# NONHOMOGENEOUS DEGENERATE QUASILINEAR PROBLEMS WITH CONVECTION

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The aim of this paper is to study the following nonlinear elliptic problem

$$
(1) \qquad \begin{cases} -div(\nu(x,u)|\nabla u|^{p-2}\nabla u) = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}
$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ and a Lipschitz boundary $\partial\Omega$. The main novelty is that the equation is driven by the degenerate $p$-Laplacian $div(\nu(x,u)|\nabla u|^{p-2}\nabla u)$, for $p \in (1,+\infty)$, with a weight generating degeneracy which in addition depends on the solution $u$. The reaction term in the above equation depends on the solution $u$ and on its gradient $\nabla u$, which is often called a convection term.

The structure that we admit for the weight $\nu$ entering problem (1) is of the form

$$
(2) \qquad \nu(x,t) = a(x)g(|t|) \text{ for a.e } x \in \Omega \text{ and for all } t \in \mathbb{R},
$$

with a positive function $a \in L^1_{loc}(\Omega)$ and a continuous function $g : [0,+\infty) \to [a_0,+\infty)$, with $a_0 > 0$. In particular, $\nu : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., $\nu(x,t)$ is measurable in $x$ for any fixed $t$ and is continuous in $t$ for any fixed $x$). The leading operator of the equation in (1) is degenerate because the ellipticity property can be lost since the part $a(x)$ in the decomposition (2) can approach zero or be unbounded. The part $g(|u|)$ of the weight $\nu(x,t)$ can be unbounded from above.

We say a few words to motivate our approach based on truncation of the unbounded weight $\nu(x,t)$. The fact that the weight $\nu(x,t)$ is unbounded from above in $t$ with no control of the growth for $g(t)$ prevents to handle the problem directly. Furthermore, there is no available function setting in which the equation can be fitted. In order to overcome this difficulty, we truncate the function $g(t)$, thus reducing the weighted problem to a manageable case for an operator theoretic treatment.

We seek the solutions to problem (1) in $W_0^{1,p}(a,\Omega)$ which is the closure of $C_c^\infty(\Omega)$ in the Banach space $W^{1,p}(a,\Omega)$ consisting of all elements $u \in L^p(\Omega)$ with $a(x)|\nabla u(x)|^p$ integrable on $\Omega$. Notice that the degeneracy part $a \in L^1_{loc}(\Omega)$ is incorporated in the definition of the underlying space. In order to handle problem (1) we assume the following hypothesis from [DrabekKufnerNicolosi(1997), page 26]:

(H1)

$$
a^{-s} \in L^1(\Omega) \quad \text{for some } s \in \left(\frac{N}{p},+\infty\right) \cap \left[\frac{1}{p-1},+\infty\right).
$$

In Section 1 it will be recalled that owing to condition (H1) there are the compact embeddings

$$
(3) \qquad W_0^{1,p_s}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega),
$$

with $1 \leq r < p_s^*$ for $p_s^* = Np_s/(N-p_s)$ if $N > p_s$ and $p_s^* = +\infty$ otherwise, where

$$
(4) \qquad p_s = \frac{ps}{s+1}.
$$

The embeddings in (3) will be used to manage the nonlinearity $f(x,u,\nabla u)$.

The right-hand side $f(x,u,\nabla u)$ of the equation in (1) is determined by a Carathéodory function $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ meaning that $f(\cdot,t,\xi)$ is measurable on $\Omega$ for each $(t,\xi)\in\mathbb{R}\times\mathbb{R}^N$ and $f(x,\cdot,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}^N$ for a.e. $x\in\Omega$. Next we identify the growth condition that $f(x,u,\nabla u)$ should satisfy.

For the sake of simplicity, we make the notational convention that for any real number $r>1$ we denote $r':=r/(r-1)$ (the Hölder conjugate of $r$).

We assume that the nonlinearity $f(x,t,\xi)$ satisfies the hypotheses:

(H2) There exist constants $c_1\ge 0$, $c_2\ge 0$, $c_3\ge 0$, and $\alpha\in(p,p_s^*)$ such that

$$|f(x,t,\xi)|\le c_1a(x)^{\frac{1}{\alpha'}}|\xi|^{\frac{p}{\alpha'}}+c_2|t|^{\alpha-1}+c_3 \text{ a.e. } x\in\Omega,\ \forall t\in\mathbb{R},\ \forall\xi\in\mathbb{R}^N.$$

(H3) There exist constants $d_1\ge 0$ and $d_2\ge 0$ with $d_1+\lambda_1^{-1}d_2<a_0$, and a function $\sigma\in L^1(\Omega)$ such that

$$f(x,t,\xi)t\le d_1a(x)|\xi|^p+d_2|t|^p+\sigma(x) \text{ for a.e. } x\in\Omega,\ \forall t\in\mathbb{R},\ \forall\xi\in\mathbb{R}^N,$$

where $\lambda_1$ denotes the first eigenvalue of the (negative) degenerated $p$-Laplacian $-\Delta_p^a$ : $W_0^{1,p}(a,\Omega)\to W_0^{1,p}(a,\Omega)^*$ with the weight $a\in L^1_{\mathrm{loc}}(\Omega)$ (see (6)-(7) below).

**Remark 0.1.** *Condition (H2) cannot be formulated without condition (H1) that provides the key number $p_s^*$ associated with $s$. The interaction of hypothesis (H3) with the function $g$ given in (2) occurs only through the lower bound $a_0$ of $g$.*

A solution to problem (1) will be understood in the following weak sense.

**Definition 0.2.** *We say that $u\in W_0^{1,p}(a,\Omega)$ is a weak solution to problem (1) if*

$$\int_\Omega \nu(x,u)|\nabla u|^{p-2}\nabla u\nabla v dx=\int_\Omega f(x,u,\nabla u)v dx \tag{5}$$

*for all $v\in W_0^{1,p}(a,\Omega)$.*

Under hypotheses (H1) and (H2) the integral in the right-hand side of (5) exists.

An essential step in our approach is to show that the set of solutions is uniformly bounded. This property is much stronger than the boundedness of each solution with a constant depending on the solution itself as given for unweighted problems in [MarinoWinkert(2019)] (see also [MarinoMotreanu(2020)]). In Theorem 2.2 below we develop a special Moser iteration adapted to our degenerate setting with convection.

Next we focus on the existence of solutions to problem (1) in the sense of Definition 0.2. Our existence result on problem (1) is formulated as follows.

