F}_e}(-3C_e - bf)) = h^1(\omega_{\mathbb{F}_e} \otimes \mathcal{O}_{\mathbb{F}_e}(3C_e + bf)) = 0$$

where the second equality follows from Serre duality whereas the third from the Kodaira vanishing theorem and the very-ampleness of $\mathcal{O}_{\mathbb{F}_e}(3C_e + bf)$, as it follows from [8, V.Cor.2.18 (a)] and from $b \geq 4e + 3$. Since all the assumptions of Theorem 3.1 are satisfied, it follows that E_u is a big vector bundle.

One can even show more, namely that E_u is actually a very-ample vector bundle. Indeed, from [8, V.Cor.2.18 (a)] and the assumption $k < 2b - 4e$, A is a very-ample line bundle. Similarly, since $k \geq \frac{3b+2-4e}{2}$ and $b \geq 4e + 3$, from [8, V.Thm.2.17 (c)] it follows that B is also very-ample. Thus $A \oplus B$ (which corresponds to the zero-vector in $\text{Ext}^1(B, A)$) is a very-ample vector bundle of rank 2. Since very-ampleness is an open condition, one deduces that E_u is very-ample, when $u \in \text{Ext}^1(B, A)$ is general.

(d) Examples of big vector bundles with rank higher than two can be easily constructed as follows. Using same notation and assumptions as in (c) above, let E_u be as in (3.3), corresponding to the general $u \in \text{Ext}^1(B, A)$. From Example (c), E_u satisfies assumptions of Theorem 3.1 and it is also very-ample. For any integer $r \geq 3$, the vector bundle $\mathcal{E}_u :=$

$\mathcal{O}_V^{r-2} \oplus E_u$ is of rank $r \geq 3$, it is globally generated, with $h^0(\mathcal{E}_u) \geq r + e + 9$ and $h^1((\det \mathcal{E}_u)^{-1}) = h^1((\det E_u)^{-1}) = 0$ (as it follows from computations in Example (c)). Then, \mathcal{E}_u is big but not ample.

Further examples of big (resp., ample or very-ample) vector bundles of rank higher than two can be easily constructed by iterating extension procedure as in Example (c), starting from E_u as in (c) and its extension via a globally generated and big (resp., ample or very-ample) line bundle L .

(e) We include here an example which shows that condition $h^1((\det E)^{-1}) = 0$ in Theorem 3.1 is actually sufficient but not necessary for bigness. In other words, we show an example of a smooth projective variety V and a big vector bundle E on it such that $h^1((\det E)^{-1}) \neq 0$. More specifically, let V be a smooth projective surface that is irregular, i.e., $q(V) \neq 0$. Fix an ample line bundle A on V and consider the rank two vector bundle on V which is defined as follows, namely $U := \mathcal{O}_V \oplus (A^{-1} \otimes A^{-1})$. By definition, U has global sections because $H^0(V, U) = H^0(V, \mathcal{O}_V) \neq 0$. Next, define the rank two vector bundle $E := U \otimes A$. As explained in [11, Example 6.1.23, p. 18], the vector bundle E is big (with notation as therein in this case $m = 1$, i.e. the first symmetric power $S^m(U)$ for which one has effectivity is for $S^1 U = U$, and $L = A$ is ample). Let us compute the determinant $\det(E) = \det(U \otimes A)$, which turns out to equal $\det(U) \otimes A^{\otimes 2} \cong \mathcal{O}_V$. Therefore $H^1(V, \det(E)^{-1}) = H^1(V, \mathcal{O}_V) \neq \{0\}$; indeed the dimension of $H^1(V, \mathcal{O}_V)$ is given by $q(V)$, which is different from 0 by our assumptions on V .

4. THE FOUR-FOLD CASE

Here we focus on the case $d = 4$, determining sufficient conditions for bigness of rank $r \geq 2$ vector bundles on a four-fold V . Preliminarily, consider the following general fact; let V be any smooth, projective variety and let E be a globally-generated, rank- r vector bundle on V , $r \geq 2$. Recall that global generation of E gives rise to the exact sequence (1.1); tensoring with E and passing to cohomology, one has the natural induced map in (1.2) and it is straightforward to observe that

$$\mu_E \text{ is injective} \Leftrightarrow h^0(M_E \otimes E) = 0. \quad (4.1)$$

With this set-up, we prove the following:

Theorem 4.1. *Let V be any smooth, irreducible projective four-fold. Let E be a globally-generated, rank- r vector bundle on V , $r \geq 2$, such that $h^0(E) \geq r + 4$. Assume further that:*

$$\begin{aligned} q(V) := h^1(\mathcal{O}_V) &= 0, \\ h^i(V, (\det E)^{-1}) &= 0, \quad 1 \leq i \leq 3, \\ h^3(V, E^\vee \otimes (\det E)^{-1}) &= 0, \\ \mu_E &\text{ is injective.} \end{aligned} \quad (4.2)$$

Then E is a big vector bundle on V .

