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Some advancements on homogeneous spaces, star-covering properties and selection principles

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in

Mathematics and Computational Sciences

Declaration of Authorship

I, Fortunato MAESANO, declare that this thesis titled, "Some advancements on homogeneous spaces, star-covering properties and selection principles" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

"A morte ogni uom su questa terra è sacro;
O presto o tardi ella s'aggiugne: e come

Uom può meglio morir, che osando impavido
Disfidar tanti orribili perigli
Pel cenere de' padri, e per i templi
De' patrii Numi

e per la dolce e tenera
Madre che in culla il carezzava al sonno,
E per la sposa che nutrisce al seno
Soavemente il caro figliuoletto [...] ?"

Canti di Roma antica, T. B. Macaulay

UNIVERSITY OF PALERMO

Abstract

Department of Mathematics and Computer Sciences

Doctor of Philosophy

Some advancements on homogeneous spaces, star-covering properties and selection principles

by Fortunato MAESANO

The aim of this work is to furnish results for the following three fields of research in general topology.

(i) The classes of n-Hausdorff m-homogeneous and n-Urysohn m-homogeneous spaces are studied. It is shown that, for every $n > 2$, there is no n-Hausdorff m-homogeneous space, $m > 1$, and there exists a 3-Urysohn homogeneous space which is not Urysohn. New upper bounds for the cardinality of n-Hausdorff homogenous and n-Urysohn homogenoeous spaces are given and for any n-Hausdorff space it is constructed an n-Hausdorff homogeneous extension which is the union of countably many n-H-closed spaces.

(ii) Some recent properties defined by relative versions of star-covering properties are considered and it is proved that some of them are in fact characterizations of the original property or, surprisingly, of known properties. Also answers some questions posed in [Bal and Kočinac, 2020] and [Kočinac and Singh, 2020] are given. Additionally some recent relative versions of Menger-type property, called set strongly star Menger and set star Menger properties, and the corresponding Hurewicz-type properties are considered, some examples distinguishing these properties are given. It is "easily" proved that the set strong star Menger and set strong star Hurewicz properties lie between countable compactness and the property of having countable extent. A consistent example of a set star Menger (set star Hurewicz) space which is not set strongly star Menger (set strongly star Hurewicz) is given, moreover, answering some questions posed in [Kočinac, Konca, and Singh, 2022] and [Singh, 2021], it is proved that the product of a set star Menger (set star Hurewicz) space with a compact space need not be set star Menger (set star Hurewicz).

(iii) The behaviour of selective separability properties in the class of Fréchet-Urysohn spaces is studied. Two examples of mH-separable Fréchet-Urysohn (hence R-separable) spaces which are not H-separable are contructed, the first one given in ZFC is a countable Hausdorff space; the second, assuming ($\mathfrak{p} = \mathfrak{c}$), is a 0-dimensional and α_4 space. By a result of Barman and Dow, under PFA, the product of two countable Fréchet-Urysohn spaces is M-separable. It is shown that the hypothesis of PFA cannot be replaced by MA. In the last section it is proved that in the Laver model, the product of any two H-separable countable spaces is mH-separable.

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BASIC NOTIONS

In this preliminary part several introductive notions are recalled. Undefined notions can be found in [Blass, 2010], [Engelking, 1989], [Hodel, 1984] and [Jech, 2003], where basic concepts can be further explored.

The acronyme "ZFC" will abbreviate "ZF+AC", the theory "ZF" obtained by the following statement, known as *Zermelo-Fraenkel axioms*:

(0) *Existence*: "$\exists x(x = x)$"

(I) *Extensionality*: "$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to x = y]$"

(II) *Comprehension scheme*: "$\forall x \forall p \forall y[\forall u(u \in y \leftrightarrow (u \in x \wedge \varphi(u, p)))]$" for a given formula φ.

(III) *Pairing*: "$\forall x \forall y \exists z(x \in z \wedge y \in z)$"

(IV) *Union*: "$\forall \mathcal{F} \exists U \forall Y \forall x[(x \in Y \wedge Y \in \mathcal{F}) \to x \in U]$"

(V) *Power set*: "$\forall X \exists P \forall z[(\forall y(y \in z \to y \in X)) \to z \in P]$"

(VI) *Infinity*: "$\exists x[\forall z(z = \emptyset \to z \in x) \wedge \forall y \in x \forall z(z = y \cup \{y\} \to z \in x)]$"

(VII) *Replacement scheme*: "$\forall A \forall p[\forall x \in A \exists! y \varphi(x, y, A, p) \to \exists Y \forall x \in A \exists y \in Y \varphi(x, y, A, p)]$" for a given formula φ.

(VIII) *Foundation*: "$\forall x[\exists y(y \in x) \to \exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y))]$"

with the addition of the *Axiom of Choice* , which states that every family $\mathcal{F}$ of disjoint nonempty sets admits a set that intersects every its element in one point; in formula

(AC) "$\forall \mathcal{F}[\forall x \in \mathcal{F}(x \neq \emptyset) \wedge \forall x \in \mathcal{F} \forall y \in \mathcal{F}(x = y \vee x \cap y = \emptyset) \to \exists S \forall x \in \mathcal{F} \exists! z(z \in S \wedge z \in x)]$"

The greek letters λ, κ will always denote cardinal numbers and κ^+ indicates the smallest cardinal greater than κ; the letters $\alpha, \beta, \gamma, \delta$ denote ordinal numbers and $\alpha + 1 = \alpha \cup \{\alpha\}$ will denote the successor ordinal of α. The symbol $\aleph_0$ denotes the smallest infinite cardinal and $\aleph_1$ the first uncountable cardinal; the symbols ω and ω_1 represent the first infinite ordinal and the first uncountable ordinal respectively. Of course $\aleph_1 = (\aleph_0)^+$, and it is possible to define $\aleph_{n+1} = (\aleph_n)^+$ for all $n \in \omega$. The letters n, m will denote integers, whose set is denoted by $\mathbb{N}$, which will often be identified with the elements of ω; for this reason the set of all finite sequences of integers will be denoted by ${}^{<\omega}\omega$.

Let A be a set. By $|A|$ it is denoted the *cardinality* of A, i.e., the least cardinal isomorphic to A; moreover if κ is a cardinal, $[A]^{\leq\kappa}$ is the collection of all subsets of A of cardinality $\leq \kappa$. Similarly, the sets $[A]^{<\kappa}$, $[A]^{\geq\kappa}$, $[A]^{>\kappa}$ are defined, while $[A]^{\kappa} = \{B : B \subseteq A \wedge |A| = \kappa\}$.

The cardinality of the set of real numbers is $2^{\aleph_0}$, referred as *continuum*, and it will be denoted with $\mathfrak{c}$.

Given a cardinal κ, the *cofinality* of λ is the cardinal $cf(\kappa) = min\{\lambda : \exists A \in [\kappa]^{\lambda} : sup(A) = \bigcup A = \kappa\}$; of course $cf(\kappa) \leq \kappa$ and if $cf(\kappa) = \kappa$, the cardinal κ is said to be *regular*, otherwise *singular*.

For two sets $A, B \in [\omega]^{\omega}$, the symbology $A \subset^* B$ stands for A to be *almost contained* in B, and means $|B \setminus A| < \aleph_0$.

The set of all functions from ω to ω (equivalently, of all infinite sequences of integers) will be denoted by ω^{ω}. Recall that for $f, g \in \omega^{\omega}$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n (and $f \leq g$ means that $f(n) \leq g(n)$ for all $n \in \omega$). A subset $B \subseteq \omega^{\omega}$ is *bounded* if there is $g \in \omega^{\omega}$ such that $f \leq^* g$ for every $f \in B$. A subset $D \subseteq \omega^{\omega}$ is *cofinal* if for each $g \in \omega^{\omega}$ there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of an unbounded subset of ω^{ω} is denoted by $\mathfrak{b}$, and the minimal cardinality of a cofinal subset of ω^{ω} is denoted by $\mathfrak{d}$. The value of $\mathfrak{d}$ does not change if one considers the relation $\leq$ instead of $\leq^*$ [van Douwen, 1984, Theorem 3.6].

A family $\mathcal{F} \subseteq [\omega]^{\omega}$ is said to have the *strong intersection property*, briefly SFIP, if every finite subfamily of it has infinite intersection. A *pseudointersection* of a family $\mathcal{F} \subseteq [\omega]^{\omega}$ is an infinite set whose all but finite elements are contained in every member of $\mathcal{F}$. The smallest cardinality of any $\mathcal{F} \subseteq [\omega]^{\omega}$ with SFIP but with no pseudointersection is denoted with $\mathfrak{p}$.

The inequality $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$ holds in ZFC.

With *"continuum hypothesis"*, briefly "CH", it is meant the following fact:

"Every infinite set of reals is either countable or in bijective correspondence with all the reals"

The latter statement is tantamount to the equality $\aleph_1 = \mathfrak{c}$ (and therefore $\aleph_1 = \mathfrak{p} = \mathfrak{b} = \mathfrak{d} = \mathfrak{c}$). It is also possible to define the *"generalized continuum hypothesis"*, briefly "GCH", which is the statement

$$"\kappa^+ = 2^{\kappa}"$$

In particular this leads to the equality "$\aleph_{n+1} = 2^{\aleph_n}$".

The symbol X will denote a non-empty topological space (X, τ), where τ is the topology on X; its elements will be referred as *points* of the space.

The following properties are classicaly known in literature as *separation axioms* of topological spaces. A topological space X is said to be

- a T_1 space if for every pair of distinct points $x, y \in X$ there exist open subsets U, V of X such that $x \in U \setminus V$ and $y \in V \setminus U$.
- an *Hausdorff* space or T_2 space if for every pair of distinct points $x, y \in X$ there exist open disjoint subsets U, V of X such that $x \in U$ and $y \in V$.
- an *Urysohn* space or $T_{2,\frac{1}{2}}$ space if for every pair of distinct points $x, y \in X$ there exist open disjoint subsets U, V of X such that $x \in \overline{U}$ and $y \in \overline{V}$.
- a *regular* space or T_3 space if it is a T_1 space and for every closed set $C \subseteq X$ and every point $x \in X \setminus C$ there exist open disjoint subsets U, V of X such that $x \in U$ and $C \subseteq V$.
- a *Tychonoff* space or $T_{3,\frac{1}{2}}$ space if it is a T_1 space and for every closed set $C \subseteq X$ and every point $x \in X \setminus C$ there exist a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(C) = 1$.
- a *normal space* or T_4 space if it is a T_1 space and for every pair of disjoint closed sets $C, D \subseteq X$ there exist open disjoint subsets U, V of X such that $C \subseteq U$ and $D \subseteq V$.
- a *hereditarily normal space* or T_5 space if it is a T_1 space and every its subset is normal.
- a *perfectly normal space* or T_6 space if it is a T_4 space and every its closed subset is the intersection of countably many open sets.

Unless specified, none of these axioms is assumed a priori.

A space X is said to be *collectionwise Hausdorff* provided that for every closed and discrete subspace D of X there exists a disjoint family $\{O_a : a \in D\}$ of open neighbourhoods of points $a \in D$. Of course every collctionwise Hausdorff space is Hausdorff.

A family $\mathcal{U}$ of open sets is a *cover* for a space X if $\bigcup \mathcal{U} = X$; a set $\mathcal{V} \subseteq \mathcal{U}$ is a *subcover* of $\mathcal{U}$ if $\bigcup \mathcal{V} = X$. An open cover $\mathcal{U}$ is

- a *γ-cover* if it is infinite and the set $\{U \in \mathcal{U} : x \in U\}$ is finite for all $x \in X$;
- an *ω-cover* if it is infinite and every finite subset of X is contained in an element of $\mathcal{U}$.

Every γ-cover is an ω-cover. The class of all covers (resp. γ-covers, ω-covers) of a topological space will be denoted by $\mathcal{O}$ (resp., by Γ and by Ω).

The following properties are known in literature as *covering properties* of topological spaces. A set is *countable* when it is finite or has cardinality $\aleph_0$. A space X is said to be:

- *compact*, briefly C, if every its open cover admits a finite subcover.
- *H-closed* if it is Hausdorff and every its open cover admits a finite subfamily whose union is a dense set; as an equivalent definition, it is an Hausdorff space closed in every Hausdorff space in which it is contained.
- *countably compact*, briefly CC, if every its countable open cover admits a finite subcover.

- *Lindelöf*, briefly *L*, if every its open cover admits a countable subcover.

Of course every compact space is also Lindelöf and countably compact, and a space is compact if and only if it is countably compact and Lindelöf.

A space X is said to be *sequential* if a subset of it is closed if and only if contains all the limits of the sequences contained in it. A space X is said to be *Fréchet-Urysohn*, briefly FU, if for every $A \subset X$ and $x \in \overline{A} \setminus A$ there exists a sequence $S \subset A \setminus \{x\}$ converging to x.

Given a space X and an element $x \in X$, a family $\mathcal{V} \subseteq \tau$ is a *local π-base for x* if for every open neighbourhood U of x there exists $V \in \mathcal{V}$ such that $V \subseteq U$; if moreover $x \in V$ for all $V \in \mathcal{V}$, the family $\mathcal{V}$ is said to be a *local base for x* . A space X said to satisfy the first countability axiom, or to be a *first-countable* space if every point admits a countable local base.

A family $\mathcal{B}$ of open sets is a *base* if for every point $x \in X$ and open set U with $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$; a *π-base* is a family $\mathcal{C}$ such that for every open set V there exists $C \in \mathcal{C}$ with $C \subseteq V$. A space X is said to satisfy the second countability axiom, or to be a *second-countable* space if it admits a countable base.

Of course every second countable space is first-countable, every first-countable space is Fréchet-Urysohn, and every Fréchet-Urysohn space is sequential.

A space is said to be *zero-dimensional* if it admits a base of clopen sets. Every zero-dimensional space is Tychonoff.

A space X is said to be *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.

A *cardinal function* is any function f assigning to every topological space X a cardinal number $f(X)$ in a way so, if X and Y are homeomorphic spaces, then $f(X) = f(Y)$. Let X be a space. The following cardinal functions are considered multiplied by ω.

The *π-character* of a point $x \in X$ is the smallest cardinality $\pi\chi(x, X)$ of local a π-base for x, and the *character* is the smallest cardinality $\chi(x, X)$ of a local base for x; then of course $\chi(x, X) \leq \pi\chi(x, X)$. The *$\pi$-character* of a space X is the caridinal $\pi\chi(X) = sup\{\pi\chi(x, X) : x \in X\}$, and the *character* of X is $\chi(X) = sup\{\chi(x, X) : x \in X\}$; then $\pi\chi(X) \leq \chi(X)$. A space X is first-countable if and only if $\chi(X) = \aleph_0$.

The *π-weight* of a space X, denoted by $\pi w(X)$, is the least cardinality of a π-base of X.

The *density* of a space X, denoted by $d(X)$, is the smallest cardinality of a dense subset of X . If $d(X) = \aleph_0$, the space is said to be *separable*. Notice that $d(X)\pi\chi(X) = \pi w(X)$.

The *cellularity* of a space X is the least cardinal $c(X)$ such that every family of pairwise disjoint open sets has cardinality lower or equal to $c(X)$. If $c(X) = \aleph_0$, the space is said to have the *countable chain condition* property, or, briefly, to be a *c.c.c.*

space.

The *extent* of a space X is the least cardinal $e(X)$ such that every closed discrete subset has cardinality lower or equal to $e(X)$.

The *Lindelöf degree* of a space X is the least cardinal $L(X)$ such that every open cover admits a subcover of cardinality $L(X)$. A space X is Lindelöf if and only if $L(X) = \aleph_0$.