**Theorem 0.3.** *Assume that the weight $\nu:\Omega\times\mathbb{R}\to\mathbb{R}$ has the structure in (2) with a positive $a\in L^1_{loc}(\Omega)$ satisfying the condition (H1) and a continuous function $g:[0,+\infty)\to[a_0,+\infty)$ with $a_0>0$. Assume also that the Carathéodory function $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ satisfies the conditions (H2) and (H3). Then Problem (1) possesses at least a bounded weak solution $u\in W_0^{1,p}(a,\Omega)$ in the sense of Definition 0.2.*

The proof of Theorem 0.3 relies on the truncation of the weight $\nu(x,u)$ described in (2) in order to drop the unboundedness from above of the part $g(|u|)$. Specifically, we consider for any $R>0$ an auxiliary problem obtained by truncating the weight $\nu(x,u)$ in $u$ from above $R$. We will be able to apply to the truncated problem the main theorem for pseudomonotone operators. As a final step, we take advantage of the boundedness of the solution set for (1) seeing that the solution of the auxiliary problem is actually a solution of the original problem (1) provided $R>0$ is sufficiently large. By a detailed example we illustrate the applicability of our main result.

There are very few papers focusing on problems of type (1). The general functional setting related to the degenerate $p$-Laplacian is presented in the monograph [DrabekKufnerNicolosi(1997)] (see also [Motreanu(2018)] for equations involving the unweighted $p$-Laplacian and convection terms). The particular case of problem (1) for which $g(|u|) \equiv 1$ (i.e., the weight does not depend on the solution $u$) was the object of [MotreanuTornatore(2021)] by using a completely different approach, namely, the method of sub-supersolution. The nondegenerate problem corresponding to (1) was investigated in [AvernaMotreanuTornatore(2016)] by means of the surjectivity theorem for pseudomonotone operators. Recently, the papers [Motreanu(2021), MotreanuNashed(2021)] deal with degenerate $(p, q)$-Laplacian problems but where the weights do not depend on the solution $u$. Until now the results regarding equation (1) driven by the degenerate $p$-Laplacian with a weight depending on the solution $u$ has been considered only when $f(x, u, \nabla u) = h(x, u)$ (i.e., without gradient dependence) and with a growth up to order $p-1$, that is $|h(x, u)| \leq c(1 + |u|^{p-1})$ for a constant $c > 0$ (refer, e.g., to [DrabekKufnerNicolosi(1997), Theorem 3.5]).

The rest of the paper consists of the following material. Section 1 comprises the necessary background regarding the weighted Sobolev spaces, degenerate $p$-Laplacian and pseudomonotone operators. Section 2 shows that (1) admits only bounded solutions. Sections 3 and 4 study the truncated problem. Section 5 contains the proof of our main result and the example.

## 1. Preliminaries

We start by recalling some facts about weighted Sobolev spaces and degenerate $p$-Laplacian that will be used subsequently. For more insight we refer to [DrabekKufnerNicolosi(1997)]. Fix a real number $p \in (1, +\infty)$, a positive function $a \in L^1_{\mathrm{loc}}(\Omega)$ and a bounded domain $\Omega \subset \mathbb{R}^N$ of Lebesgue measure $|\Omega|$, with $N \geq 3$ and a Lipschitz boundary $\partial\Omega$. We consider the weighted space

$$W^{1,p}(a, \Omega) := \{u \in L^p(\Omega) \cap W^{1,1}_{\mathrm{loc}}(\Omega) : \int_\Omega a(x)|\nabla u(x)|^p dx < \infty\}$$

which is a Banach space endowed with the norm

$$\|u\|_{W^{1,p}(a,\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall u \in W^{1,p}(a, \Omega).$$

Notice that $C^\infty_c(\Omega) \subset W^{1,p}(a, \Omega)$. The space $W^{1,p}_0(a, \Omega)$ is defined as the closure of $C^\infty_c(\Omega)$ in $W^{1,p}(a, \Omega)$ with respect to the norm $\| \cdot \|_{W^{1,p}(a,\Omega)}$. Denote by $W^{1,p}_0(a, \Omega)^*$ the dual space of $W^{1,p}_0(a, \Omega)$.

The (negative) degenerate $p$-Laplacian with the weight $a \in L^1_{\mathrm{loc}}(\Omega)$ is the nonlinear operator $-\Delta^a_p : W^{1,p}_0(a, \Omega) \to W^{1,p}_0(a, \Omega)^*$ defined by

$$\langle -\Delta^a_p(u), v\rangle := \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v dx, \ \forall u, v \in W^{1,p}_0(a, \Omega). \tag{6}$$

The definition is meaningful since by Hölder's inequality it holds

$$\begin{aligned} &\left|\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx\right| \\ &\leq \left(\int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega a(x)|\nabla v(x)|^p dx\right)^{\frac{1}{p}} < +\infty \end{aligned}$$

for all $u, v \in W^{1,p}(a, \Omega)$. The mapping $-\Delta^a_p : W^{1,p}_0(a, \Omega) \to W^{1,p}_0(a, \Omega)^*$ in (6) is continuous and bounded (in the sense that it maps bounded sets into bounded sets). The first eigenvalue of $-\Delta^a_p$ is given by

$$\lambda_1 := \inf_{u \in W^{1,p}_0(a,\Omega),\ u \neq 0} \frac{\int_\Omega a(x)|\nabla u|^p dx}{\int_\Omega |u|^p dx} \tag{7}$$

(refer to [DrabekKufnerNicolosi(1997), Lemma 3.1]).

An important embedding theorem for the space $W_0^{1,p}(a,\Omega)$ holds under hypothesis (H1). To this end, with $s$ in hypothesis (H1) we set $p_s$ as in (4). Let us observe from (4) that $p_s \geq 1$ because according to assumption (H1) we have $s \geq 1/(p-1)$. It is also seen from (4) that $p_s < p$ and $p_s/(p-p_s) = s$. Hence through Hölder's inequality and hypothesis (H1) we find the estimate

$$\int_\Omega |\nabla u(x)|^{p_s} dx = \int_\Omega (a(x)^{\frac{p_s}{p}} |\nabla u(x)|^{p_s}) a(x)^{-\frac{p_s}{p}} dx$$

$$\leq \left(\int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{p_s}{p}} \left(\int_\Omega a(x)^{-\frac{p_s}{p-p_s}} dx\right)^{\frac{p-p_s}{p}} \leq \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s+1}} \|u\|^{p_s}$$

for all $u \in W^{1,p}(a,\Omega)$. This establishes the continuous embedding of $W_0^{1,p}(a,\Omega)$ into the classical (unweighted) Sobolev space $W_0^{1,p_s}(\Omega)$, thus

$$W_0^{1,p}(a,\Omega) \hookrightarrow W_0^{1,p_s}(\Omega). \tag{8}$$

We invoke the Rellich-Kondrachov embedding theorem that guarantees the compact embedding (3) with $1 \leq r < p_s^*$ for the critical exponent $p_s^*$ (corresponding to $p_s$) given by

$$p_s^* := \begin{cases} \frac{Np_s}{N-p_s} & \text{if } N > p_s, \\ \\ +\infty & \text{if } N \leq p_s. \end{cases}$$

