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EXISTENCE AND MULTIPLICITY RESULTS FOR DOUBLE PHASE PROBLEMS WITH VARIABLE EXPONENTS

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Existence and multiplicity results for double phase problems with variable exponents

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"In mathematics, the path is often more important than the destination."

R. Lipschitz

Dedicated to my family:
Adriana, Giuseppe,
Irene, Angela,
Nonna Dina.

UNIVERSITY OF PALERMO

Abstract

Department of Mathematics and Computer Sciences

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Existence and multiplicity results for double phase problems with variable exponents

by Eleonora AMOROSO

The double phase operator is a differential operator that finds applications in several fields, among the others elasticity theory, biophysics, quantum physics, chemical reactions. The aim of this thesis is to present existence and multiplicity results for nonlinear differential equations involving the double phase operator with variable exponents, under different boundary conditions. In addition, the problems considered are parameter-dependent and an interval of parameters for which every problem admits solutions is also provided. The investigation is based on variational methods and precisely critical point theory; indeed, the main tools are the Mountain Pass theorem due to A. Ambrosetti e P. Rabinowitz, a two critical points theorem established by G. Bonanno and G. D'Aguì and the Nehari manifold method.

The first problem considered is a Dirichlet double phase problem where if the nonlinearity has a subcritical growth and verifies a superlinear condition, the existence of two nontrivial weak solutions with opposite energy sign is established. These results are contained in [6], in collaboration with G. Bonanno, G. D'Aguì and P. Winkert.

Then, a Robin double phase problem with critical growth on the boundary is studied. In particular, if the nonlinear term has a subcritical growth and satisfies the classical Ambrosetti-Rabinowitz condition, the existence of two nontrivial weak solutions with opposite energy sign is guaranteed. These results are presented in [4], in collaboration with V. Morabito.

Finally, the last part of the thesis is dedicated to the study of a nonlinear double phase problem with nonlinear Neumann boundary condition. Under very general assumptions on the nonlinearity, the existence of three nontrivial weak solutions is obtained. Specifically, a solution is nonnegative, another one is nonpositive and the third one is sign-changing with exactly two nodal domains. These results are obtained in [5], in collaboration with Á. Crespo-Blanco, P. Pucci and P. Winkert.

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Contents

Chapter 1

Introduction

The main purpose of this thesis is to present some original results on the existence and multiplicity of solutions for different nonlinear differential problems involving the double phase operator with variable exponents. Through critical point theory, the existence of two solutions for a class of double phase problems with either Dirichlet condition or Robin condition is studied. In addition, the existence of three solutions for Neumann nonlinear differential problem is investigated, also exploiting the Nehari manifold method.

Since the approach of the present thesis is variational, it is worth emphasizing the importance of Calculus of Variations in the most diverse fields of science. There are a great number of problems that are referable to the variational logic scheme and in which a functional on an assigned constraint is considered in order to study, for instance, its minimum points or stationary points or decreasing curves, according to appropriate definitions of these concepts.

To give a few examples, it is necessary to recall first the historically and scientifically important role that Calculus of Variations plays in physics, as in classical mechanics with the great variational principles that preside over the dynamics of systems, in the theory of elasticity, in quantum mechanics, in modern developments in the study of the microstructure of materials, in economics, computer science, biological or artificial neural networks, control systems, and in numerous other fields.

By applying variational methods, one can obtain more precise results than with respect to other methods, for instance on the multiplicity of solutions. In Calculus of Variations, finding the solutions of problems is equivalent to finding the critical points of certain functionals. In classical theory, this means considering a functional J defined on a Banach space that admits derivative according to Gâteaux $J' : X \to X^*$, and solving the Euler equation

$$J'(x) = 0.$$

One of the approaches to finding the critical points of a functional is to search for the points of global minimum (maximum). This type of study, which originates from the famous Weierstrass theorem on the extremes of real continuous functions, is known as *direct method* because it solves the Euler equation by investigating the properties of the functional directly. It can be summarized as follows

$$\begin{aligned} \text{coercivity} + \text{lower semicontinuity} \quad &\Longrightarrow \quad \text{existence of a minimum}, \\ \text{existence of a minimum} \quad &\Longrightarrow \quad J'(x) = 0. \end{aligned}$$

Coercivity is a compactness concept. A more general compactness concept that can be exploited in order to ensure the existence of a critical point is a compactness condition known as *Palais-Smale condition*.

The theory of critical points in infinite dimensional spaces for functionals of class C^1 developed in the second half of the 20th century. The most important result within this theory, besides the Direct Methods theorem, is certainly the Mountain Pass theorem, obtained in 1973 by A. Ambrosetti and P. Rabinowitz [2]. This theorem can be applied for functionals that are unbounded from below, differently from the Direct Methods one. This result, recalled in this thesis in Theorem 3.1.5, states that if a functional F of class C^1 satisfies the Palais-Smale condition and the mountain pass geometry, namely

$$\alpha = \max\{\gamma(0), \gamma(u_1)\} < \inf_{u \in B(0,\rho)} F(u) = \beta,$$

then it admits a critical point. Versions of the Mountain Pass theorem were already known by M. Morse and C.B. Tompkins [73], M. Shiffman [90] and R. Courant [37]. In the result of A. Ambrosetti and P. Rabinowitz it is assumed that $\alpha < \beta$, however the limiting case when $\alpha = \beta$ was studied by P. Pucci and J. Serrin [83, 84, 85]. With the aid of the dual family, N. Ghoussoub and D. Preiss [59] observed that one may get more informations on the localization of the mountain pass points. Both the Direct Methods theorem and the Mountain Pass theorem have been (and still are) used to establish the existence of at least one solution for various classes of nonlinear differential problems.

Summarizing, Direct Methods theorem implies the existence of a global minimum and Mountain Pass theorem guarantees the existence of a critical point that is not of minimum. An important chapter within critical point theory is the existence of a local minimum and it is worth mentioning the result of B. Ricceri [87] (2000). A more precise version of this result is due to G. Bonanno and P. Candito [20] in 2008. Within this context, in 2012 G. Bonanno [17] generalized these results, ensuring the existence of a local minimum for a continuously Gâteaux differentiable functional which is unbounded from below without requiring any weak continuity assumption. The variational formulation that is considered is

$$I = \Phi - \Psi,$$

where Φ and Ψ are continuously Gâteaux differentiable functionals defined on an infinite dimensional real Banach space. In fact, in the differential problems the associated energy functional is of this type. In this work, G. Bonanno presented a new definition of the Palais-Smale condition, which is more general than the usual one, a mountain pass theorem that allows the location of the critical point due to the introduced Palais-Smale condition and, as a result, several critical point theorems are then established. The main tool used for the proofs is a consequence of the classical Ekeland variational principle constructed within a nonsmooth framework. A nontrivial consequence of the local minimum theorem by G. Bonanno in combination with the Ambrosetti-Rabinowitz theorem is obtained by G. Bonanno and G. D'Aguì [23] in 2016, which ensures the existence of two nontrivial critical points.

Now, the details of the nonlinear differential problems that are the subject of this thesis are presented. In recent years, the differential operator known as "double phase operator" has found application in numerous research fields. Given a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, the double phase operator is defined as follows

$$u \mapsto -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right),$$

for every function u belonging to a suitable function space, with appropriate regularity assumptions on the variable exponents p and q and on the weight function μ. The energy functional associated with the double phase operator is given by

$$I(u) = \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right) \mathrm{d}x,$$

and its integrand, that is

$$H(x,\xi) = \frac{1}{p(x)}|\xi|^{p(x)} + \frac{\mu(x)}{q(x)}|\xi|^{q(x)} \quad \text{for all } (x,\xi) \in \Omega \times \mathbb{R}^N,$$

has unbalanced growth if μ is nonnegative and belongs to $L^\infty(\Omega)$, namely

$$b_1|\xi|^{p(x)} \leq H(x,\xi) \leq b_2\left(1 + |\xi|^{q(x)}\right) \quad \text{for a. a. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^N,\ b_1, b_2 > 0.$$

The functional I changes ellipticity on the set where the weight function μ vanishes. In fact, the energy density of I presents ellipticity in the gradient of order $q(x)$ in the set $\{x \in \Omega : \mu(x) > \tau\}$ for any fixed $\tau > 0$ and of order $p(x)$ in the set $\{x \in \Omega : \mu(x) = 0\}$. Thus, the integrand H alternates between two different phases of elliptic behavior and this is why the differential operator is called *double phase*.

This type of functionals has been first introduced in 1986 by Zhikov [97] with constant exponents. Subsequently, numerous authors have investigated problems involving this operator, which has been employed to model a range of phenomena. Among the topics, it is worth mentioning first the elasticity theory in which the double phase operator describes the behavior of strongly anisotropic materials, whose hardening properties are related to the exponents $p(\cdot)$ and $q(\cdot)$ and significantly change with the point (since they are variables) and the coefficient $\mu(\cdot)$ determines the geometry of a composite made of two different materials, see Zhikov [98]. Moreover, one can found other applications in the papers of Bahrouni-Rădulescu-Repovš [7] on transonic flows, Benci-D'Avenia-Fortunato-Pisani [14] on quantum physics and Zhikov [98] on the Lavrentiev gap phenomenon, the thermistor problem and the duality theory. For a mathematical study of such integral functionals with (p,q)-growth we refer to Baroni-Colombo-Mingione [10, 11, 12], Colombo-Mingione [35, 36], Cupini-Marcellini-Mascolo [41], De Filippis-Mingione [45], Marcellini [70, 71, 72], Ragusa-Tachikawa [86], see also the papers of Beck-Mingione [13] and De Filippis-Mingione [44] for nonautonomous integrals.

In the context of partial differential equations, the double phase operator emerges from the investigation of general reaction-diffusion equations including nonhomogeneous diffusion and transport aspects. These nonhomogeneous operators find applications in diverse fields, including biophysics, plasma physics, and chemical reactions, where the function u denotes the concentration term and the differential operator represents the diffusion coefficient.

Furthermore, the double phase operator generalizes other differential operators which have been extensively studied in the literature. Indeed, when $\inf_{\overline{\Omega}} \mu > 0$ it reduces to the (p,q)-Laplacian or $(p(\cdot),q(\cdot))$-Laplacian and for problems involving this type of operators we refer without sake of completeness to Chinnì-Sciammetta-Tornatore [32], Papageorgiou-Qin-Radulescu [76], Papageorgiou-Winkert [80], Pucci [82] and the references therein. On the other hand, if $\mu \equiv 0$ the double phase operator reduces to the p-Laplacian or $p(x)$-Laplacian, see for instance Amoroso-Bonanno-Perera [3], Barletta-Chinnì-O'Regan [9], Bonanno-Candito [19], Bonanno-Chinnì [21], Bonanno-D'Aguì [22, 24], Bonanno-D'Aguì-Papageorgiou [25], Bonanno-D'Aguì-Sciammetta [26], Bonanno-Livrea [27], Candito-Guarnotta-Perera [30], D'Aguì-Sciammetta [42] and the references therein.

In recent years, many authors have investigated double phase problems in the constant exponents context, see for instance Biagi-Esposito-Vecchi [15], Colasuonno-Perera [33], Colasuonno-Squassina [34], Crespo-Blanco-Papageorgiou-Winkert [38], DAgui-Sciammetta-Tornatore-Winkert [43], Farkas-Winkert [53], Fiscella [54], Gasiński-Papageorgiou [55], Gasiński-Winkert [56, 57], Ge-Pucci [58], Liu-Dai [68], Liu-Papageorgiou [69], Papageorgiou-Rădulescu-Repovš [77], Perera-Squassina [81], Sciammetta-Tornatore-Winkert[89], Stegliński [91], Zeng-Bai-Gasiński-Winkert [96] and the references therein.

On the other hand, there are much fewer results for the variable exponents case, see Amoroso-Bonanno-D'Aguì-Winkert [6], Amoroso-Crespo-Blanco-Pucci-Winkert [5], Amoroso-Morabito [4], Bahrouni-Rădulescu-Winkert [8], Cen-Kim-Kim-Zeng [31], Crespo-Blanco-Gasiński-Harjulehto-Winkert [40], Crespo-Blanco-Winkert [39], Leonardi-Papageorgiou [66], Liu-Pucci [67], Kim-Kim-Oh-Zeng [63], Ragusa-Tachikawa [86], Vetro-Winkert [93] and Zeng-Rădulescu-Winkert [95].

This thesis is dedicated to the the study of nonlinear differential problems involving the double phase operator with variable exponents under different boundary conditions. In particular, existence and multiplicity results are obtained under general assumptions, which in some cases (when specified) are new and optimal. In detail, the thesis is organized as follows.

In Chapter 2, some function spaces that are needed for the investigation are introduced. Specifically, the classical Lebesgue and Sobolev spaces with both constant and variable exponents are presented, together with their basic properties and embeddings. Then, an overview of general Musielak-Orlicz and Musielak-Orlicz Sobolev spaces is given and finally the function space that is involved in the study of double phase problems with variable exponents is introduced and studied in detail. In addition, a new equivalent norm for this space is presented and it has been first introduced by Amoroso-Crespo-Blanco-Pucci-Winkert [5].

In Chapter 3, some classical results of variational methods are presented, as the Direct Methods theorem and the Ambrosetti-Rabinowitz theorem, and then some recent results in the field of critical point theory due to G. Bonanno and G. D'Aguì are introduced, together with some of their consequences.

Chapter 4 is dedicated to the following parametric Dirichlet problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) &= \lambda f(x,u) \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \qquad (D_\lambda)$$

for which the existence of two bounded weak solutions is determined under very general assumptions on the nonlinear term, such as a subcritical growth and a superlinear condition. Also, a range for the parameter λ for which the problem admits such solutions is established and some special cases in which the solutions turn out to be nonnegative are presented. The main tool of this investigation is a two critical poins theorem due to G. Bonanno and G. D'Aguì. The results presented in this chapter are obtained in [6], in collaboration with G. Bonanno, G. D'Aguì and P. Winkert.

Chapter 5 deals with the study of a nonlinear parametric double phase problem under Robin boundary conditions with critical growth, that is

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)+\alpha(x)|u|^{p(x)-2}u&=\lambda f(x,u) \quad \text{in } \Omega,\\ \left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nu&=-\beta(x)|u|^{p_*(x)-2}u \quad \text{on } \partial\Omega. \end{aligned} \tag{R_λ}$$

In particular, an interval of parameters that guarantees the existence of two nontrivial weak solutions is determined and under additional conditions the solutions turn out to be nonnegative. Also for this problem, the proof is based on a two critical points theorem stated by G. Bonanno and G. D'Aguì in 2016. The results presented in this chapter are contained in [4], in collaboration with V. Morabito.

Finally, Chapter 6 is dedicated to the following double phase problem with nonlinear Neumann boundary condition

$$\begin{aligned} -\operatorname{div}\mathcal{F}(u)+|u|^{p(x)-2}u&=f(x,u) && \text{in } \Omega,\\ \mathcal{F}(u)\cdot\nu&=g(x,u)-|u|^{p(x)-2}u && \text{on } \partial\Omega, \end{aligned} \tag{N_λ}$$

for which the existence of multiple bounded solutions under very general assumptions on the nonlinearities is proved. Specifically, two constant sign solutions are obtained via a mountain-pass approach and the existence of a third solution, which is sign-changing, is proved through the Nehari manifold method. Finally, informations on the nodal domains of this sign-changing solution are given. The results presented in this chapter are obtained in [5], in collaboration with Á. Crespo-Blanco, P. Pucci and P. Winkert.

Chapter 2

Function spaces: introduction and basic properties

In this chapter we introduce some function spaces that are needed in our investigation, providing the main properties and embeddings of such spaces. Function spaces, especially Sobolev spaces, are fundamental to the analysis of differential problems using a variational approach. Sobolev spaces allow the treatment of functions that have weak derivatives up to a certain order, thus extending the classical notion of derivatives. This is crucial for studying differential equations in contexts where the solutions are not necessarily classically differentiable. In the variational approach, the differential problem is reformulated as a search for a minimum of a function defined over a Sobolev space, with existence of solutions.

In Section 2.1 we introduce the classical Lebesgue and Sobolev spaces with constant exponents, while in Section 2.2 we deal with the variable exponent case. Then, Section 2.3 gives an overview of general Musielak-Orlicz and Musielak-Orlicz Sobolev spaces and in Subsection 2.3.1 we study in detail the function space that is involved in the investigation of existence and multiplicity results of solutions for double phase problems with variable exponents. Finally, in Subsection 2.3.2 we introduce a new equivalent norm in a Musielak-Orlicz Sobolev space.

The result presented in Subsection 2.3.2 are obtained in [5, Section 3], in collaboration with Á. Crespo-Blanco, P. Pucci and P. Winkert.

2.1 Lebesgue and Sobolev spaces with constant exponents

Let $\Omega \subset \mathbb{R}^N$ be an open set and denote by $M(\Omega)$ the space of all measurable functions $u\colon \Omega \to \mathbb{R}$. For any $1 \leq r < +\infty$, $L^r(\Omega)$ indicates the usual *Lebesgue space* given by

$$L^r(\Omega) = \left\{ u \in M(\Omega) : \left(\int_\Omega |u|^r \,\mathrm{d}x \right)^{\frac{1}{r}} < +\infty \right\},$$

equipped with the norm

$$\|u\|_r = \left(\int_\Omega |u|^r \,\mathrm{d}x \right)^{\frac{1}{r}}.$$

We also define

$$L^\infty(\Omega) = \{ u \in M(\Omega) : |u| \leq C \text{ a.e. in } \Omega \text{ for some } C \geq 0 \},$$

endowed with the essential supremum norm

$$\|u\|_\infty = \inf\{C \geq 0 : |u(x)| \leq C \text{ for a.a. } x \in \Omega\}.$$

In the following, we summarize the main well-known properties of the Lebesgue spaces, see for instance the books of Brezis [29, Chapter 4] and Papageorgiou-Winkert [79, Theorem 2.3.19 and Paragraph 4.1].

Proposition 2.1.1. *The Lebesgue space $(L^r(\Omega), \|\cdot\|_r)$ is:*

- *a Banach space for any $1 \leq r \leq +\infty$;*
- *reflexive and uniformly convex for any $1 < r < +\infty$;*
- *separable for any $1 \leq r < +\infty$.*

Given $1 < r < +\infty$, we say that r' is the *conjugate exponent* of r if

$$\frac{1}{r} + \frac{1}{r'} = 1,$$

namely $r' = \frac{r}{r-1}$. Also, if $r = 1$ then $r' = +\infty$ and if $r = +\infty$ then $r' = 1$. In the following, we recall a basic and important inequality, see for instance Brezis[29, Theorem 4.6]

Proposition 2.1.2 (Hölder's inequality). *Suppose that $f \in L^r(\Omega)$ and $g \in L^{r'}(\Omega)$, with $1 \leq r \leq +\infty$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \leq \|f\|_r \|g\|_{r'}.$$

Now, we introduce the Sobolev space and its main properties. To this aim, we premit a definition and we remind that $C^1_c(\Omega)$ denotes the space of the functions of class C^1 with compact support.

Definition 2.1.3. *Given $u \in L^r(\Omega)$, we define the distributional derivative of u as follows*

$$\frac{\partial u}{\partial x_k}(\varphi) = -\int_\Omega u \frac{\partial \varphi}{\partial x_k}\,\mathrm{d}x \qquad \textit{for all } \varphi \in C^1_c(\Omega).$$

Moreover, if $\dfrac{\partial u}{\partial x_k} \in L^r(\Omega)$, then

$$\int_\Omega \frac{\partial u}{\partial x_k}\varphi\,\mathrm{d}x = -\int_\Omega u \frac{\partial \varphi}{\partial x_k}\,\mathrm{d}x \qquad \textit{for all } k = 1, \ldots, N.$$

We emphasize that if u is differentiable in the classical sense, then the distributional derivative is equivalent to the classical one. For $1 \leq r < +\infty$, we indicate with $W^{1,r}(\Omega)$ the Sobolev space defined by

$$W^{1,r}(\Omega) = \left\{u \in L^r(\Omega) : \frac{\partial u}{\partial x_k} \in L^r(\Omega) \text{ for all } k = 1, \ldots, N\right\},$$

endowed with the usual norm

$$\|u\|_{1,r} = \|u\|_r + \|\nabla u\|_r,$$

or with the equivalent norm

$$\|u\|_{W^{1,r}(\Omega)} = (\|u\|_r^r + \|\nabla u\|_r^r)^r,$$

where $\|\nabla u\|_r = \||\nabla u|\|_r$. Furthermore, in correspondence of $r = +\infty$ we introduce

$$W^{1,\infty}(\Omega) = \left\{ u \in L^\infty(\Omega) : \frac{\partial u}{\partial x_k} \in L^\infty(\Omega) \text{ for all } k = 1, \dots, N \right\},$$

equipped with the norm

$$\|u\|_{1,\infty} = \max \left\{ \|u\|_\infty, \left\| \frac{\partial u}{\partial x_1} \right\|_\infty, \left\| \frac{\partial u}{\partial x_2} \right\|_\infty, \dots, \left\| \frac{\partial u}{\partial x_N} \right\|_\infty \right\}.$$

Here, we mention the properties of such spaces, see Brezis[29, Proposition 9.1] and Papageorgiou-Winkert[79, Proposition 4.5.9].

Proposition 2.1.4. *The Sobolev space $W^{1,r}(\Omega)$ is:*

- *a Banach space for any $1 \le r \le +\infty$;*
- *reflexive for any $1 < r < +\infty$;*
- *separable for any $1 \le r < +\infty$;*
- *a Hilbert space for $r = 2$.*

An important result in this topic is the so-called Rellich-Kondrachov Theorem about the embeddings of the Sobolev spaces into the Lebesgue ones, we refer to Adams-Fournier[1, Theorem 6.3]. We denote by r^* the critical Sobolev exponent of r, given by

$$r^* = \begin{cases} \frac{Nr}{N-r} & \text{if } r < N, \\ \text{any } h \in (r, +\infty) & \text{if } r \ge N, \end{cases}$$

and by r_* the critical Sobolev exponent of r on the boundary, defined by

$$r_* = \begin{cases} \frac{(N-1)r}{N-r} & \text{if } r < N, \\ \text{any } h \in (r, +\infty) & \text{if } r \ge N, \end{cases}$$

Theorem 2.1.5 (Rellich-Kondrachov Theorem). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary $\partial\Omega$ and $1 \le r < +\infty$. Then, the following embeddings hold:*

(i) *if $1 \le r < N$, the embedding $W^{1,r}(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for all $1 \le s \le r^*$ and it is compact if $1 \le s < r^*$;*

(ii) *if $r = N$, the embedding $W^{1,r}(\Omega) \hookrightarrow L^s(\Omega)$ is continuous and compact for all $1 \le s \le +\infty$;*

(iii) *if $r > N$, the embedding $W^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ is continuous and compact.*

Moreover, let σ be the $(N-1)$-dimensional Hausdorff measure on the boundary $\partial\Omega$ and indicate with $L^r(\partial\Omega)$ the boundary Lebesgue space endowed with the usual norm $\|\cdot\|_{r,\partial\Omega} = \left(\int_{\partial\Omega} |\cdot|^r \, d\sigma\right)^{\frac{1}{r}}$. We can consider a trace operator, i.e. a continuous linear operator $\gamma_0 \colon W^{1,r}(\Omega) \to L^r(\partial\Omega)$, such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,r}(\Omega) \cap C(\overline{\Omega}).$$

In Kufner-John-Fučik[65, Theorem 6.10.5] the following result can be found.

Theorem 2.1.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary $\partial\Omega$ and $1 \le r < +\infty$. Then, the following hold:*

(i) *if $1 \le r < N$, then $\gamma_0 : W^{1,r}(\Omega) \to L^s(\partial\Omega)$ is continuous for all $1 \le s \le r_*$;*

(ii) *if $1 < r < N$, then $\gamma_0 : W^{1,r}(\Omega) \to L^s(\partial\Omega)$ is compact for all $1 \le s < r_*$;*

(iii) *if $r \ge N$, then $\gamma_0 : W^{1,r}(\Omega) \to L^s(\partial\Omega)$ is compact for all $s \ge 1$.*

Let $C_c^\infty(\Omega)$ be the space of the C^∞-functions with compact support, also called *test functions*. We denote by $W_0^{1,r}(\Omega)$ the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,r}$. However, a well-known result called Poincaré's inequality, that we recall in the following, allows us to equip the space $W_0^{1,r}(\Omega)$ with another simpler norm.

Theorem 2.1.7 (Poincaré's inequality). *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and $1 \le r < +\infty$. Then, there exists a constant C, depending from r, N, Ω, such that*

$$\|u\|_r \le C\|\nabla u\|_r \qquad \text{for all } \; u \in W_0^{1,r}(\Omega),$$

and $\|\cdot\|_{1,r,0} = \|\nabla \cdot\|_r$ is an equivalent norm on $W_0^{1,r}(\Omega)$.

2.2 Lebesgue and Sobolev spaces with variable exponents

In this section, we present the Lebesgue and Sobolev spaces with variable exponent and we recall some useful properties. In 1931, Orlicz[75] was the first who introduced the variable exponent Lebesgue space using the $\Phi-$functions theory requiring that the exponent is finite. Lately, in 1979 Sharapudinov[88] gave the definition including the infinity case and then in 1991 Kovačik-Rákosník[64] considered the higher dimensional case.

We underline that the variable exponent Lebesgue spaces are exactly Musielack-Orlicz spaces related to specific $\Phi-$functions (see Section 2.3). However, in this section we introduce them directly, avoiding this steps to make it readable. For more details, we refer to Diening-Harjulehto-Hästö-Růžička[46] and Fan-Zhao[52].

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary $\partial\Omega$ and for every $r \in M(\Omega)$, put

$$r_- := \operatorname*{essinf}_{x\in\Omega} r(x) \quad \text{and} \quad r_+ := \operatorname*{esssup}_{x\in\Omega} r(x), \tag{2.2.1}$$

and set

$$M_+(\Omega) = \{r \in M(\Omega) \,:\, r_- \ge 1\}.$$

For any $r \in M_+(\Omega)$, we introduce the modular function $\rho_{r(\cdot)}$ given by

$$\rho_{r(\cdot)}(u) = \int_\Omega |u|^{r(x)} \, dx,$$

and we define the Lebesgue space with variable exponent as follows

$$L^{r(\cdot)}(\Omega) = \{u \in M(\Omega) \,:\, \rho_{r(\cdot)}(u) < +\infty\},$$

equipped with the so-called Luxemburg norm

$$\|u\|_{r(\cdot)} = \inf\left\{\tau > 0 \,:\, \rho_{r(\cdot)}\left(\frac{u}{\tau}\right) \le 1\right\}.$$

As for the constant exponent case treated in Section 2.1, also in the variable exponent case the Lebesgue space $L^{r(\cdot)}(\Omega)$ has the following properties, see the book of Diening-Harjulehto-Hästö-Růžička [46, Theorems 3.2.7, 3.4.7, 3.4.9 and Corollary 3.4.5].

Proposition 2.2.1. *For any $r \in M_+(\Omega)$, the Lebesgue space $(L^{r(\cdot)}(\Omega), \|\cdot\|_{r(\cdot)})$ is:*

- *a Banach space;*
- *reflexive and uniformly convex when $1 < r_- \le r_+ < +\infty$;*
- *separable when $r_+ < +\infty$.*

Now, we summarize the properties about the relation between the norm $\|\cdot\|_{r(\cdot)}$ and the modular $\rho_{r(\cdot)}$, see Fan-Zhao [52].

Proposition 2.2.2. *Let $r \in C_+(\overline{\Omega})$, $u \in L^{r(\cdot)}(\Omega)$ and $\tau \in \mathbb{R}$. Then the following hold:*

(i) *If $u \neq 0$, then $\|u\|_{r(\cdot)} = \tau \iff \rho_{r(\cdot)}(\frac{u}{\tau}) = 1$;*

(ii) $\|u\|_{r(\cdot)} < 1$ *(resp. > 1, $= 1$)* $\iff \rho_{r(\cdot)}(u) < 1$ *(resp. > 1, $= 1$);*

(iii) *If $\|u\|_{r(\cdot)} < 1 \implies \|u\|_{r(\cdot)}^{r_+} \le \rho_{r(\cdot)}(u) \le \|u\|_{r(\cdot)}^{r_-}$;*

(iv) *If $\|u\|_{r(\cdot)} > 1 \implies \|u\|_{r(\cdot)}^{r_-} \le \rho_{r(\cdot)}(u) \le \|u\|_{r(\cdot)}^{r_+}$;*

(v) $\|u\|_{r(\cdot)} \to 0 \iff \rho_{r(\cdot)}(u) \to 0$;

(vi) $\|u\|_{r(\cdot)} \to 1 \iff \rho_{r(\cdot)}(u) \to 1$;

(vii) $\|u\|_{r(\cdot)} \to +\infty \iff \rho_{r(\cdot)}(u) \to +\infty$;

(viii) *If $u_n \to u$ in $L^{r(\cdot)}(\Omega) \implies \rho_{r(\cdot)}(u_n) \to \rho_{r(\cdot)}(u)$.*

The following Hölder's inequality holds, see Diening-Harjulehto-Hästö-Růžička [46, Lemma 3.2.20].

Proposition 2.2.3 (Hölder's inequality). *Let $p, q, s \in M_+(\Omega)$ such that*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \quad \text{for a.a. } x \in \Omega,$$

and suppose that $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$. Then $uv \in L^{s(\cdot)}(\Omega)$ and

$$\rho_{s(\cdot)}(uv) \le \rho_{p(\cdot)}(u) + \rho_{q(\cdot)}(v),$$
$$\|uv\|_{s(\cdot)} \le 2\|u\|_{p(\cdot)}\|v\|_{q(\cdot)}.$$

In particular, denoting by $r' \in M_+(\Omega)$ the *conjugate variable exponent* to r, that is

$$\frac{1}{r(x)} + \frac{1}{r'(x)} = 1 \quad \text{for all } x \in \Omega,$$

we have that the dual space is $L^{r(\cdot)}(\Omega)^* = L^{r'(\cdot)}(\Omega)$ (see [46, Theorem 3.2.13]) and

$$\|uv\|_1 \leq 2\|u\|_{r(\cdot)}\|v\|_{r'(\cdot)}, \tag{2.2.2}$$

for all $u \in L^{r(\cdot)}(\Omega)$ and for all $v \in L^{r'(\cdot)}(\Omega)$. Moreover, for $r_1, r_2 \in M_+(\Omega)$ with $r_1(x) \leq r_2(x)$ for a.a. $x \in \Omega$ and $1 \in L^{r(\cdot)}(\Omega)$, where

$$\frac{1}{k(\cdot)} = \max\left\{\frac{1}{q(\cdot)} - \frac{1}{p(\cdot)}, 0\right\},$$

we have the continuous embedding

$$L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega), \tag{2.2.3}$$

with

$$\|u\|_{r_1(\cdot)} \leq 2\|1\|_{k(\cdot)}\|u\|_{r_2(\cdot)} \qquad \text{for all } u \in L^{r_2(\cdot)}(\Omega),$$

which follows from (2.2.2). Furthermore, we define the weighted Lebesgue spaces with variable exponents as follows, that will be useful in the sequel. For any $r \in M_+(\Omega)$ and $\omega \in L^1(\Omega)$, $\omega \geq 0$, we introduce the modular

$$\rho_{r(\cdot),\omega}(u) = \int_\Omega \omega(x)|u|^{r(x)}\, \mathrm{d}x,$$

and we define the space

$$L^{r(\cdot)}_\omega(\Omega) = \left\{u \in M(\Omega) \,:\, \rho_{r(\cdot),\omega}(u)\, \mathrm{d}x < +\infty\right\},$$

equipped with the seminorm

$$\|u\|_{r(\cdot),\omega} = \inf\left\{\lambda > 0 \,:\, \rho_{r(\cdot),\omega}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Now, we introduce the variable exponent Sobolev space $W^{1,r(\cdot)}(\Omega)$. Given $r \in M_+(\Omega)$, we define it by

$$W^{1,r(\cdot)}(\Omega) = \left\{u \in L^{r(\cdot)}(\Omega) \,:\, |\nabla u| \in L^{r(\cdot)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,r(\cdot)} = \|u\|_{r(\cdot)} + \|\nabla u\|_{r(\cdot)},$$

where clearly $\|\nabla u\|_{r(\cdot)} = \|\,|\nabla u|\,\|_{r(\cdot)}$. In the following proposition we give the main properties of the Sobolev space $W^{1,r(\cdot)}(\Omega)$ which can be found in the book of Diening-Harjulehto-Hästö-Růžička [46, Theorem 8.1.6].