The following properties are known in literature as *selection principles* for topological spaces. The notation below was used in [Scheepers, 1996] and [Just, Miller, Scheepers, and Szeptycki, 1996]. Let X be a space and $\mathcal{A}$, $\mathcal{B}$ be collections of families of its subsets. Then:

- $S_1(\mathcal{A}, \mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle U_n : n \in \omega \rangle$ such that for each $n \in \omega$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \omega\} \in \mathcal{B}$.

- $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each $n \in \omega$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\bigcup_{n\in\omega} \mathcal{V}_n \in \mathcal{B}$.

- $U_{fin}(\mathcal{A}, \mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each $n \in \omega$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\bigcup \mathcal{V}_n : n \in \omega\} \in \mathcal{B}$.

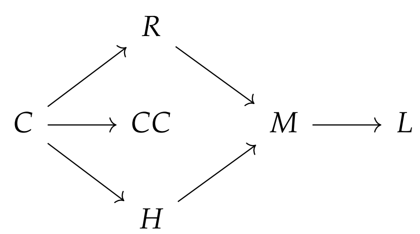
Selection principles were introudced to list a scheme properties of topological spaces which work on an "input", often sequences of topological objects, and give an "output", another topological object, selected in a way that depends on the principle. Some selection principle define known properties: a space X is said to be:

- *Rothberger*, briefly R, if satisfies $S_1(\mathcal{O}, \mathcal{O})$, i.e. , for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers there exists a sequence $\langle U_n : n \in \omega \rangle$ such that $U_n \in \mathcal{U}_n$, $n \in \omega$, and $\{U_n : n \in \omega\} \in \mathcal{O}$ (see [Rothberger, 1938, Definition 1]).

- *Menger*, briefly M, if satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$, i.e. , for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{\omega}$, $n \in \omega$, and $\bigcup_{n\in\omega} \mathcal{V}_n \in \mathcal{O}$ (see [Hurewicz, 1926, p. 402]).

- *Hurewicz*, briefly H, if satisfies $U_{fin}(\mathcal{O}, \Gamma)$, i.e. , for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{\omega}$, $n \in \omega$, and $\bigcup_{n\in\omega} \mathcal{V}_n \in \mathcal{O}$ (see [Hurewicz, 1927, p.196]).

The implications between the cited properties are summed up in the underlying diagram.



A space X is said to be *metrizable* if there exists a function $d : X^2 \to \mathbb{R}$, defined *metric* with the following properties

1. $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;
2. $d(x,y) = d(y,x)$ for all $x, y \in X$;
3. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$;

and such that the topology generated frome the sets

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

coincides with the one on the space. With abuse of notation we will call a metrizable space a *metric* space. The space X is said to be *complete* if every Cauchy sequence converges (see Engelking, 1989 for details). Every metrizable space is T_6.

The set ω^ω can be endowed with the following topology: if $s = \langle a_m : m < n \rangle$, a basic open set will be any set of the form

$$B(s) = \{\langle b_n : n \in \omega \rangle : (\forall m < n)(b_m = a_m)\}$$

The topological space obtained wil be referred as the *Baire space*. The Baire space is separable and completely metrizable by the metric $d : \omega^\omega \times \omega^\omega \to \mathbb{R}$ defined by

$$d(\langle a_n : n \in \omega \rangle \langle b_n : n \in \omega \rangle) = \frac{1}{2^{m+1}} \text{ where } m \text{ is the least integer for which } a_n \neq b_n$$

Let X be a complete metric separable space. A set $A \subseteq X$ is *analytic* if there exists a continuous function $f : \omega^\omega \to X$ such that $f(\omega^\omega) = A$.

A family of sets $\mathcal{A} \subset [\omega]^\omega$ is *almost disjoint* if the intersection of any two distinct elements is finite; if it is not included into to a strictly bigger family with this property, $\mathcal{A}$ is said to be a *maximal almost disjoint* family, briefly a "mad" family.

Given a mad family $\mathcal{A} \subset [\omega]^\omega$, a pair of disjoint $\mathcal{B}, \mathcal{C} \in [\mathcal{A}]^{<\mathfrak{c}}$ is said to be *separated* if there exists $A \subset \omega$ such that $B \subset^* A$ and $C \subset^* \omega \setminus A$ for any $B \in \mathcal{B}$ and $C \in \mathcal{C}$.

Let $\mathcal{A} \subset [\omega]^\omega$ be an almost disjoint family. Put $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$ and topologize $\Psi(\mathcal{A})$ as follows: the points of ω are isolated and a basic neighbourhood of a point $a \in \mathcal{A}$ takes the form $\{a\} \cup (a \setminus F)$, where F is a finite set. The space $\Psi(\mathcal{A})$ is known in literature as *Isbell-Mrówka* space or Ψ*-space* (see [Alexandroff and Urysohn, 1923] for the first apparition, and [Mrówka, 1955, p. 105]).

Let X be a space. With $C_p(X)$ will be denoted the space of continuous functions from X to the real line $\mathbb{R}$ endowed with the topology of pointwise conergence,i.e., the one inherited from the space $\mathbb{R}^X$ seen as a product.

Given a set A, a *partial order* $\mathcal{R}$ on A is a subset $\mathcal{R} \subseteq A \times A$ such that $a)$ if $(x,y),(y,z) \in \mathcal{R}$, then $(x,z) \in \mathcal{R}$ and $b)$ $(x,x) \notin \mathcal{R}$ for all $x \in A$; a *linear order* is a partial order such that $x < y$ or $y < x$ for all $x, y \in A$, and a *well order* is a linear order such that every subset of A admits a minimal element. Every set can be endowed of the structure of well-ordered set assuming AC.

Given a nonempty set S, a *filter* on S is a family $\mathcal{F}$ of subsets of S such that

(i) $S \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$;

(ii) if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$;

(iii) if $X \in \mathcal{F}$ and $X \subseteq Y$, then $Y \in \mathcal{F}$.

A filter is an *ultrafilter* if has the following additional property

(iv) for every subset X of S, either $X \in \mathcal{F}$ or $S \setminus X \in \mathcal{F}$.

An ultrafilter $\mathcal{F}$ on ω is a $P_{\mathfrak{c}}$*-point* if for any $\mathcal{F}' \in [\mathcal{F}]^{<\mathfrak{c}}$ there exists $F \in \mathcal{F}$ such that $F \subset^* F'$ for all $F' \in \mathcal{F}'$.

Given an almost disjoint family $\mathcal{A} \subset [\omega]^{\omega}$, it is always possible to construct a filter $\mathcal{F}(\mathcal{A})$, called *dual filter of* $\mathcal{A}$, defined as follows

$$\mathcal{F}(\mathcal{A}) = \{F : (\exists \mathcal{A}' \in [\mathcal{A}]^{<\omega})(\omega \setminus \cup \mathcal{A}' \subset^* F)\}$$

A *forcing notion* is a triple $\mathbb{P} = (\mathcal{P}, \leq, 1_{\mathcal{P}})$ where $\leq$ is a partial order of support $\mathcal{P}$ and $1_{\mathcal{P}}$ is the maximal element of $\mathcal{P}$ with respect to $\leq$. Elements of $\mathbb{P}$ will be called *forcing conditions* or just "conditions". With abuse of notation, the support of the forcing notion will be always considered to be $\mathbb{P}$ instead of $\mathcal{P}$. Two conditions $p, q \in \mathbb{P}$, are said to be *compatible* if there exists $r \in \mathbb{P}$ such that $r \leq p, q$; otherwise it will said that those are *incompatible*. A subset $G \subseteq \mathbb{P}$ is said to be a *filter* if a) every two of its elements are compatible and b) if $p \in G$ and $p \leq q$, then $q \in G$. A set $D \subseteq \mathcal{P}$ is said to be *dense* if for every $p \in \mathbb{P}$ there exists $q \in D$ such that $q \leq p$.

Given a set or a class M, the *transitive closure* of M is the set $tc(M) = \bigcup\{\bigcup^n(M) : n \in \omega\}$, where $\bigcup^0(M) = M$ and $\bigcup^{n+1}(M) = \bigcup(\bigcup^n M)$ for all $n \in \omega$; M is said to be *transitive* if $x \in M$ implies $x \subseteq M$. Of course if M is transitive, then $M = tc(M)$.

For a cardinal θ, the set H_θ will denote the set of all sets whose transitive closure has cardinality less than θ:

$$H_\theta = \{x : |tc(x)| < \theta\}$$

its existence is guaranteed in ZF (even without power axiom), but to develope properties of it the assumption AC of is needed.

Let φ be a formula written in the language of set theory, and let M be a class or a set; then M is said to be a *model* for φ if the formula φ^M obtained restricting the quantifiers of φ just to the ones contained in M is true. In this case the notation

$$M \models \varphi$$

is used.

The latter concepts were remarked with the aim to understand the tool of *forcing*. Let M be a countable transitive model of ZFC, briefly a CTM, and a formula φ. The forcing method will be used to build a set N which will result in a model of $ZFC + \varphi$ that includes M. To descrive the process of building the new model via forcing, a few more definitions are needed. Given a forcing notion $\mathcal{P} \in M$, a filter $G \subseteq \mathcal{P}$ is said to be $\mathbb{P}$*-generic over* M if it intersects every dense $D \subseteq \mathbb{P}$ which belongs to M. The existence of this kind of filter outside M is justifyied by the following result.

Lemma 1. *[Jech, 2003, Lemma 2.4] Suppose M is a CTM and $\mathbb{P} \in M$ a forcing notion. Then there are $\mathfrak{c}$-many $\mathbb{P}$-generic filters over M that are not elements of M.*

A sketch of the forcing will now be summed up. Given a forcing notion $\mathbb{P}$, a $\mathcal{P}$*-name* is defined recursively as follows:

- $\emptyset$ is a $\mathbb{P}$-name
- a relation τ such that

$$(\sigma, p) \in \tau \text{ if and only if } \sigma \text{ is a } \mathbb{P}\text{-name and } p \in \mathbb{P}$$

  is a $\mathbb{P}$-name.

If M is a CTM, we define

$$M^{\mathbb{P}} = \{\tau \in M : M \models ''\tau \text{ is a } \mathbb{P}\text{-name}''\}$$

A $\mathbb{P}$-generic filter G over M, apart trivial cases, does not belong to M (see [Jech, 2003, Example 14.2]). The notion of $\mathbb{P}$-names is used to build the model listed in the following theorem.

Theorem 1. *[Jech, 2003, Theorem 14.5] Let M be a transitive model of ZFC and $\mathbb{P} \in M$ a forcing notion. Let $G \subseteq M$ be a $\mathbb{P}$-generic filter over M. Then there exists a transitive model $M[G]$ such that:*

- *$M[G]$ is a model of ZFC*
- *$M \subseteq M[G]$ and $G \in M[G]$*
- *the ordinal numbers in the new model are the same of the starting model*
- *if N is a transitive model of ZF such that $M \subset N$ and $G \in N$, then $M[G] \subset N$*

The model $M[G]$ is called *generic extension* of M. Given a condition $p \in \mathbb{P}$ and a formula φ, the symbology

$$p \Vdash \varphi$$

read as "*p forces* φ" will mean that for all $\mathbb{P}$-generic filters $G \subseteq P$ over M with $p \in G$, the formula φ is true in $M[G]$ (see [Kunen, 1980] for further details).

In this work the following notation will be used:

- given $x \in M$, the set $\check{x} = \{(\check{y}, p) : y \in x \wedge p \in \mathbb{P}\}$ is a $\mathbb{P}$-name for x.
- given a $\mathbb{P}$-generic filter $G \subseteq P$ over M, the set $\dot{G} = \{(p, \check{p}) : p \in G\}$ is a $\mathbb{P}$-name for G.

both the definitions do not depend on the choiche of the generic filter.

Many intresting results in set theory (and topology) can be obtained by assuming additional set-theoretical axioms in a ZFC model, like CH, or some relation between cardinal characteristics. However, several set-theoretical axioms deal with forcing notions, for which the statements can include known objects of this field of interest.

Two specific axioms will be recalled, with the aim of use in this thesis.

A forcing notion is said to have the *countable chain condition* property , or, briefly, to be c.c.c., if every set of mutually incompatible elements is countable.

With *"Martin's Axiom"*, briefly "MA", it is meant the wollowing hypothesis:

"Let $\mathbb{P}$ be a c.c.c. forcing notion and $\mathcal{D}$ a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \mathfrak{c}$. Then there exists a $\mathcal{D}$-generic filter F in $\mathcal{P}$."

This assertion was proven to be a weaker version of CH, but it was also proven the existence of a ZFC model of MA+¬CH.

Given a infinite cardinal κ the following statement, known as MA_κ, will be also considered:

"Let $\mathbb{P}$ be a c.c.c. forcing notion and $\mathcal{D}$ a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$. Then there exists a $\mathcal{D}$-generic filter F in $\mathcal{P}$."

It was proven that $\mathrm{MA}_{\aleph_0}$ is true in ZFC and MA_κ implies $\kappa < \mathfrak{c}$, so the axiom is considered for $\aleph_1 \leq \kappa < \mathfrak{c}$. Of course MA implies MA_κ for all $\aleph_1 \leq \kappa < \mathfrak{c}$.

A forcing notion is said to be *proper* if for every regular cardinal $\theta > 2^{|\mathbb{P}|}$, every countable transitive submodel $M \prec H_\theta$ with $\mathbb{P} \in M$ and for every $p \in M \cap \mathbb{P}$ and every $\mathbb{P}$-generic filter $G \subseteq \mathbb{P}$ over M there exists $q \leq p$ such that $q \Vdash$ "$\dot{G} \cap M$ is $\mathbb{P}$-generic over M".

With *"Proper Forcing Axiom"*, briefly "PFA", it is meant the wollowing hypothesis:

"Let $\mathbb{P}$ be a proper forcing notion and $\mathcal{D}$ a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \mathfrak{c}$. Then there exists a $\mathcal{D}$-generic filter F in $\mathcal{P}$."

Since every c.c.c. forcing is proper, PFA is a stronger assumption than MA; moreover a consequence of this axiom is the equality $\aleph_2 = 2^{\aleph_0}$.

It is possible to build new models by forcing "repeatedly"; such technique is called *iterated forcing*: suppose that $\mathbb{P}$ is a forcing notion, M is a CTM and $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over M; the model $M[G]$ can be used as the new starting model and make the forcing work over a generic filter over $M[G]$, in order to build an extension of it. In this light, a $\mathbb{P}$-name for a forcing notion is a triple of $\mathbb{P}$-names $\dot{\mathbb{Q}} = (\dot{\mathcal{Q}}, \dot{\leq}_{\mathcal{Q}}, \dot{1_{\mathcal{Q}}})$ such that:

- for some $p \in \mathbb{P}$ we have $(\dot{1}_{\mathcal{Q}}, p) \in \dot{\mathcal{Q}}$
- $1_{\mathcal{P}} \Vdash$ "$\dot{1}_{\mathcal{Q}} \in \dot{\mathcal{Q}} \wedge (\dot{\mathcal{Q}}, \dot{\leq}_{\mathcal{Q}})$ is a partial order of with maximal element $\dot{1_{\mathcal{Q}}}$"

again, $\dot{\mathbb{Q}}$ will replace $\mathcal{Q}$ when speaking of the forcing notion. The forcing itaration will produce a forcing notion, denoted by $\mathbb{P} * \dot{\mathbb{Q}}$, defined as follows:

- conditions are the pairs $(p, \dot{q})$ such that $p \in \mathbb{P}$ and $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$
- $1_{\mathbb{P}*\mathbb{Q}} = (1_{\mathbb{P}}, 1_{\dot{\mathbb{Q}}})$
- $(p_1, q_1) \leq (p_2, q_2) \Leftrightarrow p_1 \leq p_2$ and $p_1 \Vdash \dot{q_1} \leq \dot{q_2}$.

given a $\dot{\mathbb{Q}}$-generic filter $H \subseteq \dot{\mathbb{Q}}$ over $M[G]$, this procedure will consent to force with $\dot{\mathbb{Q}}$ to build a new model $M[G * H] = (M[G])[H]$.