The assumption $s > N/p$ as postulated in condition (H1) results in $p_s^* > p$. Consequently, there holds the compact embedding

$$W_0^{1,p_s}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega).$$

Combining with (8) yields the compact embedding

$$W_0^{1,p}(a,\Omega) \hookrightarrow\hookrightarrow L^p(\Omega). \tag{9}$$

In particular, (9) ensures that

$$\|u\| := \left(\int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall u \in W_0^{1,p}(a,\Omega), \tag{10}$$

is an equivalent norm on $W_0^{1,p}(a,\Omega)$. Throughout the rest of the paper, the space $W_0^{1,p}(a,\Omega)$ will be endowed with the norm $\|\cdot\|$.

The space $W_0^{1,p}(a,\Omega)$ is a uniformly convex Banach space. Indeed, from assumption (H1) it is known that $a^{-s} \in L^1(\Omega)$ with $s \geq 1/(p-1)$. This gives $a^{-\frac{1}{p-1}} \in L^1(\Omega)$ according to

$$\int_\Omega a(x)^{-\frac{1}{p-1}} dx = \int_{\{a(x)<1\}} a(x)^{-\frac{1}{p-1}} dx + \int_{\{a(x)\geq 1\}} a(x)^{-\frac{1}{p-1}} dx$$
$$\leq \int_\Omega a(x)^{-s} dx + |\Omega| < \infty.$$

Applying [DrabekKufnerNicolosi(1997), Theorem 1.3]) renders that $W_0^{1,p}(a,\Omega)$ is uniformly convex, so reflexive.

We mention a few things about the pseudomonotone operators. For more developments we refer to [CarlLeMotreanu(2007), Chapter 2]. Let $X$ be a Banach space with the norm $\|\cdot\|$ and its dual $X^*$. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between $X$ and $X^*$, by $\to$ the strong convergence and by $\rightharpoonup$ the weak convergence. A map $A : X \to X^*$ is called bounded if it maps bounded sets into bounded sets. The map $A : X \to X^*$ is said to be coercive if

$$\lim_{\|u\|\to+\infty} \frac{\langle A(u), u\rangle}{\|u\|} = +\infty.$$

The map $A : X \to X^*$ is called pseudomonotone if for each sequence $\{u_n\} \subset X$ satisfying $u_n \rightharpoonup u$ in $X$ and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0$, it holds

$$\langle A(v), u - v\rangle \le \liminf_{n\to\infty} \langle A(u_n), u_n - v\rangle \quad \text{for all } v \in X.$$

The main theorem for pseudomonotone operators reads as follows (see, e.g., [CarlLeMotreanu(2007), Theorem 2.99]).

**Theorem 1.1.** *Let $X$ be a reflexive Banach space. If the mapping $A : X \to X^*$ is pseudomonotone, bounded and coercive, then it is surjective.*

## 2. Bounded solutions

We start with the estimate of the solution set in $W_0^{1,p}(a, \Omega)$.

**Lemma 2.1.** *Under assumptions (H1) and (H3), the set of solutions to problem (1) is bounded in $W_0^{1,p}(a, \Omega)$ with a bound depending on $g$ only through its lower bound $a_0$.*

*Proof.* Acting on (5) with the test function $v = u \in W_0^{1,p}(a, \Omega)$ results in

$$\int_\Omega \nu(x, u)|\nabla u|^{p-2}\nabla u\nabla u dx = \int_\Omega f(x, u, \nabla u)u dx.$$

Hypothesis (H3), in conjunction with (2), (7) and (10), ensures that

$$a_0\|u\|^p \le (d_1 + d_2\lambda_1^{-1})\|u\|^p + \|\sigma\|_{L^1(\Omega)}.$$

Taking into account that $d_1 + d_2\lambda_1^{-1} < a_0$, the conclusion follows.□

□

Our result on bounded solutions to problem (1) establishes that under the imposed hypotheses the solution set of problem (1) is uniformly bounded.

**Theorem 2.2.** *Assume that conditions (H1), (H2) and (H3) are fulfilled. Then there exists a constant $C > 0$ such that for each weak solution $u \in W_0^{1,p}(a, \Omega)$ to problem (1) it holds the uniform estimate $\|u\|_{L^\infty(\Omega)} \le C$. The constant $C$ depends on $g$ only through its lower bound $a_0$.*

*Proof.* Let $u \in W_0^{1,p}(a, \Omega)$ be a weak solution to problem (1). We can write $u = u^+ - u^-$, where $u^+ = \max\{u, 0\}$ (the positive part of $u$) and $u^- = \max\{-u, 0\}$ (the negative part of $u$). We have to show that $u^+$ and $u^-$ are both uniformly bounded by a constant independent of $u$. We only provide the proof for $u^+$ because in the case of $u^-$ one can argue similarly.

Our first goal is to prove that

$$u^+ \in L^r(\Omega), \quad \forall r \in [1, +\infty). \tag{11}$$

To this end we insert in (5) the test function $v = u^+u_h^{kp} \in W_0^{1,p}(a, \Omega)$, where $u_h := \min\{u^+, h\}$ with arbitrary constants $h > 0$ and $k > 0$, thus obtaining

$$\int_\Omega \nu(x, u)|\nabla u|^{p-2}\nabla u\nabla(u^+u_h^{kp})dx = \int_\Omega f(x, u, \nabla u)u^+u_h^{kp}dx. \tag{12}$$

By means of (2), the left-hand side of (12) can be estimated from below as

$$\begin{aligned}(13)\qquad &\int_\Omega \nu(x,u)|\nabla u|^{p-2}\nabla u\nabla(u^+u_h^{kp})dx\\ &=\int_\Omega a(x)g(|u|)|\nabla u|^{p-2}\nabla u(u_h^{kp}\nabla(u^+)+kpu^+u_h^{kp-1}\nabla(u_h))dx\\ &\geq a_0\left[\int_\Omega a(x)u_h^{kp}|\nabla(u^+)|^p dx+kp\int_{\{0<u<h\}}a(x)u_h^{kp}|\nabla(u^+)|^p dx\right].\end{aligned}$$