Proposition 2.2.4. *For any $r \in M_+(\Omega)$, the Sobolev space $W^{1,r(\cdot)}(\Omega)$ is*

- *a Banach space;*
- *reflexive and uniformly convex if $1 < r_- \leq r_+ < +\infty$;*
- *separable if $r_+ < +\infty$.*

For any $r \in M_+(\Omega)$, we indicate with r^* and r_* the *critical Sobolev exponents* to r, given by

$$r^*(x) = \begin{cases} \frac{Nr(x)}{N-r(x)} & \text{if } r(x) < N, \\ \infty & \text{if } r(x) \geq N, \end{cases} \qquad r_*(x) = \begin{cases} \frac{(N-1)r(x)}{N-r(x)} & \text{if } r(x) < N, \\ \infty & \text{if } r(x) \geq N. \end{cases} \tag{2.2.4}$$

Here, we present the embeddings of the Sobolev spaces $W^{1,r(\cdot)}(\Omega)$ into the Lebesgue ones, depending on the relation between the exponent $r(\cdot)$ and the dimension N of the space. To this aim, we denote by $C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ the set of all functions $s \colon \overline{\Omega} \to \mathbb{R}$ that are log-Hölder continuous, that is, there exists a constant $C > 0$ such that

$$|s(x) - s(y)| \leq \frac{C}{|\log|x-y||} \quad \text{for all } x, y \in \overline{\Omega} \text{ with } |x-y| < \frac{1}{2}.$$

Moreover, if $r \in C(\overline{\Omega})$ clearly from (2.2.1) it follows that

$$r_+ = \max_{x \in \overline{\Omega}} r(x) \quad \text{and} \quad r_- = \min_{x \in \overline{\Omega}} r(x), \tag{2.2.5}$$

and we set

$$C_+(\overline{\Omega}) = \{r \in C(\overline{\Omega}) \,:\, r_- > 1\}.$$

For the following results we refer to Diening-Harjulehto-Hästö-Růžička [46, Corollary 8.3.2] and Fan-Shen-Zhao [51] where the condition $r_+ < N$ is required and Fan [49, Proposition 2.2] where there is no restriction on r_+.

Proposition 2.2.5. *Let $r \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega}) \cap C_+(\overline{\Omega})$ and let $s \in C(\overline{\Omega})$ be such that*

$$1 \leq s(x) \leq r^*(x) \quad \textit{for all } x \in \overline{\Omega}.$$

Then, the embedding $W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is continuous. If $r \in C_+(\overline{\Omega})$, $s \in C(\overline{\Omega})$ and $1 \leq s(x) < r^*(x)$ for all $x \in \overline{\Omega}$, then the embedding above is compact.

Proposition 2.2.6. *Let $r \in C_+(\overline{\Omega})$ such that $r_- > N$. Then $W^{1,r(\cdot)}(\Omega) \subset C(\overline{\Omega})$.*

Furthermore, let σ be the $(N-1)$-dimensional Hausdorff measure on the boundary $\partial\Omega$ and denote with $L^{r(\cdot)}(\partial\Omega)$ the boundary Lebesgue space endowed with the usual norm $\|\cdot\|_{r(\cdot),\partial\Omega}$ with corresponding modular $\rho_{r(\cdot),\partial\Omega}(\cdot)$. In the following proposition we present some embedding results into the boundary Lebesgue space, see Ho-Kim-Winkert-Zhang [61, Proposition 2.5] for the continuous embedding and Fan [48, Corollary 2.4] for the compact one.

Proposition 2.2.7. *Suppose that $r \in C_+(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$. Let $s \in C(\overline{\Omega})$ be such that*

$$1 \leq s(x) \leq r_*(x) \quad \textit{for all } x \in \overline{\Omega}.$$

Then, the embedding $W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\partial\Omega)$ is continuous. If $r \in C_+(\overline{\Omega})$, $s \in C(\overline{\Omega})$ and $1 \leq s(x) < r_*(x)$ for all $x \in \overline{\Omega}$, then the embedding is compact.

Hence, we can consider a trace operator

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,r(\cdot)}(\Omega) \cap C(\overline{\Omega}), \tag{2.2.6}$$

as in Proposition 2.2.7. In this thesis, we avoid the notation of the trace operator and we consider all the restrictions of Sobolev functions to the boundary $\partial\Omega$ in the sense of traces.

Remark 2.2.8. *Note that for a bounded domain $\Omega \subset \mathbb{R}^N$ and $\gamma > N$ we have the following inclusions*

$$C^{0,1}(\overline{\Omega}) \subset W^{1,\gamma}(\Omega) \subset C^{0,1-\frac{N}{\gamma}}(\overline{\Omega}) \subset C^{0,\frac{1}{|\log t|}}(\overline{\Omega}).$$

Finally, we denote by $W_0^{1,r(\cdot)}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{1,r(\cdot)}(\Omega)$ with respect to the norm $\|\cdot\|_{1,r(\cdot)}$. Also in the variable exponent case, there exists a Poincaré's inequality that we recall in the following proposition, see Diening-Harjulehto-Hästö-Růžička [46, Theorem 8.2.4].

Theorem 2.2.9 (Poincaré's inequality). *Let $r \in M_+(\Omega) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ be such that $r_+ < \infty$. Then, there exists a constant C, depending from r, N, Ω, such that*

$$\|u\|_{r(\cdot)} \leq C\|\nabla u\|_{r(\cdot)} \qquad \text{for all } u \in W_0^{1,r(\cdot)}(\Omega),$$

and $\|\cdot\|_{1,r(\cdot),0} = \|\nabla \cdot\|_{r(\cdot)}$ is an equivalent norm on $W_0^{1,r(\cdot)}(\Omega)$.

2.3 Musielak-Orlicz and Musielak-Orlicz Sobolev spaces

This section is devoted to introduce the Musielak-Orlicz spaces, whose study is important since the weak solutions of partial differential equations are functions belonging to Musielak-Orlicz Sobolev spaces. In particular, we will focus on a specific Musielak-Orlicz Sobolev space involved in the study of our double phase problems. For more details on the general spaces we refer to the books of Musielak [74], Harjulehto-Hästö [60] and Diening-Harjulehto-Hästö-Růžička [46] as well as the papers of Colasuonno-Squassina [34] and Fan [50].

First, we recall some definitions on an important class of functions.

Definition 2.3.1.

(1) *Let $\varphi\colon [0,+\infty[\to [0,+\infty[$ be a continuous and convex function. We say that φ is a Φ-function if $\varphi(0) = 0$ and $\varphi(t) > 0$ for every $t > 0$.*

(2) *Let $\varphi\colon \Omega \times [0,+\infty[\to [0,+\infty[$ be a function. We say that φ is a generalized Φ-function if*

   (i) *$\varphi(\cdot,t)$ is measurable for all $t \geq 0$,*

   (ii) *$\varphi(x,\cdot)$ is a Φ-function for a. a. $x \in \Omega$,*

   and we denote by $\Phi(\Omega)$ the set of all generalized Φ-functions on Ω.

(3) *A function $\varphi \in \Phi(\Omega)$ is locally integrable on Ω if $\varphi(\cdot,t) \in L^1(E)$ for all $t > 0$ and for all measurable $E \subset \Omega$ with $|E| < \infty$.*

(4) *We say that $\varphi \in \Phi(\Omega)$ satisfies the (Δ_2)-condition if there exist a constant $C > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that*

$$\varphi(x,2t) \leq C\varphi(x,t) + h(x) \tag{Δ_2}$$

for a. a. $x \in \Omega$ *and for all* $t \geq 0$.

(5) *Given* $\varphi, \psi \in \Phi(\Omega)$, *we say that* φ *is weaker than* ψ, *denoted by* $\varphi \prec \psi$, *if there exist two positive constants* C_1, C_2 *and a nonnegative function* $h \in L^1(\Omega)$ *such that*

$$\varphi(x,t) \leq C_1\psi(x, C_2 t) + h(x)$$

for a. a. $x \in \Omega$ *and for all* $t \geq 0$.

For any $\varphi \in \Phi(\Omega)$, we introduce the modular function $\rho_{r(\cdot)}$ given by

$$\rho_\varphi(u) := \int_\Omega \varphi\left(x, |u|\right) \mathrm{d}x,$$

and we define the Musielak-Orlicz space $L^\varphi(\Omega)$ by

$$L^\varphi(\Omega) = \left\{u \in M(\Omega) \,:\, \text{there exists } \tau > 0 \text{ such that } \rho_\varphi(\tau u) < +\infty\right\},$$

endowed with the Luxemburg norm

$$\|u\|_\varphi := \inf\left\{\tau > 0 \,:\, \rho_\varphi\left(\frac{u}{\tau}\right) \leq 1\right\}. \tag{2.3.1}$$

In the following proposition we summarize some main results that will be useful in the sequel, see Musielak [74, Theorems 7.7, 8.5 and 8.13] and Diening-Harjulehto-Hästö-Růžička [46, Lemma 2.1.14].

Proposition 2.3.2. *Let* $\varphi, \psi \in \Phi(\Omega)$. *Then, the following hold:*

(i) $\left(L^\varphi(\Omega), \|\cdot\|_\varphi\right)$ *is a Banach space.*

(ii) *If* φ *and* ψ *are locally integrable with* $\varphi \prec \psi$, *then*

$$L^\psi(\Omega) \hookrightarrow L^\varphi(\Omega).$$

(iii) *If* φ *satisfy the* (Δ_2)*-condition, then*

$$L^\varphi(\Omega) = \left\{u \in M(\Omega) \,:\, \rho_\varphi(u) < +\infty\right\}.$$

(iv) **[Unit ball property]** *If* $u \in L^\varphi(\Omega)$, *then*

$$\rho_\varphi(u) < 1 \quad (\textit{resp.} = 1; > 1) \quad \Leftrightarrow \quad \|u\|_\varphi < 1 \quad (\textit{resp.} = 1; > 1).$$

We also need to introduce a subclass of $\Phi(\Omega)$ with "nicer" functions, namely that have better properties. This functions are called N-functions, where N stands for *nice*. Here, we give the definition and other related ones.

Definition 2.3.3. (1) *Let* $\varphi\colon [0, +\infty[\to [0, +\infty[$ *be a* Φ*-function. We say that* φ *is a* N*-function if*

$$\lim_{t \to 0^+} \frac{\varphi(t)}{t} = 0 \quad \textit{and} \quad \lim_{t \to \infty} \frac{\varphi(t)}{t} = \infty.$$

(2) *Let* $\varphi\colon \Omega \times [0, +\infty[\to [0, +\infty[$ *be a function. We say that* φ *is a generalized* N*-function if*

(i) $\varphi(\cdot, t)$ *is measurable for all* $t \geq 0$,

(ii) $\varphi(x, \cdot)$ *is a N-function for a. a.* $x \in \Omega$,

and we denote by $N(\Omega)$ *the class of all generalized N-functions on* Ω.

(3) *Given* $\varphi, \psi \in N(\Omega)$, *we say that* φ *increases essentially slower than* ψ *near infinity (and we write* $\varphi \ll \psi$*) if for any* $k > 0$

$$\lim_{t \to +\infty} \frac{\varphi(x, kt)}{\psi(x, t)} = 0 \quad \text{uniformly for a. a. } x \in \Omega.$$

(4) *We say that* $\varphi \in N(\Omega)$ *is uniformly convex if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|t - s| \leq \varepsilon \max\{t, s\} \quad \text{or} \quad \varphi\left(x, \frac{t+s}{2}\right) \leq (1 - \delta) \frac{\varphi(x, t) + \varphi(x, s)}{2}$$

for all $t, s \geq 0$ *and for a. a.* $x \in \Omega$.

Thanks to this subclass of functions, we have another property of the Musielak-Orlicz spaces $L^{\varphi}(\Omega)$ and we refer to Diening-Harjulehto-Hästö-Růžička [46, Theorems 2.4.11 and 2.4.14].

Theorem 2.3.4. *Let* $\varphi \in N(\Omega)$ *be uniformly convex and satisfying the* (Δ_2)*-condition. Then, the norm* $\| \cdot \|_{\varphi}$ *defined on* $L^{\varphi}(\Omega)$ *is uniformly convex and hence* $L^{\varphi}(\Omega)$ *is uniformly convex.*

In order to state a Hölder-type inequality, we first need the following definition.

Definition 2.3.5. *Let* $\varphi \in \Phi(\Omega)$. *The conjugate function* $\varphi^* \colon \Omega \times [0, +\infty[\to \mathbb{R}$ *of* φ *is defined by*

$$\varphi^*(x, s) = \sup_{t \geq 0} (st - \varphi(x, t))$$

for a. a. $x \in \Omega$ *and for all* $s \in [0, +\infty[$.

The next proposition can be found in Diening-Harjulehto-Hästö-Růžička [46, Lemma 2.6.5].

Proposition 2.3.6 (Hölder's inequality). *If* $\varphi \in \Phi(\Omega)$, *then*

$$\int_{\Omega} |uv| \, dx \leq 2 \|u\|_{\varphi} \|v\|_{\varphi^*}$$

for all $u \in L^{\varphi}(\Omega)$ *and for all* $v \in L^{\varphi^*}(\Omega)$.

Given $\varphi \in \Phi(\Omega)$, we define the Musielak-Orlicz Sobolev space as follows

$$W^{1,\varphi}(\Omega) = \{ u \in L^{\varphi}(\Omega) \, : \, |\nabla u| \in L^{\varphi}(\Omega) \},$$

endowed with the norm

$$\|u\|_{1,\varphi} = \|u\|_{\varphi} + \|\nabla u\|_{\varphi},$$

with $\|\nabla u\|_{\varphi} = \| |\nabla u| \|_{\varphi}$.

If $\varphi \in N(\Omega)$ is locally integrable, we denote by $W_0^{1,\varphi}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$. The following result on the properties of these spaces can be found in Musielak [74, Theorem 10.2] and Fan [50, Proposition 1.8].

Theorem 2.3.7. *Let $\varphi \in N(\Omega)$. If φ is locally integrable and*

$$\inf_{x\in\Omega} \varphi(x,1) > 0, \tag{2.3.2}$$

then the spaces $W^{1,\varphi}(\Omega)$ and $W_0^{1,\varphi}(\Omega)$ are separable Banach spaces. Moreover, if $L^\varphi(\Omega)$ is reflexive, then $W^{1,\varphi}(\Omega)$ and $W_0^{1,\varphi}(\Omega)$ are reflexive.

Remark 2.3.8. *We underline that one can construct the Lebesgue spaces $L^r(\Omega)$ and $L^{(r(x))}(\Omega)$ as Musielak-Orlicz spaces $L^\varphi(\Omega)$ by choosing $\varphi(t) = |t|^r$ for every $t \in \mathbb{R}$ and $\varphi(x,t) = |t|^{r(x)}$ for every $(x,t) \in \Omega \times \mathbb{R}$, respectively.*

2.3.1 Function spaces for double phase problems

In order to investigate the existence of weak solutions for differential double phase problems, we first need to determine the function space to which the solutions belong, namely we have to introduce the corresponding N-function. In particular, we want to study nonlinear problems involving the following double phase operator with variable exponents

$$u \longmapsto -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right), \qquad x \in \Omega, \tag{2.3.3}$$

where p, q, μ satisfy suitable assumptions that will be stated in the sequel. In this subsection we will provide the properties of the function space related to the double phase operator given in (2.3.3) and we mainly refer to the paper of Crespo-Blanco-Gasiński-Harjulehto-Winkert [40], where the authors extend, under weaker assumptions, the results of Colasuonno-Squassina [34] and Liu-Dai [68] on the properties of the function space and its embeddings, respectively.

First, we suppose that

(H1) $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $p, q \in C(\overline{\Omega})$ such that $1 < p(x) < N$ and $p(x) < q(x)$ for all $x \in \overline{\Omega}$ and $\mu \in L^1(\Omega)$ with $\mu(x) \geq 0$ for a.a. $x \in \Omega$.

Consider the nonlinear function $\mathcal{H}\colon \Omega \times [0,+\infty[\to [0,+\infty[$, given by

$$\mathcal{H}(x,t) = t^{p(x)} + \mu(x)t^{q(x)} \quad \text{for all } (x,t) \in \Omega \times [0,+\infty[.$$

It is easy to see that $\mathcal{H}$ is a generalized Φ-function if $p_-, q_- \in [1,+\infty]$, where p_-, q_- are defined in (2.2.5) and it is also a generalized N-function since $1 < p_- \leq p_+ < +\infty$ and $1 < q_- \leq q_+ < +\infty$ by (H1). In addition, $\mathcal{H}$ fulfills the (Δ_2)-condition, since we have

$$\begin{aligned} \mathcal{H}(x,2t) &= (2t)^{p(x)} + \mu(x)(2t)^{q(x)} \\ &\leq 2^{p_+}t^{p(x)} + 2^{q_+}\mu(x)t^{q(x)} \\ &\leq 2^{q_+}\mathcal{H}(x,t). \end{aligned} \tag{2.3.4}$$

Now, we denote with $L^{\mathcal{H}}(\Omega)$ the correspondent Musielak-Orlicz space, that from Proposition 2.3.2(iii) is defined by

$$L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega) : \rho_{\mathcal{H}}(u) < +\infty\},$$

where $\rho_{\mathcal{H}}(\cdot)$ is the corresponding modular, i.e.

$$\rho_{\mathcal{H}}(u) = \int_{\Omega} \mathcal{H}(x, |u|)\, \mathrm{d}x = \int_{\Omega} \left(|u|^{p(x)} + \mu(x)|u|^{q(x)} \right)\, \mathrm{d}x.$$

We endow $L^{\mathcal{H}}(\Omega)$ with the Luxemburg norm given in (2.3.1), that we recall here for completeness

$$\|u\|_{\mathcal{H}} = \inf\left\{ \tau > 0 \,:\, \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \le 1 \right\}. \tag{2.3.5}$$

In the same way, we can introduce the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$, defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\},$$

equipped with the usual norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \| \, |\nabla u| \, \|_{\mathcal{H}}$. In addition, $\mathcal{H}$ is locally integrable since for any $E \subset \Omega$ with $|E| < \infty$ and for every $t > 0$ it holds that

$$\int_E \mathcal{H}(x,t)\, \mathrm{d}x \le |E| \left(\max\left\{t^{p_-}, t^{p_+}\right\} + \|\mu\|_\infty \max\left\{t^{q_-}, t^{q_+}\right\} \right), \tag{2.3.6}$$

then we denote by $W_0^{1,\mathcal{H}}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$. Thanks to the properties of function $\mathcal{H}$, in the next proposition we present the properties of the spaces and we also give the proof, which can be found in Crespo-Blanco-Gasiński-Harjulehto-Winkert [40, Proposition 2.12].

Proposition 2.3.9. *Let (H1) be satisfied. Then, $L^{\mathcal{H}}(\Omega)$ is uniformly convex and $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are separable reflexive Banach spaces.*

Proof. From Proposition 2.3.2(i) it follows that $L^{\mathcal{H}}(\Omega)$ is complete. Also, since $\frac{\mathcal{H}(x,t)}{t}$ is almost increasing for a. a. $x \in \Omega$ and $\mathcal{H}$ satisfies (Δ_2)-contidion (see (2.3.4)), by combining Lemma 2.2.6 and Theorem 3.5.2 in [60] we obtain that $L^{\mathcal{H}}(\Omega)$ is separable. Moreover, $\mathcal{H}$ is locally integrable from (2.3.6) and it is easy to see that it satisfies condition (2.3.2). Hence, by Theorem 2.3.7 we have that $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are separable Banach spaces. Now, we want to apply Theorem 2.3.4 in order to show that $L^{\mathcal{H}}(\Omega)$ is uniformly convex. The nonlinear function $\mathcal{H}$ is a generalized N-function and from (2.3.4) we know that it satisfies the (Δ_2)-condition, so it remains to prove that it is a uniformly convex function. To this aim, let $\varepsilon > 0$ and $t, s \ge 0$ be such that $|t - s| > \varepsilon \max\{t, s\}$ and consider the function $t \mapsto t^{p_-}$, that is uniformly convex since from (H1) we have that $p_- > 1$, see Diening-Harjulehto-Hästö-Růžička [46, Remark 2.4.6]. Thus, there exists $\delta_p = \delta_p(\varepsilon, p_-) > 0$ such that

$$\left(\frac{t+s}{2}\right)^{p_-} \le (1 - \delta_p) \frac{t^{p_-} + s^{p_-}}{2},$$

and by the convexity of $t \longmapsto t^{\frac{p(x)}{p_-}}$ for $x \in \Omega$, we obtain

$$\left(\frac{t+s}{2}\right)^{p(x)} \le \left((1 - \delta_p) \frac{t^{p_-} + s^{p_-}}{2} \right)^{\frac{p(x)}{p_-}} \le (1 - \delta_p) \frac{t^{p(x)} + s^{p(x)}}{2}.$$

In the same way, we get

$$\left(\frac{t+s}{2}\right)^{q(x)} \leq (1-\delta_q)\frac{t^{q(x)}+s^{q(x)}}{2}$$

for some $\delta_q = \delta_q(\varepsilon, q_-) > 0$. Combining them together and putting $\delta = \min\{\delta_p, \delta_q\}$, we have

$$\begin{aligned}
&\left(\frac{t+s}{2}\right)^{p(x)} + \mu(x)\left(\frac{t+s}{2}\right)^{q(x)} \\
&\leq (1-\min\{\delta_p,\delta_q\})\frac{t^{p(x)}+\mu(x)t^{q(x)}+s^{p(x)}+\mu(x)s^{q(x)}}{2} \\
&= (1-\delta)\frac{\mathcal{H}(x,t)+\mathcal{H}(x,s)}{2},
\end{aligned}$$

which implies that $\mathcal{H}$ is uniformly convex and by applying Theorem 2.3.4 we obtain the uniform convexity of $L^{\mathcal{H}}(\Omega)$. Applying the Milman-Pettis Theorem, see for instance Papageorgiou-Winkert [79, Theorem 3.4.28], it follows that $L^{\mathcal{H}}(\Omega)$ is reflexive and so they are $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ from Theorem 2.3.7. □

Also in this setting, we can give a version of Proposition 2.2.2 on the relation between the norm $\|\cdot\|_{\mathcal{H}}$ and the modular $\rho_{\mathcal{H}}(\cdot)$. This result and its proof, that we also recall here, can be found in Crespo-Blanco-Gasiński-Harjulehto-Winkert [40, Proposition 2.13].

Proposition 2.3.10. *Let (H1) be satisfied, $u \in W^{1,\mathcal{H}}(\Omega)$ and $\tau > 0$. Then the following hold:*

(i) *If $u \neq 0$, then $\|u\|_{\mathcal{H}} = \tau \iff \rho_{\mathcal{H}}(\frac{u}{\tau}) = 1$;*

(ii) *$\|u\|_{\mathcal{H}} < 1$ (resp. > 1, $= 1$) $\iff \rho_{\mathcal{H}}(u) < 1$ (resp. > 1, $= 1$);*

(iii) *If $\|u\|_{\mathcal{H}} < 1 \implies \|u\|_{\mathcal{H}}^{q_+} \leqslant \rho_{\mathcal{H}}(u) \leqslant \|u\|_{\mathcal{H}}^{p_-}$;*

(iv) *If $\|u\|_{\mathcal{H}} > 1 \implies \|u\|_{\mathcal{H}}^{p_-} \leqslant \rho_{\mathcal{H}}(u) \leqslant \|u\|_{\mathcal{H}}^{q_+}$;*

(v) *$\|u\|_{\mathcal{H}} \to 0 \iff \rho_{\mathcal{H}}(u) \to 0$;*

(vi) *$\|u\|_{\mathcal{H}} \to +\infty \iff \rho_{\mathcal{H}}(u) \to +\infty$.*

(vii) *$\|u\|_{\mathcal{H}} \to 1 \iff \rho_{\mathcal{H}}(u) \to 1$.*

(viii) *If $u_n \to u$ in $L^{\mathcal{H}}(\Omega) \implies \rho_{\mathcal{H}}(u_n) \to \rho_{\mathcal{H}}(u)$.*

Proof. First, we observe that for any $u \in L^{\mathcal{H}}(\Omega)$ the mapping

$$a \mapsto \rho_{\mathcal{H}}(au) = \int_\Omega \left(|au|^{p(x)} + \mu(x)|au|^{q(x)}\right) \mathrm{d}x, \quad a \in \mathbb{R},$$

is continuous, convex, even and it is strictly increasing in $[0, +\infty[$. These properties simply the proof.

(i) From the monotonicity of $\rho_{\mathcal{H}}(\tau u)$ and from the definition of $\|\cdot\|_{\mathcal{H}}$ given in (2.3.5), it follows that

$$\|u\|_{\mathcal{H}} = \tau \iff \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) = 1.$$

(ii) Taking into account that $\rho_{\mathcal{H}}(\tau u)$ is continuous and increasing in the variable τ, the thesis follows directly from (i).

(iii) For every $u \in L^{\mathcal{H}}(\Omega)$ the following inequalities hold

$$a^{p_-}\rho_{\mathcal{H}}(u) \le \rho_{\mathcal{H}}(au) \le a^{q_+}\rho_{\mathcal{H}}(u) \quad \text{if } a > 1, \tag{2.3.7}$$
$$a^{q_+}\rho_{\mathcal{H}}(u) \le \rho_{\mathcal{H}}(au) \le a^{p_-}\rho_{\mathcal{H}}(u) \quad \text{if } 0 < a < 1. \tag{2.3.8}$$

If $\|u\|_{\mathcal{H}} = \tau < 1$, then using (2.3.7) in correspondence of $\frac{1}{\tau} > 1$, we get

$$\frac{\rho_{\mathcal{H}}(u)}{\tau^{p_-}} \le \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \le \frac{\rho_{\mathcal{H}}(u)}{\tau^{q_+}}.$$

From (i) we know that $\rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) = 1$, hence it follows that

$$\|u\|_{\mathcal{H}}^{q_+} \leqslant \rho_{\mathcal{H}}(u) \leqslant \|u\|_{\mathcal{H}}^{p_-}.$$

(iv) Follows using the same argument of (iii) and exploiting (2.3.8).

(v) Follows from (iii).

(vi) Follows from (iv).

(vii) Follows combining (iii) and (iv).

(viii) If $u_n \to u$ in $L^{\mathcal{H}}(\Omega)$, then from (v) we know that

$$\rho_{\mathcal{H}}(u_n - u) = \rho_{p(\cdot)}(u_n - u) + \rho_{q(\cdot),\mu}(u_n - u) \longrightarrow 0.$$

Since both addends are positive, it follows that $\rho_{p(\cdot)}(u_n - u) \to 0$ and by Proposition 2.2.2(v) we have that $\|u_n - u\|_{p(\cdot)} \to 0$. By the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{p_-}(\Omega)$ we obtain that $\|u_n - u\|_{p_-} \to 0$, hence $u_n \to u$ a. e. up to a subsequence (not relabeled). Moreover, it holds that

$$\begin{aligned}
&|u_n|^{p(x)} + \mu(x)|u_n|^{q(x)} \le \\
&\le 2^{p_+-1}\left(|u_n - u|^{p(x)} + |u|^{p(x)}\right) + 2^{q_+-1}\mu(x)\left(|u_n - u|^{q(x)} + |u|^{q(x)}\right) \\
&\le 2^{q_+-1}\left(|u_n - u|^{p(x)} + \mu(x)|u_n - u|^{q(x)} + |u|^{p(x)} + \mu(x)|u|^{q(x)}\right),
\end{aligned}$$

and we know that $\left\{|u_n|^{p(x)} + \mu(x)|u_n|^{q(x)}\right\}_{n\in\mathbb{N}}$ is a uniformly integrable sequence that converges a. e. to $|u|^{p(x)} + \mu(x)|u|^{q(x)}$, thanks to the a. e. convergence of $u_n \to u$. By Lebesgue-Vitali Theorem (see Bogachev [16, Theorem 4.5.4]) it follows that $\rho_{\mathcal{H}}(u_n) \to \rho_{\mathcal{H}}(u)$ through this subsequence. Recovering the whole sequence by the subsequence principle, the proof is complete.

□

Remark 2.3.11. *We observe that from Proposition 2.3.10(iii)-(iv) it follows that*

$$\min\left\{\|u\|_{\mathcal{H}}^{q_+}, \|u\|_{\mathcal{H}}^{p_-}\right\} \le \rho_{\mathcal{H}}(u) \le \max\left\{\|u\|_{\mathcal{H}}^{q_+}, \|u\|_{\mathcal{H}}^{p_-}\right\},$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$.

The following result about the main embeddings of $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ can be found in Crespo-Blanco-Gasiński-Harjulehto-Winkert [40, Propositions 2.16]. Since we assume in (H1) that $p(x) < N$ for all $x \in \overline{\Omega}$, the critical exponents to p, defined in (2.2.4), are exactly the following

$$p^*(x) := \frac{Np(x)}{N-p(x)} \quad \text{and} \quad p_*(x) := \frac{(N-1)p(x)}{N-p(x)} \quad \text{for all } x \in \overline{\Omega}.$$

Proposition 2.3.12. *Let (H1) be satisfied. Then the following embeddings hold:*

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$, $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega)$, $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega)$ *are continuous for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p(x)$ *for all* $x \in \Omega$;

(ii) *if* $p \in C_+(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, *then* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *and* $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *are continuous for* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p^*(x)$ *for all* $x \in \overline{\Omega}$;

(iii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *and* $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *are compact for* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$;

(iv) *if* $p \in C_+(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ *for some* $\gamma > N$, *then* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *and* $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *are continuous for* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p_*(x)$ *for all* $x \in \overline{\Omega}$;

(v) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *and* $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *are compact for* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < p_*(x)$ *for all* $x \in \overline{\Omega}$;

(vi) $L^{\mathcal{H}}(\Omega) \hookrightarrow L_\mu^{q(\cdot)}(\Omega)$ *is continuous;*

(vii) *if* $\mu \in L^\infty(\Omega)$, *then* $L^{q(\cdot)}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

Proof.