The latter process, which consists in a two-step iteration, can be generalized to a transfinite iteration: if α is an (infinite) ordinal, an *α-stage iteration* is a pair of sequences $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$, $\langle \dot{\mathbb{Q}}_\gamma : \beta < \alpha \rangle$ such that for every $\beta \leq \alpha$ the following statements are true:

- $\mathbb{P}_\beta = (\mathbb{P}_\beta, \leq_\beta, 1_\beta)$ is a forcing notion, whose forcing operation will be denoted by $\Vdash_\beta$.
- $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for a forcing notion $(\dot{\mathbb{Q}}_\beta, \leq_{\dot{\mathbb{Q}}_\beta}, 1_{\dot{\mathbb{Q}}_\beta})$.
- The conditions $p \in \mathbb{P}_\beta$ are sequences $\langle \dot{q}_\gamma : \gamma < \beta \rangle$ such that every $\dot{q}_\gamma$ is a name for an element of $\dot{\mathbb{Q}}_\gamma$; in particular $1_\beta = \langle 1_{\dot{\mathbb{Q}}_\gamma} : \gamma < \beta \rangle$.
- If $p, p' \in \mathbb{P}_\beta$ with $p = \langle \dot{q}_\gamma : \gamma < \beta \rangle$ and $p' = \langle \dot{q}'_\gamma : \gamma < \beta \rangle$, then $p \leq_\beta p'$ if and only if $\langle \dot{q}_\delta : \delta < \gamma \rangle \Vdash_\gamma$ "$\dot{q}_\gamma \leq \dot{q}'_\gamma$" for all $\gamma < \beta$.
- If $\beta + 1 \leq \alpha$, then $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$

Let κ be a cardinal and $\alpha \geq \kappa$ an ordinal. For a condition $p \in \mathbb{P}_\alpha$ with $p = \langle \dot{q}_\beta : \beta < \alpha \rangle$, the *support* of p is the set $supp(p) = \{\beta < \alpha : \dot{q}_\beta \neq 1_\beta\}$.

A partial order $(\mathcal{T}, \leq)$ is said to be a *tree* if the set $\{s \in \mathcal{T} : s \leq t\}$ is well-ordered by $\leq$; the elements of a tree are called *nodes*. If $s, t \in T$ are nodes and $s \leq t$, then s is said to be a *predecessor* of t, and t is said to be a *successor* of s. The forcing iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \beta < \alpha \rangle$ will said to have $<$ *κ-support* if $|supp(p)| < \kappa$ for all $p \in \mathbb{P}_\alpha$; if $\kappa = \aleph_1$, it is said to be a *countable support* iteration, if $\kappa = \aleph_0$, it is said to be a *finite support* iteration.

In [Laver, 1976, pp. 155-156], the following procedure to build a model is described. The *Laver forcing* is the set of all trees $p \subseteq {}^{<\omega}\omega$ in which there is a particular node $s_p \in p$, called *stem*, that is the maximal node which is comparable to all the other nodes and if $s_p \leq t$, then t has infinitely many successors in p. The *Laver model* is obtained from forcing $\aleph_2$-many times the laver forcing on a countable support, starting from a model of GCH. In particular, this model holds the equality $\mathfrak{b} = \mathfrak{c} = \aleph_2$, hence imples $\neg$CH.

Let $\mathcal{F}$ be a filter on ω. It is possible to consider a forcing poset $(\mathbb{M}_\mathcal{F}, \leq_\mathcal{F}, 1_\mathcal{F})$ defined as follows:

- The support is the set

$$\mathbb{M}_\mathcal{F} = \{(s, F) \in [\omega]^{<\omega} \times \mathcal{F} : max(s) < min(F)\}$$

- Let $(s, F), (t, G) \in \mathbb{M}_\mathcal{F}$. Then

$$(s, F) \leq_\mathcal{F} (t, G) \Leftrightarrow F \subseteq G, t \subseteq s, max(t) < min(s \setminus t) \text{ and } s \setminus t \subseteq G$$

- Consequently, $1_\mathcal{F} = (\emptyset, \omega)$.

Forcing with $\mathbb{M}_\mathcal{F}$ provides a generic subset $X \in [\omega]^\omega$ such that $X \subset^* F$ for all $F \in \mathcal{F}$. This forcing notion is usually called *Mathias forcing associated with $\mathcal{F}$*.

INTRODUCTION

The aim of this thesis is to give new results in the following three areas of general topology: the classes of n-Hausdorff homogeneous and n-Urysohn homogeneous spaces, covering properties and selection principles defined by stars, and selective separability properties in Fréchet-Urysohn spaces.

In the Chapter 1 the classes of n-Hausdorff homogeneous and n-Urysohn m-homogeneous spaces are studied. It is shown that, for every $n > 2$, there is no n-Hausdorff m-homogeneous space, $m > 1$, and there exists a 3-Urysohn homogeneous space which is not Urysohn. New upper bounds for the cardinality of n-Hausdorff homogenous and n-Urysohn homogenoeous spaces are given and for any n-Hausdorff space it is constructed an n-Hausdorff homogeneous extension which is the union of countably many n-H-closed spaces. The original results of this thesis presented in Chapter 1 are contained in [Bonanzinga, Giacopello and Maesano, 2023].

The n-Hausdorff and n-Urysohn properties were recently introduced in [Bonanzinga, 2013, p. 1] and [Bonanzinga, Cammaroto, and Matveev, 2011, p. 441] respectively, with the aim to generalize Hausdorff and Urysohn spaces. Intuitively in a Hausdorff space it is always possible to separate two points by open disjoint sets; the n-Hausdorff property generalize this concept, seprating n-many point by open sets whose intersection is empty. The same can be said for the Urysohn and n-Urysohn properties just applying the disjoint condition to the closure of the open sets instead.

It is intresting to study the behaviour of these new separation axioms with respect of classical topics of the general topology:

- while the Hausdorff property is productive, in [Bonanzinga, Stavrova, and Staynova, 2016, Theorem 15] it is stated that the product of m-many n-Hausdorff space is at most $(m \cdot n)$-Hausdorff, and [Bonanzinga, Stavrova, and Staynova, 2017, Example 3.2] proves the existence of a 3-Hausdorff, T_1, compact and first-countable space whose square is not 3-Hausdorff but 5-Hausdorff instead.

- every infinite Hausdorff space contains an infinite closed discrete subspace, as per [Ginsburg and Sands, 1979, Theorem 1]; [Bonanzinga, 2013, Proposition 21] shows that n-Hausdorff spaces keep the same property;

- Several bounds on the cardinality of n-Hausdorff and n-Urysohn spaces were proved, in order to generalize classical results in literature; see [Bonanzinga and Pansera, 2014] , [Bonanzinga, Cuzzupé, and Pansera, 2014] , [Bonanzinga, Stavrova, and Staynova, 2016] , [Bonanzinga, Stavrova, and Staynova, 2017], [Gotchev, 2017] and [Bonanzinga, Carlson, and Giacopello, 2023].

Roughly speaking, the cardinality of a space indicates how many elements it does contain; a cardinal bound is a way to give an approximation of this number of points for a space with good enough topological properties. Finding upper and lower cardinality bounds for topological spaces is one of the most recurrent topic of research. This thematic is strictly connected to the study of cardinal functions in topology. A cardinal function is any function defined from the class of topological spaces to the cardinals which maps homeomorphic spaces in the same cardinal. The first studies of cardinal functions date back on 1920, thanks to the russian school of topology, and received a further boost fourty years later.

The cardinal functions were introduced both as a tool to prove when two spaces are not homeomorphic, and to give (more elegant) proofs of cardinality bounds of spaces which verify not just "countable" topological properties. The cardinal functions consent to generalize known results about "countable" topological properties, as Lindelöf and separability, to the "uncountable" case.

The first problem about the cardinality of topological spaces was posed in [Alexandroff and Urysohn, 1923]:

Question 1. *Does every first countable compact topological space have cardinality* $\leq \mathfrak{c}$*?*

A positive answer was given in [Arhangel'skii, 1969], where the author proved the following result

Theorem 2. *[Arhangel'skii, 1969, Theorem 1] A first countable compact space has cardinality* $\leq \mathfrak{c}$.

This was just the first of many results in this field of research, widely analyzed by many authors. In the same article the author generalize the previous result using cardinal functions

Theorem 3. *[Arhangel'skii, 1969, Theorem 3] Let* X *be a Hausdorff space. Then* $|X| \leq 2^{L(X)\chi(X)}$.

The definitions of n-Hausdorff space and n-Urysohn space were given by the following new cardinal functions.

Definition 4. *[Bonanzinga, 2013, p. 1]* Let X be a space. The *Hausdorff number* $H(X)$ (finite or infinite) of X is the least cardinal number κ such that for every subset $A \subseteq X$ with $|A| \geq \kappa$ there exist open neighbourhoods U_a, $a \in A$, such that $\bigcap_{a \in A} U_a = \emptyset$. A space X is said *n-Hausdorff*, $n \geq 2$, if $H(X) \leq n$.

In [Gotchev, 2014, Definition 1], indipendently from Definition 4 , the cardinal function called "non Hausdorff number" was introduced.

Definition 5. *[Bonanzinga, Cammaroto, and Matveev, 2007, p. 441]* Let X be a space. The *Urysohn number* $U(X)$ (finite or infinite) of X is the least cardinal number κ such that for every subset $A \subseteq X$ with $|A| \geq \kappa$ there exist open neighbourhoods U_a, $a \in A$, such that $\bigcap_{a \in A} \overline{U_a} = \emptyset$. A space X is said *n-Urysohn*, $n \geq 2$, if $U(X) \leq n$.

A recurring problem in general topology is to prove a certain known cardinal inequality assuming weaker hypothesis or changing the cardinal functions involved with ones whose values are lesser. In this sense, in [Bonanzinga, 2013] the author proved first that 3-Hausdorff and T_1 are indipendent properties and then presented the following generalization of Theorem 3:

Theorem 6. [*Bonanzinga, 2013, Corollary 34*] *Let X be a T_1 space such that $H(X)$ is finite. Then $|X| \leq 2^{L(X)\chi(X)}$.*

Many other generalizations of Arhangel'skii inequality were proved using n-Urysohness instead of Hausdorffness or Urysohness. Hence it is clear how these axioms can be useful in the field of cardinal bounds for topological spaces. Several bounds on the cardinality of n-Hausdorff and n-Urysohn spaces are contained in [Bonanzinga and Pansera, 2014] , [Bonanzinga, Cuzzupé, and Pansera, 2014] , [Bonanzinga, Stavrova, and Staynova, 2016] , [Bonanzinga, Stavrova, and Staynova, 2017], [Gotchev, 2017] and [Bonanzinga, Carlson, and Giacopello, 2023].

Chronologically, the notion of homogeneous space belongs to the first half of XX century: a space is said to be *homogeneous* if for every pair of its points there exists a homeomorphism which maps one into the other; this class of spaces was widely analyzed in topology and set theory with many intresting results, as the following one

Theorem 7. [*van Mill, 2003, Theorem 3.6*] *There exists a compact space with countable π-weight and uncountable character which is homogeneous under MA+$\neg$CH but not so under CH.*

In the first part of the Chapter 1 a systematic study of n-Hausdorff and n-Urysohn homogeneous spaces starts. Several results and examples distinguishing these topological properties are given.

The study of the cardinalities of homogeneous spaces started one year later than the Arhangel'skii's result (Theorem 3); the same author proved the following.

Theorem 8. [*Arhangel'skii, 1970, Consequences group 2*] *Let X be a compact homogeneous space. If X is sequential, then $|X| \leq \mathfrak{c}$.*

From then, the area of cardinality bounds for homogenous spaces started to grew, collecting many elegant results. Recall the following important theorem in the theory of cardinal boundings

Theorem 9. [*Hajnal and Juhász, 1967, Theorem 5*] *Let X be a Hausdorff space. Then $|X| \leq 2^{c(X)\chi(X)}$.*

Adding the homogeneity to the hypothesis, it was possible to prove the following

Theorem 10. [*Carlson and Ridderbos, 2008, Theorem 2.3*] *Let X be a Hausdorff homogeneous space. Then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

This is in fact a better estimate in the class of homogeneous spaces since the π-character is less or equal to the character.

In [Bonanzinga, 2013], in the light of Theorem 9, the author proved the following

Theorem 11. [*Bonanzinga, 2013, Corollary 54*] *Let X be a 3-Hausdorff space. Then $|X| \leq 2^{2^{c(X)\chi(X)}}$.*

Then in [Bonanzinga, 2013, Question 55] the author asked if $|X| \leq 2^{c(X)\chi(X)}$ holds for every n-Hausdorff space X, with $n \geq 2$. In [Gotchev, 2017] the author, using the cardinal function called "non Hausdorff number" introduced in [Gotchev, 2014, Definition 2.4] independently from definitions above, gave a positive answer to the previous question.

Upper bounds on the cardinality of n-Hausdorff homogeneous and n-Urysohn homogeneous spaces are presented in paragraph 1.1. Motived by Theorem 10, the following question was considered.

Question 12. *Is $|X| \leq 2^{c(X)\pi\chi(X)}$ true for every homogeneous space X such that $H(X)$ is finite?*

The following partial answers to the previous question are given:

Theorem 13. *(Theorem 1.1.6) Let X be a 3-Hausdorff homogeneous space. Then $|X| \leq 2^{2^{c(X)\pi\chi(X)}}$.*

Theorem 14. *(Theorem 1.1.8) Let X be an n-Hausdorff homogeneous space, with $n \geq 2$. Then*

$$|X| \leq 2^{2^{\cdot^{\cdot^{2^{c(X)\pi\chi(X)}}}}}$$

where the power is made $(n-1)$-many times.

Theorem 15. *(Theorem 1.1.13) Let X be a n-Hausdorff quasiregular homogeneous space, $n \geq 2$. Then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

Also, the following result gives a positive answer to Question 12 if $H(X)$ is replaced by $U(X)$.

Theorem 16. *(Theorem 1.1.14) Let X be an n-Urysohn homogeneous space, where $n \geq 2$. Then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

A Hausdorff space is said to be *H-closed* if it is closed in every Hausdorff space in wich it is contained. The first appearence of the notion of H-closed spaces can be found in the already cited article [Alexandroff and Urysohn, 1923]. It was proved in [Porter and Woods, 1987, Theorem (n)] that every Hausdorff space can be embedded in a H-closed space which is the "larger" possible. In [Basile, Bonanzinga, Carlson, and Porter, 2019] the authors, in order to formulate an analougous result for the n-Hausdorff spaces, introduced the following property.

Definition 17. [*Basile, Bonanzinga, Carlson, and Porter, 2019, Definition 6*] Let $n \geq 2$. An n-Hausdorff space X is called *n-H-closed* if X is closed in every n-Hausdorff space Y in which X is embedded.

In the same article the authors obtained the following result

Theorem 18. [*Basile, Bonanzinga, Carlson, and Porter, 2019, Theorem 4*] *Let $n \in \omega$, $n \geq 2$. An n-Hausdorff space can be densely embedded in an n-H-closed space.*

In [Carlson, Porter, and Ridderbos, 2012], the following is proved

Theorem 19. [*Carlson, Porter, and Ridderbos, 2017, Theorem 2.3*] *Let X be a Hausdorff space. Then X can be embedded in a homogeneous space that is the countable union of H-closed spaces.*

Generalizing the previous result, in the paragraph 1.2, for any $n \geq 2$ and for any n-Hausdorff space, it is presented an n-Hausdorff homogenous extension which is the countable union of n-H-closed spaces.