Through Young's inequality, for any $\varepsilon>0$ and a constant $c(\varepsilon)>0$ we get

$$\begin{aligned}(14)\qquad &\int_\Omega a(x)^{\frac{1}{\alpha'}}|\nabla u|^{\frac{p}{\alpha'}}u_h^{kp}u^+dx=\int_\Omega (a(x)^{\frac{1}{\alpha'}}|\nabla u|^{\frac{p}{\alpha'}}u_h^{\frac{kp}{\alpha'}})(u_h^{\frac{kp}{\alpha}}u^+)dx\\ &\leq \varepsilon\int_\Omega a(x)u_h^{kp}|\nabla(u^+)|^p dx+c(\varepsilon)\int_\Omega u_h^{kp}(u^+)^\alpha dx.\end{aligned}$$

Let us note that

$$\int_\Omega u_h^{kp}(u^+)dx\leq\int_\Omega u_h^{kp}(u^+)^\alpha dx+|\Omega|.$$

Then, if $\varepsilon>0$ is sufficiently small, we derive by (12), (13), (14) and hypothesis (H2) the bound

$$\begin{aligned}(15)\qquad &\int_\Omega a(x)u_h^{kp}|\nabla(u^+)|^p dx+kp\int_{\{0<u<h\}}a(x)u_h^{kp}|\nabla(u^+)|^p dx\\ &\leq b_0\left(\int_\Omega u_h^{kp}(u^+)^\alpha dx+1\right),\end{aligned}$$

with a constant $b_0>0$. It also holds

$$\begin{aligned}(16)\qquad &\int_\Omega a(x)u_h^{kp}|\nabla(u^+)|^p dx+kp\int_{\{0<u<h\}}a(x)u_h^{kp}|\nabla(u^+)|^p dx\\ &=\frac{1}{(k+1)^p}\int_{\{u>h\}}a(x)u_h^{kp}|\nabla(u^+)|^p dx+\frac{kp+1}{(k+1)^p}\int_{\{0<u<h\}}a(x)|\nabla(u_h^{k+1})|^p dx\\ &\geq\frac{kp+1}{(k+1)^p}\int_\Omega a(x)|\nabla(u_h^k u^+)|^p dx.\end{aligned}$$

As a result, (15) and (16) imply

$$(17)\qquad \frac{kp+1}{(k+1)^p}\|u_h^k u^+\|^p\leq b_0\left(\int_\Omega u_h^{kp}(u^+)^\alpha dx+1\right).$$

Hölder's inequality using $\alpha>p$ and Sobolev embedding theorem jointly with (8) lead to

$$\begin{aligned}&\int_\Omega u_h^{kp}(u^+)^\alpha dx\leq M\int_{\{(u^+)^{\alpha-p}\leq M\}}(u_h^k u^+)^p dx\\ &+\left(\int_{\{(u^+)^{\alpha-p}>M\}}(u^+)^\alpha dx\right)^{\frac{\alpha-p}{\alpha}}\left(\int_\Omega (u_h^k u^+)^\alpha dx\right)^{\frac{p}{\alpha}}\\ &\leq M\|u_h^k u^+\|_{L^p(\Omega)}^p+C\|u_h^k u^+\|^p\left(\int_{\{(u^+)^{p_s^*-p}>M\}}(u^+)^\alpha dx\right)^{\frac{\alpha-p}{\alpha}}\end{aligned}$$

for any $M > 0$, with a constant $C > 0$ independent of $M$. Notice that

$$\lim_{M\to+\infty}\int_{\{(u^+)^{p_s^*-p}>M\}}(u^+)^\alpha dx = 0.$$

Therefore, choosing $M > 0$ large enough, (17) and the Sobolev embedding theorem entail

$$\|u_h^k u^+\|_{L^{p_s^*}(\Omega)} \le b_1(k+1)\left(\|u_h^k u^+\|^p_{L^p(\Omega)}+1\right)^{\frac{1}{p}}, \tag{18}$$

with a constant $b_1 = b_1(k, u) > 0$ depending on $k$ and $u$ (note that $u \in W_0^{1,p}(a, \Omega)$ is fixed).

Now assertion (11) results from (18) by means of a bootstrap argument. Namely, let $k_1 > 0$ be given by $(k_1+1)p = p_s^*$. It follows from (18) that $u_h^{k_1}u^+ \in L^{p_s^*}(\Omega)$ because $u \in L^{p_s^*}(\Omega)$. Inductively, setting $(k_n+1)p = (k_{n-1}+1)p_s^*$, we construct a sequence $k_n \to +\infty$ with $u_h^{k_n}u^+ \in L^{p_s^*}(\Omega)$. Letting $h \to +\infty$, Fatou's lemma ensures $u^+ \in L^{(k_n+1)p_s^*}(\Omega)$ for all $n$, which establishes (11).

Next, on the basis of (11), we prove that there exists a constant $K > 0$ such that

$$\|u^+\|_{L^r(\Omega)} \le K, \quad \forall r \ge 1, \tag{19}$$

with $K$ independent of the solution $u$ of (1). In this connection, we rely on Lemma 2.1. More precisely, we return to (17) and fix $q \in (p, \alpha)$ satisfying

$$\frac{(\alpha-p)q}{q-p} < p_s^*, \tag{20}$$

which is possible because $\alpha < p_s^*$ according to (H2). Due to (11), we are able to use Hölder's inequality getting for any $k > 0$ the estimate

$$\begin{aligned}\int_\Omega u_h^{kp}(u^+)^\alpha dx &= \int_\Omega (u^+)^{\alpha-p}(u_h^k u^+)^p dx\\ &\le \left(\int_\Omega (u^+)^{\frac{(\alpha-p)q}{q-p}}dx\right)^{\frac{q-p}{q}}\left(\int_\Omega (u_h^k u^+)^q dx\right)^{\frac{p}{q}} \le C\|u_h^k u^+\|^p_{L^q(\Omega)},\end{aligned}$$

where $C > 0$ is a constant that does not depend on the solution $u$ of (1). The independence of $C$ with respect to the solution $u$ is a consequence of Lemma 2.1 and the continuous embedding $W_0^{1,p}(a,\Omega) \hookrightarrow L^{\frac{(\alpha-p)q}{q-p}}(\Omega)$ that follows from (8) and (20). Again from Lemma 2.1 we see that the obtained constant $C$ des not depend on $g$ except for its lower bound $a_0$.