(i) Put $\mathcal{H}_{p(\cdot)}(x,t) = t^{p(x)}$ for all $t \geq 0$ and for all $x \in \overline{\Omega}$. Clearly it holds that $\mathcal{H}_{p(\cdot)} \prec \mathcal{H}$, see Definition 2.3.1 (5), hence from Proposition 2.3.2(ii) we obtain the following continuous embeddings

$$L^{\mathcal{H}}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega), \quad W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,p(\cdot)}(\Omega), \quad W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,p(\cdot)}(\Omega).$$

Thus, assertion (i) is a direct consequence of the classical embedding results for variable Lebesgue and Sobolev spaces due to the boundedness of Ω, see (2.2.3).

(ii)-(iii) Follows from (i) and Proposition 2.2.5.

(iv)-(v) Follows from (i) and Proposition 2.2.7.

(vi) Let $u \in L^{\mathcal{H}}(\Omega)$, then we have

$$\int_\Omega \mu(x)|u|^{q(x)}\,\mathrm{d}x \leq \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)}\right)\,\mathrm{d}x = \rho_{\mathcal{H}}(u).$$

Since $\rho_{\mathcal{H}}\left(\frac{u}{\|u\|_{\mathcal{H}}}\right) = 1$ whenever $u \neq 0$, we get for $u \neq 0$

$$\int_\Omega \mu(x)\left(\frac{u}{\|u\|_{\mathcal{H}}}\right)^{q(x)}\,\mathrm{d}x \leq 1,$$

which implies that

$$\|u\|_{q(\cdot),\mu} \leq \|u\|_{\mathcal{H}}.$$

(vii) For all $t \geq 0$ and for a. a. $x \in \Omega$ one has

$$\mathcal{H}(x,t) \leq \left(1 + t^{q(x)}\right) + \mu(x)t^{q(x)} \leq 1 + (1 + \|\mu\|_\infty)\, t^{q(x)},$$

so by applying Proposition 2.3.2 (ii) we complete the proof.

□

Also for the Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$ there exists a Poincaré's inequality, which allow us to equip the space with an equivalent norm. However, it is necessary to require a more restrictive assumption than (H1), namely

(H) $\Omega \subseteq \mathbb{R}^N, N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $p, q \in C(\overline{\Omega})$ such that $1 < p(x) < N$ and $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where $p^*(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}$ is the critical Sobolev exponent to $p(\cdot)$, and $\mu \in L^\infty(\Omega)$ with $\mu(x) \geq 0$ for a.a. $x \in \Omega$.

However, it is worth emphasizing that the exponents $p(\cdot)$ and $q(\cdot)$ do not need to verify a condition of the type

$$\frac{q(\cdot)}{p(\cdot)} < 1 + \frac{1}{N} \tag{2.3.9}$$

as it was needed, for example, in Kim-Kim-Oh-Zeng [63] or in Colasuonno-Squassina [34] and Liu-Dai [68] for the constant exponent case. Indeed, only assumption (H) is needed, since Crespo-Blanco-Gasiński-Harjulehto-Winkert in [40, Proposition 2.18] recently proved that the function space $W^{1,\mathcal{H}}(\Omega)$ can be equipped with the equivalent norm $\|\nabla \cdot\|_{\mathcal{H}}$ without supposing (2.3.9).

Proposition 2.3.13. *Let* (H) *be satisfied. Then the following hold:*

(i) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is a compact embedding;*

(ii) *There exists a constant* $C > 0$ *independent of* u *such that*

$$\|u\|_{\mathcal{H}} \leq C\|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega).$$

Proof.

(i) Follows by Proposition 2.3.12 (iii) and (vii).

(ii) We prove it by contradiction, so we assume that the assertion is not true. Then, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}}(\Omega)$ such that

$$\|u_n\|_{\mathcal{H}} > n\|\nabla u_n\|_{\mathcal{H}} \quad \Longleftrightarrow \quad \frac{\|\nabla u_n\|_{\mathcal{H}}}{\|u_n\|_{\mathcal{H}}} < \frac{1}{n}.$$

Let $y_n := \frac{u_n}{\|u_n\|_{\mathcal{H}}}$, then we have

$$\|y_n\|_{\mathcal{H}} = 1 \quad \text{and} \quad \|\nabla y_n\|_{\mathcal{H}} < \frac{1}{n} < 1 \quad \text{for all } n \in \mathbb{N},$$

i.e. the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Since $W_0^{1,\mathcal{H}}(\Omega)$ is a reflexive space, there exist a subsequence (not relabeled) and $y \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$y_n \rightharpoonup y \quad \text{in } W_0^{1,\mathcal{H}}(\Omega).$$

By the weak lower semicontinuity of the mapping $v \mapsto \|\nabla v\|_{\mathcal{H}}$ on $W_0^{1,\mathcal{H}}(\Omega)$ (it is a convex, continuous mapping) one has that

$$\|\nabla y\|_{\mathcal{H}} \le \liminf_{n\to\infty} \|\nabla y_n\|_{\mathcal{H}} \le \lim_{n\to\infty} \frac{1}{n} = 0,$$

hence $y = c \in \mathbb{R}$ is a constant function. Moreover, by Proposition 2.3.12 (i) we have that $y \in W_0^{1,p(\cdot)}(\Omega)$, so it must be $y = 0$ since it is the only constant function of the space. On the other hand, by the compact embedding in (i) we have that

$$y_n \to y \quad \text{in } L^{\mathcal{H}}(\Omega),$$

thus $\|y\|_{\mathcal{H}} = \lim_{n\to\infty} \|y_n\|_{\mathcal{H}} = 1$, so $y \neq 0$, which is a contradiction and this completes the proof.

□

Thanks to the previous proposition, we can endow the space $W_0^{1,\mathcal{H}}(\Omega)$ with the following equivalent norm

$$\|u\|_{1,\mathcal{H},0} = \|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega). \tag{2.3.10}$$

Finally, we have that this norm is uniformly convex on $W_0^{1,\mathcal{H}}(\Omega)$ and satisfies the Radon–Riesz (or Kadec-Klee) property with respect to the modular. This result is due to Crespo-Blanco-Gasiński-Harjulehto-Winkert in [40, Proposition 2.19].

Proposition 2.3.14. *Let* (H) *be satisfied. Then the following hold:*

(i) *The norm* $\|\cdot\|_{1,\mathcal{H},0}$ *on* $W_0^{1,\mathcal{H}}(\Omega)$ *is uniformly convex.*

(ii) *For any sequence* $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}}(\Omega)$ *such that*

$$u_n \rightharpoonup u \text{ in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \rho_{\mathcal{H}}(\nabla u_n) \to \rho_{\mathcal{H}}(\nabla u)$$

it holds that $u_n \to u$ *in* $W_0^{1,\mathcal{H}}(\Omega)$.

2.3.2 A new equivalent norm

In this subsection we prove the existence of a new and general equivalent norm in $W^{1,\mathcal{H}}(\Omega)$, which is useful since it allows the study of more double phase problems than can be dealt with the usual norm. Amoroso-Crespo-Blanco-Pucci-Winkert introduced this norm in [5, Section 3].

First, in addition to (H), we suppose the following conditions:

(H2) (i) $\delta_1, \delta_2 \in C(\overline{\Omega})$ with $1 \le \delta_1(x) \le p^*(x)$ and $1 \le \delta_2(x) \le p_*(x)$ for all $x \in \overline{\Omega}$, where

(a$_1$) $p \in C(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, if $\delta_1(x) = p^*(x)$ for some $x \in \overline{\Omega}$;

(a$_2$) $p \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$, if $\delta_2(x) = p_*(x)$ for some $x \in \overline{\Omega}$;

(ii) $\vartheta_1 \in L^\infty(\Omega)$ with $\vartheta_1(x) \geq 0$ for a.a. $x \in \Omega$;

(iii) $\vartheta_2 \in L^\infty(\partial\Omega)$ with $\vartheta_2(x) \geq 0$ for a.a. $x \in \partial\Omega$;

(iv) $\vartheta_1 \not\equiv 0$ or $\vartheta_2 \not\equiv 0$.

In the sequel we use the seminormed spaces

$$L^{\delta_1(\cdot)}_{\vartheta_1}(\Omega) = \left\{ u \in M(\Omega) \,:\, \int_\Omega \vartheta_1(x)|u|^{\delta_1(x)}\,\mathrm{d}x < \infty \right\},$$
$$L^{\delta_2(\cdot)}_{\vartheta_2}(\partial\Omega) = \left\{ u \in M(\Omega) \,:\, \int_{\partial\Omega} \vartheta_2(x)|u|^{\delta_2(x)}\,\mathrm{d}\sigma < \infty \right\},$$

with corresponding seminorms

$$\|u\|_{\delta_1(\cdot),\vartheta_1} = \inf\left\{ \tau > 0 \,:\, \int_\Omega \vartheta_1(x)\left|\frac{u}{\tau}\right|^{\delta_1(x)}\,\mathrm{d}x \leq 1 \right\},$$
$$\|u\|_{\delta_2(\cdot),\vartheta_2,\partial\Omega} = \inf\left\{ \tau > 0 \,:\, \int_{\partial\Omega} \vartheta_2(x)\left|\frac{u}{\tau}\right|^{\delta_2(x)}\,\mathrm{d}\sigma \leq 1 \right\},$$

respectively. We set

$$\|u\|^\circ_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\delta_1(\cdot),\vartheta_1} + \|u\|_{\delta_2(\cdot),\vartheta_2,\partial\Omega},$$

and

$$\|u\|^*_{1,\mathcal{H}} = \inf\left\{ \tau > 0 \,:\, \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \mu(x)\left|\frac{\nabla u}{\tau}\right|^{q(x)} \right)\mathrm{d}x \right.$$
$$\left. + \int_\Omega \vartheta_1(x)\left|\frac{u}{\tau}\right|^{\delta_1(x)}\,\mathrm{d}x + \int_{\partial\Omega} \vartheta_2(x)\left|\frac{u}{\tau}\right|^{\delta_2(x)}\,\mathrm{d}\sigma \leq 1 \right\}. \tag{2.3.11}$$

It can be easily seen that $\|\cdot\|^\circ_{1,\mathcal{H}}$ and $\|\cdot\|^*_{1,\mathcal{H}}$ are norms on $W^{1,\mathcal{H}}(\Omega)$. In the next result, we prove that they are both equivalent to the usual one.

Proposition 2.3.15. *Let (H) and (H2) be satisfied. Then, $\|\cdot\|^\circ_{1,\mathcal{H}}$ and $\|\cdot\|^*_{1,\mathcal{H}}$ are both equivalent norms on $W^{1,\mathcal{H}}(\Omega)$.*

Proof. We only prove the result when $\delta_1(x) = p^*(x)$ and $\delta_2(x) = p_*(x)$ for all $x \in \overline{\Omega}$, the other cases can be shown in a similar way. So, we suppose that $p \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$. Then, by Remark 2.2.8 we know that $p \in C(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ as well. First, for $u \in W^{1,\mathcal{H}}(\Omega) \setminus \{0\}$ we have

$$\int_\Omega \vartheta_1(x)\left(\frac{|u|}{\|u\|_{p^*(\cdot)}}\right)^{p^*(x)}\mathrm{d}x \leq \|\vartheta_1\|_\infty\, \rho_{p^*(\cdot)}\left(\frac{u}{\|u\|_{p^*(\cdot)}}\right) = \|\vartheta_1\|_\infty.$$

Hence,

$$\|u\|_{p^*(\cdot),\vartheta_1} \leq \|\vartheta_1\|_\infty \|u\|_{p^*(\cdot)}.$$

In the same way, we show that

$$\|u\|_{p_*(\cdot),\vartheta_2,\partial\Omega} \leq \|\vartheta_2\|_{\infty,\partial\Omega}\, \|u\|_{p_*(\cdot),\partial\Omega}.$$

Using these along with Proposition 2.3.12(ii), (iv), we obtain

$$\begin{aligned}\|u\|_{1,\mathcal{H}}^{\circ} &\leq \|\nabla u\|_{\mathcal{H}} + C_1\|u\|_{p^*(\cdot)} + C_2\|u\|_{p_*(\cdot),\partial\Omega}\\ &\leq \|\nabla u\|_{\mathcal{H}} + C_3\|u\|_{1,\mathcal{H}} + C_4\|u\|_{1,\mathcal{H}}\\ &\leq C_5\|u\|_{1,\mathcal{H}},\end{aligned}$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$, with positive constants C_i, $i = 1, \ldots 5$.

Next, we are going to prove that

$$\|u\|_{\mathcal{H}} \leq C_6\|u\|_{1,\mathcal{H}}^{\circ}, \tag{2.3.12}$$

for some $C_6 > 0$. We argue indirectly and assume that (2.3.12) does not hold. Then, we find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\Omega)$ such that

$$\|u_n\|_{\mathcal{H}} > n\|u_n\|_{1,\mathcal{H}}^{\circ} \quad \text{for all } n \in \mathbb{N}. \tag{2.3.13}$$

Let $y_n = \frac{u_n}{\|u_n\|_{\mathcal{H}}}$. Hence, $\|y_n\|_{\mathcal{H}} = 1$ and from (2.3.13) we get

$$\frac{1}{n} > \|y_n\|_{1,\mathcal{H}}^{\circ}. \tag{2.3.14}$$

From $\|y_n\|_{\mathcal{H}} = 1$ and (2.3.14), we know that $\{y_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\Omega)$ is bounded. Therefore, using the embeddings in Proposition 2.3.12(ii), (iv) and up to a subsequence if necessary, we may assume that

$$y_n \rightharpoonup y \quad \text{in } W^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad y_n \rightharpoonup y \quad \text{in } L^{p^*(\cdot)}(\Omega) \text{ and } L^{p_*(\cdot)}(\partial\Omega). \tag{2.3.15}$$

Furthermore, from (2.3.15) and Proposition 2.3.12(viii), we conclude that $y_n \to y$ in $L^{\mathcal{H}}(\Omega)$ and because of $\|y_n\|_{\mathcal{H}} = 1$ we have $y \neq 0$. Passing to the limit in (2.3.14) as $n \to \infty$ and using (2.3.15) along with the weak lower semicontinuity of the norm $\|\nabla \cdot\|_{\mathcal{H}}$ and of the seminorms $\|\cdot\|_{p^*(\cdot),\vartheta_1}$, $\|\cdot\|_{p_*(\cdot),\vartheta_2,\partial\Omega}$ we obtain

$$0 \geq \|\nabla y\|_{\mathcal{H}} + \|y\|_{p^*(\cdot),\vartheta_1} + \|y\|_{p_*(\cdot),\vartheta_2,\partial\Omega}. \tag{2.3.16}$$

Inequality (2.3.16) implies that $y \equiv \eta \neq 0$ is a constant and so we have the following contradiction

$$0 \geq |\eta|\|1\|_{p^*(\cdot),\vartheta_1} + |\eta|\|1\|_{p_*(\cdot),\vartheta_2,\partial\Omega} > 0,$$

because of (H2)(iv). Therefore (2.3.12) holds and we get

$$\|u\|_{1,\mathcal{H}} \leq C_7\|u\|_{1,\mathcal{H}}^{\circ},$$

for some $C_7 > 0$.

Next, we are going to show that $\|\cdot\|_{1,\mathcal{H}}^{\circ}$ and $\|\cdot\|_{1,\mathcal{H}}^{*}$ are equivalent norms in $W^{1,\mathcal{H}}(\Omega)$. For $u \in W^{1,\mathcal{H}}(\Omega)$, we obtain

$$\begin{aligned}&\int_\Omega \left(\left(\frac{|\nabla u|}{\|u\|_{1,\mathcal{H}}^{\circ}} \right)^{p(x)} + \mu(x) \left(\frac{|\nabla u|}{\|u\|_{1,\mathcal{H}}^{\circ}} \right)^{q(x)} \right) \mathrm{d}x\\ &+ \int_\Omega \vartheta_1(x) \left(\frac{|u|}{\|u\|_{1,\mathcal{H}}^{\circ}} \right)^{p^*(x)} \mathrm{d}x + \int_{\partial\Omega} \vartheta_2(x) \left(\frac{|u|}{\|u\|_{1,\mathcal{H}}^{\circ}} \right)^{p_*(x)} \mathrm{d}\sigma\end{aligned}$$

$$
\begin{aligned}
&\leq \rho_{\mathcal{H}}\left(\frac{\nabla u}{\|\nabla u\|_{\mathcal{H}}}\right)+\int_\Omega \vartheta_1(x)\left(\frac{|u|}{\|u\|_{p^*(\cdot),\vartheta_1}}\right)^{p^*(x)} \mathrm{d}x\\
&\quad+\int_{\partial\Omega} \vartheta_2(x)\left(\frac{|u|}{\|u\|_{p_*(\cdot),\vartheta_2,\partial\Omega}}\right)^{p_*(x)} \mathrm{d}\sigma\\
&=3.
\end{aligned}
$$

Thus,

$$
\|u\|_{1,\mathcal{H}}^* \leq 3\|u\|_{1,\mathcal{H}}^\circ. \tag{2.3.17}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_\Omega \left(\left(\frac{|\nabla u|}{\|u\|_{1,\mathcal{H}}^*}\right)^{p(x)}+\mu(x)\left(\frac{|\nabla u|}{\|u\|_{1,\mathcal{H}}^*}\right)^{q(x)}\right) \mathrm{d}x\\
&+\int_\Omega \vartheta_1(x)\left(\frac{|u|}{\|u\|_{1,\mathcal{H}}^*}\right)^{p^*(x)} \mathrm{d}x+\int_{\partial\Omega} \vartheta_2(x)\left(\frac{|u|}{\|u\|_{1,\mathcal{H}}^*}\right)^{p_*(x)} \mathrm{d}\sigma\\
&\leq \rho_{1,\mathcal{H}}^*\left(\frac{u}{\|u\|_{1,\mathcal{H}}^*}\right),
\end{aligned} \tag{2.3.18}
$$

where $\rho_{1,\mathcal{H}}^*$ is the corresponding modular to $\|\cdot\|_{1,\mathcal{H}}^*$ given by

$$
\begin{aligned}
\rho_{1,\mathcal{H}}^*(u) = &\int_\Omega \left(|\nabla u|^{p(x)}+\mu(x)|\nabla u|^{q(x)}\right) \mathrm{d}x+\int_\Omega \vartheta_1(x)|u|^{p^*(x)} \mathrm{d}x\\
&+\int_{\partial\Omega} \vartheta_2(x)|u|^{p_*(x)} \mathrm{d}\sigma.
\end{aligned}
$$

Note that, for $u \in W^{1,\mathcal{H}}(\Omega)$, the function $\tau \mapsto \rho_{1,\mathcal{H}}^*(\tau u)$ is continuous, convex and even and it is strictly increasing when $\tau \in [0,+\infty[$. So, by definition, we directly obtain

$$
\|u\|_{1,\mathcal{H}}^* = \tau \quad \text{if and only if} \quad \rho_{1,\mathcal{H}}^*\left(\frac{u}{\tau}\right)=1.
$$

From this and (2.3.18) we conclude that

$$
\|\nabla u\|_{\mathcal{H}} \leq \|u\|_{1,\mathcal{H}}^*, \qquad \|u\|_{p^*(\cdot),\vartheta_1} \leq \|u\|_{1,\mathcal{H}}^* \quad \text{and} \quad \|u\|_{p_*(\cdot),\vartheta_2,\partial\Omega} \leq \|u\|_{1,\mathcal{H}}^*.
$$

Therefore,

$$
\frac{1}{3}\|u\|_{1,\mathcal{H}}^\circ \leq \|u\|_{1,\mathcal{H}}^*. \tag{2.3.19}
$$

From (2.3.17) and (2.3.19) the proof is complete. □

Set

$$
r_1 := \min\left\{p_-,(\delta_1)_-,(\delta_2)_-\right\} \quad \text{and} \quad r_2 := \max\left\{q_+,(\delta_1)_+,(\delta_2)_+\right\}.
$$

In the following proposition we give the relation between the norm $\|\cdot\|_{1,\mathcal{H}}^*$ and the related modular function $\rho_{1,\mathcal{H}}^*(\cdot)$. This result is given in Amoroso-Crespo-Blanco-Pucci-Winkert [5, Proposition 3.2] and the proof is similar to that one of Proposition

2.13 given by Crespo-Blanco-Gasiński-Harjulehto-Winkert in [40], recalled in this thesis in Proposition 2.3.10.

Proposition 2.3.16. *Let* (H) *and* (H2) *be satisfied,* $u \in W^{1,\mathcal{H}}(\Omega)$ *and* $\lambda \in \mathbb{R}$. *Then the following hold:*

(i) *If* $u \neq 0$, *then* $\|u\|_{1,\mathcal{H}}^* = \lambda \iff \rho_{1,\mathcal{H}}^*(\frac{u}{\lambda}) = 1$;

(ii) $\|u\|_{1,\mathcal{H}}^* < 1$ *(resp.* $> 1, = 1$*)* $\iff \rho_{1,\mathcal{H}}^*(u) < 1$ *(resp.* $> 1, = 1$*)*;

(iii) *If* $\|u\|_{1,\mathcal{H}}^* < 1 \implies \left(\|u\|_{1,\mathcal{H}}^*\right)^{r_2} \leq \rho_{1,\mathcal{H}}^*(u) \leq \left(\|u\|_{1,\mathcal{H}}^*\right)^{r_1}$;

(iv) *If* $\|u\|_{1,\mathcal{H}}^* > 1 \implies \left(\|u\|_{1,\mathcal{H}}^*\right)^{r_1} \leq \rho_{1,\mathcal{H}}^*(u) \leq \left(\|u\|_{1,\mathcal{H}}^*\right)^{r_2}$;

(v) $\|u\|_{1,\mathcal{H}}^* \to 0 \iff \rho_{1,\mathcal{H}}^*(u) \to 0$;

(vi) $\|u\|_{1,\mathcal{H}}^* \to \infty \iff \rho_{1,\mathcal{H}}^*(u) \to \infty$;

(vii) $\|u\|_{1,\mathcal{H}}^* \to 1 \iff \rho_{1,\mathcal{H}}^*(u) \to 1$.

Chapter 3

Variational methods and Critical point theory

In this chapter, we present the classical results of variational methods, that are the Direct Methods theorem (Theorem 3.1.1) and the Mountain Pass theorem (Theorem 3.1.5), and we introduce some recent results in the field of critical point theory, namely Theorems 3.2.2 and 3.2.5 and some of their consequences.

3.1 Classical results

The general results that establish the existence of solutions to variational problems are based on the fundamental idea of combining the semicontinuity of the functional with the compactness of its domain.

This type of approach derives from the famous Weierstrass theorem on the extremes of continuous functions of real variables and it is known as "direct method" because it does not go through Euler's equations, but aims to study the properties of the functional directly. As the study of functional spaces progressed, the idea underwent successive elaborations, generalisations and refinements until, with Tonelli, it became the fundamental nucleus around which the calculus of variations gravitated.

In order to apply the direct method, one has to formulate the problem of finding a minimum in a space with a topology strong enough to make the functional semicontinuous and at the same time weak enough for a sufficiently large quantity of compact sets to exist.

A sufficient condition for the existence of the absolute minimum is the following result, usually called Direct Methods theorem.

Theorem 3.1.1 (Direct Methods Theorem)**.** *Let X be a reflexive Banach space equipped with norm $\|\cdot\|$ and let $Y \subset X$ a weakly closed subset of X. Suppose that $F\colon Y \to \mathbb{R} \cup +\infty$ is coercive on Y with respect to X, i.e.*

(i) $\displaystyle\lim_{\|u\|\to+\infty} F(u) = +\infty, \quad u \in Y$

and sequentially weakly lower semicontinuous on Y with respect to X, namely

(ii) for all $u \in Y$ and for all $\{u_n\} \subset Y$ such that $u_n \rightharpoonup u$ in X, it holds that

$$F(u) \leq \liminf_{n\to+\infty} F(u_n).$$

Then F is bounded from below on Y and it has a minimum in Y.

Proof. Set $\alpha_0 = \inf_Y F$ and let $\{u_n\}$ be a *minimizing suquence*, that is a sequence such that

$$\lim_{n \to +\infty} F(u_n) = \alpha.$$

Taking the coercivity of F into account, one has that $\{u_n\}$ is bounded in Y. Since X is reflexive, by the Eberlein-Šmulian Theorem (see [29]) we may assume that u_n weakly converges to $u \in X$. But Y is weakly closed, therefor $u \in Y$ and from the weakly semicontinuity we get that

$$F(u) \leq \liminf_{n \to +\infty} F(u_n) = \alpha_0,$$

and this completes the proof. □

As in the fundamentals of mathematical analysis, in the infinite dimensional context minima are critical points of the functional. Let us briefly recall some basic definitions.

Definition 3.1.2. *Let X be a real Banach space and denote by X^* its dual space. Let $A \subset X$ be an open set and consider the functional $F : X \to \mathbb{R}$. We say that F is Gâteaux differentiable in $x_0 \in A$ if there exists $\varphi \in X^*$ such that*

$$\lim_{t \to 0^+} \frac{F(x_0 + ty) - F(x_0)}{t} = \varphi(y) \qquad \forall\, y \in X.$$

In thi case, we say that φ is the Gâteaux derivative of F in x_0 and we set

$$F'(x_0) = \varphi(y).$$

Moreover, we say that x_0 is a critical point of F if $F'(x_0) = 0_{X^*}$.

It is easy to prove that if x_0 is a local minimum of F in X, then x_0 is a critical point of F. In the following, we provide a consequence of Theorem 3.1.1.

Corollary 3.1.3. *Let X be a reflexive Banach space. Suppose that the functional $F\colon X \to \mathbb{R}$ is coercive, sequentially weakly lower semicontinuous and Gâteaux differentiable. Then F admits at least a critical point in X.*

Now, we present the Mountain Pass theorem, which is the most famous theorem in critical point theory. It does not look for local extremes, but characterises a critical value of a functional through minimax theorems.

The basis of this result is the following compactness condition.

Definition 3.1.4. *Let $(X, \|\cdot\|)$ be a Banach space, X^* its dual and $F : X \to \mathbb{R}$ a Gâteaux differentiable functional. We say that the functional F satisfies the Palais-Smale condition at level c (in short, $(\mathrm{PS})_c$-condition), if any sequence $\{u_n\} \subseteq X$ such that*

(PS_c^1) $F(u_n) \to c \in \mathbb{R}$,

(PS_c^2) $F'(u_n) \to 0$ *in* X^* *as* $n \to \infty$,

has a strongly convergent subsequence in X.

Among the numerous versions of the Mountain Pass theorem, we initially recall the original one due to Ambrosetti and Rabinowitz ([2]).

Theorem 3.1.5 (Mountain Pass Theorem)**.** *Let X be a Banach space and let $F : X \to \mathbb{R}$ be a functional, $F \in C^1(X)$. Suppose that there exists $u_1 \in X$ such that $\|u_1\| > \rho > 0$ and*

$$\alpha = \max\{\gamma(0), \gamma(u_1)\} < \inf_{u \in B(0,\rho)} F(u) = \beta.$$

Set

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} F(u),$$

where

$$\Gamma = \{\gamma \in C([0,1], X),\ \gamma(0) = 0 \quad and \quad \gamma(1) = u_1\},$$

and suppose that F verifies the $(\mathrm{PS})_c$*-condition.*

Then F has critical value $c \geq \beta$.

Proof. First, we observe that $c < +\infty$. For any path $\gamma \in \Gamma$ it holds that $\gamma([0,1])$ is connected, $\gamma(0) = 0$ and $\gamma(1) = u_1$, therefore

$$\gamma([0,1]) \cap B(0,\rho) \neq \emptyset.$$

Hence, one has

$$\max_{u \in \gamma([0,1])} F(u) \geq \inf_{v \in B(0,\rho)} F(v) \geq \beta,$$

so $c \geq \beta$.

Now, arguing by contradiction, suppose that c is a regular value, namely $\mathbb{K}_c = \emptyset$. Then, by the Deformation Lemma ([62]) there exist $\epsilon \in]0, (\beta - \alpha)/2[$ and a deformation η such that

$$\eta(1, F^{c+\epsilon}) \subset F^{c-\epsilon}. \tag{3.1.1}$$

From the definition of c it follows that there exists some $\gamma \in \Gamma$ such that

$$\max_{u \in \gamma([0,1])} F(u) \leq c + \epsilon. \tag{3.1.2}$$

Clearly, the function $\gamma^*(t) = \eta(1, \gamma(t))$ belongs to $C([0,1]; X)$. Moreover, $\gamma^*(0) = \eta(1,0))$ and $\gamma^*(1) = \eta(1, u_1)) = u_1$ by (3.1.1) and also because $\max\{\gamma(0), \gamma(u_1)\} = \alpha < c - 2\epsilon$. Then γ^* belongs to Γ and, by (3.1.2), $\gamma([0,1]) \subset F^{c+\epsilon}$. Therefore, from (3.1.1) we obtain

$$\gamma^*([0,1]) = \eta(1, \gamma([0,1])) \subset F^{c+\epsilon},$$

that is

$$\max_{u \in \gamma^*([0,1])} F(u) \leq c - \varepsilon.$$

This is a contradiction, since $\gamma^* \in \Gamma$. □

Remark 3.1.6. *It is worth emphasizing that G. Bonanno and R. Livrea in [28] gave an alternative proof of the Ghoussoub-Preiss theorem, where the mountain pass geometry is the original assumption of Ambrosetti-Rabinowitz where also the equality is considered. The authors proved the theorem by using the ε-perturbation as introduced by Brezis-Nirenberg* *and besides the deformation lemma, other advanced tools such as the Radon measures space, sub-differential, or the theory of non-differentiable functions, are avoided.*

We recall also the result by Pucci-Serrin ([84]), giving first a definition.

Definition 3.1.7. *Let $(X, \|\cdot\|)$ be a Banach space, X^* its dual and $I : X \to \mathbb{R}$ a Gâteaux differentiable functional. We say that a functional I satisfies the Palais-Smale condition (in short* (PS)*-condition), if any sequence $\{u_n\} \subseteq X$ such that*

(PS_1) *$I(u_n)$ is bounded,*

(PS_2) *$I'(u_n) \to 0$ in X^* as $n \to \infty$,*

has a strongly convergent subsequence in X.

Clearly, if I satisfies the $(PS)_c$-condition for all $c \in \mathbb{R}$, then it satisfies the (PS)-condition.

Theorem 3.1.8. *Let X be a Banach space, $F : X \to \mathbb{R}$ a Gâteaux differentiable functional that satisfies the* (PS)*-condition and let x_0, x_1 be local minima points for F, with $x_0 \neq x_1$. Then, there exists $x_2 \in X \setminus \{x_0, x_1\}$ such that $F'(x_2) = 0$.*

The following version of the Mountain-Pass theorem is stated in the book of Papageorgiou-Rădulescu-Repovš [78, Theorem 5.4.6] and we recall the definition of the Cerami condition that is needed for the statement. In the following, for X being a Banach space, we denote by X^* its topological dual space.

Definition 3.1.9. *Given $I \in C^1(X)$, we say that I satisfies the Cerami-condition at level c,* $(C)_c$*-condition for short, if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that*

(C_c^1) *$I(u_n) \to c$ as $n \to \infty$,*

(C_c^2) *$(1 + \|u_n\|_X)\, I'(u_n) \to 0$ in X^* as $n \to \infty$,*

admits a strongly convergent subsequence in X.

If the $(C)_c$-condition holds for every $c \in \mathbb{R}$, we say that I satisfies the (C)-condition, that we can define as follows.