Theorem 20. *(Theorem 1.2.12) Let $n \geq 2$, X be an n-Hausdorff space. Then X can be embedded in an homogeneous space that is the countable union of n-H-closed spaces.*

Using this result an example of n-Hausdorff homogeneous space which is not n-Urysohn, for every $n \geq 2$ is given.

Example 21. *(Example 1.2.13) An example of an n-Hausdorff, homogeneous, not n-Urysohn space which is the countable union of n-H-closed spaces, for every $n \geq 2$.*

The Chapter 2 is devoted to the study of relative versions of some compact-type and Lindelöf-type covering properties and Menger-type and Hurewicz-type selection principles defined by stars. The original results of this thesis presented in Chapter 2 are contained in [Bonanzinga and Maesano, 2022] and [Bonanzinga, Giacopello and Maesano, 2023].

Given a space, it is possible to find the "position" of a subset of it in terms of its topological properties: from a general property $\mathcal{P}$ of the whole space it is possible to formulate a relativizaton $\mathcal{P}_{rel.}$ with respect of a subset in such a way that, when the subset coincide with the entire space, then $\mathcal{P}_{rel.}$ coincides with $\mathcal{P}$. Every global property can be relativized in several ways: the way to classify the possible variations is always trying to generalize known theorems, valid for the general property, to selected relative ones. First studies on relativizations theory are contained in [Arhangel'skii, 1989]; subsequent investigation can be found in [Arhangel'skii and Hamdy, 1989] and [Arhangel'skii, 1996].

Definition 22. Let X be a space, A a subset of it and $\mathcal{V}$ a family of open sets. The *star of A with respect of $\mathcal{V}$* is the set $st(A, \mathcal{V}) = \bigcup\{A \in \mathcal{V} : A \cap M \neq \emptyset\}$; if $x \in X$, the star of the one-point set $\{x\}$ with respect to $\mathcal{V}$ is denoted by $st(x, \mathcal{V})$ instead of $st(\{x\}, \mathcal{V})$. Furthermore, $st^1(M, \mathcal{V}) = st(M, \mathcal{V})$ and for every $n \in \omega$, $st^{n+1}(M, \mathcal{V}) = st^n(st(M, \mathcal{V}), \mathcal{V})$.

The notion of star of a set with respect of a given family of subsets belongs to the first era of research in geometry: according to [Aull and Lowen, 1997, p. 528], "this idea of 'starring' occurs prominently in the 1910 work of E.H. Moore", hence the origin of this concept is at least this ancient. The concept of star is part of the definition of developement, first used in [Chittenden and Pitcher, 1919], which was used to prove several criteria for the metrizability of topological spaces, for example [Bing, 1951, Theorem 10] and in the proof of [Stone, 1960, Theorem 1] which consists in an elegant version of the well-known [Moore, 1935, Theorem 12] (see [Section 5.4, Engelking]). Stars are also used with different terminology in the theory of simplicial complexes. Another application of stars is a characterization of paracompactness, a weaker form of compactness, exposed in [Stone, 1948]. Hence, stars can be used in arguably any topological context.

The first classes of compact-type and Lindelöf-type properties defined by stars were given in [Fleischman, 1970] , [Ikenaga and Tani, 1980], [Ikenaga, 1983] and [Matveev, 1984] using many different terminologies. Later the article [van Douwen, Reed, Roscoe, and Tree, 1991] collected and extended the works from authors and gave impulse to this field of research. Many details on compact-type and Lindelöf-type properties defined by stars can be found in [Matveev, 1998].

New results in the theory of star-covering properties are contained in [Bonanzinga and Maesano, 2021], [Kočinac, 2023] and [Basile, Bonanzinga, Maesano, and Shakmatov, 2023].

Recall the following classical star-covering properties.

Definition 23. A space X is

- *strongly star-compact*, briefly SSC, if for every open cover $\mathcal{U}$ of the space X, there exists a finite subset F of X such that $st(F,\mathcal{U}) = X$ [Fleischman, 1970, Definition 5];
- *star-compact*, briefly SC, if for every open cover $\mathcal{U}$ of the space X, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $st(\bigcup \mathcal{V},\mathcal{U}) = X$ [Sarkhel, 1986].
- *strongly star-Lindelöf*, briefly SSL, if for every open cover $\mathcal{U}$ of the space X, there exists a countable subset C of X such that $st(C,\mathcal{U}) = X$ [Ikenaga, 1990].
- *star-Lindelöf*, briefly SL, if for every open cover $\mathcal{U}$ of the space X, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $st(\bigcup \mathcal{V},\mathcal{U}) = X$ [Ikenaga, 1983].

In the paragraph 2.1, recent and new relative versions of the star-covering properties above are analyzed and distinguished by counterexamples.

Definition 24. *(Definitions 2.1.7 and 2.1.13)* A subset A of a space X is

- *relatively* SSC* in X if for every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a finite subset F of $\overline{A}$ such that $st(F,\mathcal{U}) \supset A$;
- *relatively* $\mathcal{K}$-SC* in X if for every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a compact subset K of $\overline{A}$ such that $st(K,\mathcal{U}) \supset A$
- *relatively* SC* in X if for every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $st(\bigcup \mathcal{V},\mathcal{U}) \supset A$;
- *relatively* SSL* in X if for every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a countable subset F of $\overline{A}$ such that $st(F,\mathcal{U}) \supset A$;
- *relatively* SL* in X if for every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $st(\bigcup \mathcal{V},\mathcal{U}) \supset A$;

Using the following notions the following relative star-covering properties are considered

Definition 25. *(Definitions 2.1.9, 2.1.14 and 2.1.16)* A space X is

- *set strongly star-compact*, briefly set SSC, if every nonempty subset A of X is relatively* SSC in X [Kočinac, Konca, and Singh, 2022, Definition 3].
- *set star-compact*, briefly set SC, if every nonempty subset A of X is relatively* SC in X [Kočinac, Konca, and Singh, 2022, Definition 3].
- *set $\mathcal{K}$-star-compact*, briefly set $\mathcal{K}$-SC , if every subset A of X is relatively* $\mathcal{K}$-SC in X.
- *set strongly star-Lindelöf*, briefly set SSL, if every nonempty subset A of X is relatively* SSL in X [Kočinac and Singh, 2020, Definition 5].

- *set star-Lindelöf*, briefly set SL, if every nonempty subset A of X is relatively* SL in X [Kočinac and Singh, 2020, Definition 5].

It is proved that some of the previous relative compact-type properties coincide with countable compactness under the appropriate separation axiom; indeed the following results are presented.

Proposition 26. *(Proposition 2.1.10) If X is a Hausdorff space, then the following properties are equivalent:*

(i) X is CC

(ii) X is set SSC

(iii) X is SSC

Corollary 27. *(Corollary 2.1.12) If X is a regular space, then the following properties are equivalent:*

(i) X is CC

(ii) X is set SSC

(iii) X is SSC

(iv) X is set SC

An analogous result is presented for the Lindelöf-type properties.

Proposition 28. *(Corollary 2.1.19) If X is a collectionwise Hausdorff space, then the following properties are equivalent:*

1. $e(X) = \aleph_0$
2. *X is set SSL*
3. *X is set SL*
4. *X is SSL.*

It is presented a Tychonoff collectionwise Hausdorff SL space which is not SSL, hence SL property cannot be added to the previous list (Example 2.1.20). It is also presented the following result, where a stronger hypothesis consents to add SL in the list of equivalent affirmations.

Corollary 29. *(Corollary 2.1.21) If X is a normal collectionwise Hausdorff space, then the following properties are equivalent:*

1. $e(X) = \aleph_0$
2. *X is set SSL*
3. *X is set SL*
4. *X is SSL*
5. *X is SL.*

Several examples are presented: a T_1 set SC not $\mathcal{K}$-SC space (Example 2.1.15), a Tychonoff SSL not set SL (Example 2.1.23) and a T_2 $\mathcal{K}$-SC not set SL space (Example 2.1.24).

It is proved the following result

Proposition 30. *(Corollary 2.1.28) Every c.c.c. space is set SL*

and as an application of it, a Tychonoff set SL (in fact separable) non SSL space is obtained (Example 2.1.29).

In [Bonanzinga, 1998] the a-st-L and h-cl-a-st-L properties were introduced, both weakenings of the Lindelöf property. These properties were introduced in relation to the acc property formulated in [Matveev, 1994]. In this work, the vicinance of a-st-L and h-cl-a-st-L to the set SSL property is studied (Proposition 2.1.34).

The central section of the paragraph 2.1 is devoted to answering problems posed in [Kočinac and Singh, 2020] and [Bal and Kočinac, 2020].

In [Kočinac and Singh, 2020], the author introduce the set version of the acc property, called set acc. In this work it is proved that the set property in fact concides with the original property.

Theorem 31. *(Proposition 2.1.39) A space X is acc if and only if is set acc.*

In [Aurichi, 2013] the author define the following property.

Definition 32. [*Aurichi, 2013, Definition 2.1*]A space X is *selectively c.c.c.* if for every sequence $(\mathcal{A}_n : n \in \omega)$ of maximal cellular open families in X, there is a sequence $(A_n : n \in \omega)$ such that for each $n \in \omega$, $A_n \in \mathcal{A}_n$, and $\bigcup_{n\in\omega} A_n$ is dense in X.

Later, the authors of [Bal and Kočinac, 2020] defined the selective star-c.c.c. property, and, in [Kočinac and Singh, 2020], the authors introduced the set version of the latter property, called set selective star-c.c.c. . In fact set selective star-c.c.c. and selective star-c.c.c. are equivalent, as the following shows

Proposition 33. *(Proposition 2.1.46) A space X is selectively star-c.c.c. if and only if is set selectively star-c.c.c. .*

The previous proposition answer in the negative [Kočinac and Singh, 2020, Problem 2] about the existence of a selectively star-c.c.c. not set selectively star-c.c.c. space.

In [Kočinac and Singh, 2020] the authors prove that every set SSL space is set selectively 2-star-c.c.c. , and in [Bal and Kočinac, 2020] it is proved that every Lindelöf space is selectively star-c.c.c. . Since set SSL and the property to have countable extent are equivalent, both the results are improved by the following one

Theorem 34. *(Theorem 2.1.51) Every space with countable extent is selectively star-c.c.c. .*

A classic problem in general topology is to study the behavior of the topological properties with respect the (Tychonoff) product; typical questions are: the product of two spaces with a property $\mathcal{P}$ has the property $\mathcal{P}$? If a space has the property $\mathcal{P}$, what properties must a second space possess so that the product of the two spaces has the property $\mathcal{P}$?

Regarding the compact spaces, it is well-known that the product of any number of it remains compact. This is shown in the following classical result:

Theorem 35. *[Tychonoff, 1935] Let $\{X_s\}_{s\in S}$ a family of non-empty spaces. The product $\prod_{s\in S} X_s$ is compact if and only if X_s is compact for every $s \in S$.*

The same fact does not hold for the Lindelöf nor for the countably compact spaces: in [Sorgenfrey, 1947] it is proved the existence of a Lindelöf space the square of which is not Lindelöf and in [Novák, 1949] it is proved the existence of two countably compact Tychonoff spaces whose product is not countably compact.

However, those last two properties are preserved in the product with a compact space, as the following two classical results state

Theorem 36. *[Engelking, 1989, Theorem 3.8.10] The product of a Lindelöf space with a compact space is Lindelöf.*

Theorem 37. *[Smirnov, 1951] The product of a countably compact space with a compact space is countably compact.*

In [Fleischman, 1970] the author proved that the product of a SSC space with a compact space is SSC ; later in [van Douwen, Reed, Roscoe, and Tree, 1991] the same was proved for SC spaces instead of SSC, and, in the same article, that the product of a SL space with a compact space is SL while the product of a SSL space with a compact space need not be SSL.

The last section of the paragraph 2.1 will be devoted to the study of the product of set SSL spaces with compact ones, leading to the following result

Proposition 38. *(Proposition 2.1.56) The product of a T_1 set SSL space with a compact space is set SSL.*

It is pointed out that set SSC (resp. set SC) property is not productive and it is preserved in the Hausdorff (resp. regular) product with compact spaces.

In [Scheepers, 1996] the author proposed the first systematic study of selection principles, which included many covering properties defined indipendently in the early years of 1900. Let X be a space and $\mathcal{A}, \mathcal{B}$ be collections of families of its subsets. Then:

- $S_1(\mathcal{A}, \mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle U_n : n \in \omega \rangle$ such that for each $n \in \omega$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \omega\} \in \mathcal{B}$.
- $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each $n \in \omega$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\bigcup_{n\in\omega} \mathcal{V}_n \in \mathcal{B}$.
- $U_{fin}(\mathcal{A}, \mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each $n \in \omega$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\bigcup \mathcal{V}_n : n \in \omega\} \in \mathcal{B}$.

Denote by $\mathcal{O}$ (resp., by Γ, by Ω, by $\mathcal{D}$) the class of all the open covers (resp. γ-covers, ω-covers, dense subsets) of a space. Some selection principle define known properties: a space X is said to be:

- *Rothberger*, briefly R, if satisfies $S_1(\mathcal{O}, \mathcal{O})$ [Rothberger, 1938, Definition 1].
- *Menger*, briefly M, if satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$, [Hurewicz, 1926, p. 402].

- *Hurewicz*, briefly H, if satisfies $U_{fin}(\mathcal{O},\Gamma)$ [Hurewicz, 1927, p.196].

This was the first work on this thematic, subsequently followed by [Just, Miller, Scheepers, and Szeptycki, 1996], [Scheepers, 1997] and other "Combinatorics of Open Covers", for a total of eleven papers written by Scheepers. Following this concept, in [Kočinac, 1999] and [Bonanzinga, Cammaroto, and Kočinac, 2004] the authors applied the notion of star to this already-existing structure and permitted to formulate new star topological properties and study them from a combinatorial perspective.

The following principles were defined:

- $S^*_{fin}(\mathcal{A},\mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle \mathcal{V}_n : n \in \omega\rangle$ such that for each $n \in \omega$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\bigcup_{n\in\omega}\{st(V,\mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$.
- $U^*_{fin}(\mathcal{A},\mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle \mathcal{V}_n : n \in \omega\rangle$ such that for each $n \in \omega$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{st(\bigcup \mathcal{V}_n,\mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.
- $SS^*_{fin}(\mathcal{A},\mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle F_n : n \in \omega\rangle$ of finite subsets of X such that $\{st(F_n,\mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

A space is said to be

- *strongly star-Menger*, briefly SSM, if satisfies $SS^*_{fin}(\mathcal{O},\mathcal{O})$ [Kočinac, 1999, Definition 1.4].
- *star-Menger*, briefly SM, if satisfies $S^*_{fin}(\mathcal{O},\mathcal{O})$ [Kočinac, 1999, Definition 1.4].
- *strongly star-Hurewicz*, briefly SSH, if satisfies $SS^*_{fin}(\mathcal{O},\Gamma)$ [Bonanzinga, Cammaroto, and Kočinac, 2004, p. 81].
- *star-Hurewicz*, briefly SH, if satisfies $U^*_{fin}(\mathcal{O},\Gamma)$ [Bonanzinga, Cammaroto, and Kočinac, 2004, p. 81]

The paragraph 2.2 is devoted to the study of recent relative versions of aforesaid selection principles, first introduced in [Kočinac, Konca, and Singh, 2022].

Definition 39. *(Definition 2.2.3)* A space X is

- *set strongly star-Menger* , briefly set SSM, if for each nonempty subset A of X and for each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets of X such that $\overline{A} \subset \bigcup \mathcal{U}_n$ for every $n \in \omega$, there exists a sequence $(F_n : n \in \omega)$ such that F_n, $n \in \omega$, is a finite subset of $\overline{A}$ and $A \subset \bigcup_{n\in\omega} st(F_n,\mathcal{U}_n)$ [Kočinac, Konca, and Singh, 2022, Definition 4].
- *set star-Menger* , briefly set SM, if for each nonempty subset A of X and for each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets of X such that $\overline{A} \subset \bigcup \mathcal{U}_n$ for every $n \in \omega$, there exists a sequence $(\mathcal{V}_n : n \in \omega)$ such that $\mathcal{V}_n$, $n \in \omega$, is a finite subset of $\mathcal{U}_n$ and $A \subset \bigcup_{n\in\omega} st(\bigcup \mathcal{V}_n,\mathcal{U}_n)$ [Kočinac, Konca, and Singh, 2022, Definition 4].