Inserting the preceding inequality into (17) and taking into account the integrability of any power of $u^+$ according to (11), by Fatou's lemma it turns out

$$\|u^+\|_{L^{(k+1)p_s^*}(\Omega)} \le b^{\frac{1}{k+1}}(k+1)^{\frac{1}{k+1}}\left(\|(u^+)^{k+1}\|^p_{L^q(\Omega)}+1\right)^{\frac{1}{(k+1)p}}, \tag{21}$$

with a constant $b > 0$ independent of the solution $u$ and whose dependence with respect to $g$ occurs through its lower bound $a_0$. Since the sequence $(k+1)^{\frac{1}{\sqrt{k+1}}}$ is bounded, (21) gives rise to a constant $c > 0$ such that

$$\|u^+\|_{L^{(k+1)p_s^*}(\Omega)} \le b^{\frac{1}{k+1}}c^{\frac{1}{\sqrt{k+1}}}\left(\|(u^+)^{k+1}\|^p_{L^q(\Omega)}+1\right)^{\frac{1}{(k+1)p}}. \tag{22}$$

Without loss of generality we may suppose that $\|(u^+)^{k+1}\|^p_{L^q(\Omega)} > 1$ (otherwise (19) follows). Accordingly, (22) amounts to saying that

$$\|u^+\|_{L^{(k+1)p_s^*}(\Omega)} \le C^{\frac{1}{\sqrt{k+1}}}\|u^+\|_{L^{(k+1)q}(\Omega)} \tag{23}$$

with a constant $C > 0$ independent of $k$ and of the solution $u$ and for which the dependence on $g$ reduces to the dependence on $a_0$. At this point, we again implement the Moser iteration with $(k_n+1)q = (k_{n-1}+1)p_s^*$ posing $(k_1+1)q = p_s^*$. Then (23) renders

$$\|u^+\|_{L^{(k_n+1)p_s^*}(\Omega)} \le C^{\sum_{1\le i\le n}\frac{1}{\sqrt{k_i+1}}}\|u^+\|_{L^{(k_1+1)q}(\Omega)}, \quad \forall n \ge 1. \tag{24}$$

Letting $n \to \infty$ in (24) provides (19) because the series converges and $k_n \to +\infty$ as $n \to \infty$ (note $p_s^* > q$).

Finally, passing to the limit as $r \to \infty$ in (19), the uniform boundedness of the solution set of (1) is achieved. A careful reading of the proof shows that the dependence of the uniform bound $C$ on $g$ arises just through the lower bound $a_0$ of $g$. This completes the proof. □ □

**Remark 2.3.** *In comparison to the results in* [MarinoMotreanu(2020), MarinoWinkert(2019)] *addressing bounded solutions for problems with full dependence on the gradient of the solution, we emphasize that Theorem 2.2 establishes that the whole solution set of problem (1) is (uniformly) bounded in $L^\infty(\Omega)$, which is an improvement even for unweighted problems. In line with this, we can prove that every solution of (1) belongs to $L^\infty(\Omega)$ under a growth condition weaker than (H2), namely, there exist constants $a_1 \ge 0$, $a_2 \ge 0$, and $a_3 \ge 0$ such that*

$$|f(x,t,\xi)| \le a_1 a(x)^{\frac{p_s^*-1}{p_s^*}}|\xi|^{\frac{p(p_s^*-1)}{p_s^*}} + a_2|t|^{p_s^*-1} + a_3 \text{ for a.e. } x \in \Omega,\ \forall t \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

*We also remark that in* [DrabekKufnerNicolosi(1997), Theorem 3.5] *it is proven the uniform boundedness for the solution set of an equation of type (1) with a right-hand side of the form $h(x,u)$ (so, independent of the gradient $\nabla u$) and exhibiting a growth of order $p-1$ in $u$.*

## 3. Truncated weight and associated operator

For any number $R > 0$ we consider the following truncation of the weight $\nu(x,u)$ in problem (1):

$$\nu_R(x,t) = a(x)g_R(|t|), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where

$$g_R(t) = \begin{cases} g(t) & \text{if} \quad t \in [0,R] \\ g(R) & \text{if} \quad t > R \end{cases} \tag{25}$$

The next proposition focuses on the properties of the degenerate $p$-Laplacian associated to the truncated weight $\nu_R(x,u)$.

**Proposition 3.1.** *Given $R > 0$, let $\mathcal{A}_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ be defined by*

$$\langle \mathcal{A}_R(u), v\rangle = \int_\Omega \nu_R(x,u)|\nabla u|^{p-2}\nabla u \nabla v dx,\ \forall u,v \in W_0^{1,p}(a,\Omega). \tag{26}$$

*Then the following assertions hold:*

(i) *$\mathcal{A}_R$ is well defined and bounded;*

(ii) *$\mathcal{A}_R$ has the $S_+$ property, that is, any sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and*

$$\limsup_{n\to\infty}\langle \mathcal{A}_R(u_n), u_n - u\rangle \le 0 \tag{27}$$

*satisfies $u_n \to u$ in $W_0^{1,p}(a,\Omega)$.*

(iii) *$\mathcal{A}_R$ is continuous.*

*Proof.* (i) By (2), (25), the continuity of $g$ and Hölder's inequality, we have

$$
\begin{aligned}
&\left|\int_\Omega \nu_R(x,u)|\nabla u|^{p-2}\nabla u\nabla v dx\right| \leq \int_\Omega \nu_R(x,u)|\nabla u|^{p-1}|\nabla v| dx \\
=\ & \int_{\{|u|\leq R\}} a(x)g(|u(x)|)|\nabla u|^{p-1}|\nabla v|dx + \int_{\{|u|>R\}} g(R)a(x)|\nabla u|^{p-1}|\nabla v|dx \\
\leq\ & \max_{t\in[0,R]} g(t)\int_\Omega a(x)|\nabla u|^{p-1}|\nabla v|dx \\
\leq\ & \max_{t\in[0,R]} g(t)\left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega a(x)|\nabla v|^p dx\right)^{\frac{1}{p}} \\
=\ & \max_{t\in[0,R]} g(t)\|u\|^{p-1}\|v\|.
\end{aligned}
$$

for all $u,v\in W_0^{1,p}(a,\Omega)$. The operator $\mathcal{A}_R$ in (26) is thus well defined and bounded.