Definition 3.1.10. *Given $I \in C^1(X)$, we say that I satisfies the Cerami-condition,* (C)*-condition for short, if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that*

(C_1) *$\{I(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded,*

(C_2) *$(1 + \|u_n\|_X)\, I'(u_n) \to 0$ in X^* as $n \to \infty$,*

admits a strongly convergent subsequence in X.

Theorem 3.1.11. *Let X be a Banach space and suppose $\varphi \in C^1(X), u_0, u_1 \in X$ with $\|u_1 - u_0\| > \delta > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} \leq \inf\{\varphi(u) \,:\, \|u - u_0\| = \delta\} = m_\delta,$$
$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t)) \quad \textit{with} \quad \Gamma = \{\gamma \in C([0,1], X) \,:\, \gamma(0) = u_0, \gamma(1) = u_1\},$$

and φ satisfies the $(C)_c$*-condition. Then $c \geq m_\delta$ and c is a critical value of φ. Moreover, if $c = m_\delta$, then there exists $u \in B_\delta(u_0)$ such that $\varphi'(u) = 0$.*

3.2 Recent critical point theorems

In this section, we present some of the most recent results in the field of critical point theory. In particular, we mention the results that we are exploited in the study of some problems in the next chapters. Indeed, we recall a local minimum theorem due to Bonanno [17, 18] and a two critical points theorem established by Bonanno-D'Aguì in [23].

First, we present a local minimum theorem established by Bonanno in 2012 [17] and we premit a definition of a compactness conditions the is needed in the sequel. Let $(X, \|\cdot\|)$ be a Banach space, X^* its dual and $I : X \to \mathbb{R}$ a Gâteaux differentiable functional.

Definition 3.2.1. *Let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions, put*

$$I = \Phi - \Psi,$$

and fix $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$. We say that I verifies the Palais-Smale condition cut off lower at r_1 and upper at r_2 (in short ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition) if any sequence $\{u_n\}$ such that (PS$_1$), (PS$_2$) hold and

(PS_l^u) $(\gamma) r_1 < \Phi(u_n) < r_2 \forall n \in \mathbb{N}$,

has a convergent subsequence.

Clearly, if $r_1 = -\infty$ and $r_2 = +\infty$ it coincides with the classical (PS)-condition. Moreover, if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(\mathrm{PS})^{[r_2]}$, while if $r_1 \in \mathbb{R}$ and $r_2 = +\infty$ it is denoted by ${}^{[r_1]}(\mathrm{PS})$.

Furthermore, if $I = \Phi - \Psi$ satisfies ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition, then it satisfies ${}^{[\rho_1]}(\mathrm{PS})^{[\rho_2]}$-condition for all $\rho_1, \rho_2 \in [-\infty, +\infty]$ such that $r_1 \le \rho_1 < \rho_2 \le r_2$. So, in particular, if $I = \Phi - \Psi$ satisfies the classical (PS)-condition, then it satisfies ${}^{[\rho_1]}(\mathrm{PS})^{[\rho_2]}$-condition for all $\rho_1, \rho_2 \in [-\infty, +\infty]$ with $\rho_1 < \rho_2$.

Now, we present a result by Bonanno [17, Theorem 3.1] that ensures the existence of a local minimum for a given functional.

Theorem 3.2.2. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Put*

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$, with $r_1 < \Phi(x_0) < r_2$, such that

$$\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) \le r_2 - \Phi(x_0) + \Psi(x_0), \tag{3.2.1}$$

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \le r_1 - \Phi(x_0) + \Psi(x_0). \tag{3.2.2}$$

Moreover, assume that I satisfies ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition.

Then, there is $u_0 \in \Phi^{-1}(]r_1, r_2[)$ such that

$$I(u_0) \le I(u) \quad \text{for all } u \in \Phi^{-1}(]r_1, r_2[) \quad \text{and} \quad I'(u_0) = 0.$$

Remark 3.2.3. *We remind that if I satisfies (PS)-condition, then it verifies also ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition for any $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$. Hence, if the functional I satisfies the (PS)-condition and the algebraic inequalities given in (3.2.1) and (3.2.2), then the conclusion of Theorem 3.2.2 hold true.*

Now, we point out a consequence of the previous local minimum theorem, which can be found in Bonanno [18, Theorem 2.3].

Theorem 3.2.4. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{3.2.3}$$

and, for each

$$\lambda \in \Lambda_r := \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r[)} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(\mathrm{PS})^{[r]}$-condition.

Then, for each $\lambda \in \Lambda_r$ there is $u_\lambda \in \Phi^{-1}(]0, r[)$ (hence, $u_\lambda \neq 0$) such that

$$I_\lambda(u_\lambda) \leq I_\lambda(u) \quad \text{for all } u \in \Phi^{-1}(]0, r[) \quad \text{and} \quad I'_\lambda(u_\lambda) = 0.$$

An important multiliplicity result that we exploit in our study is a two critical point theorem established by Bonanno-D'Aguì in 2016 [23, Theorem 2.1], which is a nontrivial consequence of the local minimum theorem given in Theorem 3.2.4 in combination with the Ambrosetti-Rabinowitz Theorem (Theorem 3.1.5).

Theorem 3.2.5. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{3.2.4}$$

and for all $\lambda \in \Lambda_r = \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ admits at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

Proof. Fix λ as in the conclusion. Since I_λ satisfies (PS)-condition, then it verifies also $(\mathrm{PS})^{[r]}$-condition. Moreover, by (3.2.4) one has

$$\frac{\sup_{u \in \Phi^{-1}([-\infty, r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},$$

with $0 < \Phi(\tilde{u}) < r$. Therefore, from Theorem 3.2.4 it follows that there exits $u_{\lambda,1} \in \Phi^{-1}(]0,r[)$ (hence, $u_{\lambda,1} \neq 0$) such that

$$I_\lambda\left(u_{\lambda,1}\right) \leq I_\lambda(u) \quad \text{for all } u \in \Phi^{-1}(]0,r[) \quad \text{and} \quad I'_\lambda\left(u_{\lambda,1}\right) = 0.$$

Furthermore, we observe that one also has

$$I_\lambda\left(u_{\lambda,1}\right) \leq I_\lambda(u) \quad \text{for all } u \in \Phi^{-1}(]-\infty,r]) \quad \text{and} \quad I_\lambda\left(u_{\lambda,1}\right) < 0.$$

In fact, since $\lambda > \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}$, it holds that $\Phi(\tilde{u}) - \lambda\Psi(\tilde{u}) < 0 = \Phi(0) - \lambda\Psi(0)$, for which $I_\lambda\left(u_{\lambda,1}\right) \leq I_\lambda(\tilde{u}) < I_\lambda(0) = 0$. Moreover, for all $\bar{u} \in X$ such that $\Phi(\bar{u}) = r$, taking into account that $\lambda < \frac{r}{\sup_{u\in\Phi^{-1}(j-\infty,r])}\Psi(u)}$, one has

$$\Phi(\bar{u}) - \lambda\Psi(\bar{u}) \geq \Phi(\bar{u}) - \lambda \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) > \Phi(\bar{u}) - r = 0,$$

that is $I_\lambda(\bar{u}) > I_\lambda(0) > I_\lambda\left(u_{\lambda,1}\right)$. So, our claim is proved.

Now, since I_λ is unbounded from below there is $\bar{u}_{\lambda,2} \in X$ such that

$$I_\lambda\left(\bar{u}_{\lambda,2}\right) < I_\lambda\left(u_{\lambda,1}\right)$$

Clearly, being $u_{\lambda,1}$ a global minimum for I_λ in $\Phi^{-1}(]-\infty,r])$, must be $\Phi\left(\bar{u}_{\lambda,2}\right) > r$. It is easy to verify that all the assumptions of the Mountain Pass Theorem are satisfied, hence there exists $u_{\lambda,2} \in X$ such that

$$I'_\lambda\left(u_{\lambda,2}\right) = 0 \quad \text{and} \quad I_\lambda\left(u_{\lambda,2}\right) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1]) : \gamma(0) = u_{\lambda,1}, \gamma(1) = \bar{u}_{\lambda,2}\}$.

We now claim that $I_\lambda\left(u_{\lambda,2}\right) > 0$. To this end, set

$$k = r - \lambda \sup_{u\in\Phi^{-1}([-\infty,r])} \Psi(u),$$

and then observe that, since $\lambda < \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}$, one has $k > 0$. Now, let $\gamma \in \Gamma$. Since $\Phi(\gamma(0)) < r$ and $\Phi(\gamma(1)) > r$, there is $\bar{t} \in]0,1[$ such that $\Phi(\gamma(\bar{t})) = r$. So, setting $\bar{u} = \gamma(\bar{t})$, one has

$$\Phi(\bar{u}) - \lambda\Psi(\bar{u}) \geq \Phi(\bar{u}) - \lambda \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) = r - \lambda \sup_{u\in\Phi^{-1}([]-\infty,r])} \Psi(u) = k,$$

that is $I_\lambda(\gamma(\bar{t})) > k$. It follows that $\max_{t\in[0,1]} I_\lambda(\gamma(t)) > k$ for each $\gamma \in \Gamma$. Hence, one has

$$I_\lambda\left(u_{\lambda,2}\right) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) \geq k > 0,$$

for which our claim is proved and the conclusion is achieved. □

A version of this result can be obtained taking into account Remark 2.2 [23].

Theorem 3.2.6. *Let X be a real Banach space and let $\Phi, \Psi\colon X \to \mathbb{R}$ be two continously Gâteaux differentiable functionals such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that Φ *is coercive and there exist* $r \in \mathbb{R}$ *and* $\tilde{u} \in X$*, with* $0 < \Phi(\tilde{u}) < r$*, such that*

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{3.2.5}$$

and, for each $\lambda \in \Lambda_r := \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)} \right[$*, the functional* $I_\lambda = \Phi - \lambda\Psi$ *satisfies the* (C)*-condition and it is unbounded from below.*

Then, for each $\lambda \in \Lambda_r$*, the functional* I_λ *admits at least two nontrivial critical points* $u_{\lambda,1}, u_{\lambda,2}$ *such that* $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

Proof. One can assume the (C)-condition instead of (PS)-condition, provided that the coercivity of Φ is assumed. Indeed, it is enough to observe that (PS)-condition and (C)-condition coincide for bounded sequences and so, taking into account that Φ is coercive, also (C)-condition implies the $(PS)^{[r]}$-condition for all $r > 0$. Therefore, the same proof of Theorem (3.2.5) ensures our claim, by applying the version of mountain pass theorem with the (C)-condition (see, for instance, Theorem 3.1.11). □

Chapter 4

Dirichlet double phase problem

This chapter is devoted to the study of a nonlinear double phase problem with variable exponents under Dirichlet boundary condition. In particular, through critical point theory we determine the existence of two bounded weak solutions under very general assumptions on the nonlinear term, such as a subcritical growth and a superlinear condition. Moreover, we state some special cases in which the solutions turn out to be nonnegative.

The results presented in this chapter are obtained in [6], in collaboration with G. Bonanno, G. D'Aguì and P. Winkert.

4.1 The problem

Consider the following boundary value problem with a nonlinear differential equation involving the double phase operator with variable exponents and Dirichlet boundary condition

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) &= \lambda f(x,u) \quad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{D_λ}$$

where the domain Ω, the exponents p, q and the weight μ verify assumption (H), $\lambda > 0$ is a parameter and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies subcritical growth and a certain behavior at $\pm\infty$.

For the sake of completeness, we recall here hypothesis (H), which has been introduced in Chapter 2, and we introduce the assumptions on the perturbation in problem (D_λ).

Suppose the following hypotheses:

(H) $\Omega \subseteq \mathbb{R}^N, N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $p, q \in C(\overline{\Omega})$ such that $1 < p(x) < N$ and $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where $p^*(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}$ is the critical Sobolev exponent to $p(\cdot)$, and $\mu \in L^\infty(\Omega)$ with $\mu(x) \geq 0$ for a.a. $x \in \Omega$,

(H_f^D) Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $F(x,t) = \int_0^t f(x,s)\, ds$ for all $x \in \Omega$ be such that the following hold:

(f_1^D) f is a Carathéodory function, that is, $x \to f(x,t)$ is measurable for all $t \in \mathbb{R}$ and $x \to f(x,t)$ is continuous for a.a. $x \in \Omega$;

(f_2^D) there exist $\ell \in C_+(\overline{\Omega})$ with $\ell_+ < (p_-)^*$ and $\kappa_1 > 0$ such that

$$|f(x,t)| \leq \kappa_1 \left(1 + |t|^{\ell(x)-1}\right)$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$;

(f_3^D)

$$\lim_{t\to\pm\infty} \frac{F(x,t)}{|t|^{q_+}} = +\infty$$

uniformly for a.a. $x \in \Omega$;

(f_4^D) there exists $\zeta \in C_+(\overline{\Omega})$ with

$$\zeta_- \in \left((\ell_+ - p_-)\frac{N}{p_-}, \ell_+ \right)$$

and $\zeta_0 > 0$ such that

$$0 < \zeta_0 \le \liminf_{t\to\pm\infty} \frac{f(x,t)t - q_+F(x,t)}{|t|^{\zeta(x)}}$$

uniformly for a.a. $x \in \Omega$.

Here, we give some remarks on the assumptions.

Remark 4.1.1. *We obeserve that* (f_3^D) *is weaker than requiring that*

$$\lim_{t\to\pm\infty} \frac{f(x,t)}{|t|^{q_+}} = +\infty$$

uniformly for a.a. $x \in \Omega$. *On the other hand, using* q_+ *is stronger than using* $q(\cdot)$ *as exponent.*

Remark 4.1.2. *It is worth noting that the condition on* ζ *in* (f_4^D) *is well defined since from* (f_2^D) *we have* $\ell_+ < (p_-)^*$ *and so it holds*

$$(\ell_+ - p_-)\frac{N}{p_-} = \ell_+\frac{N}{p_-} - (p_-)^*\frac{N-p_-}{p_-} < \ell_+\frac{N}{p_-} - \ell_+\frac{N-p_-}{p_-} = \ell_+.$$

We underline that this boundedness is needed for the interpolation argument in the claim of Lemma 4.3.1 and it is required only for ζ_- *and not for the whole exponent* ζ. *Indeed, this is one of the advantages of the variable exponent setting. Furthermore, another advantage is that one can choose different exponents for going to* $\pm\infty$ *(see for example* (h_4^N) *for the Neumann problem in Chapter 6), however in this case we choose the same exponent just for simplicity.*

4.2 Variational framework

The differential operator in (D_λ) is the so-called double phase operator with variable exponents given by

$$-\operatorname{div}\left(|\nabla(\cdot)|^{p(x)-2}\nabla(\cdot) + \mu(x)|\nabla(\cdot)|^{q(x)-2}\nabla(\cdot) \right),$$

and since we deal with Dirichlet boundary conditions, we search for solutions u of problem (D_λ) in the Musielack-Orlicz Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$, introduced in Chapter 2, endowed with the norm $\|\cdot\|_{1,\mathcal{H},0}$ (see Proposition 2.3.13 and formula (2.3.10)).

We refer to subsection 2.3.1 for the properties of the space $W_0^{1,\mathcal{H}}(\Omega)$ and in particular we will exploit the embeddings in (i) and (iii) of Proposition 2.3.12, which clearly hold also under assumption (H) that is stronger than (H1).

Now, for any $r \in C(\overline{\Omega})$ for which the continuous embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ hold (see Proposition 2.3.12), we denote by $\tilde{c}_r$ the best constant for which one has

$$\|u\|_{r(\cdot)} \leq \tilde{c}_r \|u\|_{1,\mathcal{H},0}. \tag{4.2.1}$$

It is well known that $u \in W_0^{1,\mathcal{H}}(\Omega)$ is called a weak solution of problem (D_λ) if

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u \right) \cdot \nabla v \, \mathrm{d}x = \lambda \int_\Omega f(x,u)v \, \mathrm{d}x, \tag{4.2.2}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. In order to establish results on the existence of two nontrivial weak solution for (D_λ), we define the functionals $\Phi, \Psi, I_\lambda \colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)} \right) \mathrm{d}x, \quad \Psi(u) = \int_\Omega F(x,u(x)) \, \mathrm{d}x,$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$, where I_λ is the so-called energy functional related to the problem. We know that Φ and Ψ are Gâteaux differentiable (see [40, Proposition 3.1]) with the following derivatives

$$\begin{aligned}
\langle \Phi'(u), v\rangle &= \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u \right) \cdot \nabla v \, \mathrm{d}x, \\
\langle \Psi'(u), v\rangle &= \int_\Omega f(x,u)v \, \mathrm{d}x, \\
\langle I_\lambda'(u), v\rangle &= \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u \right) \cdot \nabla v \, \mathrm{d}x - \lambda \int_\Omega f(x,u)v \, \mathrm{d}x,
\end{aligned}$$

for all $u, v \in W_0^{1,\mathcal{H}}(\Omega)$, where $\langle \cdot\,, \cdot \rangle$ is the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$. Hence, from (4.2.2) it follows that u is a weak solution of (D_λ) if and only if u is a critical point of I_λ, i.e.

$$u \in W_0^{1,\mathcal{H}}(\Omega) \text{ weak solution of } (D_\lambda) \quad \Longleftrightarrow \quad \langle I_\lambda'(u), v\rangle = 0 \quad \text{for all } v \in W_0^{1,\mathcal{H}}(\Omega).$$

In the next proposition we summarize the properties of the operator $\Phi' \colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$, see [40, Theorem 3.3] which is a generalization of [68, Proposition 3.1] in the variable exponent case.

Proposition 4.2.1. *Let* (H) *be satisfied. Then, the operator* Φ' *is continuous, bounded, strictly monotone, coercive, a homeomorphism and it satisfies the* (S_+)*−property, that is,*

$$\text{if} \quad u_n \rightharpoonup u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle \leq 0,$$

then $u_n \to u$ *in* $W_0^{1,\mathcal{H}}(\Omega)$.

Our main tool is is a two critical point theorem due to Bonanno-D'Aguì in [23, Theorem 2.1 and Remark 2.2], recalled in Theorem 3.2.6 in Chapter 3.

4.3 Main results

In this section, we present our main result on the existence of two nontrivial solutions for the Dirichlet double phase problem with variational exponents given in (D_λ). From now on, in this chapter, we put

$$X = W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \|\cdot\|_X = \|\cdot\|_{1,\mathcal{H},0} = \|\nabla \cdot\|_{\mathcal{H}}.$$

First, we present the following Lemma that we will use in the proof of the main result.

Lemma 4.3.1. *Let* (H) *and* (H_f^D) *be satisfied. Then, the functional* I_λ *satisfies the* (C)-*condition for all* $\lambda > 0$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}}(\Omega)$ be a sequence such that (C_1) and (C_2) from Definition 3.1.10 hold. We provide the proof in three steps.

Claim 1. $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{\zeta_-}(\Omega)$.

First, it is easy to show that using (f_1^D) and (f_4^D) we get that

$$f(x,t)t - q_+F(x,t) \geq c_1|t|^{\zeta_-} - c_2 \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R}, \tag{4.3.1}$$

with some constants $c_1, c_2 > 0$. Moreover, from C_1 we have that there exists a constant $M > 0$ such that for all $n \in \mathbb{N}$ one has $|I_\lambda(u_n)| \leq M$, so

$$\left| \int_\Omega \left(\frac{|\nabla u_n|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u_n|^{q(x)}}{q(x)} \right) \mathrm{d}x - \lambda \int_\Omega F(x,u_n)\, \mathrm{d}x \right| \leq M,$$

which, multiplying by q_+, leads to

$$\rho_{\mathcal{H}}(\nabla u_n) - \lambda \int_\Omega q_+F(x,u_n)\, \mathrm{d}x \leq c_3, \tag{4.3.2}$$

for some $c_3 > 0$ and for all $n \in \mathbb{N}$. Besides, from (C_2), there exists $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0^+$ such that

$$|\langle I'_\lambda(u_n), v\rangle| \leq \frac{\varepsilon_n\|v\|_X}{1+\|u_n\|_X} \quad \text{for all } n \in \mathbb{N} \text{ and for all } v \in X. \tag{4.3.3}$$

Choosing $v = u_n$, one has

$$\left| \int_\Omega \left(|\nabla u_n|^{p(x)} + \mu(x)|\nabla u_n|^{q(x)} \right) \mathrm{d}x - \lambda \int_\Omega f(x,u_n)u_n\, \mathrm{d}x \right| < \varepsilon_n,$$

which implies

$$-\rho_{\mathcal{H}}(\nabla u_n) + \lambda \int_\Omega f(x,u_n)u_n\, \mathrm{d}x < \varepsilon_n \tag{4.3.4}$$

for all $n \in \mathbb{N}$. Adding (4.3.2) and (4.3.4) we obtain

$$\int_\Omega (f(x,u_n)u_n - q_+F(x,u_n))\, \mathrm{d}x < c_4$$

for all $n \in \mathbb{N}$ with some constant $c_4 > 0$. Using this along with (4.3.1) we derive

$$\int_\Omega \left(c_1|u_n|^{\zeta_-} - c_2\right) \, dx < c_4,$$

which gives

$$\|u_n\|_{\zeta_-}^{\zeta_-} < c_5 \quad \text{for all } n \in \mathbb{N}$$

with some $c_5 > 0$. Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{\zeta_-}(\Omega)$ and so Claim 1 is proved.

Claim 2. $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X.

From (f_2^D) and (f_4^D), we have that

$$\zeta_- < \ell_+ < (p_-)^*.$$

Hence, there exists $s \in (0,1)$ such that

$$\frac{1}{\ell_+} = \frac{s}{(p_-)^*} + \frac{1-t}{\zeta_-}, \tag{4.3.5}$$

and using the interpolation inequality, see Papageorgiou-Winkert [79, Proposition 2.3.17 p.116], one has

$$\|u_n\|_{\ell_+} \le \|u_n\|_{(p_-)^*}^s \|u_n\|_{\zeta_-}^{1-s} \quad \text{for all } n \in \mathbb{N}.$$

From Claim 1, it follows that

$$\|u_n\|_{\ell_+} \le c_6 \|u_n\|_{(p_-)^*}^s \tag{4.3.6}$$

for some $c_6 > 0$ and for all $n \in \mathbb{N}$. Again, from (4.3.3) with $v = u_n$, we get

$$\rho_{\mathcal{H}}(\nabla u_n) - \lambda \int_\Omega f(x,u_n)u_n \, dx < \varepsilon_n. \tag{4.3.7}$$

We may assume $\|u_n\|_X \ge 1$ for all $n \in \mathbb{N}$, otherwise we are done. Then, using Proposition 2.3.10(iv), (4.3.7), (f_2^D) and (4.3.6), we derive that

$$\begin{aligned}
\|u_n\|_X^{p_-} &\le \rho_{\mathcal{H}}(\nabla u_n) < \varepsilon_n + \lambda \int_\Omega f(x,u_n)u_n \, dx \\
&\le \lambda \kappa_1 \left(\|u_n\|_1 + \|u_n\|_{\ell_+}^{\ell_+}\right) + \varepsilon_n \\
&\le c_7 \left(1 + c_6^{\ell_+} \|u_n\|_{(p_-)^*}^{s\ell_+}\right) + \varepsilon_n
\end{aligned}$$

with $c_7 > 0$. Hence, taking the embedding $X \hookrightarrow L^{(p_-)^*}(\Omega)$ into account, we have

$$\|u_n\|_X^{p_-} \le c_8 \left(1 + \|u_n\|_X^{s\ell_+}\right) + \varepsilon_n,$$

for all $n \in \mathbb{N}$ and for some $c_8 > 0$. From (4.3.5) and (f_4^D), it follows that

$$\begin{aligned}
s\ell_+ &= \frac{(p_-)^*)(\ell_+ - \zeta_-)}{(p_-)^*) - \zeta_-} = \frac{Np_-(\ell_+ - \zeta_-)}{Np_- - N\zeta_- + \zeta_- p_-} \\
&< \frac{Np_-(\ell_+ - \zeta_-)}{Np_- - N\zeta_- + p_-(\ell_+ - p_-)\frac{N}{p_-}} = p_-,
\end{aligned}$$

and this shows our second claim.

Claim 3. $u_n \to u$ in X up to a subsequence.

Since $\{u_n\}_{n\in\mathbb{N}} \subset X$ is bounded (Claim 2) and X is a reflexive space, there exists a subsequence, not relabeled, that converges weakly in X and strongly in $L^{\ell_+}(\Omega)$, that is,

$$u_n \rightharpoonup u \quad \text{in X} \quad \text{and} \quad u_n \to u \quad \text{in } L^{\ell_+}(\Omega).$$

Using this to (4.3.3) with $v = u_n - u$ and passing to the limit as $n \to \infty$, we obtain

$$\langle \Phi'(u_n), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

Since Φ' satisfies the (S_+)-property, see Proposition 4.2.1, the proof is complete. □

Now, we state our main result. For this purpose, let

$$R := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega).$$

Then there exists $x_0 \in \Omega$ such that the ball with center x_0 and radius $R > 0$ belongs to Ω, that is,

$$B(x_0, R) \subseteq \Omega.$$

We indicate with ω_R the the Lebesgue measure of $B(x_0, R)$ in $\mathbb{R}^N$ given by

$$\omega_R := |B(x_0, R)| = \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})} R^N,$$

and we put

$$\delta := \frac{\min\left\{R^{p_-}, R^{q_+}\right\}\, p_-}{\max\left\{1, \|\mu\|_\infty\right\}\, \omega_R\, (2^N - 1)2^{q_+ + 1 - N}}.$$

Furthermore, for any $r, \eta \in \mathbb{R}^+$, we define

$$\alpha(r) := \kappa_1 \frac{\tilde{c}_1 \max\left\{(q_+ r)^{\frac{1}{p_-}}, (q_+ r)^{\frac{1}{q_+}}\right\} + \bar{c}_\ell \max\left\{(q_+ r)^{\frac{\ell_+}{p_-}}, (q_+ r)^{\frac{\ell_-}{q_+}}\right\}}{r}, \tag{4.3.8}$$

$$\beta(\eta) := \delta \frac{\displaystyle\int_{B(x_0, \frac{R}{2})} F(x, \eta)\, dx}{\max\left\{\eta^{p_-}, \eta^{q_+}\right\}}, \tag{4.3.9}$$

where $\bar{c}_\ell = \max\left\{\tilde{c}_\ell^{\ell_-}, \tilde{c}_\ell^{\ell_+}\right\}$ and $\tilde{c}_1$, $\tilde{c}_\ell$, κ_1, ℓ are given in (4.2.1) and (f_2^D), respectively.

Theorem 4.3.2. *Assume that* (H) *and* (H_f^D) *hold. Furthermore, suppose that there exist two positive constants r, η satisfying*

$$\max\left\{\eta^{p_-}, \eta^{q_+}\right\} < \delta r, \tag{4.3.10}$$

such that

(h_1^D) $F(x,t) \geq 0$ *for a.a.* $x \in \Omega$ *and for all* $t \in [0, \eta]$;

(h_2^D) $\alpha(r) < \beta(\eta)$,

as defined in (4.3.8) *and* (4.3.9)*. Then, for each* $\lambda \in \Lambda$*, where*

$$\Lambda := \left] \frac{1}{\beta(\eta)}, \frac{1}{\alpha(r)} \right[,$$

problem (D_λ) *admits at least two nontrivial bounded weak solutions* $u_{\lambda,1}, u_{\lambda,2} \in W_0^{1,\mathcal{H}}(\Omega)$ *with opposite energy sign.*

Proof. Our aim is to apply Theorem 3.2.6. Let $(X, \|\cdot\|_X), \Phi, \Psi$ be as introduced in Section 4.2 and note that they already fulfill the required assumptions needed in Theorem 3.2.6. In particular, from Proposition 2.3.10(vi) and (f_3^D) follows that Φ is coercive and I_λ is unbounded from below, respectively.

Now, fix $\lambda \in \Lambda$, which is nonempty because of (h_2^D), and consider $\tilde{u} \in X$ defined by

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R), \\ \dfrac{2\eta}{R}(R - |x - x_0|) & \text{if } x \in B(x_0, R) \setminus B\left(x_0, \frac{R}{2}\right), \\ \eta & \text{if } x \in B\left(x_0, \frac{R}{2}\right). \end{cases}$$

Clearly, $\tilde{u} \in X$. We show that $0 < \Phi(\tilde{u}) < r$. Indeed, using (4.3.10), it follows that

$$\begin{aligned} \Phi(\tilde{u}) &= \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} \left(\frac{1}{p(x)} \left(\frac{2\eta}{R}\right)^{p(x)} + \frac{\mu(x)}{q(x)} \left(\frac{2\eta}{R}\right)^{q(x)} \right) \mathrm{d}x \\ &\le \frac{2^{q_+}}{p_-} \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} \left(\left(\frac{\eta}{R}\right)^{p(x)} + \mu(x) \left(\frac{\eta}{R}\right)^{q(x)} \right) \mathrm{d}x \\ &\le \frac{2^{q_+}}{p_-} \frac{\max\{1, \|u\|_\infty\}}{\min\{R^{p_-}, R^{q_+}\}} \max\{\eta^{p_-}, \eta^{q_+}\} \cdot 2 \cdot \left(\omega_R - \omega_{\frac{R}{2}}\right) \\ &= \frac{1}{\delta} \max\{\eta^{p_-}, \eta^{q_+}\} < r. \end{aligned}$$

Now, we prove (3.2.5). From assumption (h_1^D), we obtain

$$\begin{aligned} \Psi(\tilde{u}) &= \int_{B\left(x_0,\frac{R}{2}\right)} F(x,\eta)\,\mathrm{d}x + \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} F\left(x, \frac{2\eta}{R}(R - |x - x_0|)\right) \mathrm{d}x \\ &\ge \int_{B\left(x_0,\frac{R}{2}\right)} F(x,\eta)\,\mathrm{d}x. \end{aligned}$$

Hence,

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \ge \delta \frac{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,\eta)\,\mathrm{d}x}{\max\{\eta^{p_-}, \eta^{q_+}\}}. \tag{4.3.11}$$

On the other hand, fix $u \in X$ such that $\Phi(u) < r$. Then, one has

$$q_+ r > q_+ \Phi(u) > \rho_{\mathcal{H}}(\nabla u) \ge \min\left\{\|u\|_X^{p_-}, \|u\|_X^{q_+}\right\},$$

which implies that

$$\Phi^{-1}(]-\infty, r]) \subseteq \left\{u \in X : \|u\|_X \le \max\left\{(q_+ r)^{\frac{1}{p_-}}, (q_+ r)^{\frac{1}{q_+}}\right\}\right\}.$$

Furthermore, we have

$$\begin{aligned}
&\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) \\
&\le \sup_{u\in\Phi^{-1}(]-\infty,r])} \kappa_1 \int_\Omega \left(|u|+|u|^{\ell(x)}\right)\,\mathrm{d}x \\
&= \sup_{u\in\Phi^{-1}(]-\infty,r])} \kappa_1 \left(\|u\|_1+\rho_{\ell(\cdot)}(u)\right) \\
&\le \sup_{u\in\Phi^{-1}(]-\infty,r])} \kappa_1 \left(\|u\|_1+\max\left\{\|u\|_{\ell(\cdot)}^{\ell_-},\|u\|_{\ell(\cdot)}^{\ell_+}\right\}\right) \\
&\le \sup_{u\in\Phi^{-1}(]-\infty,r])} \kappa_1 \left(\tilde{c}_1\|u\|_X+\tilde{c}_\ell\max\left\{\|u\|_X^{\ell_-},\|u\|_X^{\ell_+}\right\}\right) \\
&\le \kappa_1\left(\tilde{c}_1\max\left\{(q_+r)^{\frac{1}{p_-}},(q_+r)^{\frac{1}{q_+}}\right\}+\tilde{c}_\ell\max\left\{(q_+r)^{\frac{\ell_+}{p_-}},(q_+r)^{\frac{\ell_-}{q_+}}\right\}\right).
\end{aligned}$$

Then, taking (h_2^D) and (4.3.11) into account, we get

$$\begin{aligned}
&\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} \\
&\le \frac{\kappa_1\left(\tilde{c}_1\max\left\{(q_+r)^{\frac{1}{p_-}},(q_+r)^{\frac{1}{q_+}}\right\}+\tilde{c}_\ell\max\left\{(q_+r)^{\frac{\ell_+}{p_-}},(q_+r)^{\frac{\ell_-}{q_+}}\right\}\right)}{r} \\
&< \delta\,\frac{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,\eta)\,\mathrm{d}x}{\max\{\eta^{p_-},\eta^{q_+}\}} \le \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},
\end{aligned}$$

namely hypothesis (3.2.5) is satisfied. Hence, along with Lemma 4.3.1, Theorem 3.2.6 ensures the existence of two nontrivial weak solutions $u_{\lambda,1}, u_{\lambda,2} \in W_0^{1,\mathcal{H}}(\Omega)$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$. Finally, from Crespo-Blanco-Winkert [39, Theorem 3.1] it follows that $u_{\lambda,1}, u_{\lambda,2}$ belong to $L^\infty(\Omega)$. This finishes the proof. □

Corollary 4.3.3. *Suppose that all assumptions of Theorem 4.3.2 are satisfied. Moreover, assume that*

$$f(x,0)\ge 0 \quad \text{and} \quad f(x,t)=f(x,0) \quad \text{for a.a. } x\in\Omega \text{ and for all } t<0.$$

Then, problem (D_λ) admits at least two nontrivial and nonnegative bounded weak solutions $u_{\lambda,1}, u_{\lambda,2} \in W_0^{1,\mathcal{H}}(\Omega)$ with opposite energy sign.