- *set strongly star-Hurewicz* (briefly, set SSH) if for each nonempty subset $A \subseteq X$ and for each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}_n$ for every $n \in \omega$, there exists a sequence $(F_n : n \in \omega)$ such that F_n, $n \in \omega$, is a finite subset of $\overline{A}$ and $\forall x \in A$, $x \in st(F_n, \mathcal{U}_n)$ for all but finitely many $n \in \omega$ [Kočinac, Konca, and Singh, 2022, Definition 5].
- *set star-Hurewicz* (briefly, set SH) if for each nonempty subset $A \subseteq X$ and for each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}_n$ for every $n \in \omega$, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that $\mathcal{V}_n$, $n \in \omega$, is a finite subset of $\mathcal{U}_n$ and $\forall x \in A$, $x \in st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many $n \in \omega$ [Kočinac, Konca, and Singh, 2022, Definition 5].

In [Sakai, 2014, Example 3.1], using [Matveev, 2002, Theorem 1], it is proved that the extent of a Tychonoff SM (in fact SC) space can be arbitrarily large; Theorem 2.2.8 shows that the extent of a regular set SM space cannot exceed $\mathfrak{c}$; this result can be used to construct Example 2.2.10: a Tychonoff space which distinguish SM and set SM properties (in fact it distinguishes SC and set SC, and also SH and set SH properties).

Motivated by the results in [Sakai, 2014], the following is proved.

Proposition 40. *(Propositions 2.2.11 and 2.2.12) Every set SL (set SSL) space of cardinality $< \mathfrak{d}$ is set SM (set SSM).*

By the previous proposition it is obtained a characterization of set SSM property.

Corollary 41. *(Corollary 2.2.13) Let X be a T_1 space of cardinality less than $\mathfrak{d}$. The following affirmations are equivalent:*

1. *X is set SSM*
2. *$e(X) = \aleph_0$.*

In [Sakai, 2014] it is proved the proposition below

Theorem 42. *[Sakai, 2014, Proposition 2.9] The following statements are equivalent.*

1. *$\omega_1 = \mathfrak{d}$;*
2. *if X is a regular SSM space, then $e(X) \leq \aleph_0$.*

The list of equivalent affirmations will be enlareged in this work; in fact the following result will be given

Theorem 43. *(Theorem 2.2.16) The following statements are equivalent*

1. *$\omega_1 = \mathfrak{d}$;*
2. *if X is a regular SSM space, then $e(X) \leq \aleph_0$;*
3. *for regular spaces of cardinality less than $< \mathfrak{d}$, set SSM and SSM are equivalent properties.*
4. *for regular spaces of cardinality $< \mathfrak{d}$, set SSL and SSL are equivalent properties.*
5. *every closed subspace of a regular (set) SSM space X such that $|X| < \mathfrak{d}$ is (set) SSM.*

A consistent example of a Tychonoff set SM non set SSM space will be exopsed. Using this example, a partial answer to [Kočinac, Konca, and Singh, 2022, Problem 1] is given.

In [Kočinac, 1999] the author proved that the product of a SM space with a compact space is SM. Using [Bonanzinga and Matveev, 2001, Lemma 2.3], Matveev noted that assuming $\omega_1 < \mathfrak{d}$, if $X = \Psi(\mathcal{A})$ with $|\mathcal{A}| = \omega_1$ and Y is a compact space such that $c(Y) > \aleph_0$, then the product $X \times Y$ is not SSL, hence not SSM; therefore he gave a consistent example of a not SSM space which is the product of a SSM space and a compact space.

In the last part of the chapter, the following questinos are considered.

Question 44. *(Kočinac, Konca, and Singh, 2022) Is the product of a set SSM space with a compact space a set SSM space?*

Question 45. *(Kočinac, Konca, and Singh, 2022) Is the product of a set SM space with a compact space a set SM space?*

A partial answer to Question 44 and a positive answer to Question 45 are presented.

Proposition 46. *(Corollary 2.2.29) The T_1 product of a set SSM space with cardinality less than $\mathfrak{d}$ and a compact space is set SSM.*

Example 47. *(Example 2.2.30) There exists a set SM space X and a compact space Y with $c(Y) > \aleph_0$ such that $X \times Y$ is not set SM.*

Analogous results are obtained for the classes of set SSH and set SH spaces.

The Chapter 3, the last of this work, pertains the study of the selective separability properties: M-separability and H-separability. In the first two paragraphs of this chapter, Fréchet-Uryson property in countable spaces is distinguished from H-separability by two examples. The first example (paragraph 3.1) is an Hausdorff space without isolated points and it is contructed in ZFC; the second example (paragraph 3.2) is obtained under the assumption ($\mathfrak{p} = \mathfrak{c}$), and is a zero-dimensional α_4 space. In paragraph 3.3, the non-productivity of M-separable property is obtained under some set-theoretical axiom stronger than MA; lastly, in paragraph 3.4, it is proved that the product of countable H-separable spaces is mH-separable in the Laver model. The original results of this thesis presented in Chapter 3 are contained in [Bardyla, Maesano and Zdomskyy, 2023].

Selective separability properties were first introduced in [Scheepers, 1999], applying the selection principles S_1 and S_{fin} to the family $\mathcal{D}$ of the dense subsets of a space, in order to prove the following results :

Theorem 48. *[Scheepers, 1999, Theorem 13] Let X be a separable metric space. The following affirmations are equivalent*

- *X satisfy $S_1(\Omega, \Omega)$*
- *$C_p(X)$ satisfy $S_1(\mathcal{D}, \mathcal{D})$*

Theorem 49. *[Scheepers, 1999, Theorem 35] Let X be a separable metric space. The following affirmations are equivalent*

- *X satisfy $S_{fin}(\Omega, \Omega)$*
- *$C_p(X)$ satisfy $S_{fin}(\mathcal{D}, \mathcal{D})$*

By [Just, Miller, Scheepers, and Szeptycki, 1996, theorems 3.8 and 3.9], this is equivalent to say that every finite power of a metric separable space X satisfies a selection principle if and only if the space of realvalued continuous function $C_p(X)$ satisfies a corresponding selective separabilty property.

In [Bella, Bonanzinga, Matveev, and Tkachuk, 2008], the authors studied in depth $S_{fin}(\mathcal{D}, \mathcal{D})$, calling this property *selective separability* , and proved the following interesting results

Proposition 50. *[Bella, Bonanzinga, Matveev, and Tkachuk, 2008, Proposition 2.3.(1)] Given a space X, if $\pi w(X) = \aleph_0$, then X is selectively separable.*

Proposition 51. *[Bella, Bonanzinga, Matveev, and Tkachuk, 2008, Proposition 2.4] A compact space X is selectively separable if and only if $\pi w(X) = \aleph_0$.*

Theorem 52. *[Bella, Bonanzinga, Matveev, and Tkachuk, 2008, Theorem 2.26] Suppose X is selectively separable and $\pi w(Y) = \aleph_0$. Then $X \times Y$ is selectively separable.*

Also the following questions were posed

Question 53. *[Bella, Bonanzinga, Matveev, and Tkachuk, 2008, Problem 3.7] Suppose that X and Y are selectively separable spaces. Must $X \times Y$ be selectively separable?*

Question 54. *[Bella, Bonanzinga, Matveev, and Tkachuk, 2008, Problem 3.8] Suppose that $X = A \cup B$ and the spaces A and B are selectively separable. Must X be selectively separable?*

In [Bella, Bonanzinga, and Matveev, 2009] the selective separable property was renamed as M-separability, and, in analogy with Menger, Hurewicz and Rothberger selection principles, the following definitions were given

Definition 55. *[Bella, Bonanzinga, and Matveev, 2009]*A space X is said to be

- *H-separable* if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there are finite sets $F_n \subset D_n$, $n \in \omega$, such that every nonempty open set of X meets all but finitely many F_n.
- *R-separable* if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there points $x_n \in D_n$, $n \in \omega$, such that $\{x_n : n \in \omega\}$ is dense in X.

The following question was posed

Question 56. *[Bella, Bonanzinga, and Matveev, 2009, Question 34] Does there exists a ZFC example of M-separable space which is not H-separable?*

In [Gruenhage and Sakai, 2011] the authors gave a positive answer to Question 54.

Theorem 57. *[Gruenhage and Sakai, 2011, Theorem 2.2] The finite union of selectively separable spaces is selectively separable*

In the proof of the latter result, the following lemma was crucial

Lemma 58. [*Gruenhage and Sakai, 2011, Lemma 2.1*] *A space is M-separable if and only if for every decreasing sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there are finite sets $F_n \subset D_n$, $n \in \omega$, such that $\bigcup\{F_n : n \in \omega\}$ is dense in X.*

In the view of this result, the authors introduced the following properties (with different terminology), which consist in some weaker forms of the selective separability properties mentioned above

Definition 59. [*Gruenhage and Sakai, 2011, Definition 2.6*]A space X is said to be

- *mH-separable* if for every decreasing sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there are finite sets $F_n \subset D_n$, $n \in \omega$, such that every nonempty open set of X meets all but finitely many F_n.
- *mR-separable* if for every decreasing sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there points $x_n \in D_n$, $n \in \omega$, such that $\{x_n : n \in \omega\}$ is dense in X .

In the same article the following problem is posed.

Question 60. [*Gruenhage and Sakai, 2011, Question 2.10 (1)*] *Is there a mR-separable (resp., mH-separable) space which is not R-separable (resp., H-separable)?*

The class of Fréchet-Urysohn spaces is one of the most studied not just in topology, but even in analysis: a space is said to be *Fréchet-Urysohn*, briefly FU, if every point in the closure of a set admits a sequence converging to it which is included into the set.

The authors of [Barman and Dow, 2011] expressed their surprise upon discovering the following result

Theorem 61. [*Barman and Dow, 2011, Theorem 2.9*] *A separable FU space is M-separable.*

This result was improved later in the same year.

Corollary 62. [*Gruenhage and Sakai, 2011, Corollary 4.2*] *A separable FU space is R-separable.*

It is proved in this work that H-separability behaves differently; the following result is presented in paragraph 3.1

Theorem 63. *(Theorem 3.1.1) There exists a countable Hausdorff FU space X without isolated points which is not H-separable.*

This answers positively both questions 56 and 60 (in fact countable FU spaces are mH-separable by [Gruenhage and Sakai, 2011, Lemma 2.7(2) and Corollary 4.2]).

The behaviour of the productivity of FU property under ZFC and set-theoretical assumptions was widely analyzed during XX century: it was proved that FU property is not inherithed in the product of compact FU spaces and countable FU spaces (see [Gruenhage, 1978], [Simon, 1980] and [Tamano, 1986] for some interesting results). Motivated by the search of an additional property that would allow the product of FU spaces to keep FU property, in [Arhangel'skii, 1972] the author introduced the following definitions.

Definition 64. [*Arhangel'skii, 1972, p. 267*] *Given a space X and $x \in X$, denote by Γ_x the set of all $A \in [X \setminus \{x\}]^\omega$ which converge to x. A point $x \in X$ has the property:*

(α_1) *if for each* $\langle S_n : n \in \omega \rangle \in [\Gamma_x]^\omega$, *there is* $S \in \Gamma_x$ *such that* $S_n \subset^* S$ *for all* $n \in \omega$;

(α_2) *if for each* $\langle S_n : n \in \omega \rangle \in [\Gamma_x]^\omega$, *there is* $S \in \Gamma_x$ *such that* $S_n \cap S$ *is infinite for all* $n \in \omega$;

(α_3) *if for each* $\langle S_n : n \in \omega \rangle \in [\Gamma_x]^\omega$, *there is* $S \in \Gamma_x$ *such that* $S_n \cap S$ *is infinite for infinitely many* $n \in \omega$;

(α_4) *if for each* $\langle S_n : n \in \omega \rangle \in [\Gamma_x]^\omega$, *there is* $S \in \Gamma_x$ *such that* $S_n \cap S \neq \emptyset$ *for infinitely many* $n \in \omega$.

A space X is an α_i *space, where* $i \in \{1, 2, 3, 4\}$, *if it is an* α_i *space at each* $x \in X$.

In paragraph 3.2 it is proved an improvement of Theorem 63 under a strong equality of cardinal characteristics of continuum.

Theorem 65. ($\mathfrak{p} = \mathfrak{c}$) *There exists a countable zero-dimensional* α_4 *FU space without isolated points which is not H-separable.*

Since every α_i-property imply the next one, Proposition 3.2.4 shows that it is not possible to obtain an example of a countable zero-dimensional FU space without isolated points endowed with α_1 or α_2 property which is not H-separable.

In the second decade of XXI century several results were obtained using set-theoretical assumptions in the problem of the productivity of FU countable (hence M-separable) spaces. Using two different complementary assumptions, the following two results were given.

Theorem 66. [*Barman and Dow*, 2011, *Theorem 2.24*] *(CH) There exist two countable FU spaces whose product may not be M-separable.*

Theorem 67. [*Barman and Dow*, 2012, *Theorem 3.3*] *(PFA) The product of two countable FU spaces is M-separable.*

It is worth noting that the proof given by the authors provides a stronger result: their argument proves that, under PFA, the product of two countable FU space is in fact mR-separable.

Theorem 67 motivates the contents of paragraph 3.3. It is presented the following set-theoretical assumption

Definition 68. (Definition 3.3.1) Let τ_0 be a topology on ω turning it into a space homeomorphic to the rationals $\mathbb{Q}$. Define $(*_\mathbb{Q})$ to stand for the following statement:

MA + there exists a mad family $\mathcal{A} \subset [\omega]^\omega$ such that:

(1) every disjoint pair $\mathcal{A}', \mathcal{A}'' \in [\mathcal{A}]^{<\mathfrak{c}}$ is separated;

(2) Every $A \in \mathcal{A}$ is either closed discrete or a convergent sequence in $\langle \omega, \tau_0 \rangle$.

Consider the statement $(*_\mathbb{Q})$ without (2). This weaker statement was proved to be consistent with $\aleph_2 = \mathfrak{c}$ (with different terminology) in [Dow and Shelah, 2012, Theorem 2.1]. In this work this last result is extended (Theorem 3.3.7), showing the consistency of $(*_\mathbb{Q})$. This assumption will be used to prove the main result of the paragraph 3.3.

Theorem 69. *(Theorem 3.3.9)* $(*_{\mathbb{Q}})$ *There exist two countable regular (hence zero-dimensional) FU spaces without isolated points whose product is not M-separable.*

As a consequence of this result, MA is not sufficient to prove Theorem 67, and hence that the same result cannot be obtained by assumptions on cardinal characteristics of the continuum.

The productivity of selective separability properties was recently investigated in several classical models of the set-theory. In [Repovš and Zdomskyy, 2020] it is proved that in the Miller model the product of every M-separable space of functions $C_p(X)$, where X is metrizable and separable, with an M-separable countable space Y is M-separable. In [Repovš and Zdomskyy, 2018] it is proved that in the Laver model the product of any two H-separable spaces is M-separable.

Following this approach, in the last paragraph of the chapter it is proved the following improvement of the result in [Repovš and Zdomskyy, 2018].

Theorem 70. *(Theorem 3.4.4) In the Laver model, the product of two countable H-separable spaces is mH-separable.*

CHAPTER 1

n-HAUSDORFF HOMOGENEOUS AND n-URYSOHN HOMOGENEOUS SPACES

All the new contributions obtained by the author of the thesis included in this chapter are contained in the article [Bonanzinga, Carlson, Giacopello and Maesano, 2023].