Proof. Reasoning similarly as in the proof of Theorem 3.1, for general $W \subseteq H^0(E)$ of dimension $r + 4$, one has the exact sequence

$$0 \rightarrow N \rightarrow W \otimes \mathcal{O}_V \xrightarrow{ev_W} E \rightarrow 0, \quad (4.3)$$

where N is a vector bundle of rank 4. Dualizing (4.3) shows that N^\vee is globally generated.

Let $\sigma \in H^0(N^\vee)$ be a general section; then the zero-locus $V(\sigma) \subset V$ is a zero-dimensional scheme of length $0 \leq c_4(N^\vee) = c_4(N)$, the equality following from (2.3). Thus, from (2.2), one gets $0 \leq c_4(N) = s_4(E)$, where $s_4(E)$ is as in (2.1)–(2.2).

If $0 < s_4(E)$, (2.9) and the nefness of E (cf. Rem. 2.1) give $n(E) - r + 1 \geq 4$, i.e. $n(E) \geq r + 3$. In such a case, from (2.8) one concludes that $n(E) = r + 3 = \dim(V) + (r - 1)$, which implies that E is a big vector bundle.

One is therefore left to show that, under assumptions (4.2), the case $s_4(E) = 0$ cannot occur. Assume by contradiction that $s_4(E) = 0$, so $c_4(N) = c_4(N^\vee) = 0$. This implies that $\sigma \in H^0(N^\vee)$ general as above is no-where vanishing on V , giving rise to the exact sequence

$$0 \rightarrow \mathcal{O}_V \xrightarrow{\cdot\sigma} N^\vee \rightarrow F \rightarrow 0, \quad (4.4)$$

where F is a rank-3 vector bundle. Dualizing (4.4), one gets

$$0 \rightarrow F^\vee \rightarrow N \rightarrow \mathcal{O}_V \rightarrow 0, \quad (4.5)$$

i.e. $N \in \text{Ext}^1(\mathcal{O}_V, F^\vee) \cong H^1(F^\vee)$. If we show that $h^1(F^\vee) = 0$, then $N = \mathcal{O}_V \oplus F^\vee$ which, plugged into (3.2), gives

$$0 \rightarrow \mathcal{O}_V \oplus F^\vee \rightarrow W \otimes \mathcal{O}_V \cong \mathcal{O}_V^{\oplus(r+4)} \rightarrow E \rightarrow 0$$

from which one deduces

$$0 \rightarrow F^\vee \rightarrow \mathcal{O}_V^{\oplus(r+3)} \rightarrow E \rightarrow 0.$$

Condition $h^1(F^\vee) = 0$ would therefore imply $h^0(E) \leq r + 3$, contradicting the assumptions.

The rest of the proof is therefore concerned to showing that conditions in (4.2) guarantee $h^1(F^\vee) = 0$. To do this, consider

$$0 \rightarrow \bigwedge^2 F^\vee \xrightarrow{\alpha_2} \bigwedge^2 N \xrightarrow{\beta_2} F^\vee \rightarrow 0, \quad (4.6)$$

deduced from (4.5) and [8, II.5, Ex. 5.16(d), p.127]. Then (4.6) gives:

$$h^1(F^\vee) = 0 \Leftrightarrow \begin{cases} H^1(\bigwedge^2 F^\vee) \xrightarrow{H^1(\alpha_2)} H^1(\bigwedge^2 N) & \text{surjective, and} \\ H^2(\bigwedge^2 F^\vee) \xrightarrow{H^2(\alpha_2)} H^2(\bigwedge^2 N) & \text{injective.} \end{cases} \quad (4.7)$$