(ii) Let $\{u_n\}\subset W_0^{1,p}(a,\Omega)$ be a sequence as required. First, we note by Hölder's inequality that

$$
\begin{aligned}
&\int_\Omega g_R(|u_n|)a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n-u)dx \\
&\geq a_0\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n-u)dx \\
&\geq a_0\int_\Omega a(x)|\nabla u_n|^p dx + a_0\int_\Omega a(x)|\nabla u|^p dx \\
&-a_0\int_\Omega (a(x)^{\frac{p-1}{p}}|\nabla u_n|^{\frac{p-1}{p}})(a(x)^{\frac{1}{p}}|\nabla u|)dx \\
&-a_0\int_\Omega (a(x)^{\frac{p-1}{p}}|\nabla u|^{\frac{p-1}{p}})(a(x)^{\frac{1}{p}}|\nabla u_n|)dx \\
&\geq a_0\|u_n\|^p + a_0\|u\|^p - a_0\left(\int_\Omega a(x)|\nabla u_n|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{1}{p}} \\
&-a_0\left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega a(x)|\nabla u_n|^p dx\right)^{\frac{1}{p}} \\
&= a_0(\|u_n\|^p + \|u\|^p - \|u_n\|^{p-1}\|u\| - \|u\|^{p-1}\|u_n\|) \\
&= a_0(\|u_n\| - \|u\|)(\|u_n\|^{p-1} - \|u\|^{p-1}).
\end{aligned}
$$

By (26), (10) and the preceding inequality, we obtain

$$\begin{aligned}
(28)\qquad & \langle \mathcal{A}_R(u_n) - \mathcal{A}_R(u), u_n - u\rangle \\
= \; & \int_\Omega g_R(|u_n|)a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n - u)dx \\
+ \; & \int_\Omega (g_R(|u_n|) - g_R(|u|))a(x)|\nabla u|^{p-2}\nabla u \nabla(u_n - u)dx \\
\geq \; & a_0(\|u_n\| - \|u\|)(\|u_n\|^{p-1} - \|u\|^{p-1}) \\
+ \; & \int_\Omega (g_R(|u_n|) - g_R(|u|))a(x)|\nabla u|^{p-2}\nabla u \nabla(u_n - u)dx.
\end{aligned}$$

The assumptions $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and (27) imply

$$(29)\qquad \limsup_{n\to\infty} \langle \mathcal{A}_R(u_n) - \mathcal{A}_R(u), u_n - u\rangle \leq 0.$$

We also have

$$(30)\qquad \lim_{n\to\infty} \int_\Omega (g_R(|u_n|) - g_R(|u|))a(x)|\nabla u|^{p-2}\nabla u \nabla(u_n - u)dx = 0.$$

Indeed, through Hölder's inequality and the boundedness of $\{u_n\}$ in $W_0^{1,p}(a,\Omega)$ (note that $u_n \rightharpoonup u$), there is a constant $C > 0$ such that

$$\begin{aligned}
& \left| \int_\Omega (g_R(|u_n|) - g_R(|u|))a(x)|\nabla u|^{p-2}\nabla u \nabla(u_n - u)dx \right| \\
\leq \; & C \left( \int_\Omega |g_R(|u_n|) - g_R(|u|)|^{\frac{p}{p-1}} a(x)|\nabla u|^p dx \right)^{\frac{p-1}{p}}.
\end{aligned}$$

Then (30) is achieved by applying Lebesgue's dominated convergence theorem on the basis of the continuity of $g$ and the strong convergence $u_n \to u$ in $L^p(\Omega)$.

Combining (28), (29) and (30) we are led to $\lim_{n\to+\infty} \|u_n\| = \|u\|$. The space $W_0^{1,p}(a,\Omega)$ is uniformly convex (see Section 1), whence $u_n \to u$ in $W_0^{1,p}(a,\Omega)$.

(iii) Let $u_n \to u$ in $W_0^{1,p}(a,\Omega)$. Proceeding as in (28), we infer that

$$\begin{aligned}
& |\langle \mathcal{A}_R(u_n) - \mathcal{A}_R(u), v\rangle| \\
& \leq \left| \int_\Omega g_R(|u_n|)a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla v dx \right| \\
& + \left| \int_\Omega (g_R(|u_n|) - g_R(|u|))a(x)|\nabla u|^{p-2}\nabla u \nabla v dx \right| \\
& \leq \max_{t\in[0,R]} g(t) \left( \int_\Omega a(x) \left| |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \|v\| \\
& + \left( \int_\Omega |g_R(|u_n|) - g_R(|u|)|^{\frac{p}{p-1}} a(x)|\nabla u|^p dx \right)^{\frac{p-1}{p}} \|v\|
\end{aligned}$$

for every $v \in W_0^{1,p}(a,\Omega)$. Arguing as in part (ii), we arrive at $\mathcal{A}_R(u_n) \to \mathcal{A}_R(u)$ in $W_0^{1,p}(a,\Omega)^*$, which establishes the desired conclusion. $\square$

## 4. Operator formulation for the auxiliary problem

Corresponding to the truncation in (25), we state the auxiliary problem

$$\begin{cases} -\mathrm{div}(\nu_R(x,u)|\nabla u|^{p-2}\nabla u) = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{31}$$

Our approach to study problem (31) is based on the theory of pseudomonotone operators. In this respect, we introduce a Nemytskii-type operator by

$$\langle \mathcal{N}(u), v\rangle = \int_\Omega f(x,u(x),\nabla u(x))v(x)dx \text{ for all } u,v \in W_0^{1,p}(a,\Omega). \tag{32}$$

**Lemma 4.1.** *Assume that conditions (H1) and (H2) hold. Then the map $\mathcal{N} : W_0^{1,p}(a,\Omega) \to L^{\alpha'}(\Omega)$ in (32) is well defined, continuous and bounded.*

*Proof.* Assumption (H2) in conjunction with (32) yields

$$|\langle \mathcal{N}(u), v\rangle| \le \int_\Omega (c_1 a(x)^{\frac{1}{\alpha'}}|\nabla u|^{\frac{p}{\alpha'}}|v| + c_2|u|^{\alpha-1}|v| + c_3|v|)dx, \ \forall u,v \in W_0^{1,p}(a,\Omega).$$

Using Hölder's inequality and (10), one obtains the estimate

$$|\langle \mathcal{N}(u), v\rangle| \le c_1\|u\|^{\frac{p}{\alpha'}}\|v\|_{L^\alpha(\Omega)} + c_2\|u\|^{\alpha-1}_{L^\alpha(\Omega)}\|v\|_{L^\alpha(\Omega)} + c_3\|v\|_{L^1(\Omega)} \tag{33}$$

for all $u,v \in W_0^{1,p}(a,\Omega)$. Hence $\mathcal{N}(u) \in L^{\alpha'}(\Omega)$ whenever $u \in W_0^{1,p}(a,\Omega)$, due to the density of $W_0^{1,p}(a,\Omega)$ in $L^\alpha(\Omega)$. Moreover, it is clear from (33) that $\mathcal{N} : W_0^{1,p}(a,\Omega) \to L^{\alpha'}(\Omega)$ is bounded.