Proof. Since all the assumptions are satisfied, we can apply Theorem 4.3.2. We only need to prove that the solutions $u_{\lambda,1}, u_{\lambda,2}$ are nonnegative. Since $u_{\lambda,1}$ is a weak solution of (D_λ), from (4.2.2) one has $\langle I_\lambda'(u_{\lambda,1}), v\rangle = 0$ for every $v \in X$. Choosing $v = -u_{\lambda,1}^- = -\max\{-u_{\lambda,1}, 0\} \in W_0^{1,\mathcal{H}}(\Omega)$, see [40, Proposition 2.17(iii)], we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_{\lambda,1}|^{p(x)-2}\nabla u_{\lambda,1}+\mu(x)|\nabla u_{\lambda,1}|^{q(x)-2}\nabla u_{\lambda,1}\right)\cdot\nabla\left(-u_{\lambda,1}^-\right)\,\mathrm{d}x \\
&= \lambda\int_\Omega f(x,u_{\lambda,1})\left(-u_{\lambda,1}^-\right)\,\mathrm{d}x,
\end{aligned}$$

which leads to

$$-\rho_{\mathcal{H}}(\nabla u_{\lambda,1}^-) \geq 0.$$

But the previous inequality implies that

$$\min\left\{\|u_{\lambda,1}^-\|_X^{p_-}, \|u_{\lambda,1}^-\|_X^{q_+}\right\} \leq \rho_{\mathcal{H}}(\nabla u_{\lambda,1}^-) \leq 0,$$

which gives $\|u_{\lambda,1}^-\|_X = 0$. Then, $u_{\lambda,1}^- = 0$ and $u_{\lambda,1} \geq 0$. With the same argument we obtain $u_{\lambda,2} \geq 0$ and the proof is complete. □

Now we consider the special case when the nonlinear term is nonnegative.

Theorem 4.3.4. *Assume that* (H) *and* (H_f^D) *hold. Furthermore, suppose that f is nonnegative and*

$$\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^{p_-}} = +\infty \qquad (h_3^D)$$

Then, for each $\lambda \in]0, \lambda^*[$, with

$$\lambda^* = \sup_{r>0} \frac{1}{\alpha(r)},$$

where $\alpha(r)$ is given in (4.3.8)*, problem* (D_λ) *admits at least two nontrivial and nonnegative bounded weak solutions $u_{\lambda,1}$, $u_{\lambda,2} \in W_0^{1,\mathcal{H}}(\Omega)$ with opposite energy sign.*

Proof. We observe that (h_3^D) implies that

$$\begin{aligned}\limsup_{\eta\to 0^+} \beta(\eta) &= \limsup_{\eta\to 0^+} \delta \frac{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,\eta)\,\mathrm{d}x}{\max\{\eta^{p_-},\eta^{q_+}\}} \\ &\geq \delta\,\omega_{\frac{R}{2}} \limsup_{\eta\to 0^+} \frac{\inf_{x\in\Omega} F(x,\eta)}{\eta^{p_-}} = +\infty.\end{aligned} \tag{4.3.12}$$

Then, fixing $\lambda \in]0, \lambda^*[$, there exists $r > 0$ such that

$$\lambda < \frac{1}{\alpha(r)} = \frac{r}{\kappa_1\left(\tilde{c}_1 \max\left\{(q_+r)^{\frac{1}{p_-}}, (q_+r)^{\frac{1}{q_+}}\right\} + \tilde{c}_\ell \max\left\{(q_+r)^{\frac{\ell_+}{p_-}}, (q_+r)^{\frac{\ell_-}{q_+}}\right\}\right)}.$$

Moreover, from (4.3.12), there is $\eta > 0$ small enough such that

$$\delta\,\omega_{\frac{R}{2}} \frac{\inf_{x\in\Omega} F(x,\eta)}{\eta^{p_-}} > \frac{1}{\lambda},$$

implying that $\alpha(r) < \beta(\eta)$. Applying Theorem 4.3.2 and arguing as in the proof of Corollary 4.3.3, we achieve our goal. □

Finally, we provide an example of a function that satisfies our assumptions.

Example 4.3.5. *Consider* $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x,t) = \begin{cases} |t|^{\alpha(x)-2}t & \text{if } |t| < 1, \\ |t|^{\beta(x)-2}t\,(\log|t|+1) & \text{if } |t| \geq 1, \end{cases}$$

where $\alpha, \beta \in C(\overline{\Omega})$ *such that* $q_+ < \beta(x) < (p_-)^*$ *for all* $x \in \overline{\Omega}$ *and*

$$\frac{\beta_+}{p_-} - \frac{\beta_-}{N} < 1.$$

Then, f *satisfies assumptions* ($\mathrm{H}_{\mathrm{f}}^{D}$) *with* $\zeta(x) = \beta(x)$ *for all* $x \in \overline{\Omega}$ *and* $\ell(x) = \beta(x) + \sigma$ *for all* $x \in \overline{\Omega}$*, with* $\sigma > 0$ *small enough such that*

$$\begin{aligned} &\ell_+ < (p_-)^*, \\ &\frac{\ell_+}{p_-} - \frac{\beta_-}{N} < 1. \end{aligned}$$

Moreover, we can apply Theorem 4.3.4 at $\tilde{f}(x,t) = |f(x,t)|$ *for every* $(x,t) \in \Omega \times \mathbb{R}$*, requiring also that* $\alpha(x) < p_-$ *for all* $x \in \overline{\Omega}$*.*

Chapter 5

Robin double phase problem

This chapter deals with the study of a nonlinear parametric differential problem involving the double phase operator with variable exponents and Robin boundary conditions with critical growth. In particular, we determine an interval of parameters that guarantees the existence of two nontrivial weak solutions and we state the hypotheses under which the solutions turn out to be nonnegative. The main tool of our investigastion is a two critical points theorem.

The results presented in this chapter are obtained in [4], in collaboration with V. Morabito.

5.1 The problem

Let $\Omega \subseteq \mathbb{R}^N, N \geq 2$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and denote with $\nu(x)$ the unit normal of Ω at every point $x \in \partial\Omega$. We study the following nonlinear differential equation involving the double phase operator with variable exponents under nonlinear Robin boundary conditions

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) + \alpha(x)|u|^{p(x)-2}u &= \lambda f(x,u) \quad \text{in}\, \Omega, \\ \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nu &= -\beta(x)|u|^{p_*(x)-2}u \quad \text{on}\, \partial\Omega, \end{aligned} \tag{R_λ}$$

where $\lambda > 0, p, q, \mu$ satisfy assumptions stated in the sequel, α, β are nonnegative functions belonging to suitable spaces and $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function with a subcritical growth that satisfies the well-known Ambrosetti-Rabinowitz condition. In particular, recalling that

$$p^*(x) := \frac{Np(x)}{N-p(x)} \quad \text{and} \quad p_*(x) := \frac{(N-1)p(x)}{N-p(x)} \quad \text{for all } x \in \overline{\Omega},$$

are the critical Sobolev exponent of p and the critical exponent to p on $\partial\Omega$, respectively, we emphasize that we allow a critical growth on the boundary. Clearly, if $\beta \equiv 0$, then (R_λ) becomes the corresponding homogeneous Neumann problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) + \alpha(x)|u(x)|^{p(x)-2}u &= \lambda f(x,u) \quad \text{in}\ \Omega, \\ \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nu &= 0 \quad \text{on}\ \partial\Omega. \end{aligned}$$

For our purpose, we require more restrictive hypotheses than (H) on the exponents. Now, we introduce them and we also state the assumptions on the weight functions. Indeed, we assume the following:

(H^R) (i) $p \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$, $q \in C(\overline{\Omega})$ such that

$$1 < p(x) < N, \quad p(x) < q(x) < p_*(x) \quad \text{for all } x \in \overline{\Omega},$$

(ii) $\mu \in L^\infty(\Omega)$ with $\mu \geq 0$ a.e. in Ω,

(iii) $\alpha \in L^\infty(\Omega)$ with $\alpha \geq 0$ a.e. in Ω and $\alpha \not\equiv 0$,

(iv) $\beta \in L^\infty(\partial\Omega)$ with $\beta \geq 0$ a.e. in $\partial\Omega$.

(H$_f^R$) Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $F(x,t) = \int_0^t f(x,s)\, ds$ for all $x \in \Omega$ be such that:

(f$_1^R$) f is L^1-Carathéodory, i.e.

(i) $x \to f(x,t)$ is measurable for all $t \in \mathbb{R}$;

(ii) $t \to f(x,t)$ is continuous for a.a. $x \in \Omega$;

(iii) for all $s > 0$ the function $\sup_{|t| \leq s} |f(\cdot,t)|$ belongs to $L^1(\Omega)$;

(f$_2^R$) there exist $k_1, k_2 > 0$ and $\ell \in C_+(\overline{\Omega})$ with $\ell_+ < (p^*)_-$, such that

$$|f(x,t)| \leq k_1 + k_2 |t|^{\ell(x)-1}$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$;

(AR) there exist $\eta > (p_*)_+$, $s > 0$ such that

$$0 < \eta F(x,t) \leq t f(x,t)$$

for all $x \in \Omega$ and for all $|t| \geq s$.

5.2 Variational framework

Since we deal with a nonlinear Robin double phase problem, we search for solutions u of problem (R_λ) in the Musielack-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$, introduced in Chapter 2. We need to equipe the space with an equivalent norm, which is a particular case of the new general one introduced in subsection 2.3.2. To this aim we consider the seminormed spaces

$$L^{p(\cdot)}_\alpha(\Omega) = \left\{ u \in M(\Omega) \,:\, \int_\Omega \alpha(x) |u|^{p(x)}\, dx < \infty \right\},$$
$$L^{p_*(\cdot)}_\beta(\partial\Omega) = \left\{ u \in M(\Omega) \,:\, \int_{\partial\Omega} \beta(x) |u|^{p_*(x)}\, d\sigma < \infty \right\},$$

endowed with the seminorms

$$\|u\|_{p(\cdot),\alpha} = \inf \left\{ \tau > 0 \,:\, \int_\Omega \alpha(x) \left| \frac{u}{\tau} \right|^{p(x)} dx \leq 1 \right\},$$
$$\|u\|_{p_*(\cdot),\beta,\partial\Omega} = \inf \left\{ \tau > 0 \,:\, \int_{\partial\Omega} \beta(x) \left| \frac{u}{\tau} \right|^{p_*(x)} d\sigma \leq 1 \right\},$$

whose corresponding modular are given by

$$\rho_{p(\cdot),\alpha}(u) = \int_\Omega \alpha(x)|u|^{p(x)}\,\mathrm{d}x \quad \text{for all} \quad u \in L^{p(\cdot)}_\alpha(\Omega),$$
$$\rho_{p_*(\cdot),\beta,\partial\Omega}(u) = \int_{\partial\Omega} \beta(x)|u|^{p_*(x)}\,\mathrm{d}\sigma \quad \text{for all} \quad u \in L^{p_*(\cdot)}_\beta(\partial\Omega),$$

respectively. We recall that in Section 2.2 we introduced a trace operator

$$\gamma_0 \colon W^{1,r(\cdot)}(\Omega) \to L^{k(\cdot)}(\partial\Omega) \quad \text{for every } k \in C(\overline{\Omega}) \text{ with } 1 \le k(x) \le r_*(x)\ \forall x \in \overline{\Omega}.$$

So, we avoid the notation of the trace map and we consider all the restrictions of Sobolev functions to the boundary $\partial\Omega$ in the sense of traces, see Proposition 2.2.7 for more details. Now, we endow the space $W^{1,\mathcal{H}}(\Omega)$ with the following equivalent norm

$$\|u\| = \inf\left\{ \tau > 0 : \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \mu(x)\left|\frac{\nabla u}{\tau}\right|^{q(x)} \right) \mathrm{d}x \right.$$
$$\left. + \int_\Omega \alpha(x)\left|\frac{u}{\tau}\right|^{p(x)} \mathrm{d}x + \int_{\partial\Omega} \beta(x)\left|\frac{u}{\tau}\right|^{p_*(x)} \mathrm{d}\sigma \le 1 \right\},$$

which is obtained by $\|\cdot\|^*_{1,\mathcal{H}}$, defined in (2.3.11), by choosing $\theta_1 \equiv \alpha, \delta_1 \equiv p, \theta_2 \equiv \beta$ and $\delta_2 \equiv p_*$. Clearly, assumption (H2) is satisfied since we assume (H^R), so the results presented in subsection 2.3.2 hold true. In particular, the norm $\|\cdot\|$ and the corresponding modular function $\rho(\cdot)$, given by

$$\rho(u) = \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)} \right) \mathrm{d}x + \int_\Omega \alpha(x)|u|^{p(x)}\,\mathrm{d}x + \int_{\partial\Omega} \beta(x)|u|^{p_*(x)}\,\mathrm{d}\sigma,$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$, are related as in Proposition 2.3.16. For reader's convenience, we recall such propoperties in our case in the following proposition.

Proposition 5.2.1. *Let (H^R) be satisfied, $u \in W^{1,\mathcal{H}}(\Omega)$ and $\tau > 0$. Then the following hold:*

(i) *If $u \neq 0$, then $\|u\| = \tau \iff \rho(\frac{u}{\tau}) = 1$;*

(ii) $\|u\| < 1$ *(resp.* $> 1, = 1$*)* $\iff \rho(u) < 1$ *(resp.* $> 1, = 1$*);*

(iii) *If* $\|u\| < 1 \implies \|u\|^{(p_*)_+} \le \rho(u) \le \|u\|^{p_-}$*;*

(iv) *If* $\|u\| > 1 \implies \|u\|^{p_-} \le \rho(u) \le \|u\|^{(p_*)_+}$*;*

(v) $\|u\| \to 0 \iff \rho(u) \to 0$*;*

(vi) $\|u\| \to \infty \iff \rho(u) \to \infty$*;*

(vii) $\|u\| \to 1 \iff \rho(u) \to 1$*.*

We refer to subsection 2.3.1 for the properties of the space $W^{1,\mathcal{H}}(\Omega)$, which clearly hold also under assumption (H^R) that is more restrictive than (H) and (H1). Moreover, we prove two other embeddings that are useful in our treatment.

Proposition 5.2.2. *Let (H^R) be satisfied. Then the following embeddings hold:*

(i) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{p(\cdot)}_\alpha(\Omega)$ *is continuous;*

(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{p_*(\cdot)}_{\beta}(\partial\Omega)$ *is continuous.*

Proof. Fix $u \in W^{1,\mathcal{H}}(\Omega)$ with $u \neq 0$, then it holds that

$$\begin{aligned}\rho_{p(\cdot),\alpha}(u) =& \int_\Omega \alpha(x)|u|^{p(x)}\,\mathrm{d}x\\ \leq& \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)} + \alpha(x)|u|^{p(x)}\right)\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)|u|^{p_*(x)}\,\mathrm{d}\sigma\\ =&\rho(u).\end{aligned}$$

From Proposition 5.2.1(i) it follows that

$$\rho_{p(\cdot),\alpha}\left(\frac{u}{\|u\|}\right) \leq \rho\left(\frac{u}{\|u\|}\right) = 1.$$

Hence, we have

$$\|u\|_{p(\cdot),\alpha} \leq \|u\|,$$

and (i) is proved. The embedding in (ii) can be proved similarly. □

Our aim is to establish the existence of two nontrivial weak solutions for the problem (R_λ). We say that $u \in W^{1,\mathcal{H}}(\Omega)$ is a nontrivial weak solution of (R_λ) if the following holds

$$\begin{aligned}&\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x + \int_\Omega \alpha(x)|u|^{p(x)-2}uv\,\mathrm{d}x\\ &+\int_{\partial\Omega}\beta(x)|u|^{p_*(x)-2}uv\,\mathrm{d}\sigma = \lambda\int_\Omega f(x,u)v\,\mathrm{d}x\end{aligned}$$

for all $v \in W^{1,\mathcal{H}}(\Omega)$. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between $W^{1,\mathcal{H}}(\Omega)$ and its dual space $W^{1,\mathcal{H}}(\Omega)^*$ and we introduce the nonlinear operator $J : W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ defined by

$$\begin{aligned}\langle J(u), v\rangle =& \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x\\ &+ \int_\Omega \alpha(x)|u|^{p(x)-2}uv\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)|u|^{p_*(x)-2}uv\,\mathrm{d}\sigma.\end{aligned}$$

In the next proposition we give the properties of the operator J which has been proved in Amoroso-Crespo-Blanco-Pucci-Winkert [5, Proposition 3.3] for a more general operator.

Proposition 5.2.3. *Let* (H^R) *be satisfied. Then, the operator* $J : W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ *is bounded, continuous, strictly monotone, coercive, a homeomorphism and of type* (S_+)*, that is*

$$u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty}\langle J(u_n), u_n - u\rangle \leq 0$$

imply $u_n \to u$ *in* $W^{1,\mathcal{H}}(\Omega)$.

Furthermore, we define the functionals $\Phi, \Psi, I_\lambda : W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned}\Phi(u) =& \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)} + \alpha(x)\frac{|u|^{p(x)}}{p(x)}\right)\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)\frac{|u|^{p_*(x)}}{p_*(x)}\,\mathrm{d}\sigma,\\ \Psi(u) =& \int_\Omega F(x,u)\,\mathrm{d}x,\\ I_\lambda(u) =& \Phi(u) - \lambda\Psi(u),\end{aligned} \tag{5.2.1}$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$ and I_λ is the so-called energy functional associated to our problem (R_λ).

Proposition 5.2.4. *Let* (H^R) *be satisfied. Then, the functional* Φ *is well-defined and of class* C^1 *with* $\Phi'(u) = J(u)$.

Proof. For any $u \in W^{1,\mathcal{H}}(\Omega)$ one has

$$0 \leq \frac{\rho(u)}{(p_*)_+} < \Phi(u) < \frac{\rho(u)}{p_-} < \infty,$$

which implies that Φ is well defined in $W^{1,\mathcal{H}}(\Omega)$. Proving that J is the Gâteaux derivative of Φ means showing that

$$\frac{\Phi(u+tv)-\Phi(u)}{t} \xrightarrow{t\to 0} \langle J(u), v\rangle,$$

for all $u, v \in W^{1,\mathcal{H}}(\Omega)$. In [40, Proposition 3.1] the authors demonstrate that for $t \in \mathbb{R}$ and for all $u, v \in W^{1,\mathcal{H}}(\Omega)$

$$\int_\Omega \left(\frac{|\nabla u + t\nabla v|^{p(x)} - |\nabla u|^{p(x)}}{tp(x)} + \mu(x)\frac{|\nabla u + t\nabla v|^{q(x)} - |\nabla u|^{q(x)}}{tq(x)} \right) \mathrm{d}x$$
$$\xrightarrow{t\to 0} \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u \right) \cdot \nabla v \,\mathrm{d}x,$$

hence it remains to prove that for $t \in \mathbb{R}$ and for all $u, v \in W^{1,\mathcal{H}}(\Omega)$ the following hold

$$\int_\Omega \alpha(x)\frac{|u+tv|^{p(x)} - |u|^{p(x)}}{tp(x)} \,\mathrm{d}x \xrightarrow{t\to 0} \int_\Omega \alpha(x)|u|^{p(x)-2}uv \,\mathrm{d}x, \tag{5.2.2}$$

$$\int_{\partial\Omega} \beta(x)\frac{|u+tv|^{p_*(x)} - |u|^{p_*(x)}}{tp_*(x)} \,\mathrm{d}\sigma \xrightarrow{t\to 0} \int_{\partial\Omega} \beta(x)|u|^{p_*(x)-2}uv \,\mathrm{d}\sigma. \tag{5.2.3}$$

First, we consider (5.2.2). From the Mean Value Theorem one has that there exists $\varepsilon = \varepsilon(x,t) \in (0,1)$ such that

$$\int_\Omega \alpha(x)\frac{|u+tv|^{p(x)} - |u|^{p(x)}}{tp(x)} \,\mathrm{d}x = \int_\Omega \alpha(x)|u+\varepsilon tv|^{p(x)-2}(u+\varepsilon tv)v \,\mathrm{d}x. \tag{5.2.4}$$

In order to apply the Dominated Convergence Theorem, we estimate that

$$\alpha(x)|u+\varepsilon tv|^{p(x)-2}(u+\varepsilon tv)v \leq \alpha(x)|u+\varepsilon tv|^{p(x)-1}|v|$$
$$\leq 2^{p_+-1}\alpha(x)\left(|u|^{p(x)-1} + \varepsilon|t||v|^{p(x)-1}\right)|v| \in L^1(\Omega),$$

using the embedding in Proposition 5.2.2(i), Hölder's inequality for variable exponents (Proposition 2.2.3) and Proposition 5.2.1(iii)-(iv). Indeed, we get

$$\begin{aligned}\int_\Omega \alpha(x)|u|^{p(x)-1}|v|\,\mathrm{d}x =& \int_\Omega \alpha(x)^{\frac{p(x)-1}{p(x)}}|u|^{p(x)-1}\alpha(x)^{\frac{1}{p(x)}}|v|\,\mathrm{d}x\\ \leq& 2\left\|\alpha(\cdot)^{\frac{p(\cdot)-1}{p(\cdot)}}|u|^{p(\cdot)-1}\right\|_{\frac{p(\cdot)}{p(\cdot)-1}}\left\|\alpha(\cdot)^{\frac{1}{p(\cdot)}}|v|\right\|_{p(\cdot)}\\ \leq& 2\left(\rho_{\frac{p(\cdot)}{p(\cdot)-1}}\left(\alpha(\cdot)^{\frac{p(\cdot)-1}{p(\cdot)}}|u|^{p(\cdot)-1}\right)\right)^a\left(\rho_{p(\cdot)}\left(\alpha(\cdot)^{\frac{1}{p(\cdot)}}|v|\right)\right)^b\\ =& 2\left(\rho_{p(\cdot),\alpha}(u)\right)^a\left(\rho_{p(\cdot),\alpha}(v)\right)^b<\infty,\end{aligned}$$

for some $a, b > 0$ and similarly for $\alpha(\cdot)|v|^{p(\cdot)-1}|v| \in L^1(\Omega)$. So, passing to the limit for $t \to 0$ in (5.2.4), by the Dominated Convergence Theorem we obtain (5.2.2). In the same way we can prove (5.2.3), using now the embedding in Proposition 5.2.2(ii) and this completes the proof that $\Phi'(u) = J(u)$ for every $u \in W^{1,\mathcal{H}}(\Omega)$.

Finally, the C^1-property follows from the continuity of J, see Proposition 5.2.3. □

In addition, we note that Ψ is of class C^1 from the Carathéodory assumption on f, then I_λ is of class C^1 and its derivative is the following

$$\begin{aligned}&\langle I_\lambda'(u),v\rangle\\ =&\langle\Phi'(u),v\rangle-\lambda\langle\Psi'(u),v\rangle\\ =&\int_\Omega\left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x+\int_\Omega\alpha(x)|u|^{p(x)-2}uv\,\mathrm{d}x\\ &+\int_{\partial\Omega}\beta(x)|u|^{p_*(x)-2}uv\,\mathrm{d}\sigma-\lambda\int_\Omega f(x,u)v\,\mathrm{d}x,\end{aligned}$$

for all $u, v \in W^{1,\mathcal{H}}(\Omega)$. Hence, u is a nontrivial weak solution of (R_λ) if and only if u is a critical point of I_λ, i.e.

$$u \in W_0^{1,\mathcal{H}}(\Omega) \text{ weak solution of } (D_\lambda) \quad\Longleftrightarrow\quad \langle I_\lambda'(u),v\rangle=0 \quad \text{for all } v\in W^{1,\mathcal{H}}(\Omega).$$

So, we study the problem through critical point theory since discussing the existence of weak solutions for (R_λ) is equivalent to study the existence of critical point of the energy functional I_λ associated to our problem. Our main tool of investigation is is a two critical point theorem due to Bonanno-D'Aguì in [23, Theorem 2.1], recalled in Theorem 3.2.5 in Chapter 3.

5.3 Main results

In this section, we present our main result on the existence of two nontrivial solutions for the nonlinear Robin double phase problem with variable exponents given by (R_λ). Since our purpose is to apply Theorem 3.2.5 to the functionals Φ and Ψ defined in (5.2.1), we prove the following result on the required properties of the energy functional I_λ related to problem (R_λ).

Lemma 5.3.1. *Let* (H^R) *and* (H_f^R) *be satisfied. Then, for every* $\lambda > 0$ *the energy functional* I_λ *given in* (5.2.1) *satisfies the* (PS)*-condition and is unbounded from below.*

Proof. First, we prove that I_λ fulfills the (PS)-condition for any $\lambda > 0$.