We first show the injectivity of the map $H^2(\alpha_2)$. Since F^\vee is of rank 3, from [8, II.5, Ex. 5.16(d), p.127], one has

$$\bigwedge^2 F^\vee \cong F \otimes (\det F^\vee) = F \otimes (\det F)^{-1}.$$

Moreover (4.5) gives $(\det F)^{-1} \cong \det N$ whereas (4.3) gives $\det N \cong (\det E)^{-1}$, i.e. $\det F \cong \det E$. Thus, the previous isomorphism reads $\bigwedge^2 F^\vee \cong F \otimes (\det E)^{-1}$, so the map $H^2(\alpha_2)$ reads $H^2(F \otimes (\det E)^{-1}) \xrightarrow{H^2(\alpha_2)} H^2(\bigwedge^2 N)$. Tensoring (4.4) by $(\det E)^{-1}$ gives:

$$0 \rightarrow (\det E)^{-1} \rightarrow N^\vee \otimes (\det E)^{-1} \rightarrow F \otimes (\det E)^{-1} \rightarrow 0. \quad (4.8)$$

Since, from (4.2) we have $h^i((\det E)^{-1}) = 0$ for $1 \leq i \leq 3$, (4.8) gives

$$H^i(N^\vee \otimes (\det E)^{-1}) \cong H^i(F \otimes (\det E)^{-1}), \quad 1 \leq i \leq 2.$$

Dualizing (4.3) and tensoring with $(\det E)^{-1}$ gives:

$$0 \rightarrow E^\vee \otimes (\det E)^{-1} \rightarrow W^\vee \otimes (\det E)^{-1} \rightarrow N^\vee \otimes (\det E)^{-1} \rightarrow 0.$$

Since $h^2((\det E)^{-1}) = h^3(E^\vee \otimes (\det E)^{-1}) = 0$ from (4.2), the previous exact sequence gives $h^2(N^\vee \otimes (\det E)^{-1}) = 0$ which from the isomorphism above implies $h^2(F \otimes (\det E)^{-1}) = 0$, proving the injectivity of $H^2(\alpha_2)$.

Concerning the surjectivity of $H^1(\alpha_2)$, consider the exact sequence (4.3). From [8, II.5, Ex. 5.16(d), p.127], (4.3) gives rise to a filtration

$$0 \subset G^2 \subset G^1 \subset G^0 = \bigwedge^2 W \otimes \mathcal{O}_V \cong \mathcal{O}_V^{\oplus \binom{r+4}{2}},$$

where

$$G^2 \cong \bigwedge^2 N, \quad G^1/G^2 \cong N \otimes E, \quad G^0/G^1 \cong \bigwedge^2 E.$$

In other words, from (4.3) one deduces the following exact sequences:

$$0 \rightarrow \bigwedge^2 N \rightarrow G^1 \rightarrow N \otimes E \rightarrow 0 \quad \text{and} \quad 0 \rightarrow G^1 \rightarrow \bigwedge^2 W \otimes \mathcal{O}_V \rightarrow \bigwedge^2 E \rightarrow 0. \quad (4.9)$$

Passing to cohomology in the second exact sequence in (4.9) and using assumption $q(V) = h^1(\mathcal{O}_V) = 0$, we get

$$0 \rightarrow H^0(G^1) \rightarrow \bigwedge^2 W \xrightarrow{\lambda_{E|}} H^0(\bigwedge^2 E) \xrightarrow{\pi} H^1(G^1) \rightarrow 0. \quad (4.10)$$

Claim 4.2. *The map $\lambda_{E|}$ in (4.10) is injective. In particular, one has*

$$H^0(G^1) = 0 \quad \text{and} \quad H^1(G^1) \cong \frac{H^0(\bigwedge^2 E)}{\bigwedge^2 W}.$$

Proof of Claim 4.2. Consider the map $\mu_E : H^0(E)^{\otimes 2} \rightarrow H^0(E^{\otimes 2})$ as in (1.2). On the one hand, one has

$$H^0(E)^{\otimes 2} = \bigwedge^2 H^0(E) \oplus \text{Sym}^2(H^0(E))$$

and the map μ_E then splits as $\mu_E = \lambda_E \oplus \sigma_E$, where

$$\lambda_E := \mu_E|_{\bigwedge^2 H^0(E)} : \bigwedge^2 H^0(E) \rightarrow H^0(E^{\otimes 2}) \quad \text{and} \quad \sigma_E := \mu_E|_{\text{Sym}^2(H^0(E))} : \text{Sym}^2(H^0(E)) \rightarrow H^0(E^{\otimes 2}).$$