Consider the Banach space

$$L^p(a,\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega a(x)|u(x)|^p dx < \infty\}.$$

It is seen from (32) that the mapping $\mathcal{N}$ can be expressed as the composition $\mathcal{N} = \mathcal{N}_a \circ \Gamma_a$, where $\Gamma_a : u \in W_0^{1,p}(a,\Omega) \mapsto (u,\nabla u) \in L^\alpha(\Omega) \times L^p(a,\Omega)^N$ and $\mathcal{N}_a : (v,(v_1,...,v_N)) \in L^\alpha(\Omega) \times L^p(a,\Omega)^N \mapsto f(x,v,v_1,...,v_N) \in L^{\alpha'}(\Omega)$. The continuity of the mapping $\Gamma_a$ readily follows from the definition of the space $W_0^{1,p}(a,\Omega)$. The continuity of the mapping $\mathcal{N}_a$ is the direct consequence of Krasnoselskii's theorem for weighted spaces (refer to [DrabekKufnerNicolosi(1997), Theorem 1.1] applied for the weights therein $w_0 = 1$ and $w_1 = \cdots = w_N = a$). Therefore the mapping $\mathcal{N}$ is continuous. □

Corresponding to an $R > 0$, we introduce the mapping $A_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ as

$$A_R = \mathcal{A}_R - \mathcal{N}, \tag{34}$$

with the operators $\mathcal{A}_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ in (26) and $\mathcal{N} : W_0^{1,p}(a,\Omega) \to L^{\alpha'}(\Omega)$ in (32). The mapping $A_R$ is well defined because $L^{\alpha'}(\Omega) \subset W_0^{1,p}(a,\Omega)^*$.

**Remark 4.2.** *One has that $u \in W_0^{1,p}(a,\Omega)$ is a (weak) solution to problem (31) if and only if it solves the equation $A_R(u) = 0$ with $A_R$ given in (34).*

**Proposition 4.3.** *Assume that the weight $\nu : \Omega \times \mathbb{R} \to \mathbb{R}$ has the structure in (2) with a positive $a \in L^1_{loc}(\Omega)$ satisfying condition (H1) and a continuous function $g : [0,+\infty) \to [a_0,+\infty)$ with $a_0 > 0$. If $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying condition (H2), then the operator $A_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ in (34) is bounded and pseudomonotone for every $R > 0$.*

*Proof.* By Proposition 3.1 and Lemma 4.1 it is known that the mapping $A_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ is bounded.

Let us show that $A_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ is a pseudomonotone operator. To this end, let $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and

$$\limsup_{n\to\infty} \langle A_R(u_n), u_n - u\rangle \le 0. \tag{35}$$

There holds

$$\lim_{n\to\infty} \langle \mathcal{N}(u_n), u_n - u\rangle = 0 \tag{36}$$

as can be noticed from Lemma 4.1 since

$$|\langle \mathcal{N}(u_n), u_n - u\rangle| \le \|\mathcal{N}(u_n)\|_{L^{\alpha'}(\Omega)} \|u_n - u\|_{L^\alpha(\Omega)}$$

and $u_n \to u$ in $L^\alpha(\Omega)$ (refer to the compact embedding of $W_0^{1,p}(a,\Omega)$ into $L^\alpha(\Omega)$ and that $\mathcal{N}(u_n)$ is bounded in $L^{\alpha'}(\Omega)$).

On the basis of (34) and (36), we note that (35) reduces to (27). We are thus enabled to apply Proposition 3.1 (ii) obtaing the strong convergence $u_n \to u$ in $W_0^{1,p}(a,\Omega)$. In view of the continuity of the maps $\mathcal{A}_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ and $\mathcal{N} : W_0^{1,p}(a,\Omega) \to L^{\alpha'}(\Omega)$ for which we address to Proposition 3.1 (iii) and Lemma 4.1, we infer that $A_R(u_n) \to A_R(u)$ in $W_0^{1,p}(a,\Omega)^*$ and $\langle A_R(u_n), u_n\rangle \to \langle A_R(u), u\rangle$, thus $A_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ is pseudomonotone. □ □

We turn our attention to show that the operator $A_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ in (34) is coercive.

**Proposition 4.4.** *Assume the weight $\nu : \Omega \times \mathbb{R} \to \mathbb{R}$ is as in Proposition 4.3 and that the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ fulfills the condition (H3). Then the operator $A_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ is coercive for every $R > 0$, which reads as*

$$\lim_{\|u\|\to\infty} \frac{\langle A_R(u), u\rangle}{\|u\|} = +\infty. \tag{37}$$

*Proof.* The proof is carried out by making use of hypothesis (H3) that implies

$$\begin{aligned}
\langle A_R(u), u\rangle &= \int_\Omega \nu_R(x,u)|\nabla u|^p dx - \int_\Omega f(x,u,\nabla u)u\,dx \\
&\ge a_0 \int_\Omega a(x)|\nabla u|^p dx - \int_\Omega (d_1 a(x)|\nabla u|^p + d_2|u|^p + \sigma(x))dx \\
&\ge (a_0 - d_1 - d_2\lambda_1^{-1})\|u\|^p - \|\sigma\|_{L^1(\Omega)},\ \forall u \in W_0^{1,p}(a,\Omega).
\end{aligned}$$

In the last inequality we have used the variational characterization (7) of the first eigenvalue $\lambda_1$ of $-\Delta_p^a$ as well as the definition of the norm in (10). Since $p > 1$ and $d_1 + \lambda_1^{-1} d_2 < a_0$, we infer that (37) holds true. □

Now we are able to prove the solvability of auxiliary problem (31).