Fix $\lambda > 0$ and let $\{u_n\} \subseteq W^{1,\mathcal{H}}(\Omega)$ be such that (PS$_1$) and (PS$_2$) hold. We observe that

$$\begin{aligned}\eta I_\lambda(u_n) - I'_\lambda(u_n)(u_n) &= \eta\Bigg(\int_\Omega \left(\frac{|\nabla u_n|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u_n|^{q(x)}}{q(x)} + \alpha(x)\frac{|u_n|^{p(x)}}{p(x)}\right) \mathrm{d}x \\ &\quad + \int_{\partial\Omega} \beta(x)\frac{|u_n|^{p_*(x)}}{p_*(x)} \,\mathrm{d}\sigma - \lambda \int_\Omega F(x,u_n)\,\mathrm{d}x\Bigg) \\ &\quad - \rho(u_n) + \lambda\int_\Omega f(x,u_n)u_n\,\mathrm{d}x \\ &> \frac{\eta}{(p_*)_+}\rho(u_n) - \rho(u_n) + \lambda\int_\Omega \left(f(x,u_n)u_n - \eta F(x,u_n)\right)\mathrm{d}x \\ &= \left(\frac{\eta}{(p_*)_+} - 1\right)\rho(u_n) + \lambda\int_\Omega \left(f(x,u_n)u_n - \eta F(x,u_n)\right)\mathrm{d}x\end{aligned}$$

for all $n \in \mathbb{N}$, where $\left(\frac{\eta}{(p_*)_+} - 1\right) > 0$ since $\eta > (p_*)_+$. Thanks to Proposition 5.2.1 (iii)-(iv), one has

$$\min\left\{\|u_n\|^{p_-}, \|u_n\|^{(p_*)_+}\right\} \le \rho(u_n) \le \max\left\{\|u_n\|^{p_-}, \|u_n\|^{(p_*)_+}\right\} \qquad \text{for all } n \in \mathbb{N}.$$

Therefore,

$$\begin{aligned}\eta I_\lambda(u_n) - I'_\lambda(u_n)(u_n) > &\left(\frac{\eta}{(p_*)_+} - 1\right)\min\left\{\|u_n\|^{p_-}, \|u_n\|^{(p_*)_+}\right\} \\ &- \lambda\int_\Omega \left(\eta F(x,u_n) - f(x,u_n)u_n\right)\mathrm{d}x.\end{aligned} \tag{5.3.1}$$

We observe that

$$\int_\Omega \left(\eta F(x,u_n) - f(x,u_n)u_n\right)\mathrm{d}x \le A, \tag{5.3.2}$$

for some $A \ge 0$ and for all $n \in \mathbb{N}$. Indeed, by (AR)-condition one has

$$\eta F(x,t) - tf(x,t) \le 0 \quad \text{for all } |t| \ge s \tag{5.3.3}$$

and from hypothesis (f_1^R) follows that

$$|F(x,t)| \le \sup_{|t|\le s}|f(x,t)|s \le s\cdot h_s(x) \quad \text{for all } |t| \le s \tag{5.3.4}$$

with $h_s \in L^1(\Omega)$. Putting together (5.3.3) and (5.3.4) we obtain (5.3.2) with $A = s(\eta+1)\|h_s\|_1 \ge 0$. Then, using (5.3.2) in (5.3.1) we have

$$\eta I_\lambda(u_n) - I'_\lambda(u_n)(u_n) > \left(\frac{\eta}{(p_*)_+} - 1\right)\min\left\{\|u_n\|^{p_-}, \|u_n\|^{(p_*)_+}\right\} - \lambda A,$$

which yields to

$$\left(\frac{\eta}{(p_*)_+} - 1\right)\min\left\{\|u_n\|^{p_-}, \|u_n\|^{(p_*)_+}\right\} < \eta I_\lambda(u_n) - I'_\lambda(u_n)(u_n) + \lambda A, \tag{5.3.5}$$

for all $n \in \mathbb{N}$. From (PS_1) and (PS_2) we have that there exist $M > 0$ and $\{\varepsilon_n\} \subseteq \mathbb{R}^+$, with $\varepsilon_n \to 0^+$, such that

$$I_\lambda(u_n) \leq M \quad \text{and} \quad |I'_\lambda(u_n)(v)| \leq \varepsilon_n \|v\|, \tag{5.3.6}$$

for all $v \in W^{1,\mathcal{H}}(\Omega)$ and for all $n \in \mathbb{N}$. Choosing $v = u_n$ in (5.3.6) and combining it with (5.3.5), we get

$$\left(\frac{\eta}{(p_*)_+} - 1\right) \min\left\{\|u_n\|^{p_-}, \|u_n\|^{(p_*)_+}\right\} < \eta M + \varepsilon_n \|u_n\| + \lambda A,$$

and this proves that u_n is bounded in $W^{1,\mathcal{H}}(\Omega)$. Since $W^{1,\mathcal{H}}(\Omega)$ is a reflexive Banach space, there exists a subsequence, renamed u_n, that weakly converges in $W^{1,\mathcal{H}}(\Omega)$. Hence, taking into account Proposition 2.3.12(iii) and that $\ell_+ < (p^*)_-$ from (f_2^R), it holds that

$$u_n \rightharpoonup u \text{ in } W^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^{\ell(\cdot)}(\Omega). \tag{5.3.7}$$

Moreover, choosing $v = u_n - u$ in (5.3.6), from (f_2^R) and the convergence properties in (5.3.7) we obtain

$$\langle \Phi'(u_n), u_n - u \rangle \to 0 \quad \text{as} \quad n \to +\infty.$$

Proposition 5.2.3 ensures that Φ' satisfies the (S_+)-property, which implies that

$$u_n \to u \quad \text{in} \quad W^{1,\mathcal{H}}(\Omega)$$

and the (PS)-condition is proved. Finally, we prove the unboundedness from below of I_λ for every $\lambda > 0$. Indeed, from the (AR)-condition one has

$$F(x,t) \geq \frac{\min\{F(x,s), F(x,-s)\}}{s^\eta}|t|^\eta = \frac{C_1(x)}{s^\eta}|t|^\eta \quad \text{for all } x \in \Omega, |t| \geq s, \tag{5.3.8}$$

with $C_1(x) > 0$ for all $x \in \Omega$ and $C_1 \in L^1(\Omega)$ since f is L^1-Carathéodory (see (f_1^R)). Also, for all $x \in \Omega$ and $|t| \leq s$ it holds that

$$\begin{aligned} F(x,t) &\geq \min_{|\xi|\leq s} F(x,\xi) \geq \min_{|\xi|\leq s} F(x,\xi) + \frac{C_1(x)}{s^\eta}|t|^\eta - C_1(x) \\ &= \frac{C_1(x)}{s^\eta}|t|^\eta - \left(C_1(x) - \min_{|\xi|\leq s} F(x,\xi)\right) = \frac{C_1(x)}{s^\eta}|t|^\eta - C_2(x), \end{aligned} \tag{5.3.9}$$

where $C_2(x) > 0, C_2 \in L^1(\Omega)$ by construction. So, combining (5.3.8) and (5.3.9) together, it follows that

$$F(x,t) \geq \frac{C_1(x)}{s^\eta}|t|^\eta - C_2(x) \qquad \text{for all } x \in \Omega, \text{ for all } t \in \mathbb{R}.$$

Fixing $u \in W^{1,\mathcal{H}}(\Omega)$ such that $\|u\| \neq 0$ and $h \in \mathbb{R}^+$, one has

$$\begin{aligned} I_\lambda(hu) &\leq \frac{1}{p_-} h^{(p_*)_+} \rho(u) - \frac{\lambda}{s^\eta} \int_\Omega C_1(x)|hu|^\eta \, dx + \lambda \int_\Omega C_2(x) \, dx \\ &\leq \frac{1}{p_-} h^{(p_*)_+} \rho(u) - \frac{\lambda}{s^\eta} \inf_\Omega C_1 \, h^\eta \|u\|_\eta^\eta + \lambda \|C_2\|_1. \end{aligned}$$

Passing to the limit for $h \to \infty$, we achieve our aim and the proof is complete. □

Now, we give our main result concerning the existence of at least two nontrivial weak solutions for problem (R_λ). For this purpose, set

$$\delta = \max\left\{\|\alpha\|_\infty|\Omega|, \|\beta\|_{\infty,\partial\Omega}|\partial\Omega|\right\}, \tag{5.3.10}$$

where $\|\cdot\|_{\infty,\partial\Omega} = \|\cdot\|_{L^\infty(\partial\Omega)}$ and $|\partial\Omega|$ denotes the measure of $\partial\Omega$ in $\mathbb{R}^{N-1}$. Also, for any $r, \omega > 0$ put

$$\begin{aligned} A(r) = &\frac{k_1 c_1 \max\left\{ (r(p_*)_+)^{\frac{1}{p_-}}, (r(p_*)_+)^{\frac{1}{(p_*)_+}} \right\}}{r} \\ &+ \frac{\frac{k_2}{\ell_-}\,\bar{c}\max\left\{ (r(p_*)_+)^{\frac{\ell_-}{p_-}}, (r(p_*)_+)^{\frac{\ell_+}{(p_*)_+}} \right\}}{r}, \end{aligned} \tag{5.3.11}$$

$$B(\omega) = \frac{\int_\Omega F(x,\omega)\,\mathrm{d}x}{\delta\max\left\{\omega^{(p_*)_+}, \omega^{p_-}\right\}}, \tag{5.3.12}$$

where c_1 is the embedding constant of $W^{1,\mathcal{H}}(\Omega)$ in $L^1(\Omega)$ and $\bar{c} = \max\{c_\ell^{\ell_-}, c_\ell^{\ell_+}\}$, with c_ℓ being the embedding constant of $W^{1,\mathcal{H}}(\Omega)$ in $L^{\ell(\cdot)}(\Omega)$.

Theorem 5.3.2. *Let* (H^R) *and* (H_f^R) *be satisfied. Suppose that there exist $r, \omega > 0$ such that*

$$\max\left\{\omega^{(p_*)_+}, \omega^{p_-}\right\} < \frac{r}{\delta}, \tag{5.3.13}$$

and

$$A(r) < B(\omega). \tag{5.3.14}$$

Then, for each $\lambda \in \Lambda$, with

$$\Lambda := \left]\frac{1}{B(\omega)}, \frac{1}{A(r)}\right[, \tag{5.3.15}$$

problem (R_λ) admits at least two nontrivial weak solutions with opposite energy sign.

Proof. The proof is based on Theorem 3.2.5. Let $W^{1,\mathcal{H}}(\Omega), \Phi, \Psi$ be as in Section 5.2 and note that they verify all the regularity assumptions required in Theorem 3.2.5. In addition, Lemma 5.3.1 ensures the (PS)-condition and the unboundedness of I_λ for every $\lambda > 0$. So, it remains to prove only condition (3.2.4). Let $\tilde{u} \equiv \omega \in W^{1,\mathcal{H}}(\Omega)$, with $\omega > 0$ as in (5.3.13). It holds that $0 < \Phi(\omega) < r$, indeed one has

$$\begin{aligned} 0 < \Phi(\omega) &= \int_\Omega \alpha(x)\omega^{p(x)}\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)\omega^{p_*(x)}d\sigma \leq \\ &\leq \max\left\{w^{p_-}, \omega^{p_+}\right\}\|\alpha\|_\infty|\Omega| \\ &\quad + \max\left\{\omega^{(p_*)_-}, \omega^{(p_*)_+}\right\}\|\beta\|_{\infty,\partial\Omega}|\partial\Omega| \\ &\leq \max\left\{\omega^{(p_*)_+}, \omega^{p_-}\right\}\delta \\ &< r. \end{aligned} \tag{5.3.16}$$

Then, we consider $u \in W^{1,\mathcal{H}}(\Omega)$ such that $\Phi(u) \leq r$ and using Proposition 5.2.1(iii)-(iv) we note that

$$r \geq \Phi(u) \geq \frac{1}{(p_*)_+}\rho(u) \geq \frac{1}{(p_*)_+}\min\left\{\|u\|^{p_-}, \|u\|^{(p_*)_+}\right\}.$$

Therefore,

$$\Phi^{-1}\left(]-\infty,r]\right) \subseteq \left\{u \in W^{1,\mathcal{H}}(\Omega) : \|u\| < \max\left\{(r(p_*)_+)^{\frac{1}{p_-}}, (r(p_*)_+)^{\frac{1}{(p_*)_+}}\right\}\right\}. \tag{5.3.17}$$

Furthermore, using assumption (f_2^R), Proposition 2.2.2, Proposition 2.3.12 and (5.3.17), we get

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) &\leq \sup_{u\in\Phi^{-1}(]-\infty,r])} \int_\Omega \left(k_1|u| + k_2\frac{|u|^{\ell(x)}}{\ell(x)}\right) \mathrm{d}x \\
&\leq \sup_{u\in\Phi^{-1}(]-\infty,r])} \left(k_1\|u\|_1 + \frac{k_2}{\ell_-}\rho_{\ell(\cdot)}(u)\right) \mathrm{d}x \\
&\leq \sup_{u\in\Phi^{-1}(]-\infty,r])} \left(k_1c_1\|u\| + \frac{k_2}{\ell_-}\,\bar{c}\max\left\{\|u\|^{\ell_-}, \|u\|^{\ell_+}\right\}\right) \\
&\leq k_1c_1 \max\left\{(r(p_*)_+)^{\frac{1}{p_-}}, (r(p_*)_+)^{\frac{1}{(p_*)_+}}\right\} \\
&\quad + \frac{k_2}{\ell_-}\,\bar{c}\max\left\{(r(p_*)_+)^{\frac{\ell_-}{p_-}}, (r(p_*)_+)^{\frac{\ell_+}{(p_*)_+}}\right\}.
\end{aligned}$$

Hence, exploiting (5.3.14) and (5.3.16), from the previous inequality we derive that

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} \leq A(r) < B(\omega) \leq \frac{\Psi(\omega)}{\Phi(\omega)} \qquad \text{and} \qquad \Lambda \subset \Lambda_r.$$

Then, hypothesis (3.2.4) is satisfied and Theorem 3.2.5 ensures the existence of two nontrivial weak solutions $u_{\lambda,1}, u_{\lambda,2} \in W^{1,\mathcal{H}}(\Omega)$ for any $\lambda \in \Lambda$ with opposite energy sign, namely such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$. □

In order to find information about the sign of the solutions, we may assume

$$f(x,t) = f(x,0) \qquad \text{for all } x \in \Omega, \text{ for all } t < 0.$$

Indeed, we have the following result.

Lemma 5.3.3. *Let $u \in W^{1,\mathcal{H}}(\Omega)$ be a weak solution of problem (R_λ). If $f(x,0) \geq 0$ for a.a. $x \in \Omega$, then u is nonnegative.*

Proof. Since u is a weak solution of problem (R_λ), it holds that $\langle I'_\lambda(u), v\rangle = 0$, namely

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla v \,\mathrm{d}x + \int_\Omega \alpha(x)|u|^{p(x)-2}uv \,\mathrm{d}x \\
&+ \int_{\partial\Omega} \beta(x)|u|^{p_*(x)-2}uv \,\mathrm{d}\sigma - \lambda\int_\Omega f(x,u)v \,\mathrm{d}x = 0,
\end{aligned}$$

for every $v \in W^{1,\mathcal{H}}(\Omega)$. Arguing by contradiction, we suppose that

$$\{x \in \Omega\colon u(x) < 0\} \neq \emptyset.$$

Choosing $v = u^- = \max\{-u, 0\}$, we obtain

$$-\rho(u^-) = \lambda\int_\Omega f(x,u)u^- \,\mathrm{d}x = \lambda\int_{\{u<0\}} f(x,0)u^- \,\mathrm{d}x \geq 0.$$

From Proposition 5.2.1 it follows that

$$\min\left\{\|u^-\|^{p_-}, \|u^-\|^{(p_*)_+}\right\} \le \rho(u^-) \le 0,$$

so it must be $\|u^-\| = 0$. Thus, $u^- \equiv 0$ which implies that $u \ge 0$ and this completes the proof. □

Combining Theorem 5.3.2 and Lemma 5.3.3, we get the following existence result of two nonnegative weak solutions.

Corollary 5.3.4. *Let $f(x,0) \ge 0$ for a.a. $x \in \Omega$ and assume hypotheses (H^R), (f_1^R) and (f_2^R). Suppose that*

(AR^+) *there exist $\eta > (p_*)_+$, $s > 0$ such that*

$$0 < \eta F(x,t) \le t f(x,t) \quad \textit{for all } x \in \Omega \textit{ and for all } t \ge s,$$

and there exist $r, \omega > 0$ such that

$$\max\left\{\omega^{(p_*)_+}, \omega^{p_-}\right\} < \frac{r}{\delta} \qquad \textit{and} \qquad A(r) < B(\omega),$$

where δ, $A(r)$ and $B(\omega)$ are given by (5.3.10), (5.3.11) and (5.3.12), respectively. Then, for each $\lambda \in \Lambda$, defined in (5.3.15), problem (R_λ) admits at least two nontrivial and nonnegative weak solutions with opposite energy sign.

Furthermore, we consider some particular cases. In the following Theorem, we establish an existence result when the nonlinear term is nonnegative, requiring also a particular behavior near zero.

Theorem 5.3.5. *Let (H^R), (f_1^R), (f_2^R) and (AR^+) be satisfied. Suppose that f is nonnegative and*

$$\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^{p_-}} = +\infty. \tag{5.3.18}$$

Then, for each $\lambda \in]0, \lambda^*[$, with

$$\lambda^* = \sup_{r>0} \frac{1}{A(r)},$$

problem (R_λ) admits at least two nontrivial and nonnegative weak solutions with opposite energy sign.

Proof. Fix $\lambda \in]0, \lambda^*[$. Then, there exist $r > 0$ such that

$$\lambda < \frac{1}{A(r)}. \tag{5.3.19}$$

Moreover, from assumption (5.3.18) it follows that

$$\begin{aligned}\limsup_{\omega\to 0^+} B(\omega) &= \limsup_{\omega\to 0^+} \frac{\int_\Omega F(x,\omega)\,\mathrm{d}x}{\delta \max\left\{\omega^{(p_*)_+}, \omega^{p_-}\right\}} \\ &\ge \frac{|\Omega|}{\delta} \limsup_{\omega\to 0^+} \frac{\inf_{x\in\Omega} F(x,\omega)}{\omega^{p_-}} = +\infty.\end{aligned}$$

Therefore, there exists $\omega > 0$ small enough such that

$$\frac{|\Omega|}{\delta}\frac{\inf\limits_{x\in\Omega} F(x,\omega)}{\omega^{p_-}} > \frac{1}{\lambda},$$

which together with (5.3.19) implies that $A(r) < B(\omega)$. Then, the result follows from Corollary 5.3.4. □

Here, we provide an example of application of the previous result.

Example 5.3.6. *Consider the following function*

$$f(x,t) = h_1(x) + h_2(x)|t|^{\xi(x)-1} \qquad \text{for all } (x,t) \in \Omega \times \mathbb{R}, \tag{5.3.20}$$

with $h_1, h_2 \in L^\infty(\Omega), \operatorname{essinf}_\Omega h_i > 0$ *for* $i = 1,2$ *and* $\xi \in C_+(\overline{\Omega})$ *with*

$$(p_*)_+ < \xi(x) < (p^*)_- \quad \text{for all } x \in \overline{\Omega}.$$

We observe that (f_1^R) *holds and* (f_2^R) *is satisfied by choosing* $k_i = \|h_i\|_\infty$ *for* $i = 1,2$ *and* $\ell(x) = \xi(x) + \varepsilon$ *for all* $x \in \overline{\Omega}$, *with* $\varepsilon \geq 0$ *small enough such that* $\ell_+ < (p^*)_-$. *Moreover, since*

$$F(x,t) = h_1(x)t + \frac{h_2(x)}{\xi(x)}t^{\xi(x)} \quad \text{for all } x \in \Omega, t > 0,$$

it follows that (AR^+) *is verified for any* $(p_*)_+ < \eta \leq \xi_-$ *and we have*

$$\lim_{t\to 0^+} \frac{\inf\limits_{x\in\Omega} F(x,t)}{t^{p_-}} = +\infty,$$

so assumption (5.3.18) *is verified. Then, Theorem* 5.3.5 *ensures that for each* $\lambda \in]0,\lambda^*[$ *problem* (R_λ) *with the nonlinearity given in* (5.3.20) *admits at least two nontrivial and nonnegative weak solutions with opposite energy sign.*

Chapter 6

Neumann double phase problem

In this chapter we consider a variable exponent double phase problem with a nonlinear boundary condition and we prove the existence of multiple bounded solutions under very general assumptions on the nonlinearities. To be more precise, we get two constant sign solutions via a mountain-pass approach, in particular one is nonnegative and the other one is nonpositive, and we determine the existence of a third solution, which is sign-changing, through the Nehari manifold method. We also give informations on the nodal domains of this sign-changing solution.

The results presented in this chapter are obtained in [5], in collaboration with Á. Crespo-Blanco, P. Pucci and P. Winkert.

6.1 The problem

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$ and denoting with $\nu(x)$ the outer unit normal of Ω at $x \in \partial\Omega$, we study the following problem

$$\begin{aligned} -\operatorname{div} \mathcal{F}(u) + |u|^{p(x)-2}u &= f(x,u) && \text{in } \Omega, \\ \mathcal{F}(u) \cdot \nu &= g(x,u) - |u|^{p(x)-2}u && \text{on } \partial\Omega, \end{aligned} \tag{N}$$

where $\operatorname{div} \mathcal{F}(u)$ is the variable exponent double phase operator given by

$$\mathcal{F}(u) := |\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u,$$

and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ as well as $g\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions which are superlinear with respect to the second argument, see the precise conditions in ($\mathrm{H}^N_{f,g}$).

Regarding the exponents p, q and the weight μ, we have to strengthen the hypotheses in (H) as follows:

($\mathbf{H}^N$) $p, q \in C(\overline{\Omega})$ such that $1 < p(x) < N$ and $p(x) < q(x) < (p_-)_*$ for all $x \in \overline{\Omega}$ and $\mu \in L^\infty(\Omega)$ with $\mu(x) \geq 0$ for a.a. $x \in \Omega$.

Next, we state the required assumptions on the nonlinearities:

($\mathbf{H}^N_{f,g}$) Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $g\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions and $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ and $G(x,t) = \int_0^t g(x,s)\,\mathrm{d}s$ be such that the following hold:

(h_1^N) there exist $\ell, \kappa \in C_+(\overline{\Omega})$ and $K_1, K_2 > 0$ with $\ell_+ < (p_-)^*$ and $\kappa_+ < (p_-)_*$ such that

$$|f(x,t)| \leq K_1 \left(1 + |t|^{\ell(x)-1}\right) \quad \text{for a.a. } x \in \Omega,$$
$$|g(x,t)| \leq K_2 \left(1 + |t|^{\kappa(x)-1}\right) \quad \text{for a.a. } x \in \partial\Omega,$$

and for all $t \in \mathbb{R}$;

(h_2^N)

$$\lim_{t \to \pm\infty} \frac{F(x,t)}{|t|^{q_+}} = \infty \quad \text{uniformly for a.a. } x \in \Omega,$$
$$\lim_{t \to \pm\infty} \frac{G(x,t)}{|t|^{q_+}} = \infty \quad \text{uniformly for a.a. } x \in \partial\Omega;$$

(h_3^N)

$$\lim_{t \to 0} \frac{F(x,t)}{|t|^{p(x)}} = 0 \quad \text{uniformly for a.a. } x \in \Omega,$$
$$\lim_{t \to 0} \frac{G(x,t)}{|t|^{p(x)}} = 0 \quad \text{uniformly for a.a. } x \in \partial\Omega;$$

(h_4^N) there exist $\alpha, \beta, \zeta, \theta \in C_+(\overline{\Omega})$ with

$$\min\{\alpha_-, \beta_-\} \in \left((\ell_+ - p_-)\frac{N}{p_-}, \ell_+\right),$$
$$\min\{\zeta_-, \theta_-\} \in \left((\kappa_+ - p_-)\frac{N-1}{p_- - 1}, \kappa_+\right),$$

and $K_3 > 0$ such that

$$0 < K_3 \leq \liminf_{t \to \infty} \frac{f(x,t)t - q_+ F(x,t)}{|t|^{\alpha(x)}},$$
$$0 < K_3 \leq \liminf_{t \to -\infty} \frac{f(x,t)t - q_+ F(x,t)}{|t|^{\beta(x)}},$$

uniformly for a.a. $x \in \Omega$ and $K_4 > 0$ such that

$$0 < K_4 \leq \liminf_{t \to \infty} \frac{g(x,t)t - q_+ G(x,t)}{|t|^{\zeta(x)}},$$
$$0 < K_4 \leq \liminf_{t \to -\infty} \frac{g(x,t)t - q_+ G(x,t)}{|t|^{\theta(x)}},$$

uniformly for a.a. $x \in \partial\Omega$;

(h_5^N) the functions

$$t \mapsto \frac{f(x,t)}{|t|^{q_+-1}} \quad \text{and} \quad t \mapsto \frac{g(x,t)}{|t|^{q_+-1}}$$

are increasing in $(-\infty, 0)$ and in $(0, \infty)$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.

Here, we give some important remarks on the assumptions.

Remark 6.1.1. *We obeserve that* (h_2^N) *and* (h_3^N) *are weaker than the corresponding hypotheses obtained replacing F and G with f and g, respectively. However, using* q_+ *is stronger than using* $q(\cdot)$ *as exponent and similarly using* $p(\cdot)$ *is weaker than using* p_+ *but is stronger than using* p_- *as exponent.*

Remark 6.1.2. *The conditions on the exponents in* (h_4^N) *are well defined since from* (h_1^N) *we have* $\ell_+ < (p_-)^*$ *and* $\kappa_+ < (p_-)_*$ *and the following hold*

$$(\ell_+ - p_-)\frac{N}{p_-} = \ell_+\frac{N}{p_-} - (p_-)^*\frac{N-p_-}{p_-} < \ell_+\frac{N}{p_-} - \ell_+\frac{N-p_-}{p_-} = \ell_+,$$

$$(\kappa_+ - p_-)\frac{N-1}{p_- - 1} = \kappa_+\frac{N-1}{p_- - 1} - (p_-)_*\frac{N-p_-}{p_- - 1} < \kappa_+\frac{N-1}{p_- - 1} - \kappa_+\frac{N-p_-}{p_- - 1} = \kappa_+.$$

These boundedness conditions are the precise ones that are needed for the interpolation argument in Proposition 6.3.4 and they are sharp. Precisely, the upper bound with ℓ_+ *and* κ_+ *is due to Crespo-Blanco-Winkert [39, see Remark 4.2], while the lower bound on the boundary, i.e.* $(\kappa_+ - p_-)\frac{N-1}{p_- -1}$*, is established by Amoroso-Crespo-Blanco-Pucci-Winkert [5]. We also underline that one advantage of the variable exponent setting is that these bounds are required only for the infimum of the exponents and not for the whole ones and also that one can choose different exponents for going to* $\pm\infty$*.*

Remark 6.1.3. *We note that assumption* (h_3^N) *together with the continuity of* $f(x,\cdot)$ *and* $g(x,\cdot)$ *implies that*

$$f(x,0) = 0 \quad \text{for a.a. } x \in \Omega \quad \text{and} \quad g(x,0) = 0 \quad \text{for a.a. } x \in \partial\Omega. \tag{6.1.1}$$

Moreover, in Lemma 4.4 of Crespo-Blanco-Winkert [39], the authors summarize the properties that the nonlinear term of the equation (i.e. function f*) verifies as consequences of the previous assumptions. Clearly, as the nonlinear function* g *satisfies similar hypotheses on the boundary, it also verifies the same properties on* $\partial\Omega$*.*

6.2 Variational framework

Our aim is to investigate the existence of multiple solutions of problem (N) which involves the double phase operator with variable exponents denoted by $\mathcal{F}$. Therefore, consider the Musielack-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$, introduced in Chapter 2, endowed with the following norm

$$\|u\| = \inf\left\{\tau > 0 : \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \mu(x)\left|\frac{\nabla u}{\tau}\right|^{q(x)}\right) \mathrm{d}x \right. \\ \left. + \int_\Omega \left|\frac{u}{\tau}\right|^{p(x)} \mathrm{d}x + \int_{\partial\Omega} \left|\frac{u}{\tau}\right|^{p(x)} \mathrm{d}\sigma \leq 1\right\}, \tag{6.2.1}$$

induced by the modular

$$\rho(u) = \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)}\right) \mathrm{d}x + \int_\Omega |u|^{p(x)} \mathrm{d}x + \int_{\partial\Omega} |u|^{p(x)} \mathrm{d}\sigma,$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$. For sake of completeness, we remind that in Section 2.2 we introduced a trace operator that allows us to avoid the notation of the trace map and to consider all the restrictions of Sobolev functions to the boundary $\partial\Omega$ in the sense of traces (we refer to Proposition 2.2.7 for more details).

The norm introduced in (6.2.1) derives from $\|\cdot\|^*_{1,\mathcal{H}}$ defined in (2.3.11) (subsection 2.3.2) by choosing $\vartheta_1 \equiv \vartheta_2 \equiv 1$ and $\delta_1 \equiv \delta_2 \equiv p$. For reader's convenience, we give here the relationship between the modular $\rho(\cdot)$ and the norm $\|\cdot\|$, that derive from Proposition 5.2.1.

Proposition 6.2.1. *Let* (H^N) *be satisfied,* $u \in W^{1,\mathcal{H}}(\Omega)$ *and* $\lambda \in \mathbb{R}$. *Then the following hold:*

(i) *If* $u \neq 0$, *then* $\|u\| = \lambda \quad \Longleftrightarrow \quad \rho(\frac{u}{\lambda}) = 1$;

(ii) $\|u\| < 1$ *(resp.* $> 1, = 1$*)* $\quad \Longleftrightarrow \quad \rho(u) < 1$ *(resp.* $> 1, = 1$*)*;

(iii) *If* $\|u\| < 1 \quad \Longrightarrow \quad \|u\|^{q_+} \leq \rho(u) \leq \|u\|^{p_-}$;

(iv) *If* $\|u\| > 1 \quad \Longrightarrow \quad \|u\|^{p_-} \leq \rho(u) \leq \|u\|^{q_+}$;

(v) $\|u\| \to 0 \quad \Longleftrightarrow \quad \rho(u) \to 0$;

(vi) $\|u\| \to +\infty \quad \Longleftrightarrow \quad \rho(u) \to +\infty$;

(vii) $\|u\| \to 1 \quad \Longleftrightarrow \quad \rho(u) \to 1$;

Furthermore, we underline that in subsection 2.3.1 we provide the properties of the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$, which hold also under assumption (H^N) that is stronger than (H) and (H1). Now, denote by $\langle \cdot\,, \cdot \rangle$ the duality pairing between $W^{1,\mathcal{H}}(\Omega)$ and its dual space $W^{1,\mathcal{H}}(\Omega)^*$ and by $A\colon W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ the nonlinear operator defined for all $u, v \in W^{1,\mathcal{H}}(\Omega)$ by

$$
\begin{aligned}
\langle A(u), v\rangle = &\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nabla v \,\mathrm{d}x \\
&+ \int_\Omega |u|^{p(x)-2}uv\,\mathrm{d}x + \int_{\partial\Omega} |u|^{p(x)-2}uv\,\mathrm{d}\sigma.
\end{aligned}
$$

In the following proposition we give the properties of this operator, which has been proved in Amoroso-Crespo-Blanco-Pucci-Winkert [5, Proposition 3.3] for a more general operator.

Proposition 6.2.2. *Let* (H^N) *be satisfied. Then, the operator* $A\colon W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ *is bounded, continuous, strictly monotone, coercive, a homeomorphism and of type* (S_+), *that is,*

$$
\text{if} \quad u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0,
$$

then $u_n \to u$ *in* $W^{1,\mathcal{H}}(\Omega)$.

Our aim is to establish results on the existence of weak solutions for problem (N), namely functions $u \in W^{1,\mathcal{H}}(\Omega)$ such that

$$
\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nabla v \,\mathrm{d}x + \int_\Omega |u|^{p(x)-2}uv\,\mathrm{d}x \\
&= \int_\Omega f(x,u)v\,\mathrm{d}x + \int_{\partial\Omega} g(x,u)v\,\mathrm{d}\sigma - \int_{\partial\Omega} |u|^{p(x)-2}uv\,\mathrm{d}\sigma,
\end{aligned}
$$

for every $v \in W^{1,\mathcal{H}}(\Omega)$. In particular, these weak solutions are critical points of the energy functional $I\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ associated to the problem (N) given by

$$I(u) = \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x) \frac{|\nabla u|^{q(x)}}{q(x)} \right) \,\mathrm{d}x + \int_\Omega \frac{|u|^{p(x)}}{p(x)} \,\mathrm{d}x$$
$$+ \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)} \,\mathrm{d}\sigma - \int_\Omega F(x,u) \,\mathrm{d}x - \int_{\partial\Omega} G(x,u) \,\mathrm{d}\sigma,$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$. Therefore, we study the problem through critical point theory and in particular by a mountain-pass approach and the Nehari manifold method.

Finally, we present a version of the Quantitative Deformation Lemma, which can be found in Willem [94, Lemma 2.3] and that is used in our treatment.

Lemma 6.2.3. *Let X be a Banach space , $\varphi \in C(X;\mathbb{R}), \emptyset \neq S \subset X, c \in \mathbb{R}, \varepsilon, \delta > 0$ such that*

$$\|\varphi'(u)\|_* \geq \frac{8\varepsilon}{\delta} \qquad \text{for all } u \in \varphi^{-1}\left([c-2\varepsilon, c+2\varepsilon]\right) \cap S_{2\delta},$$

where $S_r = \{u \in X : d(u,S) = \inf_{u_0 \in S} \|u - u_0\| < r\}$ for any $r > 0$. Then there exists $\eta \in C([0,1] \times X; X)$ such that

(i) *$\eta(t,u) = u$, if $t = 0$ or if $u \notin \varphi^{-1}\left([c-2\varepsilon, c+2\varepsilon]\right) \cap S_{2\delta}$,*

(ii) *$\varphi(\eta(1,u)) \leq c - \varepsilon$ for all $u \in \varphi^{-1}((-\infty, c+\varepsilon]) \cap S$,*

(iii) *$\eta(t,\cdot)$ is an homeomorphism of X for all $t \in [0,1]$,*

(iv) *$\|\eta(t,u) - u\| \leq \delta$ for all $u \in X$ and $t \in [0,1]$,*

(v) *$\varphi(\eta(\cdot,u))$ is decreasing for all $u \in X$,*

(vi) *$\varphi(\eta(t,u)) < c$ for all $u \in \varphi^{-1}((-\infty, c]) \cap S_\delta$ and $t \in (0,1]$.*

6.3 Main results

In this section, we prove the esxistence of two bounded constant sign solutions (nonpositive and nonnegative) trhough a version (Theorem 3.1.11) of the classical Mountain Pass Theorem (Theorem 3.1.5) and we get the existence of a bounded sign-changing solution by using an appropriate subset of the corresponding Nehari manifold along with the Brouwer degree and the Quantitative Deformation Lemma (Lemma 6.2.3).

We present here the main results and we give the proof in the next subsections in a constructive way, i.e. providing preliminary results that lead to the thesis by combining them.

Theorem 6.3.1. *Let (H^N) and ($\mathrm{H}^N_{f,g}$) be satisfied. Then, there exist three nontrivial weak solutions $u_0, v_0, w_0 \in W^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ of problem (N) such that $u_0 \geq 0$, $v_0 \leq 0$ and w_0 is sign-changing.*

Furthermore, we derive information about the number of nodal domains of the sign-changing solution, that is the number of maximal regions where it has constant sign. The usual definition of nodal domains of a function deals with a continuous function. Nevertheless, we do not know whether our solutions are continuous.