On the other hand, since $E^{\otimes 2} = \bigwedge^2 E \oplus \text{Sym}^2(E)$, then

$$H^0(E^{\otimes 2}) = H^0(\bigwedge^2 E) \oplus H^0(\text{Sym}^2(E));$$

therefore, more precisely one has

$$\lambda_E : \bigwedge^2 H^0(E) \rightarrow H^0(\bigwedge^2 E) \quad \text{and} \quad \sigma_E : \text{Sym}^2(H^0(E)) \rightarrow H^0(\text{Sym}^2(E)).$$

By assumption (4.2), the map μ_E is injective, so λ_E is also injective. Since $W \subseteq H^0(E)$, then $\bigwedge^2 W \subseteq \bigwedge^2 H^0(E)$ and the map $\lambda_{E|}$ in (4.10) is nothing but the restriction of λ_E to $\bigwedge^2 W$, proving the injectivity of $\lambda_{E|}$. The rest of the claim easily follows from (4.10). \square

From the first exact sequence in (4.9), one gets

$$0 \rightarrow H^0(\bigwedge^2 N) \xrightarrow{\gamma_1} H^0(G^1) \xrightarrow{\gamma_2} H^0(N \otimes E) \xrightarrow{\gamma_3} H^1(\bigwedge^2 N) \xrightarrow{\gamma_4} H^1(G^1) \xrightarrow{\gamma_5} H^1(N \otimes E) \rightarrow \dots \quad (4.11)$$

From Claim 4.2, (4.11) reduces to

$$0 \rightarrow H^0(N \otimes E) \xrightarrow{\gamma_3} H^1(\bigwedge^2 N) \xrightarrow{\gamma_4} H^1(G^1) \xrightarrow{\gamma_5} H^1(N \otimes E) \rightarrow \dots;$$

on the other hand, tensoring (4.3) by E and passing to cohomology, one gets also:

$$0 \rightarrow H^0(N \otimes E) \rightarrow W \otimes H^0(E) \xrightarrow{\mu_{E|}} H^0(E \otimes E) \xrightarrow{\psi} H^1(N \otimes E) \rightarrow \dots, \quad (4.12)$$

where $\mu_{E|} := \mu_{E|_{W \otimes H^0(E)}}$ as $W \otimes H^0(E) \subseteq H^0(E)^{\otimes 2}$. Since μ_E is injective by assumptions in (4.2), $\mu_{E|}$ is injective too. Therefore one has $h^0(N \otimes E) = 0$, which reduces (4.11) and (4.12), respectively, to

$$H^1(\bigwedge^2 N) \xrightarrow{\gamma_4} H^1(G^1) \xrightarrow{\gamma_5} H^1(N \otimes E) \rightarrow \dots \quad \text{and} \quad W \otimes H^0(E) \xrightarrow{\mu_{E|}} H^0(E \otimes E) \xrightarrow{\psi} H^1(N \otimes E) \rightarrow \dots \quad (4.13)$$

Claim 4.3. *The map γ_5 is injective.*

Proof of Claim 4.3. From Claim 4.2, the map π in (4.10) induces an isomorphism

$$\frac{H^0(\bigwedge^2 E)}{\text{Im}(\lambda_{E|})} = \frac{H^0(\bigwedge^2 E)}{\bigwedge^2 W} \xrightarrow{\bar{\pi} \cong} H^1(G^1).$$

Composing with γ_5 , one gets

$$\frac{H^0(\bigwedge^2 E)}{\text{Im}(\lambda_{E|})} = \frac{H^0(\bigwedge^2 E)}{\bigwedge^2 W} \xrightarrow{\gamma_5 \circ \bar{\pi}} H^1(N \otimes E)$$

which is compatible with the injection

$$\frac{H^0(E^{\otimes 2})}{\text{Im}(\mu_{E|})} = \frac{H^0(E^{\otimes 2})}{W \otimes H^0(E)} \xrightarrow{\bar{\psi}} H^1(N \otimes E)$$

induced by ψ as in (4.13). Since $W \subseteq H^0(E)$, write $H^0(E) = W \oplus U$ when $W \subsetneq H^0(E)$. Therefore

$$W \otimes H^0(E) = W^{\otimes 2} \oplus W \otimes U = \bigwedge^2 W \oplus \text{Sym}^2(W) \oplus W \otimes U.$$