**Theorem 4.5.** *Assume that the weight $\nu : \Omega \times \mathbb{R} \to \mathbb{R}$ has the structure in (2) with a positive $a \in L^1_{loc}(\Omega)$ satisfying the condition (H1) and a continuous function $g : [0,+\infty) \to [a_0,+\infty)$ with $a_0 > 0$. If $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the conditions (H2) and (H3), then problem (31) has a weak solution $u_R \in W_0^{1,p}(a,\Omega)$ for every $R > 0$.*

*Proof.* We are going to apply Theorem 1.1 to the operator $A_R : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ in (34) with any fixed $R > 0$. Proposition 4.3 entails that this operator is bounded and pseudomonotone, whereas by Proposition 4.4 it is coercive. Therefore it is allowed to apply Theorem 1.1, which provides a solution $u_R \in W_0^{1,p}(a,\Omega)$ for the operator equation $A_R(u_R) = 0$. Invoking Remark 4.2, $u_R$ represents a weak solution to equation (31). The proof is complete. □

## 5. Proof of Theorem 0.3 and example

**Proof of Theorem 0.3.** Theorem 2.2 ensures that the entire set of solutions of problem (1) is uniformly bounded, that is, there exists a constant $C > 0$ such that $\|u\|_{L^\infty(\Omega)} \le C$ for all weak solutions $u \in W_0^{1,p}(a,\Omega)$ to problem (1). The truncated problem (31) satisfies exactly the same hypotheses, and with the same coefficients, as the original problem (1) with $g_R$ in place of $g$. It is essential to note that the inequality $d_1 + \lambda_1^{-1} d_2 < a_0$ assumed in hypothesis (H3) is independent of $R > 0$. Consequently, Theorem 2.2 applies to the truncated problem (31) involving the truncation $g_R$ and produces the same uniform bound $C > 0$ for the solution set of (31) with any $R > 0$. Actually, the statements of Theorem 2.2 and Lemma 2.1 show that the uniform bound $C > 0$ for the solution set depends on the function $g$ only through the lower bound $a_0$ of $g$, which is the same for each truncation $g_R$ as seen from (25). In particular, we have that the solution $u_R \in W^{1,p}(a,\Omega)$ to problem (31) provided by Theorem 4.5 satisfies the estimate $\|u_R\|_{L^\infty(\Omega)} \le C$ whenever $R > 0$.

Now choose $R \ge C$. Then the estimate $\|u_R\|_{L^\infty(\Omega)} \le C$ and (25) imply

$$g_R(|u_R(x)|) = g(|u_R(x)|) \quad \text{for all } x \in \Omega,$$

hence due to (2),

$$\nu_R(x, u_R(x)) = \nu(x, u_R(x)) \quad \text{for all } x \in \Omega.$$

It follows that the solution $u_R \in W^{1,p}(a,\Omega)$ to the auxiliary problem (31) is a bounded weak solution to the original problem (1), which completes the proof of Theorem 0.3. □

We illustrate by an example the applicability of Theorem 0.3.

**Example 5.1.** Consider on the unit open ball $B = \{x \in \mathbb{R}^3 : |x| < 1\}$ the Dirichlet problem:

$$\left\{ \begin{array}{ll} -div\left(\frac{a_0}{|x|^{\frac{1}{2}}} e^{|u|} \frac{1}{|\nabla u|^{\frac{1}{3}}} \nabla u\right) = \frac{c_1}{|x|^{\frac{1}{5}}} \frac{u}{1+u^2} |\nabla u|^{\frac{5}{6}} + c|u|^{\frac{1}{2}} + c_0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{array} \right. \tag{38}$$

with constants $a_0 > 0$, $c_1 \ge 0$, $c \ge 0$, and $c_0 \ge 0$, satisfying $a_0 > c_1 + \lambda_1^{-1}(c + c_0)$, where $\lambda_1$ stands for the first eigenvalue of $-\Delta_p^a$ with $p = 5/3$ and $a(x) = 1/|x|^{\frac{1}{2}} \in L^1(B)$. Our goal is to apply Theorem 0.3 with $g(t) = a_0 e^t$ for $t \ge 0$ and

$$f(x,t,\xi) = \frac{c_1}{|x|^{\frac{1}{5}}} \frac{t}{1+t^2} |\xi|^{\frac{5}{6}} + c|t|^{\frac{1}{2}} + c_0$$

for $(x,t,\xi) \in (B \setminus \{0\}) \times \mathbb{R} \times \mathbb{R}^3$. To this end, we choose

$$s = 3 \in \left(\frac{N}{p}, +\infty\right) \cap \left[\frac{1}{p-1}, +\infty\right) = \left[\frac{3}{2}, +\infty\right).$$

Since $a(x)^{-s} = |x|^{\frac{3}{2}} \in L^\infty(B)$, it follows that condition (H1) is verified.

We find that

$$p_s = \frac{ps}{s+1} = \frac{5}{4} \text{ and } p_s^* = \frac{Np_s}{N - p_s} = \frac{15}{7}.$$

Hence it is allowed to take $\alpha = \alpha' = 2 \in (\frac{5}{3}, \frac{15}{7})$. Then, noting that $|x|^{\frac{1}{4}} \le |x|^{\frac{1}{5}}$ in $B$, we get

$$|f(x,t,\xi)| \le \frac{c_1}{|x|^{\frac{1}{4}}}|\xi|^{\frac{5}{6}} + c(|t|+1) + c_0 = c_1 a(x)^{\frac{1}{\alpha'}}|\xi|^{\frac{p}{\alpha'}} + c_2|t|^{\alpha-1} + c_3$$

for $(x,t,\xi) \in (B \setminus \{0\}) \times \mathbb{R} \times \mathbb{R}^3$, with $c_2 = c$ and $c_3 = c + c_0$. Hypothesis (H2) is thus fulfilled. In the same way, we infer that

$$f(x,t,\xi)t \le \frac{c_1}{|x|^{\frac{1}{2}}}(|\xi|^{\frac{5}{3}}+1) + c(|t|^{\frac{5}{3}}+1) + c_0(|t|^{\frac{5}{3}}+1) = d_1 a(x)|\xi|^p + d_2|t|^p + \sigma(x)$$

for $(x,t,\xi) \in (B \setminus \{0\}) \times \mathbb{R} \times \mathbb{R}^3$, with $d_1 = c_1$, $d_2 = c + c_0$, and $\sigma(x) = \frac{c_1}{|x|^{\frac{1}{2}}} + c + c_0 \in L^1(B)$. Thanks to the assumption $a_0 > c_1 + \lambda_1^{-1}(c + c_0)$, hypothesis (H3) is satisfied, too. We are thus in a position to apply Theorem 0.3 to problem (38) ensuring the existence of a bounded weak solution.

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