Definition 1.1. [*Hausdorff, 1914*]A space X is said to be *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.

Definition 1.2. [*Bonanzinga, 2013, p. 1*] Let X be a space. The *Hausdorff number* $H(X)$ (finite or infinite) of X is the least cardinal number κ such that for every subset $A \subseteq X$ with $|A| \geq \kappa$ there exist open neighbourhoods U_a, $a \in A$, such that $\bigcap_{a \in A} U_a = \emptyset$. A space X is said *n-Hausdorff*, $n \geq 2$, if $H(X) \leq n$.

Definition 1.3. [*Bonanzinga, Cammaroto, and Matveev, 2007, p. 441*] Let X be a space. The *Urysohn number* $U(X)$ (finite or infinite) of X is the least cardinal number κ such that for every subset $A \subseteq X$ with $|A| \geq \kappa$ there exist open neighbourhoods U_a, $a \in A$, such that $\bigcap_{a \in A} \overline{U_a} = \emptyset$. A space X is said *n-Urysohn*, $n \geq 2$, if $U(X) \leq n$.

Of course, with $|X| \geq 2$, X is Hausdorff (resp. Urysohn) if and only if $H(X) = 2$ (resp. $U(X) = 2$).

In [Bonanzinga, 2013], examples of $(n + 1)$-Hausdorff spaces which are not n-Hausdorff, for every $n \geq 2$, and an example of a space X such that $H(X) = \omega$ and $H(X) \neq n$, for each $n \geq 2$, are given. Also, in [Bonanzinga, Cammaroto, and Matveev, 2007] examples of Hausdorff $(n + 1)$-Urysohn spaces which are not n-Urysohn were given for every $n \geq 2$. Recall moreover that examples of $(n + 1)$-Urysohn spaces which are not n-Urysohn are presented in [Bonanzinga, Cammaroto, and Matveev, 2011].

Definition 1.4. [*Carlson, Porter, and Ridderbos, 2017, Definition 3.1*] A space X is 2-homogeneous if for every $x_1, x_2, y_1, y_2 \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

In general it is possible give the definition of n-homogeneous space for any n. Notice that 1-homogeneity coincides with the definition of homogeneity. Of course, if a space is $n+1$-homogeneous, then it is m-homogeneous for every $m = 1, ..., n$.

In the following Theorem 1.11 proves that n-Hausdorff, $n > 2$, non Hausdorff spaces are not m-homogeneous, $m > 1$, and give an example (Example 1.13) of a 3-Urysohn homogeneous non Urysohn space.

Definition 1.5. [*Engelking, 1989*] A space is said to be *hyperconnected* (or *nowhere Hausdorff*) if the intersection of any two nonempty open sets is nonempty.

Definition 1.6. [*Engelking, 1989*] A space is said to be *nowhere Urysohn* if the intersection of the closures of every pair of nonempty open sets is nonempty.

Proposition 1.7. [*Bonanzinga, Giacopello, M., Carlson, 2023, Proposition 2.1*] *A non Hausdorff* 2*-homogeneous space is hyperconnected.*

Proof. Let X be a non Hausdorff 2-homogeneous space. Suppose that there are two nonempty open subset V_1 and V_2 of X such that $V_1 \cap V_2 = \emptyset$. Fix two points $y_1 \in V_1$ and $y_2 \in V_2$. Since X is not Hausdorff there exist two points $x_1, x_2 \in X$ such that for every open neighbourhood U_1 of x_1 and U_2 of x_2, one has that $U_1 \cap U_2 \neq \emptyset$. Define the homeomorphism $h : X \to X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$. Of course $h^{-1}(V_1) \cap h^{-1}(V_2) \neq \emptyset$. Pick a point $x \in h^{-1}(V_1) \cap h^{-1}(V_2)$, then $h(x) \in V_1 \cap V_2$, a contradiction. □

Proposition 1.8. [*Bonanzinga, Giacopello, M., Carlson, 2023, Proposition 2.2*] *A non Urysohn* 2*-homogeneous space is nowhere Urysohn.*

Proof. The proof is similar to the one of Proposition 1.7. One just needs to consider that if $h : X \to X$ is a homeomorphism, then $h(\overline{A}) = \overline{h(A)}$ for each $A \subseteq X$. □

The following proposition follows directly from the definition.

Proposition 1.9. [*Bonanzinga, Giacopello, M., Carlson, 2023, Proposition 2.3*] *A space X is hyperconnected if and only if for every finite $A \subseteq X$, $|A| = n$, $n \geq 2$, and for every choice of neighbourhoods U_a, $a \in A$, $\bigcap_{a \in A} U_a \neq \emptyset$.*

By Proposition 1.9 one can easily show the following.

Proposition 1.10. [*Bonanzinga, Giacopello, M., Carlson, 2023, Proposition 2.4*] *Let $n \geq 2$. Any n-Hausdorff space is not hyperconnected.*

Theorem 1.11. [*Bonanzinga, Giacopello, M., Carlson, 2023, Theorem 2.5*] *There is no n-Hausdorff non Hausdorff m-homogeneous space for every $n > 2$ and every $m > 1$.*

Proof. It follows directly from propositions 1.10 and 1.7. □

The following example shows that there exist 3-Hausdorff homogeneous spaces.

Example 1.12. [*Bonanzinga, Giacopello, M., Carlson, 2023, Example 2.6*] *A countable* 3*-Hausdorff homogeneous space.*

Consider the space X of natural numbers with the topology generated by the base $\{\{n, n+1\} : n \text{ is even}\}$. X is a 3-Hausdorff homogeneous space. △

Note that the space in the previous example is a homogeneous space which is not 2-homogeneous.

The analogues of Proposition 1.10 and Theorem 1.11 for n-Urysohn spaces do not hold, as the following example shows.

Example 1.13. [*Bonanzinga, Giacopello, M., Carlson, 2023, Example 2.7*] *A 3-Urysohn homogeneous space which is not Urysohn.*

Consider the well known "irrational slope space", also called Bing's Tripod space (see [Steen and Seebach, 1978, Example 75]). This space is n-homogeneous, $n \geq 1$ [Banakh, Banakh, O., and Stelmakh, 2021], and 3-Urysohn. △

Examples 1.12 and 1.13 give examples of an $(n+1)$-Hausdorff homogenous non n-Hausdorff space, and of an $(n+1)$-Urysohn homogenous non n-Urysohn space for $n = 2$. Note that the space of Example 1.12 is 3-Urysohn, and the construction can be generalized to obtain $(n+1)$-Urysohn non n-Hausdorff spaces for each $n \geq 2$. This shows that even in the class of homogeneous spaces $(n+1)$-Hausdorff (resp., $(n+1)$-Urysohn) spaces need not be n-Hausdorff (resp., n-Urysohn), with $n \geq 2$. In the last part of the chapter, for any $n \geq 2$ and for any n-Hausdorff space, it is presented an n-Hausdorff homogenous extension which is the countable union of n-H-closed spaces. Using this result it is possible to construct the Example 1.2.13, giving an n-Hausdorff homogeneous space which is not n-Urysohn, for every $n \geq 2$.

In [Bonanzinga, 2013], the author gives an example of an ω-Hausdorff space which is not n-Hausdorff for every $n \geq 2$. A countable ω-Hausdorff homogeneous space which is not n-Hausdorff for every $n \geq 2$ is now presented.

Example 1.14. [*Bonanzinga, Giacopello, M., Carlson, 2023, Example 2.10*] *There is a countable T_1 hyperconnected (hence not n-Hausdorff for every $n \geq 2$) space, which is ω-Hausdorff and homogeneous.*

In [Bonanzinga, Stavrova, and Staynova, 2017], the following space is constructed. Let $X = \mathbb{Z} \times \mathbb{Z}$ and $\mathcal{B} = \{U_{j,k}, V_{j,k} : j,k \in \mathbb{Z}\}$ is the subbase for the topology, where

$$U_{j,k} = \{(x,y) \in \mathbb{Z}^2 : x > j \text{ or } y > k\}$$

$$V_{j,k} = \{(x,y) \in \mathbb{Z}^2 : x < j \text{ or } y < k\}.$$

This is a T_1 hyperconnected, hence not n-Hausdorff space for every $n \geq 2$ which is ω-Hausdorff, homogeneous, first countable, Lindelöf. △

1.1 On the cardinality of n-Hausdorff and n-Urysohn homogeneous spaces.

In [Carlson and Ridderbos, 2008], the authors proved the following result.

Theorem 1.1.1. [*Carlson and Ridderbos, 2008, Theorem 2.3*] *Let X be a homogeneous Hausdorff space. Then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

Question 1.1.2. [*Bonanzinga, Giacopello, M., Carlson, 2023*] *Is $|X| \leq 2^{c(X)\pi\chi(X)}$ true for every homogeneous space X such that $H(X)$ is finite?*

The theorems 1.1.6, 1.1.8 and 1.1.13 below give partial answers to the previous question and Theorem 1.1.14 shows that the answer is positive if $H(X)$ is replaced by $U(X)$.

Recall that a family $\mathcal{U}$ of open sets of a space X is *point-finite* if for every $x \in X$, the set $\{U \in \mathcal{U} : x \in U\}$ is finite [Engelking, 1989, p. 44]. In [Tkachuck, 1983, Definition 2] it was defined the cardinal function $p(X) = sup\{|\mathcal{U}| : \mathcal{U}$ is a point-finite family in $X\}$.

In [Bonanzinga, 2013], the author generalized the previous definitions, introducing the following :

Definition 1.1.3. [*Bonanzinga, 2013, Definition 1*] A family $\mathcal{U}$ of open sets of a space X is *point-($\leq n$) finite*, where $n \in \mathbb{N}$, if for every $x \in X$, the set $\{U \in \mathcal{U} : x \in U\}$ has cardinality $\leq n$. For each $n \in \mathbb{N}$, define

$$p_n(X) = sup\{|\mathcal{U}| : \mathcal{U} \text{ is a point-}(\leq n) \text{ finite family in } X\}.$$

Proposition 1.1.4. [*Bonanzinga, 2013, Proposition 2*] *Let X be a topological space. Then $p_n(X) = c(X)$ for every $n \in \mathbb{N}$.*

Theorem 1.1.5. [*Erdös and Rado, 1956, Theorem 39*] *Let κ be a cardinal number and $f : [(2^{2^\kappa})^+]^3 \to \kappa$ a function, then there exists a subset $H \in [(2^{2^\kappa})^+]^{\kappa^+}$ such that $f \restriction [H]^3$ is constant.*

Theorem 1.1.6. [*Bonanzinga, Giacopello, M., Carlson, 2023, Theorem 3.4*] *Let X be a 3-Hausdorff homogeneous space. Then*

$$|X| \leq 2^{2^{c(X)\pi\chi(X)}}$$

Proof. Let $c(X)\pi\chi(X) = \kappa$. By Proposition 1.1.4, $p_2(X) \leq \kappa$. Suppose that $|X| \geq (2^{2^\kappa})^+$. For every triple $x_1, x_2, x_3 \in X$ of distinct points select neighbouroods $U_i(x_1, x_2, x_3)$ of x_i for $i = 1, 2, 3$ such that $\bigcap_{i=1}^3 U_i(x_1, x_2, x_3) = \emptyset$. Fix a point $p \in X$ and a local π-base $\mathcal{B}$ for p with $|\mathcal{B}| = \kappa$. Since the space is homogeneous, there exists a family $\{h_x\}_{x \in X}$ of homeomorphisms $h_x : X \to X$ such that $h_x(p) = x$ for every $x \in X$. Fix distinct points $x_1, x_2, x_3 \in X$ and observe that the set $\bigcap_{i=1}^3 h_{x_i}^{-1}(U_i(x_1, x_2, x_3))$ is an open neighbourhood of p; since $\mathcal{B}$ is a π-base, there is a non empty $B(x_1, x_2, x_3) \in \mathcal{B}$ such that $B(x_1, x_2, x_3)$ is contained in it. Consider now the function $f : [X]^3 \to \mathcal{B}$ defined by $f(\{x_1, x_2, x_3\}) = B(x_1, x_2, x_3)$. Then by Theorem 1.1.5 there is $Z \in [X]^{\kappa^+}$ and $B \in \mathcal{B}$ such that $f \restriction [Z]^3 = \{B\}$. Now, the family $\{h_z(B) : z \in Z\}$ is point-(≤ 2) finite in X. To see this, suppose by way of contradiction that there exists $x_0 \in X$ such that $|\{h_z(B) : x_0 \in h_z(B)\}| = 3$. So there are $z_1, z_2, z_3 \in X$ such that $x_0 \in h_{z_i}(B)$, $i = 1, 2, 3$. This implies

$$x_0 \in h_{z_i}(B) \subseteq h_{z_i}(\bigcap_{i=1}^{3} h_{z_i}^{-1}(U_i(z_1, z_2, z_3))) \subseteq h_{z_i}(h_{z_i}^{-1}(U_i(z_1, z_2, z_3))) = U_i(z_1, z_2, z_3).$$

Then, $x_0 \in \bigcap_{i=1}^{3} U_i(z_1, z_2, z_3) \neq \emptyset$, a contradiction. Furthermore, $\{h_z(B) : z \in Z\}$ has cardinality exactly κ^+. Otherwise there exists $z_0 \in Z$ s.t. $|\{z \in Z : h_z(B) = h_{z_0}(B)\}| = \kappa^+$. As before, from $h_{z_0}(B) \subseteq U_i(z_1, z_2, z_3)$ for every triple of elements in $\{z \in Z : h_z(B) = h_{z_0}(B)\}$, a contradiction. Thus $p_2(X) = \kappa^+$, a contradiction with $p_2(X) \leq \kappa$. This concludes the proof. □

Recall the following result.

Theorem 1.1.7. [*Erdös and Rado, 1956, Theorem 39*] *Let κ be a cardinal number, $n \geq 3$ and $f : [(2^{2^{\cdot^{\cdot^{\cdot^{2^\kappa}}}}})^+]^n \to \kappa$ a function (where the power is made $(n-1)$-many times), then there exists a subset $H \in [(2^{2^{\cdot^{\cdot^{\cdot^{2^\kappa}}}}})^+]^{\kappa^+}$ such that $f \restriction [H]^n$ is constant.*

Theorem 1.1.8. [*Bonanzinga, Giacopello, M., Carlson, 2023, Theorem 3.6*] *Let X be an n-Hausdorff homogeneous space, with $n \geq 2$. Then*

$$|X| \leq 2^{2^{\cdot^{\cdot^{2^{c(X)\pi\chi(X)}}}}}$$

where the power is made $(n-1)$-many times.

Proof. Similar to the proof of the previous theorem using Theorem 1.1.7 instead of Theorem 1.1.5. □

In [Carlson, Porter, and Ridderbos, 2012], the authors proved the following result.

Theorem 1.1.9. [*Carlson, Porter, and Ridderbos, 2012, Theorem 5.3(1)*] *If X is an n-Hausdorff homogeneous space, with $n \geq 2$, then $|X| \leq d(X)^{\pi\chi(X)}$.*

Definition 1.1.10. [*Velichko, 1966, Definition 2*]Let X be a space. For $A \subseteq X$, the *θ-closure* of A is defined by

$$cl_\theta(A) = \{x \in X : \overline{V} \cap A \neq \emptyset \text{ for every open set } V \ni x\}$$

A set $A \subseteq X$ is *θ-dense* if $cl_\theta(A) = X$. The *θ-density* $d_\theta(X)$ of X is defined as the least cardinality of a θ-dense subset of X.