Since $H^0(E^{\otimes 2}) = H^0(\bigwedge^2 E) \oplus H^0(\text{Sym}^2(E))$ and μ_E , λ_E and σ_E are injective, then

$$\frac{H^0(E^{\otimes 2})}{W \otimes H^0(E)} = \frac{H^0(\bigwedge^2 E)}{\bigwedge^2 W \oplus ((W \otimes U) \cap \bigwedge^2 H^0(E))} \oplus \frac{H^0(\text{Sym}^2(E))}{\text{Sym}^2(W) \oplus ((W \otimes U) \cap \text{Sym}^2(H^0(E)))}.$$

The injectivity of $\bar{\psi}$ implies the injectivity of its restriction

$$\frac{H^0(\bigwedge^2 E)}{\bigwedge^2 W \oplus ((W \otimes U) \cap \bigwedge^2 H^0(E))} \xrightarrow{\bar{\psi}|} H^1(N \otimes E).$$

If we prove that $\bar{\psi}|$ coincides with $\gamma_5 \circ \bar{\pi}$, then $\gamma_5 \circ \bar{\pi}$ is therefore injective, so is γ_5 . For these purposes, it suffices to show that $\bigwedge^2 W \oplus ((W \otimes U) \cap \bigwedge^2 H^0(E))$ is equal to $\bigwedge^2 W$.

To do this, observe that in $H^0(E) \otimes H^0(E)$ the elements of $\wedge^2 H^0(E)$ correspond to skew-symmetric matrices with $h^0(E)$ rows and $h^0(E)$ columns. Since, as above, $H^0(E) = W \oplus U$, such skew-symmetric matrices have the following type, namely:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (4.14)$$

where $A^T = -A, D^T = -D, B^T + C = 0$ and where A is a square matrix with $r + 4$ rows and D is a square matrix with $h^0(E) - r - 4$ rows.

Let us now describe the elements of $W \otimes H^0(E)$ in $H^0(E) \otimes H^0(E)$; these correspond to square matrices of the following form, namely:

$$\begin{pmatrix} L & 0 \\ M & 0 \end{pmatrix}, \quad (4.15)$$

where L is a square matrix with $r + 4$ rows and where M is a matrix with $h^0(E) - r - 4$ rows and $r + 4$ columns. By (4.14) and (4.15), an element in $\wedge^2 W \oplus ((W \otimes U) \cap \wedge^2 H^0(E))$ has to be a skew-symmetric matrix in $M_{r+4}(\mathbb{C})$, namely an element in $\wedge^2 W$. This implies that $((W \otimes U) \cap \wedge^2 H^0(E)) = \{0\}$, as claimed.

Therefore, the map $\overline{\psi}_1$ coincides with the map $\gamma_5 \circ \overline{\pi}$, which is therefore injective. Since $\overline{\pi}$ is an isomorphism, one has that γ_5 is injective. \square

Claim 4.3 and (4.13) give $H^1(\wedge^2 N) = (0)$, proving the surjectivity of $H^1(\alpha_2)$ as in (4.7), which completes the proof of Theorem 4.1. \square

4.1. Consequences and examples. In this section we discuss some direct consequences of Theorem 4.1, showing how this main result can be related to several aspects like Fano varieties, Lazarsfeld-Mukai bundles, etcetera.

(a) To start with, let V be any smooth, projective variety and denote by T_V its tangent bundle. As proved, for instance, in [9, Proposition 4.1], if T_V is nef and big, then V is a Fano manifold, i.e., $\det(T_V) = -K_V$ is ample. In loc. cit., the author poses Question 4.5., i.e.,

If V is Fano with nef tangent bundle T_V , is it true that T_V is big?

As explained in [9, p. 1550098-8], the affirmative answer to the previous question has been proved up to dimension 3.

Theorem 4.1 allows us to answer this question in dimension four in some cases. More precisely, the following holds.

Proposition 4.4. *Let V be a smooth projective four-fold. Assume V is a Fano manifold and T_V is globally generated with $h^0(T_V) \geq 9$. Then T_V is a big vector bundle on V .*

Proof. To prove the proposition, it suffices to verify conditions (4.2).