Therefore, we use the definition proposed by Crespo-Blanco-Winkert [39, Section 6] that we recall in the following.

Definition 6.3.2. *Let $u \in W^{1,\mathcal{H}}(\Omega)$ and A be a Borelian subset of Ω with $|A| > 0$. We say that A is a nodal domain of u if*

(i) *$u \geq 0$ a.e. on A or $u \leq 0$ a.e. on A;*

(ii) *$0 \neq u\mathbb{1}_A \in W^{1,\mathcal{H}}(\Omega)$;*

(iii) *A is minimal w.r.t.* (i) *and* (ii)*, i.e., if $B \subseteq A$ with B being a Borelian subset of Ω, $|B| > 0$ and B satisfies* (i) *and* (ii)*, then $|A \setminus B| = 0$.*

For our purposes, we need to require one more assumption on the nonlinearities:

(h_6^N) the functions $t \mapsto f(x,t)t - q_+F(x,t)$ and $t \mapsto g(x,t)t - q_+G(x,t)$ are decreasing in $]-\infty, 0]$ and increasing in $[0, +\infty[$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.

Theorem 6.3.3. *Let (H^N), ($H_{f,g}^N$) and (h_6^N) be satisfied. Then, there exist three nontrivial weak solutions $u_0, v_0, w_0 \in W^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ of problem (N) such that*

$$u_0 \geq 0, \quad v_0 \leq 0, \quad w_0 \text{ being sign-changing with two nodal domains.}$$

6.3.1 Costant sign solutions

For any $h \in \mathbb{R}$ let

$$h^+ = \max\{h, 0\} \quad \text{and} \quad h^- = \max\{-h, 0\},$$

then one has that

$$h = h^+ - h^- \quad \text{and} \quad |h| = h^+ + h^-.$$

Also, from [40, Proposition 2.17] we know that, under assumption (H1), so also under hypothesis (H^N), if $u \in W^{1,\mathcal{H}}(\Omega)$ then $u^\pm \in W^{1,\mathcal{H}}(\Omega)$. Since we are first interested in constant sign solutions, we consider the positive and negative truncations of the functional I, that are $I_\pm \colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} I_\pm(u) = &\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)} \right) \mathrm{d}x + \int_\Omega \frac{|u|^{p(x)}}{p(x)}\, \mathrm{d}x \\ &+ \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)}\, \mathrm{d}\sigma - \int_\Omega F(x, \pm u^\pm)\, \mathrm{d}x - \int_{\partial\Omega} G(x, \pm u^\pm)\, \mathrm{d}\sigma, \end{aligned}$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$, where we have taken (6.1.1) into account.

Our existence result is based on a version of the Mountain-Pass Theorem given by Papageorgiou-Rădulescu-Repovš [78, Theorem 5.4.6] and recalled in Theorem 3.1.11. First, we give preliminary results in order to verify the required assumptions and we start with the compactness condition on the functional.

Proposition 6.3.4. *Let (H^N), (h_1^N) and (h_4^N) be satisfied. Then, the functionals $I_\pm$ satisfy the* (C)*-condition.*

Proof. We show the proof for I_+, the case for I_- works in the same way. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}}(\Omega)$ be a sequence such that (C_1) and (C_2) from Definition 3.1.10 hold. From (C_2), there exists $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0^+$ such that

$$\left|\langle I'_+(u_n), v\rangle\right| \leq \frac{\varepsilon_n \|v\|}{1+\|u_n\|} \quad \text{for all } n \in \mathbb{N} \text{ and for all } v \in W^{1,\mathcal{H}}(\Omega). \tag{6.3.1}$$

Choosing $v = -u_n^- \in W^{1,\mathcal{H}}(\Omega)$, one has

$$\rho(-u_n^-) - \int_\Omega f(x, +u_n^+)(-u_n^-)\,\mathrm{d}x - \int_{\partial\Omega} g(x, +u_n^+)(-u_n^-)\,\mathrm{d}\sigma \leq \varepsilon_n,$$

for all $n \in \mathbb{N}$, which leads to $\rho(-u_n^-) \to 0$ as $n \to \infty$, since the supports of $+u_n^+$ and $-u_n^-$ do not overlap. From Proposition 6.2.1(v) it follows that

$$-u_n^- \to 0 \quad \text{in } W^{1,\mathcal{H}}(\Omega). \tag{6.3.2}$$

Claim 1: $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded in $L^{\alpha_-}(\Omega)$ and in $L^{\zeta_-}(\partial\Omega)$.
From (C_1) we have that there exists a constant $M_1 > 0$ such that for all $n \in \mathbb{N}$ one has $|I_+(u_n)| \leq M_1$, that is

$$\frac{1}{q_+}\rho(u_n^+) - \int_\Omega F(x, u_n^+)\,\mathrm{d}x - \int_{\partial\Omega} G(x, u_n^+)\,\mathrm{d}\sigma \leq M_1 - \frac{1}{q_+}\rho(-u_n^-),$$

which, taking (6.3.2) into account, leads to

$$\rho(u_n^+) - \int_\Omega q_+ F(x, u_n^+)\,\mathrm{d}x - \int_{\partial\Omega} q_+ G(x, u_n^+)\,\mathrm{d}\sigma \leq M_2, \tag{6.3.3}$$

for all $n \in \mathbb{N}$ and for some $M_2 > 0$. Testing (6.3.1) for $v = u_n^+$, we have

$$-\rho(u_n^+) + \int_\Omega f(x, u_n^+)u_n^+\,\mathrm{d}x + \int_{\partial\Omega} g(x, u_n^+)u_n^+\,\mathrm{d}\sigma \leq \varepsilon_n, \tag{6.3.4}$$

for all $n \in \mathbb{N}$. Adding (6.3.3) and (6.3.4) we obtain

$$\begin{aligned} &\int_\Omega \left(f(x, u_n^+)u_n^+ - q_+ F(x, u_n^+)\right)\,\mathrm{d}x \\ &+ \int_{\partial\Omega} \left(g(x, u_n^+)u_n^+ - q_+ G(x, u_n^+)\right)\,\mathrm{d}\sigma \leq M_3, \end{aligned} \tag{6.3.5}$$

for all $n \in \mathbb{N}$, with $M_3 > 0$. Without loss of generality, we can assume $\alpha_- \leq \beta_-$ and $\zeta_- \leq \theta_-$. From (h_4^N), there exist $\hat{K}_3, \tilde{K}_3, \hat{K}_4, \tilde{K}_4 > 0$ such that for all $t \in \mathbb{R}$ the following hold

$$\begin{aligned} f(x,t)t - q_+ F(x,t) &\geq \hat{K}_3 |t|^{\alpha_-} - \tilde{K}_3 \quad \text{for a.a. } x \in \Omega, \\ g(x,t)t - q_+ G(x,t) &\geq \hat{K}_4 |t|^{\zeta_-} - \tilde{K}_4 \quad \text{for a.a. } x \in \partial\Omega. \end{aligned}$$

Exploiting these relations in (6.3.5), we derive

$$\hat{K}_3 \|u_n^+\|_{\alpha_-}^{\alpha_-} + \hat{K}_4 \|u_n^+\|_{\zeta_-,\partial\Omega}^{\zeta_-} \leq M_4,$$

which gives

$$\|u_n^+\|_{\alpha_-} \leq M_5 \quad \text{and} \quad \|u_n^+\|_{\zeta_-,\partial\Omega} \leq \tilde{M}_5 \quad \text{for all } n \in \mathbb{N}$$

for some $M_5, \tilde{M}_5 > 0$ and Claim 1 is achieved.

Claim 2: $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}}(\Omega)$.
From (h_1^N) and (h_4^N), we have that

$$\alpha_- < \ell_+ < (p_-)^* \quad \text{and} \quad \zeta_- < \kappa_+ < (p_-)_*.$$

Hence, there exist $s, \tau \in (0,1)$ such that

$$\frac{1}{\ell_+} = \frac{s}{(p_-)^*} + \frac{1-s}{\alpha_-} \quad \text{and} \quad \frac{1}{\kappa_+} = \frac{\tau}{(p_-)_*} + \frac{1-\tau}{\zeta_-}, \tag{6.3.6}$$

and applying the interpolation inequality, see Papageorgiou-Winkert [79, Proposition 2.3.17 p.116], we obtain

$$\begin{aligned} \|u_n^+\|_{\ell_+} &\le \|u_n^+\|_{(p_-)^*}^s \|u_n^+\|_{\alpha_-}^{1-s}, \\ \|u_n^+\|_{\kappa_+,\partial\Omega} &\le \|u_n^+\|_{(p_-)_*,\partial\Omega}^\tau \|u_n^+\|_{\zeta_-,\partial\Omega}^{1-\tau}, \end{aligned}$$

for all $n \in \mathbb{N}$. Taking Claim 1 into account, one has

$$\|u_n^+\|_{\ell_+} \le M_6 \|u_n^+\|_{(p_-)^*}^s \quad \text{and} \quad \|u_n^+\|_{\kappa_+,\partial\Omega} \le \tilde{M}_6 \|u_n^+\|_{(p_-)_*,\partial\Omega}^\tau, \tag{6.3.7}$$

for some $M_6, \tilde{M}_6 > 0$ and for all $n \in \mathbb{N}$. Again, from (6.3.1) with $v = u_n^+$, using (h_1^N), it follows that

$$\rho(u_n^+) \le \varepsilon_n + K_1 \int_\Omega \left(|u_n^+| + |u_n^+|^{\ell(x)}\right) \, dx + K_2 \int_{\partial\Omega} \left(|u_n^+| + |u_n^+|^{\kappa(x)}\right) \, d\sigma. \tag{6.3.8}$$

We may assume that $\|u_n^+\| \ge 1$ for all $n \in \mathbb{N}$, otherwise we are done. Then, using Proposition 6.2.1(iv), (6.3.8) and (6.3.7), we derive that

$$\begin{aligned} \|u_n^+\|^{p_-} \le \rho(u_n^+) &\le \varepsilon_n + K_1 \left(\|u_n^+\|_1 + \|u_n^+\|_{\ell_+}^{\ell_+}\right) + K_2 \left(\|u_n^+\|_{1,\partial\Omega} + \|u_n^+\|_{\kappa_+,\partial\Omega}^{\kappa_+}\right) \\ &\le \varepsilon_n + M_7 \left(1 + \|u_n^+\|_{(p_-)^*}^{s\ell_+}\right) + \tilde{M}_7 \left(1 + \|u_n^+\|_{(p_-)_*,\partial\Omega}^{\tau\kappa_+}\right), \end{aligned}$$

with $M_7, \tilde{M}_7 > 0$. Then, considering the embeddings $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \hookrightarrow L^{(p_-)^*}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \hookrightarrow L^{(p_-)_*}(\partial\Omega)$, we get

$$\|u_n^+\|^{p_-} \le \varepsilon_n + M_8 \left(1 + \|u_n^+\|^{s\ell_+} + \|u_n^+\|^{\tau\kappa_+}\right),$$

for all $n \in \mathbb{N}$ and for some $M_8 > 0$. From (6.3.6), the definition of $(p_-)^*$ and (h_4^N), one has

$$\begin{aligned} s\ell_+ &= \frac{(p_-)^*(\ell_+ - \alpha_-)}{(p_-)^* - \alpha_-} = \frac{Np_-(\ell_+ - \alpha_-)}{Np_- - N\alpha_- + p_-\alpha_-} \\ &< \frac{Np_-(\ell_+ - \alpha_-)}{Np_- - N\alpha_- + p_-(\ell_+ - p_-)\frac{N}{p_-}} = p_-. \end{aligned}$$

Similarly, from (6.3.6), the definition of $(p_-)_*$ and (h_4^N), we have

$$\zeta_- > \frac{\zeta_-}{p_-} + (\kappa_+ - p_-)\frac{N-1}{p_-},$$

which implies

$$\tau\kappa_+ = \frac{(p_-)_*(\kappa_+ - \zeta_-)}{(p_-)_* - \zeta_-} = \frac{(N-1)p_-(\kappa_+ - \zeta_-)}{(N-1)p_- - N\zeta_- + p_-\zeta_-}$$
$$< \frac{(N-1)p_-(\kappa_+ - \zeta_-)}{(N-1)p_- - N\zeta_- + p_-\left(\frac{\zeta_-}{p_-} + (\kappa_+ - p_-)\frac{N-1}{p_-}\right)} = p_-.$$

This completes the proof of Claim 2.

Claim 3: $u_n \to u$ in $W^{1,\mathcal{H}}(\Omega)$ up to a subsequence.
From (6.3.2) and Claim 2, it follows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}}(\Omega)$. Since $W^{1,\mathcal{H}}(\Omega)$ is a reflexive space, there exists a weakly convergent subsequence in $W^{1,\mathcal{H}}(\Omega)$, not relabeled, such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}}(\Omega).$$

Then, as by (h_1^N) and (6.3.1) in correspondence of $v = u_n - u$, it holds

$$\langle I'_+(u_n), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

The f and g terms are strongly continuous (see for example [39, Lemma 4.4]), hence their limit vanishes and we derive

$$\langle A(u_n), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

As A satisfies the (S_+)-property, see Proposition 4.2.1, the proof is complete. □

The following results are needed to verify the so-called mountain-pass geometry.

Proposition 6.3.5. *Let* (H^N), (h_1^N) *and* (h_3^N) *be satisfied. Then, there exist constants* $C_i > 0, i = 1, \dots, 5$ *such that*

$$I(u), I_\pm(u) \geq \begin{cases} C_1\|u\|^{q_+} - C_2\|u\|^{\ell_-} - C_3\|u\|^{\kappa_-} & \textit{if } \|u\| \leq \min\{1, C_4, C_5\}, \\ C_1\|u\|^{p_-} - C_2\|u\|^{\ell_+} - C_3\|u\|^{\kappa_+} & \textit{if } \|u\| \geq \max\{1, C_4, C_5\}. \end{cases}$$

Proof. We give the proof only for the functional I, the proof for $I_\pm$ is similar. From assumptions (h_1^N) and (h_3^N) it follows that for all $\varepsilon > 0$ there exist $c_\varepsilon, \tilde{c}_\varepsilon > 0$ such that

$$\begin{aligned} |F(x,t)| &\leq \frac{\varepsilon}{p(x)}|t|^{p(x)} + c_\varepsilon |t|^{\ell(x)} \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R}, \\ |G(x,t)| &\leq \frac{\varepsilon}{p(x)}|t|^{p(x)} + \tilde{c}_\varepsilon |t|^{\kappa(x)} \quad \text{for a.a. } x \in \partial\Omega \text{ and for all } t \in \mathbb{R}. \end{aligned} \tag{6.3.9}$$

Let $u \in W^{1,\mathcal{H}}(\Omega)$ be fixed. Using (6.3.9), Proposition 2.2.2, the embedding $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{\ell(\cdot)}(\Omega)$ with constant C_ℓ and the embedding $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{\kappa(\cdot)}(\partial\Omega)$ with constant $C_{\kappa,\partial\Omega}$ one has

$$\begin{aligned} I(u) \geq &\frac{1}{q_+}\rho_{\mathcal{H}}(\nabla u) + \frac{1}{p_+}\rho_{p(\cdot)}(u) + \frac{1}{p_+}\rho_{p(\cdot),\partial\Omega}(u) \\ &- \frac{\varepsilon}{p_-}\rho_{p(\cdot)}(u) - c_\varepsilon \rho_{\ell(\cdot)}(u) - \frac{\varepsilon}{p_-}\rho_{p(\cdot),\partial\Omega}(u) - \tilde{c}_\varepsilon \rho_{\kappa(\cdot),\partial\Omega}(u) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q_+}\rho_{\mathcal{H}}(\nabla u) + \left(\frac{1}{p_+} - \frac{\varepsilon}{p_-}\right)\rho_{p(\cdot)}(u) + \left(\frac{1}{p_+} - \frac{\varepsilon}{p_-}\right)\rho_{p(\cdot),\partial\Omega}(u) \\
&\quad - c_\varepsilon \rho_{\ell(\cdot)}(u) - \tilde{c}_\varepsilon \rho_{\kappa(\cdot),\partial\Omega}(u) \\
&\geq \min\left\{\frac{1}{q_+}, \frac{1}{p_+} - \frac{\varepsilon}{p_-}\right\}\rho(u) \\
&\quad - c_\varepsilon \max\left\{\|u\|_{\ell(\cdot)}^{\ell_-}, \|u\|_{\ell(\cdot)}^{\ell_+}\right\} - \tilde{c}_\varepsilon \max\left\{\|u\|_{\kappa(\cdot),\partial\Omega}^{\kappa_-}, \|u\|_{\kappa(\cdot),\partial\Omega}^{\kappa_+}\right\} \\
&\geq \min\left\{\frac{1}{q_+}, \frac{1}{p_+} - \frac{\varepsilon}{p_-}\right\}\rho(u) \\
&\quad - c_\varepsilon \max\left\{C_\ell^{\ell_-}\|u\|^{\ell_-}, C_\ell^{\ell_+}\|u\|^{\ell_+}\right\} - \tilde{c}_\varepsilon \max\left\{C_{\kappa,\partial\Omega}^{\kappa_-}\|u\|^{\kappa_-}, C_{\kappa,\partial\Omega}^{\kappa_+}\|u\|^{\kappa_+}\right\}.
\end{aligned}
$$

Choosing $\varepsilon \in \left(0, \frac{(q_+-p_+)p_-}{p_+q_+}\right)$ and taking

$$
C_1 = \frac{1}{q_+}, \quad C_4 = \frac{1}{C_\ell} \quad \text{and} \quad C_5 = \frac{1}{C_{\kappa,\partial\Omega}},
$$

our statement follows from Proposition 6.2.1(iii)-(iv) and by setting

$$
\begin{aligned}
C_2 &= c_\varepsilon C_\ell^{\ell_-} \quad \text{and} \quad C_3 = \tilde{c}_\varepsilon C_{\kappa,\partial\Omega}^{\kappa_-} \quad && \text{if } \|u\| \leq \min\{1, C_4, C_5\}, \\
C_2 &= c_\varepsilon C_\ell^{\ell_+} \quad \text{and} \quad C_3 = \tilde{c}_\varepsilon C_{\kappa,\partial\Omega}^{\kappa_+} \quad && \text{if } \|u\| \geq \max\{1, C_4, C_5\}.
\end{aligned}
$$

□

The following result is a direct consequence of Proposition 6.3.5.

Proposition 6.3.6. *Let* (H^N), (h_1^N) *and* (h_3^N) *be satisfied with* $q_+ < \ell_-, \kappa_-$. *Then there exists* $\delta > 0$ *such that*

$$
\inf_{\|u\|=\delta} I(u) > 0 \quad \text{and} \quad \inf_{\|u\|=\delta} I_\pm(u) > 0,
$$

or alternatively, there exists $\lambda > 0$ *such that* $I(u) > 0$ *for* $0 < \|u\| < \lambda$.

Proposition 6.3.7. *Let* (H^N), (h_1^N) *and* (h_2^N) *be satisfied. Then,* $I(su) \to -\infty$ *as* $s \to \pm\infty$ *for every* $u \in W^{1,\mathcal{H}}(\Omega) \setminus \{0\}$. *Moreover,* $I_\pm(su) \to -\infty$ *as* $s \to \pm\infty$ *for all* $u \in W^{1,\mathcal{H}}(\Omega) \setminus \{0\}$ *such that* $u \geq 0$ *a.e. in* Ω.

Proof. We give the proof only for the functional I, since if $u \geq 0$ a.e. in Ω then $I_\pm(su) = I(su)$ for $\pm s > 0$. Fix $s, \varepsilon \in \mathbb{R}$ and $u \in W^{1,\mathcal{H}}(\Omega)$ such that $|s| \geq 1, \varepsilon \geq 1$ and $u \neq 0$. From (h_1^N) and (h_2^N) it follows that

$$
\begin{aligned}
|F(x,t)| &\geq \frac{\varepsilon}{q_+}|t|^{q_+} - c_\varepsilon \quad && \text{for a.a. } x \in \Omega, \\
|G(x,t)| &\geq \frac{\varepsilon}{q_+}|t|^{q_+} - c_\varepsilon \quad && \text{for a.a. } x \in \partial\Omega,
\end{aligned}
$$

see also [39, Lemma 4.4]. Then, using the previous inequalities, one has

$$
\begin{aligned}
I(su) &\leq \frac{|s|^{p_+}}{p_-}\left(\rho_{p(\cdot)}(\nabla u) + \rho_{p(\cdot)}(u) + \rho_{p(\cdot),\partial\Omega}(u)\right) + c_\varepsilon\left(|\Omega| + |\partial\Omega|\right) \\
&\quad + |s|^{q_+}\left[\frac{\rho_{q(\cdot),\mu}(\nabla u)}{q_-} - \frac{\varepsilon}{q_+}\left(\|u\|_{q_+}^{q_+} + \|u\|_{q_+,\partial\Omega}^{q_+}\right)\right].
\end{aligned}
$$

Noting that $\|u\|_{q_+} < \infty$ and $\|u\|_{q_+,\partial\Omega} < \infty$ since $q_+ < l_- < (p_-)^*$ and $q_+ < \kappa_- < (p_-)_*$, we can choose ε large enough such that the third term is negative and $I(su) \to -\infty$ as $|s| \to \infty$. □

Finally, we state the main result of this section.

Theorem 6.3.8. *Let* (H^N), (h_1^N)–(h_4^N) *be satisfied. Then, there exist two nontrivial weak solutions* $u_0, v_0 \in W^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ *of problem* (N) *such that* $u_0 \geq 0$ *and* $v_0 \leq 0$ *a.e. in* Ω*.*

Proof. Thanks to Proposition 6.3.4, 6.3.6 and 6.3.7, we can apply Theorem 3.1.11 to both functionals $I_\pm$. Then, there exist $u_0, v_0 \in W^{1,\mathcal{H}}(\Omega)$ such that $I'_+(u_0) = 0$ and $I'_-(v_0) = 0$, namely u_0, v_0 are weak solutions of problem (N). In particular, from Proposition 6.3.6 it follows that

$$I_+(u_0) \geq \inf_{\|u\|=\delta} I_+(u) > 0 = I_+(0),$$

which implies $u_0 \neq 0$. Analogously, $I_-(v_0) > 0$ and $v_0 \neq 0$. Finally, since $\langle I'_+(u_0), v\rangle = 0$ for every $v \in W^{1,\mathcal{H}}(\Omega)$, we can choose $v = -u_0^-$ and this leads to

$$\rho(-u_0^-) = \int_\Omega f(x, u_0^+)u_0^- \,\mathrm{d}x + \int_{\partial\Omega} g(x, u_0^+)u_0^- \,\mathrm{d}\sigma = 0.$$

From Proposition 6.2.1 it follows that $-u_0^- = 0$ a.e. in Ω, hence $u_0 \geq 0$ a.e. in Ω. Similarly, we can test $\langle I'_-(v_0), v_0^+\rangle = 0$ and derive that $v_0 \leq 0$ a.e in Ω. Finally, we know that u_0 and v_0 are bounded functions because we can apply Theorem 4.1 of Amoroso-Crespo-Blanco-Pucci-Winkert [5], which is a boundedness result for a more general class of problems. □

6.3.2 Sign-changing solution

We indicate with $\mathcal{N}$ the Nehari manifold of I, defined by

$$\mathcal{N} = \left\{ u \in W^{1,\mathcal{H}}(\Omega) \,:\, \langle I'(u), u\rangle = 0,\, u \neq 0 \right\}.$$

Clearly, any nontrivial weak solution of (N) belongs to $\mathcal{N}$, because the weak solutions of (N) are exactly the critical points of I. Since we are interested in sign-changing solutions, we introduce the following subset of $\mathcal{N}$

$$\mathcal{N}_0 = \left\{ u \in W^{1,\mathcal{H}}(\Omega) \,:\, \pm u^\pm \in \mathcal{N} \right\}.$$

For an overview on the method of the Nehari manifold, we refer to the book chapter of Szulkin-Weth [92].

First, we prove some properties of the Nehari manifold $\mathcal{N}$ (Proposition 6.3.9) and of the energy functional I restricted to $\mathcal{N}$ (Proposition 6.3.10).

Proposition 6.3.9. *Let* (H^N), (h_1^N)–(h_3^N) *and* (h_5^N) *be satisfied. Then, for any* $u \in W^{1,\mathcal{H}}(\Omega) \setminus \{0\}$*, there exists a unique* $s_u > 0$ *such that* $s_u u \in \mathcal{N}$*.*
Moreover, one has

$$I(s_u u) > 0 \quad \textit{and} \quad I(s_u u) > I(su) \quad \textit{for all } s > 0 \textit{ with } s \neq s_u.$$

and

$$\partial_s I(su) > 0 \quad \text{for } 0 < s < s_u \quad \text{and} \quad \partial_s I(su) < 0 \quad \text{for } s > s_u.$$

Proof. For any fixed $u \in W^{1,\mathcal{H}}(\Omega) \setminus \{0\}$ we define $\phi_u \colon [0, \infty[\to \mathbb{R}$ as follows

$$\phi_u(s) = I(su) \quad \text{for all } s \in [0, \infty[.$$

Clearly, ϕ_u belongs to $C([0, \infty[)$ and $C^1((0, \infty[)$. From Propositions 6.3.6 and 6.3.7 we derive that there exist $\delta, M > 0$ such that

$$\phi_u(s) > 0 \quad \text{for } 0 < t < \delta \quad \text{and} \quad \phi_u(s) < 0 \quad \text{for } t > M.$$

Then, applying the extreme value theorem, we get in particular that ϕ_u admits a local maximum, i.e., there exists $0 < s_u \leq M$ such that

$$\sup_{s \in [0,\infty)} \phi_u(s) = \max_{s \in [0,M]} \phi_u(s) = \phi_u(s_u).$$

Since s_u is also a critical point of ϕ_u, in combination with $\phi_u'(s) = \langle I'(su), u \rangle$ for every $s \geq 0$, one has

$$\phi_u'(s_u) = \langle I'(s_u u), u \rangle = 0 \quad \Longrightarrow \quad s_u u \in \mathcal{N}.$$

Claim: s_u is unique.
From assumption (h_5^N) we have that

$$\begin{aligned}
s &\mapsto \frac{f(x,su)}{s^{q_+-1}|u|^{q_+-1}} \text{ increasing} \Rightarrow s \mapsto \frac{f(x,su)u}{s^{q_+-1}} \text{ increasing in } \{x \in \Omega : u(x) > 0\},\\
s &\mapsto \frac{f(x,su)}{s^{q_+-1}|u|^{q_+-1}} \text{ decreasing} \Rightarrow s \mapsto \frac{f(x,su)u}{s^{q_+-1}} \text{ increasing in } \{x \in \Omega : u(x) < 0\},\\
s &\mapsto \frac{g(x,su)}{s^{q_+-1}|u|^{q_+-1}} \text{ increasing} \Rightarrow s \mapsto \frac{g(x,su)u}{s^{q_+-1}} \text{ increasing in } \{x \in \partial\Omega : u(x) > 0\},\\
s &\mapsto \frac{g(x,su)}{s^{q_+-1}|u|^{q_+-1}} \text{ decreasing} \Rightarrow s \mapsto \frac{g(x,su)u}{s^{q_+-1}} \text{ increasing in } \{x \in \partial\Omega : u(x) < 0\}.
\end{aligned}$$

Multiplying the equation $\phi_u'(s) = \langle I'(su), u \rangle = 0$ with $s > 0$, which is a necessary condition for $su \in \mathcal{N}$, by $1/s^{q_+-1}$, we obtain

$$\begin{aligned}
&\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{s^{q_+-p(x)}} + \frac{\mu(x)|\nabla u|^{q(x)}}{s^{q_+-q(x)}} \right) \mathrm{d}x + \int_\Omega \frac{|u|^{p(x)}}{s^{q_+-p(x)}} \,\mathrm{d}x + \int_{\partial\Omega} \frac{|u|^{p(x)}}{s^{q_+-p(x)}} \,\mathrm{d}\sigma\\
&= \int_\Omega \frac{f(x,su)u}{s^{q_+-1}} \,\mathrm{d}x + \int_{\partial\Omega} \frac{g(x,su)u}{s^{q_+-1}} \,\mathrm{d}\sigma.
\end{aligned} \tag{6.3.10}$$

As functions of s, the left-hand side is strictly decreasing, because it is so in the sets $\{x \in \Omega : \nabla u \neq 0\}$, $\{x \in \Omega : u \neq 0\}$ and $\{x \in \partial\Omega : u \neq 0\}$ and at least decreasing in the rest (we recall that $p(x) < q(x) \leq q_+$ for all $x \in \overline{\Omega}$), while from the previous comments the right-hand side is increasing. Consequently, there can be at most one single value $s_u > 0$ for which the equation holds, namely there exists a unique $s_u > 0$ such that $s_u u \in \mathcal{N}$.