Theorem 1.1.11. [*Carlson, Porter, and Ridderbos, 2012, Theorem 5.3(2)*] *Let X be an n-Urysohn homogeneous space, where $n \geq 2$. Then $|X| \leq d_\theta(X)^{\pi\chi(X)}$.*

Theorem 1.1.12. [*Carlson, 2007, Theorem 2.4*] *Let X be a space. Then $d_\theta(X) \leq \pi\chi(X)^{c(X)}$.*

Recall that a space is *quasiregular* if every nonempty open set contains a nonempty regular closed set [Engelking, 1989].

Theorem 1.1.13. [*Bonanzinga, Giacopello, M., Carlson, 2023, Theorem 3.8*] *Let X be an n-Hausdorff quasiregular homogeneous space with $n \geq 2$, then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

Proof. As X is n-Hausdorff and homogeneous, $|X| \leq d(X)^{\pi\chi(X)}$ by Theorem 1.1.9. As X is quasiregular, it follows that $d_\theta(X) = d(X)$. By Theorem 1.1.12, $d(X) \leq \pi\chi(X)^{c(X)}$. Thus, $|X| \leq d(X)^{\pi\chi(X)} \leq \left(\pi\chi(X)^{c(X)}\right)^{\pi\chi(X)} = 2^{c(X)\pi\chi(X)}$. □

By Theorems 1.1.11 and 1.1.12, it is obtained the following.

Theorem 1.1.14. [*Bonanzinga, Giacopello, M., Carlson, 2023, Theorem 3.12*] *Let X be an n-Urysohn homogeneous space, where $n \geq 2$. Then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

1.2 A homogenous extension of an n-Hausdorff space

In [Basile, Bonanzinga, Carlson, and Porter, 2019] the following is introduced to generalize H-closed property in the context of n-Hausdorff spaces.

Definition 1.2.1. [*Basile, Bonanzinga, Carlson, and Porter, 2019, Definition 6*] Let $n \geq 2$. An n-Hausdorff space X is called *n-H-closed* if X is closed in every n-Hausdorff space Y in which X is embedded.

In the following it is presented (Theorem 1.2.12 below) a homogeneous extension of an n-Hausdorff space, $n \geq 2$, which is a countable union of n-H-closed spaces; using this result, the Example 1.2.13 consisting in an n-Hausdorff homogeneous non n-Urysohn space is given. This is in fact a generalization of the following result.

Theorem 1.2.2. [*Carlson, Porter, and Ridderbos, 2017, Theorem 2.3*] *Let X be a Hausdorff space. Then X can be embedded in a homogeneous space that is the countable union of H-closed spaces.*

Given a space X and an ultrafilter $\mathcal{U} \subseteq \mathcal{P}(X)$, we will say that $\mathcal{U}$ is an *open ultrafilter* if its base consists of open sets. Define $a\mathcal{U} = \bigcap\{\overline{U} : U \in \mathcal{U}\}$. For an n-Hausdorff space X, with $n \geq 2$, an open ultrafilter $\mathcal{U}$ on X is said to be *full* if $|a\mathcal{U}| = n-1$.

Theorem 1.2.3. [*Basile, Bonanzinga, Carlson, and Porter, 2019, Theorem 3*] *Let $n \geq 2$, and X be a space. The following are equivalent:*

(a) X is n-Hausdorff

(b) if $\mathcal{U}$ is an open ultrafilter of X, then $|a\mathcal{U}| \leq n-1$.

Theorem 1.2.4. [*Basile, Bonanzinga, Carlson, and Porter, 2019*] *Let $n \geq 2$, and X be an n-Hausdorff space. The following are equivalent:*

(a) X is n-H-closed

(b) every open ultrafilter on X is full.

Recall the following construction, made in [Basile, Bonanzinga, Carlson, and Porter, 2019, Extension contruction Technique 2]. Let $n \geq 2$, X be an n-Hausdorff space and $\mathfrak{U} = \{\mathcal{U} : \mathcal{U}$ is an open ultrafilter such that $|a\mathcal{U}| < n-1\}$. Enumerate $\mathfrak{U}$ by $\mathfrak{U} = \{\mathcal{U}_\alpha : \alpha \in |\mathfrak{U}|\}$. For each $\alpha \in |\mathfrak{U}|$, let $k\alpha = (n-1) - |a\mathcal{U}_\alpha|$ and $\{p_{\alpha i} : 1 \leq i \leq k\alpha\}$ be a set of distinct points disjoint from X. Let $Y = X \cup \{p_{\alpha i} : 1 \leq i \leq k\alpha, \alpha \in |\mathfrak{U}|\}$. A set V is defined to be open in Y if $V \cap X$ is open in X and if $p_{\alpha i} \in V$ for $1 \leq i \leq k\alpha$, $V \cap X \in \mathcal{U}_\alpha$. The space Y is an n-Hausdorff space.

In the following results it is used the notation of the previous contruction.

Proposition 1.2.5. [*Basile, Bonanzinga, Carlson, and Porter, 2019, Proposition 9(a)*] *For every $\alpha \in |\mathfrak{U}|$,*

$$\mathcal{U}_\alpha = \{V \cap X : p_{\alpha i} \in V \in \tau(Y) \text{ for some } 1 \leq i \leq k\alpha\},$$

where $\tau(Y)$ is the topology on Y.

By the previous proposition the space Y has the property that every open ultrafilter on Y is full. Indeed the points $p_{\alpha i}$, $1 \leq i \leq k\alpha$, added to the space X, are in the closure of each element of $\mathcal{U}_\alpha$. Therefore the space Y is n-H-closed.

Definition 1.2.6. [*Basile, Bonanzinga, Carlson, and Porter, 2019, Remark 3*] Let $n \geq 2$, S and T be n-H-closed extensions of an n-Hausdorff space X. The extension S is *projectively larger* than T if there is a continuous surjection $f : S \to T$ such that $f(x) = x$ for $x \in X$.

Notice that this projectively larger function may not be unique.

Theorem 1.2.7. [*Basile, Bonanzinga, Carlson, and Porter, 2019, Theorem 5*] *Let $n \geq 2$, X be n-Hausdorff space and Y be the n-H-closed extension of X constructed above. If Z is an n-H-closed extension of X, there is a continuous surjection $f : Y \to Z$ such that $f(x) = x$ for all $x \in X$.*

Theorem 1.2.7 shows that the n-H-closed extension Y of X is projectively larger than every n-H-closed extension of X. Moreover, the space Y has an interesting unique property as it is noted in the next result.

Theorem 1.2.8. [*Basile, Bonanzinga, Carlson, and Porter*, 2019, *Theorem 6*] *Let* $n \geq 2$, *X be an n-Hausdorff space and Y be the n-H-closed extension of X described above. Let $f : Y \to Y$ be a continuous surjection such that $f(x) = x$ for all $x \in X$. Then f is a homeomorphism.*

Remark 1.2.9. In the class of Hausdorff spaces the function in Definition 1.2.6 is unique [Basile, Bonanzinga, Carlson, and Porter, 2019]. Sometimes this is a problem in non-Hausdorff spaces. The n-H-closed space Y constructed before for an n-Hausdorff space X is a projective maximum, that is Y is projectively larger than every n-H-closed extention and given a continuous surjection $f : Y \to Y$ such that $f(x) = x$ for every $x \in X$, then f is a homeomorphism. For the future such Y will be denoted by n-kX and it is said to be the *n-Katětov extension* of X.

In [Uspenskiĭ, 1983] it is shown that for any space X there exists a cardinal κ and a nonempty subspace $Z \subseteq X^\kappa$ such that $X \times Z$ is homogeneous. The space Z is found by selecting a set A such that $\kappa = |A| \geq |X|$ and letting $Z = \{f \in {}^A X : \text{for each } x \in X, |f^{-1}(x)| = \kappa\}$, where ${}^A X$ is the space of all functions from A to X. Both Z and $X \times Z$ are homogeneous and homeomorphic. For the present construction put $\mathbf{H}(X) = X \times Z$ and consider X as a subspace of $\mathbf{H}(X)$.

Lemma 1.2.10. [*Carlson, Porter, and Ridderbos*, 2017, *Lemma 2.1*] *Let X be a space and $h : X \to X$ be a homeomorphism and let id_Z be the identity function on Z. Then the function $h \times id_Z : \mathbf{H}(X) \to \mathbf{H}(X)$ is also a homeomorphism that extends h.*

Lemma 1.2.11. [*Bonanzinga, Giacopello, M., Carlson*, 2023, *Lemma 4.11*] *Let $n \geq 2$, X be an n-Hausdorff space and $h : X \to X$ be a homeomorphism. Then there is a homeomorphism n-$kh : n$-$kX \to n$-kX that extends h.*

Proof. Let $p \in n\text{-}kX \setminus X$, then $p = p_{\alpha i}$ for some $\alpha \in |\mathfrak{U}|$ and for some $i = 1, ..., k\alpha$. The set $\mathcal{V} = \{h(U) : U \in \mathcal{U}_\alpha\}$ is an open ultrafilter on X and since $|a\mathcal{U}_\alpha| = |a\mathcal{V}|$, there exists $\beta \in |\mathfrak{U}|$ such that $\mathcal{V} = \mathcal{U}_\beta$. Define $n\text{-}kh(p_{\alpha i}) = p_{\beta i}$ for every $i = 1, ..., k\alpha = k\beta$. For $x \in X$, define $n\text{-}kh(x) = h(x)$. The function n-kh is clearly a homeomorphism that extends h. □

Theorem 1.2.12. [*Bonanzinga, Giacopello, M., Carlson*, 2023, *Theorem 4.12*] *Let $n \geq 2$, X be an n-Hausdorff space. Then X can be embedded in an homogeneous space that is the countable union of n-H-closed spaces.*

Proof. Let $H_1 = \mathbf{H}(n\text{-}kX)$. If H_m is defined, let's define $H_{m+1} = \mathbf{H}(n\text{-}kH_m)$ and $H = \bigcup_{m \in \mathbb{N}} H_m$. A subset $U \subseteq H$ is open in H if $U \cap H_m \in \tau(H_m)$ for every $m \in \mathbb{N}$. The space H is the countable union of n-H-closed spaces. It is sufficent to prove that H is homogeneous. Let $p, q \in H$. Since $H_m \subseteq H_{m+1}$, there exists $m \in \mathbb{N}$ such that $p, q \in H_m$. Each H_m is homogeneous, then there exist a homeomorphism $h : H_m \to H_m$ such that $h(p) = q$. By Lemma 1.2.11 there exists a homeomorphism $n\text{-}kh : n\text{-}kH_m \to n\text{-}kH_m$ that extends h. By Lemma 1.2.10 the function $n\text{-}kh \times id_Z : H_{m+1} \to H_{m+1}$ is a homeomorphism. Put $n\text{-}kh = h_1$. By induction h can be extended to $h_k : H_{m+k} \to H_{m+k}$ for every $k \in \mathbb{N}$. The function $g = \bigcup_{k \in \mathbb{N}} h_k : H \to H$ extends h and it is a homeomorphism on H. Then H is homogeneous. □

Example 1.2.13. [*Bonanzinga, Giacopello, M., Carlson*, 2023, *Example 4.13*] *An example of an n-Hausdorff, homogeneous, not n-Urysohn space which is the countable union of n-H-closed spaces, for every $n \geq 2$.*

Let's take an n-Hausdorff, not n-Urysohn space X (for example see [Bonanzinga, 1998, Example 4]), $n \geq 2$. Then, by Theorem 1.2.12, X can be embedded in an n-Hausdorff, homogeneous space Y which is the countable union of n-H-closed spaces. Furthermore Y is not n-Urysohn, since X is a non-n-Urysohn subset of it. $\triangle$

CHAPTER 2

SOME RELATIVE COVERING PROPERTIES DEFINED BY STARS

All the new contributions obtained by the author of the thesis included in this chapter are contained in the articles [Bonanzinga and Maesano, 2022] and [Bonanzinga, Giacopello and Maesano, 2023].

Definition 2.1. A space X is

- *strongly star-compact*, briefly SSC, if for every open cover $\mathcal{U}$ of the space X, there exists a finite subset F of X such that $st(F,\mathcal{U}) = X$ [Fleischman, 1970, Definition 5];
- *star-compact*, briefly SC, if for every open cover $\mathcal{U}$ of the space X, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $st(\bigcup\mathcal{V},\mathcal{U}) = X$ [Sarkhel, 1986].

In [van Douwen, Reed, Roscoe, and Tree, 1991] several useful results are proved for compact-type properties defined by stars.

Theorem 2.2. *[van Douwen, Reed, Roscoe, and Tree, 1991, theorems 2.1.4 and 2.1.5] Let X be a T_2 space. The following affirmations are equivalent*

- *X is CC*
- *X is SSC*

Definition 2.3. A space X is

- *strongly star-Lindelöf*, briefly SSL, if for every open cover $\mathcal{U}$ of the space X, there exists a countable subset C of X such that $st(C,\mathcal{U}) = X$ [Ikenaga, 1990];
- *star-Lindelöf*, briefly SL, if for every open cover $\mathcal{U}$ of the space X, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $st(\bigcup\mathcal{V},\mathcal{U}) = X$ [Ikenaga, 1983].

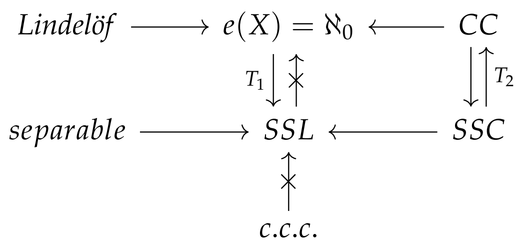
In [Bonanzinga, 1998] the Lindelöf-type star-properties are deeply studied. Several result concern the bond between these properties and the property for the space to possess countable extent. Recall the following

Proposition 2.4. *[Bonanzinga, 1998, Proposition 1.1] Let X be a T_1 space with $e(X) = \aleph_0$. Then X is SSL.*

In the same article it is pointed out that the converse of the previous proposition is not true: every Isbell-Mrówka space is separable, hence SSL, and has uncountable extent.

Proposition 2.5. [*Bonanzinga, 1998, Corollary 3.9*] *Let X be a collectionwise Hausdorff SSL space. Then $e(X) = \aleph_0$.*

The following diagram is also given.



Recall also the following result in [Matveev, 2002], exposed with different terminology

Theorem 2.6. [*Matveev, 2002, Theorem 1*] *For every infinite cardinal κ there is a Tychonoff SSL space X such that $e(X) \geq \kappa$.*

Recall that the product of a SC (SSC) space with a compact space is SC (SSC) ([Fleischman, 1970], [van Douwen, Reed, Roscoe, and Tree, 1991]); further the product of a SL space with a compact space is SL ([van Douwen, Reed, Roscoe, and Tree, 1991]) while the product of a SSL space with a compact space need not be SSL ([van Douwen, Reed, Roscoe, and Tree, 1991, Example 3.3.4]).

In [Ikenaga and Tani, 1980] the following property is defined.

Definition 2.7. [*Ikenaga and Tani, 1980*] A space X is *$\mathcal{K}$-star-compact*, briefly $\mathcal{K}$-*SC*, if for every open cover $\mathcal{U}$ of the space X, there exists a compact subset K of X such that $st(K, \mathcal{U}) = X$.

In paragraph 2.1, results on relative versions of compact-type and Lindelöf-type properties are presented.

In [Kočinac, 1999] and [Bonanzinga, Cammaroto, and Kočinac, 2004] selection principles defined by stars are introduced. Let X be a space and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be collections of families of its subsets. Then:

- $S^*_{fin}(\mathcal{A}, \mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each $n \in \omega$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\bigcup_{n\in\omega}\{st(V, \mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$.