Since V is Fano, then $h^{p,0}(V) = 0$ for $p \neq 0$, hence $q(V) = 0$. Moreover, the following holds:

$$H^i(V, \det(T_V)^{-1}) = H^i(V, K_V) \simeq H^{4-i}(V, \mathcal{O}_V)^\vee = \{0\}.$$

As for $h^3(V, E^\vee \otimes \det(E)^{-1}) = 0$, with $E = T_V$, we have

$$H^3(V, T_V^\vee \otimes K_V) \simeq H^1(V, T_V)^\vee.$$

We want to show that this is zero. By [12, Proposition 2.1], V is a homogeneous variety as its tangent bundle T_V is globally generated. As recalled, for instance, in loc. cit., Theorem 2.2., any homogeneous manifold is isomorphic to the product $A \times Y_1 \times \dots \times Y_k$, where A is an abelian variety whereas Y_i is a rational homogeneous manifold, i.e., the quotient of a simple Lie group G_i by a parabolic subgroup P_i . Since V is Fano, so V is isomorphic to a product of rational homogeneous manifolds; more precisely, there cannot be an abelian factor in the product mentioned before, otherwise in that case one would have $q(V) \neq 0$, a contradiction. By [13, Proposition 11.6], we have $H^1(G/P, T_{G/P}) = \{0\}$ which implies $H^1(V, T_V) = \{0\}$ as desired; in particular also the third requirement in (4.2) is fulfilled.

Finally, it remains to check the injectivity of the map μ_{T_V} as in (1.2) (with $E = T_V$). As recalled in (4.1), this is equivalent to requiring that $H^0(V, M_{T_V} \otimes T_V) = \{0\}$. First, notice that the rank of M_{T_V} is greater than 4 as it

follows from the assumption $h^0(T_V) \geq 9$ and the fact that, by (1.1), $rk(M_{T_V}) = h^0(V, T_V) - 4$. Now, suppose, by contradiction, that there exists a non-zero section $\sigma \in H^0(V, M_{T_V} \otimes T_V)$. This yields

$$\mathcal{O}_V \xrightarrow{\sigma} M_{T_V} \otimes T_V,$$

which is equivalent to an injective map

$$M_{T_V}^\vee \hookrightarrow T_V$$

(cf. [8, Prop. 6.3(c), p. 234 and Prop. 6.7, p.235]). This would imply that $M_{T_V}^\vee$ has rank less than or equal to 4, which is a contradiction. Therefore, all the requirements in (4.2) are fulfilled and the proposition is completely proved. \square

Remark 4.5. Proposition 4.4 answers Question 4.5 in [9] when T_V is globally generated and has at least 9 global sections, as any globally generated bundle is nef.

(b) In the same light of Example 3.1-(e), we want to show that Theorem 4.1 gives sufficient, but not necessary, cohomological conditions for bigness. We discuss here an example which has been inspired by [2, Prop. 2], concerning rank-two Lazarsfeld-Mukai vector bundles on suitable smooth, projective surfaces. Here we will instead consider suitable Lazarsfeld-Mukai vector bundles on smooth, projective four-folds. For simplicity in what follows, with a small abuse of notation, we will identify line bundles with associated Cartier divisors using interchangeably multiplicative and additive notation.

Let V be a smooth, projective four-fold. We assume $q(V) = 0$ and we take any ample, globally generated line bundle L on V such that $h^0(V, L) := x \geq 4$.

Let Y be a general element in the linear system $|2L|$; by Bertini's theorem, Y is smooth. Moreover, the exact sequence

$$0 \rightarrow \mathcal{O}_V(-2L) \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_Y \rightarrow 0 \quad (4.16)$$

ensures that $h^0(\mathcal{O}_Y) = 1$; indeed, one has $h^0(\mathcal{O}_V(-2L)) = 0 = h^1(\mathcal{O}_V(-2L))$, the second equality following from the ampleness of L , Serre duality $h^1(\mathcal{O}_V(-2L)) = h^3(K_V + 2L)$ and the Kodaira vanishing theorem; thus Y , being smooth, is also irreducible.

Let $A := \mathcal{O}_Y(L)$. Since L is globally generated, then A is globally generated on Y . It therefore makes sense to consider the vector bundle E on V , defined by the exact sequence:

$$0 \rightarrow E^\vee \rightarrow H^0(A) \otimes \mathcal{O}_V \xrightarrow{ev} A \rightarrow 0, \quad (4.17)$$

which is called the *Lazarsfeld-Mukai vector bundle* associated to the pair (Y, A) .