Finally, since $s \mapsto \phi_u'(s)$ is strictly decreasing (see (6.3.10) and the comments above) and $\phi_u'(s_u) = 0$, it follows that

$$\phi_u'(s) > 0 \quad \text{for } 0 < s < s_u, \quad \text{and} \quad \phi_u'(s) < 0 \quad \text{for } s > s_u.$$

Thus s_u is a strict maximum for ϕ_u and this completes the proof. □

Proposition 6.3.10. *Let* (H^N), (h$_1^N$)–(h$_3^N$) *and* (h$_5^N$) *be satisfied. Then, the functional* $I|_{\mathcal{N}}$ *is sequentially coercive, namely for any sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}$ *such that* $\|u_n\| \xrightarrow{n\to\infty} \infty$ *one has* $I(u_n) \xrightarrow{n\to\infty} \infty$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}$ be a sequence such that $\|u_n\| \xrightarrow{n\to\infty} \infty$ and put

$$y_n = \frac{u_n}{\|u_n\|} \quad \text{for all } n \in \mathbb{N}. \tag{6.3.11}$$

Since $\{y_n\}$ is bounded in the reflexive space $W^{1,\mathcal{H}}(\Omega)$, there exists a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ and $y \in W^{1,\mathcal{H}}(\Omega)$ such that

$$y_{n_k} \rightharpoonup y \quad \text{in } W^{1,\mathcal{H}}(\Omega).$$

Claim: $y = 0$.
By contradiction, suppose that $y \neq 0$. As $\|u_n\| \to \infty$, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ one has $\|u_{n_k}\| \geq 1$ and

$$\begin{aligned} I(u_{n_k}) &\leq \frac{1}{p_-}\rho(u_{n_k}) - \int_\Omega F(x,u_{n_k})\,dx - \int_{\partial\Omega} G(x,u_{n_k})\,d\sigma \\ &\leq \frac{1}{p_-}\|u_{n_k}\|^{q_+} - \int_\Omega F(x,u_{n_k})\,dx - \int_{\partial\Omega} G(x,u_{n_k})\,d\sigma, \end{aligned}$$

where we have used Proposition 6.2.1(iv). Dividing by $\|u_{n_k}\|^{q_+}$ and taking (6.3.11) into account, we obtain

$$\frac{I(u_{n_k})}{\|u_{n_k}\|^{q_+}} \leq \frac{1}{p_-} - \int_\Omega \frac{F(x,u_{n_k})}{|u_{n_k}|^{q_+}}|y_{n_k}|^{q_+}\,dx - \int_{\partial\Omega} \frac{G(x,u_{n_k})}{|u_{n_k}|^{q_+}}|y_{n_k}|^{q_+}\,d\sigma. \tag{6.3.12}$$

Now, we observe that if f and g fulfill (h$_1^N$) and (h$_2^N$), then there exist $M_9, M_{10} > 0$ such that

$$\begin{aligned} F(x,t) &> -M_9 \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R}, \\ G(x,t) &> -M_{10} \quad \text{for a.a. } x \in \partial\Omega \text{ and for all } t \in \mathbb{R}. \end{aligned} \tag{6.3.13}$$

Setting $\Omega_0 = \{x \in \Omega : y(x) = 0\}$, by using (6.3.13), (h$_2^N$) and Fatou's Lemma, we get

$$\begin{aligned} &\lim_{k\to\infty} \int_\Omega \frac{F(x,u_{n_k})}{|u_{n_k}|^{q_+}}|y_{n_k}|^{q_+}\,dx \\ &= \lim_{k\to\infty} \left(\int_{\Omega\setminus\Omega_0} \frac{F(x,u_{n_k})}{|u_{n_k}|^{q_+}}|y_{n_k}|^{q_+}\,dx + \int_{\Omega_0} \frac{F(x,u_{n_k})}{\|u_{n_k}\|^{q_+}} \right) \\ &\geq \int_{\Omega\setminus\Omega_0} \left(\lim_{k\to\infty} \frac{F(x,u_{n_k})}{|u_{n_k}|^{q_+}}|y_{n_k}|^{q_+} \right) dx - \lim_{k\to\infty} \frac{M_9|\Omega_0|}{\|u_{n_k}\|^{q_+}} \\ &= \infty. \end{aligned}$$

Analogously, for $\Sigma_0 = \{x \in \partial\Omega : y(x) = 0\}$, we have

$$\begin{aligned}
&\lim_{k\to\infty} \int_{\partial\Omega} \frac{G(x,u_{n_k})}{|u_{n_k}|^{q_+}} |y_{n_k}|^{q_+} \, d\sigma \\
&= \lim_{k\to\infty} \left(\int_{\partial\Omega\setminus\Sigma_0} \frac{G(x,u_{n_k})}{|u_{n_k}|^{q_+}} |y_{n_k}|^{q_+} \, d\sigma + \int_{\Sigma_0} \frac{G(x,u_{n_k})}{\|u_{n_k}\|^{q_+}} \right) \\
&\geq \int_{\partial\Omega\setminus\Sigma_0} \left(\lim_{k\to\infty} \frac{G(x,u_{n_k})}{|u_{n_k}|^{q_+}} |y_{n_k}|^{q_+} \right) d\sigma - \lim_{k\to\infty} \frac{M_{10}|\Sigma_0|}{\|u_{n_k}\|^{q_+}} \\
&= \infty.
\end{aligned}$$

Hence, passing to the limit as $k \to \infty$ in (6.3.12), it follows that

$$\lim_{k\to\infty} \frac{I(u_{n_k})}{\|u_{n_k}\|^{q_+}} = -\infty,$$

which is a contradiction with $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}$ that implies $I(u_n) > 0$ for all $n \in \mathbb{N}$ (see Proposition 6.3.9). Thus, the proof of our claim is complete.

Recall that $u_{n_k} \in \mathcal{N}$ for every $k \in \mathbb{N}$, from Proposition 6.3.9 it follows that $I(u_{n_k}) \geq I(su_{n_k})$ for every $s > 0, s \neq 1$ and for all $k \in \mathbb{N}$. Fixing $s > 1$ and using Proposition 6.2.1(iv), one has

$$\begin{aligned}
I(u_{n_k}) &\geq I(sy_{n_k}) \\
&\geq \frac{1}{p_-}\rho(sy_{n_k}) - \int_\Omega F(x,sy_{n_k}) \, dx - \int_{\partial\Omega} G(x,sy_{n_k}) \, d\sigma \\
&\geq \frac{1}{p_-}\|sy_{n_k}\|^{p_-} - \int_\Omega F(x,sy_{n_k}) \, dx - \int_{\partial\Omega} G(x,sy_{n_k}) \, d\sigma \\
&= \frac{s^{p_-}}{p_-} - \int_\Omega F(x,sy_{n_k}) \, dx - \int_{\partial\Omega} G(x,sy_{n_k}) \, d\sigma.
\end{aligned}$$

Moreover, as a consequence of the assumptions on the nonlinear functions f and g, it follows that the integral terms are strongly continuous (see for example [39, Lemma 4.4]). Since $sy_{n_k} \rightharpoonup 0$, we derive that there exists $k_1 \in \mathbb{N}$ such that

$$I(u_{n_k}) \geq \frac{s^{p_-}}{p_-} - 1 \quad \text{for all } k \geq k_1.$$

From the arbitrariness of $s > 1$, we get $I(u_{n_k}) \to \infty$ as $k \to \infty$, which implies that $I(u_n) \xrightarrow{n\to\infty} \infty$ and our statement is achieved. □

Now, we are able to prove the existence of a minimizer of I restricted to $\mathcal{N}_0$.

Proposition 6.3.11. *Let* (H^N), (h_1^N)–(h_3^N) *and* (h_5^N) *be satisfied. Then*

$$\inf_{u\in\mathcal{N}} I(u) > 0 \quad \text{and} \quad \inf_{u\in\mathcal{N}_0} I(u) > 0.$$

Proof. Fix $u \in \mathcal{N}$. Then, from Proposition 6.3.9 we have that $I(u) \geq I(su)$ for all $s > 0, s \neq 1$. In particular, applying Proposition 6.3.6, it follows that

$$I(u) \geq I\left(\frac{\delta}{\|u\|}u\right) \geq \inf_{\|u\|=\delta} I(u) > 0 \quad \text{for all } u \in \mathcal{N},$$

that implies

$$\inf_{u\in\mathcal{N}} I(u) > 0.$$

Now, fix $u \in \mathcal{N}_0$. Since by definition $\pm u^{\pm} \in \mathcal{N}$, we get

$$I(u) = I(u^+) + I(-u^-) \geq 2 \inf_{u\in\mathcal{N}} I(u) > 0 \quad \text{for all } u \in \mathcal{N}_0,$$

so we obtain

$$\inf_{u\in\mathcal{N}_0} I(u) > 0.$$

□

Proposition 6.3.12. *Let* (H^N), (h_1^N)–(h_3^N) *and* (h_5^N) *be satisfied. Then, there exists* $w_0 \in \mathcal{N}_0$ *such that*

$$I(w_0) = \inf_{u\in\mathcal{N}_0} I(u).$$

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_0$ be a minimizing sequence, that is, $I(u_n) \searrow \inf_{u\in\mathcal{N}_0} I(u)$. As $u_n \in \mathcal{N}_0$, then $\pm u_n^{\pm} \in \mathcal{N}$ and $I(\pm u_n^{\pm}) > 0$ for all $n \in \mathbb{N}$ (see Proposition 6.3.9). Moreover, since $I(u_n) = I(u_n^+) + I(-u_n^-)$ for every $n \in \mathbb{N}$ and from Proposition 6.3.10, one has that $\{\pm u_n^{\pm}\}_{n\in\mathbb{N}}$ are both bounded. Then, there exist subsequences $\{\pm u_{n_k}^{\pm}\}_{k\in\mathbb{N}}$ and $v_1, v_2 \in W^{1,\mathcal{H}}(\Omega)$ such that

$$\begin{aligned} u_{n_k}^+ &\rightharpoonup v_1 \quad \text{in } W^{1,\mathcal{H}}(\Omega) \quad \text{with } v_1 \geq 0,\\ u_{n_k}^- &\rightharpoonup v_2 \quad \text{in } W^{1,\mathcal{H}}(\Omega) \quad \text{with } v_2 \geq 0 \text{ and } v_1 v_2 = 0. \end{aligned}$$

Claim: $v_1, v_2 \neq 0$.
Arguing by contradiction, suppose that $v_1 = 0$. Recalling that $u_{n_k}^+ \in \mathcal{N}$ implies that

$$\langle I'(u_{n_k}^+), u_{n_k}^+\rangle = 0,$$

one has

$$\rho(u_{n_k}^+) - \int_\Omega f(x, u_{n_k}^+)(u_{n_k}^+)\, dx - \int_{\partial\Omega} g(x, u_{n_k}^+)(u_{n_k}^+)\, d\sigma = 0.$$

From the Carathéodory assumption on the nonlinearities f and g and by (h_1^N), it follows that the two integral terms are strongly continuous (see [39, Lemma 4.4]), thus $\rho(u_{n_k}^+) \to 0$ as $k \to \infty$. By Proposition 6.2.1(v), we get $u_{n_k}^+ \to 0$ in $W^{1,\mathcal{H}}(\Omega)$ and

$$0 < \inf_{u\in\mathcal{N}} I(u) \leq I(u_{n_k}^+) \to I(0) = 0 \quad \text{as } k \to \infty,$$

that is a contradiction. Analogously we prove that $v_2 \neq 0$ and our claim is true. Now, using Proposition 6.3.9, there exist $s_1, s_2 > 0$ such that $s_1 v_1, s_2 v_2 \in \mathcal{N}$. We put

$$w_0 = s_1 v_1 - s_2 v_2 = w_0^+ - w_0^-,$$

hence $w_0 \in \mathcal{N}_0$. Finally, it remains to prove that $I(w_0) = \inf_{u\in\mathcal{N}_0} I(u)$. It is worth noticing that the modular is convex and continuous, thus sequentially weakly lower semicontinuous, and the nonlinear terms are strongly continuous. Hence, I is sequentially weakly lower semicontinuous and this leads to

$$\begin{aligned}\inf_{u\in\mathcal{N}_0} I(u) &= \lim_{k\to\infty} I(u_{n_k}) = \lim_{k\to\infty}\left(I(u_{n_k}^+) + I(-u_{n_k}^-)\right)\\ &\geq \liminf_{k\to\infty}\left(I(s_1 u_{n_k}^+) + I(-s_2 u_{n_k}^-)\right)\\ &\geq I(s_1 v_1) + I(-s_2 v_2)\\ &= I(w_0^+) + I(-w_0^-)\\ &= I(w_0) \geq \inf_{u\in\mathcal{N}_0} I(u).\end{aligned}$$

The proof is complete. □

Now, we prove that the minimizer obtained in Proposition 6.3.12 is a critical point of the functional I.

Proposition 6.3.13. *Let* (H^N), (h_1^N)–(h_3^N) *and* (h_5^N) *be satisfied and let* $w_0 \in \mathcal{N}_0$ *such that* $I(w_0) = \inf_{u\in\mathcal{N}_0} I(u)$. *Then,* w_0 *is a critical point of the functional* I.

Proof. First, we observe something that will be useful in the sequel. Recalling that $\pm w_0^{\pm} \neq 0$ and indicating with C_{p_-} the constant of the embedding $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{p_-}(\Omega)$, we have that

$$\|w_0 - v\| \geq C_{p_-}^{-1}\|w_0 - v\|_{p_-} \geq \begin{cases} C_{p_-}^{-1}\|w_0^-\|_{p_-} & \text{if } v^- = 0,\\ C_{p_-}^{-1}\|w_0^+\|_{p_-} & \text{if } v^+ = 0,\end{cases}$$

for all $v \in W^{1,\mathcal{H}}(\Omega)$. Thus, taking

$$0 < \delta_0 < \min\left\{C_{p_-}^{-1}\|w_0^+\|_{p_-}, C_{p_-}^{-1}\|w_0^-\|_{p_-}\right\},$$

we have the following implication

$$\text{if } \|w_0 - v\| < \delta_0, \quad \text{then} \quad v^+ \neq 0 \neq v^-. \tag{6.3.14}$$

Now, arguing by contradiction, suppose that $I'(w_0) \neq 0$. Then there exists $\gamma, \delta_1 > 0$ such that

$$\|I'(u)\|_* \geq \gamma \quad \text{for all } u \in W^{1,\mathcal{H}}(\Omega) \text{ with } \|u - w_0\| < 3\delta_1. \tag{6.3.15}$$

Put

$$\delta = \min\left\{\frac{\delta_0}{2}, \delta_1\right\}. \tag{6.3.16}$$

From the continuity of the map defined by $(s,t) \mapsto s w_0^+ - t w_0^-$ for every $(s,t) \in [0,\infty[^2$, we have that for every $\delta > 0$ there exists $\lambda > 0$ such that

$$\|s w_0^+ - t w_0^- - w_0\| < \delta, \tag{6.3.17}$$

for all $(s,t) \in [0,\infty[^2$ with $\max\{|s-1|, |t-1|\} < \lambda$. Let

$$D = (1-\lambda, 1+\lambda)^2, \quad m_0 = \max_{(s,t)\in\partial D} I(s w_0^+ - t w_0^-),$$

and

$$c = \inf_{u \in \mathcal{N}_0} I(u). \tag{6.3.18}$$

We emphasize that for any $(s,t) \in [0,\infty[^2 \setminus \{(1,1)\}$, using Proposition 6.3.9, one has

$$\begin{aligned} I(sw_0^+ - tw_0^-) &= I(sw_0^+) + I(-tw_0^-) \\ &< I(w_0^+) + I(-w_0^-) = I(w_0) = \inf_{u \in \mathcal{N}_0} I(u), \end{aligned} \tag{6.3.19}$$

which implies that $m_0 < c$. In order to use the same notation of the Quantitative Deformation Lemma given in Lemma 6.2.3, we set

$$S = B(w_0, \delta), \quad \varepsilon = \min\left\{\frac{c - m_0}{4}, \frac{\gamma\,\delta}{8}\right\},$$

and δ, c as in (6.3.16) and (6.3.18), respectively. We also notice that by the definition of S it follows that $S_\delta = B(w_0, 2\delta)$ and $S_{2\delta} = B(w_0, 3\delta)$. From (6.3.15), we get

$$\|I'(u)\|_* \geq \gamma \geq \frac{8\varepsilon}{\delta} \quad \text{for all } u \in S_{2\delta},$$

so all the assumptions of Lemma 6.2.3 are verified. Hence, there exists a mapping $\eta \in C\left([0,1] \times W^{1,\mathcal{H}}(\Omega), W^{1,\mathcal{H}}(\Omega)\right)$ such that

(i) $\eta(t,u) = u$, if $t = 0$ or if $u \notin I^{-1}\left([c - 2\varepsilon, c + 2\varepsilon]\right) \cap S_{2\delta}$,

(ii) $I(\eta(1,u)) \leq c - \varepsilon$ for all $u \in I^{-1}\left(]-\infty, c + \varepsilon]\right) \cap S$,

(iii) $\eta(t,\cdot)$ is an homeomorphism of $W^{1,\mathcal{H}}(\Omega)$ for all $t \in [0,1]$,

(iv) $\|\eta(t,u) - u\| \leq \delta$ for all $u \in W^{1,\mathcal{H}}(\Omega)$ and $t \in [0,1]$,

(v) $I(\eta(\cdot,u))$ is decreasing for all $u \in W^{1,\mathcal{H}}(\Omega)$,

(vi) $I(\eta(t,u)) < c$ for all $u \in I^{-1}\left(]-\infty, c]\right) \cap S_\delta$ and $t \in]0,1]$.

Afterwards, we consider $h\colon [0,\infty[^2 \to W^{1,\mathcal{H}}(\Omega)$ defined by

$$h(s,t) = \eta(1, sw_0^+ - tw_0^-) \quad \text{for all } (s,t) \in [0,\infty[^2,$$

which has the following properties:

(vii) $h \in C\left([0,\infty[^2, W^{1,\mathcal{H}}(\Omega)\right)$,

(viii) $I(h(s,t)) < c - \varepsilon$ for all $(s,t) \in D$, by (ii), (6.3.17) and (6.3.19),

(ix) $h(D) \subseteq S_\delta$, by (iv) and (6.3.17),

(x) $h(s,t) = sw_0^+ - tw_0^-$ for all $(s,t) \in \partial D$,

where the last one follows from (i) and

$$I(sw_0^+ - tw_0^-) \leq m_0 + c - c < c - \left(\frac{c - m_0}{2}\right) \leq c - 2\varepsilon \quad \text{for all } (s,t) \in \partial D.$$

Now, we define two mappings $H_0, H_1\colon (0,\infty)^2 \to \mathbb{R}^2$ given by

$$H_0(s,t) = \left(\langle I'(sw_0^+), w_0^+\rangle \,,\, \langle I'(-tw_0^-), -w_0^-\rangle \right),$$
$$H_1(s,t) = \left(\frac{1}{s}\langle I'(h^+(s,t)), h^+(s,t)\rangle \,,\, \frac{1}{t}\langle(-h^-(s,t)), -h^-(s,t)\rangle \right),$$

which are clearly continuous. From Proposition 6.3.9 it follows that

$$\begin{aligned} &\langle I'(sw_0^+), w_0^+\rangle \begin{cases} > 0 & \text{for all } 0 < s < 1, \\ < 0 & \text{for all } s > 1, \end{cases} \\ &\langle I'(-tw_0^-), -w_0^-\rangle \begin{cases} > 0 & \text{for all } 0 < t < 1, \\ < 0 & \text{for all } t > 1. \end{cases} \end{aligned} \tag{6.3.20}$$

Given $A \subseteq \mathbb{R}^N$ open and bounded and $g \in C(A, \mathbb{R}^N)$, we denote by $\deg(g, A, y)$ the Brouwer degree over A of g at the value $y \in \mathbb{R}^N \setminus g(\partial A)$. From the Cartesian product property of the Brouwer degree (see the book of Dinca-Mawhin[47, Lemma 7.1.1 and Theorem 7.1.1]) we get

$$\begin{aligned} \deg(H_0, D, 0) = \deg&\left(\langle I'(sw_0^+), w_0^+\rangle \,,\, (1-\lambda, 1+\lambda) \,,\, 0 \right) \\ &\times \deg\left(\langle I'(-tw_0^-), -w_0^-\rangle \,,\, (1-\lambda, 1+\lambda) \,,\, 0 \right), \end{aligned}$$

and by (6.3.20) and Proposition 1.2.3 of Dinca-Mawhin[47], we obtain

$$\deg(H_0, D, 0) = (-1)(-1) = 1.$$

We observe that (x) implies $H_0|_{\partial D} = H_1|_{\partial D}$, so as the Brouwer degree depends on the boundary ([47, Corollary 1.2.7]), we have

$$\deg(H_1, D, 0) = \deg(H_0, D, 0) = 1,$$

and by the solution property ([47, Corollary 1.2.5]) it follows that there exists $(s_0, t_0) \in D$ such that $H_1(s_0, t_0) = (0, 0)$, namely

$$\langle I'(h^+(s_0,t_0)), h^+(s_0,t_0)\rangle = 0 = \langle I'(-h^-(s_0,t_0)), -h^-(s_0,t_0)\rangle.$$

Finally, by (ix)

$$\|h(s_0,t_0) - w_0\| \le 2\delta \le \delta_0,$$

which, taking (6.3.14) into account, leads to

$$h^+(s_0,t_0) \neq 0 \quad \text{and} \quad -h^-(s_0,t_0) \neq 0.$$

Thus, $h(s_0, t_0) \in \mathcal{N}_0$, that is a contradiction with

$$I(h(s_0,t_0)) < c - \varepsilon = \inf_{u \in \mathcal{N}_0} I(u) - \varepsilon,$$

obtained by (viii). This completes the proof. □

Now, we are able to prove the first main result.

Proof of Theorem 6.3.1. Combining Theorem 6.3.8 with Propositions 6.3.12 and 6.3.13, we get the existence of three weak solutions for problem (N). We further know that they are bounded functions thanks to Theorem 4.1 of Amoroso-Crespo-Blanco-Pucci-Winkert [5].

In the last part of this subsection, we provide information on the nodal domain of the sign-changing solution.

Proposition 6.3.14. *Let* (H^N), ($H^N_{f,g}$) *and* (h^N_6) *be satisfied. Then, any sign-changing weak solution of problem* (N)*, which is also a minimizer of* $I|_{\mathcal{N}_0}$*, has exactly two nodal domains.*

Proof. Let w_0 be such that $I(w_0) = \inf_{u \in \mathcal{N}_0} I(u)$, fix any $\widetilde{w_0}$ representative of w_0 and set

$$\Omega_\pm = \{x \in \Omega : \pm\widetilde{w_0}(x) > 0\}.$$

As $w_0 \mathbb{1}_{\Omega_\pm} = \pm\widetilde{w_0}^\pm$ a.e. in Ω, it follows that Ω_+ and Ω_- satisfy conditions (i) and (ii) of Definition 6.3.2. By contradiction, we prove that they are also minimal. We assume, without loss of generality, that there exist Borelian subsets A_1, A_2 of Ω, with $A_1 \cap A_2 = \emptyset$, $|A_1| > 0$ and $|A_2| > 0$, such that $\Omega_- = A_1 \dot\cup A_2$ and A_1 satisfies (i) and (ii) of Definition 6.3.2. Moreover, it holds

$$\begin{aligned} w_0 \mathbb{1}_{A_2} &= \widetilde{w_0} \mathbb{1}_{A_2} < 0 \quad \text{a.e. in } A_2,\\ w_0 \mathbb{1}_{A_2} &= w_0 \mathbb{1}_{\Omega_-} - w_0 \mathbb{1}_{A_1} \in W^{1,\mathcal{H}}(\Omega), \end{aligned}$$

thus A_2 also satisfies (i) and (ii). Summarizing, we have

$$\mathbb{1}_{\Omega_+} w_0 \geq 0, \quad \mathbb{1}_{A_1} w_0 \leq 0, \quad \mathbb{1}_{A_2} w_0 \leq 0 \quad \text{a.e. in } \Omega, \tag{6.3.21}$$

and

$$w_0 = \mathbb{1}_{\Omega_+} w_0 + \mathbb{1}_{A_1} w_0 + \mathbb{1}_{A_2} w_0 \quad \text{a.e. in } \Omega.$$

Setting $y_1 = \mathbb{1}_{\Omega_+} w_0 + \mathbb{1}_{A_1} w_0$ and $y_2 = \mathbb{1}_{A_2} w_0$, from (6.3.21) we have $y_1^+ = \mathbb{1}_{\Omega_+} w_0$ and $-y_1^- = \mathbb{1}_{A_1} w_0$. Since $I'(w_0) = 0$ and as the supports of $y_1^+, -y_1^-$ and y_2 do not overlap, one has

$$0 = \langle I'(w_0), y_1^+\rangle = \langle I'(y_1^+), y_1^+\rangle.$$

Hence $y_1^+ \in \mathcal{N}$ and analogously, $-y_1^- \in \mathcal{N}$. Therefore, $y_1 \in \mathcal{N}_0$. With the same argument one can show that $\langle I'(y_2), y_2\rangle = 0$. Then, from these properties, we obtain

$$\begin{aligned} I(y_2) &= I(y_2) - \frac{1}{q_+}\langle I'(y_2), y_2\rangle \\ &\geq \left(\frac{1}{p_+} - \frac{1}{q_+}\right)\rho_{p(\cdot)}(\nabla y_2) + \left(\frac{1}{p_+} - \frac{1}{q_+}\right)\rho_{p(\cdot)}(y_2) + \left(\frac{1}{p_+} - \frac{1}{q_+}\right)\rho_{p(\cdot),\partial\Omega}(y_2) \\ &\quad + \int_\Omega \left(\frac{1}{q_+} f(x, y_2) y_2 - F(x, y_2)\right) \mathrm{d}x + \int_{\partial\Omega} \left(\frac{1}{q_+} g(x, y_2) y_2 - G(x, y_2)\right) \mathrm{d}\sigma, \end{aligned}$$

which leads to

$$I(y_2) > 0,$$

because of $p_+ < q_+$, $y_2 \neq 0$ and (h^N_6). Finally, we get

$$\inf_{u \in \mathcal{N}_0} I(u) = I(w_0) = I(y_1) + I(y_2) > I(y_1) \geq \inf_{u \in \mathcal{N}_0} I(u),$$

which is a contradiction and this completes the proof. □

Finally, we are able to prove the second main result.

Proof of Theorem 6.3.3. Combining Theorem 6.3.1 and Proposition 6.3.14, we get the existence of three bounded weak solutions of problem (N) such that one is nonnegative, one is nonpositive and one is sign-changing with two nodal domains.

List of Symbols

a.e.	almost everywhere
a.a.	almost all
w.r.t.	with respect to
$C(\overline{\Omega})$	space of continuoua functions in $\overline{\Omega}$
$C^1_C(\Omega)$	space of C^1-functions with compact support
$C^\infty_C(\Omega)$	space of C^∞-functions with compact support
$C^{0,\alpha}(\overline{\Omega})$	space of Hölder continuous functions with exponent α
$C^{0,\frac{1}{\lvert \log t\rvert}}(\overline{\Omega})$	set of log-Hölder continuous functions
$C_+(\overline{\Omega})$	set of continuous functions in $\overline{\Omega}$ that are greater or equal than 1
$M(\Omega)$	space of measurable functions in Ω
$M_+(\Omega)$	set of measurable functions in Ω that are greater or equal than 1
X^*	dual space of X
$\langle \cdot, \cdot \rangle$	duality pairing between a space X and its dual X^*
$L^r(\Omega)$	Lebesgue space with constant exponent r
$L^r(\partial\Omega)$	Lebesgue space on the boundary with constant exponent r
$W^{1,r}(\Omega)$	Sobolev space with constant exponent r
$W^{1,r}_0(\Omega)$	completion of $C^\infty_C(\Omega)$ in $W^{1,r}(\Omega)$
$L^{r(\cdot)}(\Omega)$	Lebesgue space with variable exponent $r(\cdot)$
$L^{r(\cdot)}_\omega(\Omega)$	weighted Lebesgue space with variable exponent $r(\cdot)$ and weight $\omega(\cdot)$
$L^\varphi(\Omega)$	Musielak-Orlicz space
$W^{1,r(\cdot)}(\Omega)$	Sobolev space with variable exponent $r(\cdot)$
$W^{1,r(\cdot)}_0(\Omega)$	completion of $C^\infty_C(\Omega)$ in $W^{1,r(\cdot)}(\Omega)$
$W^{1,\varphi}(\Omega)$	Musielak-Orlicz Sobolev space
$W^{1,\varphi}_0(\Omega)$	completion of $C^\infty_0(\Omega)$ in $W^{1,\varphi}(\Omega)$
$\lVert \cdot \rVert_r$	usual norm in $L^r(\Omega)$
$\lVert \cdot \rVert_{r,\partial\Omega}$	usual norm in $L^r(\partial\Omega)$
$\lVert \cdot \rVert_{1,r}$	usual norm in $W^{1,r}(\Omega)$
$\lVert \cdot \rVert_{1,r,0}$	equivalent norm in $W^{1,r}_0(\Omega)$
$\lVert \cdot \rVert_{r(\cdot)}$	usual norm in $L^{r(\cdot)}(\Omega)$
$\lVert \cdot \rVert_{r(\cdot),\omega}$	seminorm in $L^{r(\cdot)}_\omega(\Omega)$
$\lVert \cdot \rVert_{1,r(\cdot)}$	usual norm in $W^{1,r(\cdot)}(\Omega)$
$\lVert \cdot \rVert_{1,r(\cdot),0}$	equivalent norm in $W^{1,r(\cdot)}_0(\Omega)$
$\lVert \cdot \rVert_\varphi$	Luxemburg norm (usual norm) in $L^\varphi(\Omega)$
$\lVert \cdot \rVert_{1,\varphi}$	usual norm in $W^{1,\varphi}(\Omega)$
$\lVert \cdot \rVert_{1,\mathcal{H},0}$	equivalent norm in $W^{1,\mathcal{H}}_0(\Omega)$
$\rho_{r(\cdot)}$	modular function related to $r(\cdot)$
$\rho_{r(\cdot),\omega}$	modular function related to $r(\cdot)$ with weight $\omega(\cdot)$
ρ_φ	modular function related to the Φ-function φ

r'	conjugate exponent of r
$r'(\cdot)$	conjugate exponent of $r(\cdot)$
r^*	critical Sobolev exponent of r
r_*	critical Sobolev exponent of r on the boundary
$r^*(\cdot)$	critical Sobolev exponent of $r(\cdot)$
$r_*(\cdot)$	critical Sobolev exponent of $r(\cdot)$ on the boundary
r_-	essential inf of $r(\cdot)$ in Ω
r_+	essential sup of $r(\cdot)$ in Ω
$\Phi(\Omega)$	set of all generalized Φ-functions on Ω
$\phi \prec \psi$	ϕ weaker than ψ
$N(\Omega)$	set of all generalized N-functions on Ω
$\varphi \ll \psi$	φ increases essentially slower than ψ near infinity
φ^*	conjugate function of the Φ-function φ
∂_s	partial derivative with respect to the variable s
$\mathbb{1}_A$	indicator or characteristic function

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List of publications

This thesis represents the collection of the following author's papers:

1. E. Amoroso, G. Bonanno, G. D'Aguì, P. Winkert, *Two solutions for Dirichlet double phase problems with variable exponents*, Advanced Nonlinear Studies (2024), https://doi.org/10.1515/ans-2023-0134

2. E. Amoroso, V. Morabito, *Nonlinear Robin problem with double phase variable exponent operator*, Discrete and Continuous Dynamical Systems Series S (2024), doi: 10.3934/dcdss.2024047.

3. E. Amoroso, Á. Crespo-Blanco, P. Pucci, P. Winkert, *Superlinear elliptic equations with unbalanced growth and nonlinear boundary condition*, submitted.

In addition, there is a list of other author's works on the study of existence and multiplicity of solutions for different type of problems through critical point theory:

1. E. Amoroso, G. Bonanno, G. D'Aguì, S. De Caro, S. Foti, D. O'Regan, A. Testa, *Second order differential equations for the power converters dynamical performance analysis*, Mathematical Methods in the Applied Sciences, **45 (9)** (2022), pp. 5573–5591.

2. E. Amoroso, P. Candito, G. D'Aguì, *Two positive solutions for a nonlinear Robin problem involving the discrete p-Laplacian*, Dolomites Res. Notes Approx., **15 (5)** (2022), pp. 1–7.

3. E. Amoroso, G. Bonanno, G. D'Aguì, S. Foti, *Multiple solutions for nonlinear Sturm-Liouville differential equations with possibly negative variable coefficients*, Nonlinear Analysis: Real World Applications, **69** (2023), 15 pp.

4. E. Amoroso, P. Candito, J. Mawhin, *Existence of a priori bounded solutions for discrete two-point boundary value problems*, Journal of Mathematical Analysis and Applications, **519 (2)** (2023), 18 pp.

5. E. Amoroso, G. Bonanno, K. Perera, *Nonlinear elliptic p-Laplacian equations in the whole space*, Nonlinear Analysis, **23** (2023), 113364.

6. E. Amoroso, A. Sciammetta, P. Winkert, *Anisotropic $(\vec{p},\vec{q})$-Laplacian problems with superlinear nonlinearities*, Communications in Analysis and Mechanics, **16(1)** (2024), 1–23.

7. E. Amoroso, G. D'Aguì, V. Morabito, *On a complete parametric Sturm-Liouville problem with sign changing coefficients*, AIMS Mathematics, **9(3)** (2024), 6499–6512.

8. E. Amoroso, C. Colaiacomo, G. D'Aguì, P. Vergallo, *A second order Hamiltonian neural model*, Applied Mathematics Letter (2024), https://doi: 10.1016/j.aml.2024.109295.