Lemma 4.6. *E is big.*

Proof. If one dualizes (4.17), one gets

$$0 \rightarrow H^0(A)^\vee \otimes \mathcal{O}_V \rightarrow E \rightarrow \mathcal{E}xt^1(A, \mathcal{O}_V) \rightarrow 0; \quad (4.18)$$

the sheaf $\mathcal{E}xt^1(A, \mathcal{O}_V)$ is supported on Y and $\mathcal{E}xt^1(A, \mathcal{O}_V) \simeq \mathcal{E}xt^1(\mathcal{O}_Y, \mathcal{O}_V) \otimes A^{-1}$. Similarly, dualizing (4.16), we get

$$0 \rightarrow \mathcal{O}_V \rightarrow \mathcal{H}om(\mathcal{O}_V(-2L), \mathcal{O}_V) \simeq \mathcal{O}_V(Y) \rightarrow \mathcal{E}xt^1(\mathcal{O}_Y, \mathcal{O}_V) \rightarrow 0,$$

i.e. $\mathcal{E}xt^1(\mathcal{O}_Y, \mathcal{O}_V) \simeq \mathcal{O}_Y(Y)$. Thus, (4.18) yields

$$0 \rightarrow H^0(A)^\vee \otimes \mathcal{O}_V \rightarrow E \rightarrow A \rightarrow 0, \quad (4.19)$$

since $\mathcal{E}xt^1(A, \mathcal{O}_V) \simeq \mathcal{E}xt^1(\mathcal{O}_Y, \mathcal{O}_V) \otimes A^{-1} \simeq \mathcal{O}_Y(Y) \otimes A^{-1} \simeq \mathcal{O}_Y(2L) \otimes \mathcal{O}_Y(-L) \simeq \mathcal{O}_Y(L) = A$.

Tensoring (4.19) by $\mathcal{O}_V(-L)$ gives

$$0 \rightarrow H^0(A)^\vee \otimes \mathcal{O}_V(-L) \rightarrow E(-L) \rightarrow \mathcal{O}_Y \rightarrow 0,$$

from which we deduce $h^0(E(-L)) = 1$, as $h^0(\mathcal{O}_V(-L)) = 0 = h^1(\mathcal{O}_V(-L))$ (the latter equality is a consequence of Serre duality, Kodaira vanishing theorem and the ampleness of L). Since L is ample and $E(-L)$ is effective, by [11, Example 6.1.23, p. 18] it follows that E is big. \square

We want to show that, nonetheless, the pair (V, E) as above satisfies all but one of the assumptions in Theorem 4.1. As for the regularity of V , we have $q(V) = 0$ by assumption. Moreover, the fact that E is globally generated follows from (4.19) and the global generation of A .

Let us show that $h^i((\det E)^{-1}) = 0$, for any $i \in \{1, 2, 3\}$; if we tensor (4.16) by $\mathcal{O}_V(L)$, we get

$$0 \rightarrow \mathcal{O}_V(-L) \rightarrow \mathcal{O}_V(L) \rightarrow \mathcal{O}_Y(L) = A \rightarrow 0. \quad (4.20)$$

Combining (4.17) and (4.20), we get $c_1(E) = 2L$. Therefore,

$$h^i((\det E)^{-1}) = h^i(-2L) = h^{4-i}(K_V + 2L) = 0, \quad i \in \{1, 2, 3\},$$

the latter equality due to the Kodaira vanishing theorem and the ampleness of L .

As for condition $h^3(E^\vee \otimes (\det E)^{-1}) = 0$, if we tensor (4.17) by $\mathcal{O}_V(-2L)$, we get the following exact sequence:

$$0 \rightarrow E^\vee \otimes \mathcal{O}_V(-2L) \rightarrow H^0(A) \otimes \mathcal{O}_V(-2L) \rightarrow A \otimes \mathcal{O}_V(-2L) \rightarrow 0.$$

By the definition of A and the fact $c_1(E) = 2L$, we have therefore

$$0 \rightarrow E^\vee \otimes (\det E)^{-1} \rightarrow H^0(A) \otimes \mathcal{O}_V(-2L) \rightarrow \mathcal{O}_Y(-L) \rightarrow 0;$$

in order to show that $h^3(E^\vee \otimes (\det E)^{-1}) = 0$, it suffices to prove that $h^2(\mathcal{O}_Y(-L)) = 0 = h^3(\mathcal{O}_V(-2L))$. Notice indeed that $h^2(\mathcal{O}_Y(-L)) = h^1(K_Y + L|_Y) = 0$, as it follows from Serre duality, the Kodaira vanishing theorem and the ampleness of $L|_Y = A$ on Y . Similarly, $h^3(\mathcal{O}_V(-2L)) = h^1(K_V + 2L) = 0$.

From (4.19), one has $rk(E) = h^0(Y, A)$; moreover, since $q(V) = 0$, (4.19) also gives $h^0(V, E) = 2h^0(Y, A) = 2rk(E)$. From (4.20) and $h^0(\mathcal{O}_V(-L)) = 0 = h^1(\mathcal{O}_V(-L))$, it follows that $h^0(Y, A) = h^0(V, L) = x$ so $rk(E) = x$ and $h^0(E) = 2x$. Since $x \geq 4$ by assumption, one has that also condition $h^0(E) \geq rk(E) + 4$ is satisfied.

Finally, notice that, from (1.1) and the fact that $h^0(V, E) = 2rk(E)$, one has $rk(M_E) = rk(E)$; we want to show more precisely that

$$M_E \simeq E^\vee. \quad (4.21)$$

If the previous isomorphism holds true, then we will have

$$h^0(M_E \otimes E) = h^0(E^\vee \otimes E) = h^0(\mathcal{E}nd(E)) \geq 1$$

(as, for any vector bundle, one has $\mathbb{C}^* \subseteq H^0(\mathcal{E}nd(E))$) so from (4.1), μ_E will be not injective.

To prove (4.21), from the cohomology sequence associated to (4.19) and $q(V) = 0$, we get

$$0 \rightarrow H^0(A)^\vee \rightarrow H^0(E) \rightarrow H^0(A) \rightarrow 0.$$

Plugging it within (1.1), (4.17) and (4.19), we get the following commutative diagram

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & H^0(A)^\vee \otimes \mathcal{O}_V & = & H^0(A)^\vee \otimes \mathcal{O}_V & \\ & & & \downarrow & & \downarrow & \\ 0 \rightarrow & M_E & \rightarrow & H^0(E) \otimes \mathcal{O}_V & \longrightarrow & E & \rightarrow 0 \\ & & & \downarrow & & \downarrow & \\ 0 \rightarrow & E^\vee & \rightarrow & H^0(A) \otimes \mathcal{O}_V & \longrightarrow & A & \rightarrow 0 \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & \end{array}$$

which proves (4.21).

REFERENCES

- [1] T. Bauer, S.J. Kovács, A. Küronya, E. C. Mistretta, T. Szemberg, S. Urbinati, On positivity and base loci of vector bundles, *Eur. J. of Math.* **1** (2015), n. 2, 229–249.
- [2] A. Beauville, An ampleness criterion for rank 2 vector bundles on surfaces, *arXiv:1806.00243 [math.AG] 1Jun2018*, 1–4.
- [3] F. Campana, T. Peternell, Projective manifolds whose tangent bundles are numerically effective, *Math. Ann.* **289** (1991), 169–187.
- [4] J.-P. Demailly, *Analytic methods in algebraic geometry. Surveys of Modern Mathematics*, **1**, International Press, Somerville, MA; Higher Education Press, Beijing, 2012.
- [5] J.-P. Demailly, T. Peternell, M. Schneider, Compact complex manifolds with numerically effective tangent bundles, *J. Alg. Geom.* **3** (1994), 295–345.
- [6] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3 Folge, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 1998.

- [7] M. Green, P. Griffiths, Positivity of vector bundles and Hodge theory, *arXiv:1803.07405 [math.AG]* 10Oct2018, 1–97.
- [8] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math. **52**, Springer-Verlag, New York, 1977.
- [9] J.-C. Hsiao, A remark on bigness of the tangent bundle of a smooth projective variety and D -simplicity of its section rings, *J. Algebra and its Applications*, **14** (2015), no. 7, 1550098 (10 pages).
- [10] R. Lazarsfeld, *Positivity in Algebraic Geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3 Folge, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 2004.
- [11] R. Lazarsfeld, *Positivity in Algebraic Geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3 Folge, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 2004.
- [12] R. Muñoz, G. Occhetta, L. E. Solà Conde, K. Watanabe, J. A. Wiśniewski, A survey on the Campana-Peternell conjecture, *Rend. Ist. Mat. dell'Università di Trieste* **47** (2015) 127–185.
- [13] D.-M. Snow, *Homogeneous Vector Bundles*, Notes available at <https://www3.nd.edu/~snow/Papers/HomogVB.pdf>

GILBERTO BINI, DIPARTIMENTO DI MATEMATICA “F. ENRIQUES”, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA C. SALDINI 50, 20133 MILANO – ITALY

E-mail address: `gilberto.bini@unimi.it`

FLAMINIO FLAMINI, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI ROMA “TOR VERGATA”, VIALE DELLA RICERCA SCIENTIFICA 1, 00133 ROMA – ITALY

E-mail address: `flamini@mat.uniroma2.it`