- $U^*_{fin}(\mathcal{A}, \mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each $n \in \omega$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{st(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

- $SS^*_{fin}(\mathcal{A}, \mathcal{B})$ denotes the following statement: for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle F_n : n \in \omega \rangle$ of finite subsets of X such that $\{st(F_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

Denote the class of all covers (resp. γ-covers) of X by $\mathcal{O}$ (resp., by Γ). The space X is said to be:

- *strongly star-Menger*, briefly SSM, if satisfies $SS^*_{fin}(\mathcal{O},\mathcal{O})$, i.e. , for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle F_n : n \in \omega \rangle$ of finite subsets of the space such that $\{st(F_n,\mathcal{U}_n) : n \in \omega\} \in \mathcal{O}$ [Kočinac, 1999, Definition 1.4];
- *star-Menger*, briefly SM, if satisfies $S^*_{fin}(\mathcal{O},\mathcal{O})$, i.e. , for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$, $n \in \omega$, and $\bigcup_{n\in\omega}\{st(V,\mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{O}$ [Kočinac, 1999, Definition 1.4];
- *strongly star-Hurewicz*, briefly SSH, if satisfies $SS^*_{fin}(\mathcal{O},\Gamma)$, i.e. , for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle F_n : n \in \omega \rangle$ of finite subsets of the space such that $\{st(F_n,\mathcal{U}_n) : n \in \omega\} \in \Gamma$ [Bonanzinga, Cammaroto, and Kočinac, 2004, p. 81];
- *star-Hurewicz*, briefly SH, if satisfies $U^*_{fin}(\mathcal{O},\Gamma)$, i.e. , for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$, $n \in \omega$, and $\{st(\bigcup \mathcal{V}_n,\mathcal{U}_n) : n \in \omega\} \in \Gamma$ [Bonanzinga, Cammaroto, and Kočinac, 2004, p. 81].

Recall also the following results which are crucial in this context.

Theorem 2.8. [*Song, 2013, Example 2.4*] *For every infinite cardinal κ, there exists a T_1 SSM space X such that $e(X) \geq \kappa$.*

Proposition 2.9. [*Sakai, 2014, Corollary 2.2*] *Every closed and discrete subspace of a regular SSM space has cardinality less than $\mathfrak{c}$. Hence a regular SSM space has extent less or equal to $\mathfrak{c}$.*

Proposition 2.10. [*Bonanzinga and Matveev, 2009, Proposition 2*] *Let $\mathcal{A}$ be an almost disjoint family. The following affirmations are equivalent:*

- *$\Psi(\mathcal{A})$ is SSM*
- *$|\mathcal{A}| < \mathfrak{d}$.*

Proposition 2.11. [*Bonanzinga and Matveev, 2009, Proposition 3*] *Let $\mathcal{A}$ be an almost disjoint family. The following affirmations are equivalent:*

- *$\Psi(\mathcal{A})$ is SSH*
- *$|\mathcal{A}| < \mathfrak{b}$.*

In paragraph 2.2, generalizations of the following results for some relative versions of the selection principles defined by stars will be presented.

Proposition 2.12. [*Sakai, 2014, Proposition 1.7*] *Every SL (SSL) space of cardinality less than $\mathfrak{d}$ is SM (SSM).*

Proposition 2.13. [*Casas-de la Rosa, Garcia-Balan, and Szeptycki, 2019, Corollary 3.10*] *Every SL (SSL) space of cardinality less than $\mathfrak{b}$ is SH (SSH).*

Theorem 2.14. [*Sakai, 2014, Corollary 2.6*] *If X is a regular SM space such that $w(X) = \mathfrak{c}$, then every closed and discrete subspace of X has cardinality less than $\mathfrak{c}$. Hence $e(X) \leq \mathfrak{c}$.*

Theorem 2.15. [*Sakai, 2014, Proposition 2.9*] *The following statements are equivalent.*

1. *$\omega_1 = \mathfrak{d}$;*

2. *if X is a regular SSM space, then $e(X) \leq \omega$.*

Theorem 2.16. [*Kočinac, 1999, Theorem 2.13*] *If X is a SM space and Y is a compact space, then $X \times Y$ is a SM space.*

Example 2.17. [*Kočinac, 1999, Example 2.13*] *There exist two spaces X,Y such that X is SSM, Y is SSL and $X \times Y$ is not SSM.*

In [Bonanzinga and Matveev, 2001, Corollary 2.4] it is proved that the product of a space with uncountable extent with a space having uncountable cellularity is not SSL, hence not SSM; then it is given a consistent example of a not SSM space which is the product of a SSM space and a compact space.

In the last section of paragraph 2.2, partial answers for the following questions are presented.

Question 2.18. [*Kočinac, Konca, and Singh, 2022, Problem 2*] *Is the product of a set SSM space with a compact space a set SSM space?*

Question 2.19. [*Kočinac, Konca, and Singh, 2022, Problem 2*] *Is the product of a set SM space with a compact space a set SM space?*

Partial answers to the previous questions are given also for the classes of set SH and set SSH spaces.

2.1 Some relative compact-type and Lindelöf type properties defined by stars

In [Bonanzinga and Pansera, 2007] the following relative versions of compact-type properties defined by stars were considered.

Definition 2.1.1. [*Bonanzinga and Pansera, 2007, p. 236*]A subspace Y of a space X is:

- *relatively SC in X* if for every open cover $\mathcal{U}$ of X there is a finite subset $\mathcal{V} \subset \mathcal{U}$ such that $st(\bigcup \mathcal{V}, \mathcal{U}) \supset Y$.
- *relatively (*) in X* if for every open cover $\mathcal{U}$ of X there is a compact subset $K \subset X$ such that $st(K, \mathcal{U}) \supset Y$.

In similar way it is possible to formulate the following properties of subsets of a topological space.

Definition 2.1.2. [*Bonanzinga, M., 2022*]A subspace Y of a space X is:

- *relatively finite (*) in X* if for every open cover $\mathcal{U}$ of X there is a finite subset $K \subset X$ such that $st(K, \mathcal{U}) \supset Y$.
- *relatively countable (*) in X* if for every open cover $\mathcal{U}$ of X there is a countable subset $C \subset X$ such that $st(C, \mathcal{U}) \supset Y$.

In the following propositions, characterizations of SC, $\mathcal{K}$-SC and SSC properties using relative versions of them are given.

Proposition 2.1.3. [*Bonanzinga, M., 2022, Propoistion 1.1*] *The following are equivalent for a space X:*

1. *X is SC;*

2. *for every $A \subset X$, A is relatively SC in X.*

Proof. The implication 1. $\Rightarrow$ 2. is obvious.2. $\Rightarrow$ 1. Suppose, by contraddiction, there is an open cover $\mathcal{U}$ such that for every finite subfamily $\mathcal{V} \subseteq \mathcal{U}$, $St(\bigcup \mathcal{V}, \mathcal{U}) \neq X$. Fixed a finite subfamily $\mathcal{V} \subseteq \mathcal{U}$, call $Y = X \setminus St(\bigcup \mathcal{V}, \mathcal{U})$; since Y is relatively SC in X, there is a finite subfamily $\mathcal{V}_0 \subseteq \mathcal{U}$ such that $Y \subset St(\bigcup \mathcal{V}_0, \mathcal{U})$. Define $\mathcal{V}' = \mathcal{V}_0 \cup \mathcal{V}$. Then, $\mathcal{V}'$ is a finite subfamily of $\mathcal{U}$ such that $X = St(\bigcup \mathcal{V}', \mathcal{U})$; a contradiction. □

Proposition 2.1.4. [*Bonanzinga, M.*, 2022, *Proposition 1.2*] *The following are equivalent for a space X:*

1. *X is $\mathcal{K}$-SC;*

2. *for every $A \subset X$, A is relatively (*) in X;*

3. *for every nonempty subset A of X and every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a compact subset K of X such that $st(K, \mathcal{U}) \supset A$.*

Proof. The implications 1. $\Leftrightarrow$ 2. and 3. $\Rightarrow$ 1. are obvious. 1. $\Rightarrow$ 3.. Assume that X is a $\mathcal{K}$-SC space. Let $A \subseteq X$ be a non-empty subset and $\mathcal{U}$ a family of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}$. Define

$$\mathcal{U}' = \mathcal{U} \cup \{X \setminus \overline{A}\}.$$

Clearly, $\mathcal{U}'$ is an open cover of X. Since the space is $\mathcal{K}$-SC, there is a compact subset K of X such that $st(K, \mathcal{U}') = X$. Fix $x \in A$. Thus

$$x \in st(K, \mathcal{U}') \Leftrightarrow st(x, \mathcal{U}') \cap K \neq \emptyset$$

Then

$$st(x, \mathcal{U}') = \bigcup \{U \in \mathcal{U}' : x \in U\} = \bigcup \{U \in \mathcal{U} : x \in U\} = st(x, \mathcal{U})$$

So

$$x \in st(K, \mathcal{U}') \Leftrightarrow st(x, \mathcal{U}') \cap K \neq \emptyset \Leftrightarrow st(x, \mathcal{U}) \cap K \neq \emptyset \Leftrightarrow x \in st(K, \mathcal{U}).$$

Then $A \subset st(K, \mathcal{U})$. □

The proof of the following two propositions are similar to the previous one, this omitted.

Proposition 2.1.5. [*Bonanzinga, M.*, 2022, *Proposition 1.3*] *The following are equivalent for a space X:*

1. *X is SSC;*

2. *for every $A \subset X$, A is relatively finite (*) in X;*

3. *for every nonempty subset A of X and every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a finite subset F of X such that $st(F, \mathcal{U}) \supset A$.*

Proposition 2.1.6. [*Bonanzinga, M.*, 2022, *Proposition 1.4*] *The following are equivalent for a space X:*

1. *X is SSL;*

2. *for every $A \subset X$, A is relatively countable (*) in X;*

3. *for every nonempty subset A of X and every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a countable subset C of X such that $st(C, \mathcal{U}) \supset A$.*

Notice that in propositions 2.1.4, 2.1.5 and 2.1.6, the affirmation 3. the compact, resp. finite or countable, sets are required to be subsets of X. Asking that the finite, resp. finite or countable, sets must be a subsets of the closure of A, the authors in [Kočinac, Konca, and Singh, 2022] introduced the following relative versions of it.

Definition 2.1.7. [*Bonanzinga, M.*, 2022] A subset A of a space X is

- *relatively* SSC* in X if for every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a finite subset F of $\overline{A}$ such that $st(F, \mathcal{U}) \supset A$;
- *relatively* SC* in X if for every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $st(\bigcup \mathcal{V}, \mathcal{U}) \supset A$;
- *relatively* SSL* in X if for every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a countable subset F of $\overline{A}$ such that $st(F, \mathcal{U}) \supset A$;
- *relatively* SL* in X if for every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $st(\bigcup \mathcal{V}, \mathcal{U}) \supset A$;

The following result is obvious.

Proposition 2.1.8. [*Bonanzinga, M.*, 2022, *Propoistion 2.1*] *Let A be a subspace of a space X. If A is relatively* SC in X, then A is relatively SC in X.*

The converse of the previous proposition is not true; see Example 2.1.24, where a SC non set SC space is given.

Using definitions above, in [Kočinac, Konca, and Singh, 2022], the authors introduced the following relative versions of SC and SSC (with different terminology).

Definition 2.1.9. [*Kočinac, Konca, and Singh*, 2022, *Definition 3*] A space X is

- *set strongly star-compact*, briefly set SSC, if every nonempty subset A of X is relatively* SSC in X.
- *set star-compact*, briefly set SC, if every nonempty subset A of X is relatively* SC in X.

Recall that in the class of Hausdorff spaces, a space is CC iff it is SSC; by this fact and since CC is hereditary with respect to closed sets, it is clear that CC implies set SSC.

Proposition 2.1.10. [*Bonanzinga, M.*, 2022, *Proposition 2.2*] *If X is a Hausdorff space, then the following properties are equivalent:*

(i) X is CC

(ii) X is set SSC

(iii) X is SSC

The list of equivalent conditions can be enlarged at the cost of a higher separation axiom.

Proposition 2.1.11. [*Bonanzinga, Giacopello, M.*, 2023, *Proposition 1.3*] *If X is a regular space, set SC and CC are equivalent properties.*

Proof. Of course, every CC space is set SC. Now, let X be a regular set SC space. By contradiction, assume there exists a closed and discrete subspace $D = \{x_n : n \in \omega\}$ of X. By regularity, there exists a disjoint family $\mathcal{U} = \{U_n : n \in \omega\}$ of open subsets of X such that $x_n \in U_n$, for every $n \in \omega$. Then $D \subseteq \bigcup \mathcal{U}$ but for every finite subfamily $\mathcal{V}$ of $\mathcal{U}$, $D \not\subset st(\bigcup \mathcal{V}, \mathcal{U})$; a contradiction. □

Corollary 2.1.12. [*Bonanzinga, Giacopello, M.*, 2023] *If X is a regular space, then the following properties are equivalent:*

(i) X is CC

(ii) X is set SSC

(iii) X is SSC

(iv) X is set SC

In a similar way the following relative version of $\mathcal{K}$-SC property is considered.

Definition 2.1.13. [*Bonanzinga, M.*, 2022] A subset A of a space X is *relatively* $\mathcal{K}$-SC* in X if for every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there exists a compact subset K of $\overline{A}$ such that $st(K, \mathcal{U}) \supset A$

And a covering property can be defined.

Definition 2.1.14. [*Bonanzinga, M.*, 2022, *Definition 2.2*] A space X is *set $\mathcal{K}$-star-compact*, briefly *set $\mathcal{K}$-SC*, if every subset A of X is relatively* $\mathcal{K}$-SC in X.

The examples below will distinguish all the properties stated up to now.

Example 2.1.15. [*Bonanzinga, M.*, 2022, *Example 2.2*] *A T_1 set SC not $\mathcal{K}$- SC space (hence not set $\mathcal{K}$- SC and set SSC).*

In [Song, 2007, Example 2.2] it is considered the set $X = \omega_1 \cup A$, where $A = \{a_\alpha : \alpha \in \omega_1\}$ is a set of cardinality ω_1, topologized as follows: ω_1 has the usual order topology and is an open subspace of X; a basic neighborhood of a point $a_\alpha \in A$ takes the form

$$O_\beta(a_\alpha) = \{a_\alpha\} \cup (\beta, \omega_1), \text{ where } \beta < \omega_1.$$

In [Song, 2007] it is proved that X is SC not $\mathcal{K}$- SC. It is enough to prove that X is set SC. Fix a nonempty subset $B \subseteq X$ and a family $\mathcal{U}$ of open sets of X such that $\overline{B} \subseteq \bigcup \mathcal{U}$; it is possible to assume that $\mathcal{U}$ consists of basic open sets. Define the sets $B_1 = B \cap \omega_1$ and $B_2 = B \cap A$; obviously $B = B_1 \cup B_2$. Since ω_1 is countably compact (hence set SC) and $\overline{B_1} \subseteq \overline{B}$, it is possible to find a finite subfamily $\mathcal{V}' \subset \mathcal{U}$ such that $B_1 \subset St(\bigcup \mathcal{V}', \mathcal{U})$. Fix $a_\gamma \in B_2$ and consider an open set $U_\gamma \in \mathcal{U}$ such that $a_\gamma \in U_\gamma$. Then, $B_2 \subseteq St(U_\gamma, \mathcal{U})$. Define $\mathcal{V} = \mathcal{V}' \cup \{U_\gamma\}$. The family $\mathcal{V}$ is a finite subfamily of $\mathcal{U}$ such that $B \subseteq St(\cup \mathcal{V}, \mathcal{U})$ and then X is set SC. △

Definition 2.1.16. [*Kočinac and Singh*, 2020, *Definition 5*] A space X is

- *set strongly star-Lindelöf*, briefly set SSL, if every nonempty subset A of X is relatively* SSL in X.
- *set star-Lindelöf*, briefly set SL, if every nonempty subset A of X is relatively* SL in X.

Proposition 2.1.17. [*Bonanzinga, M., 2022, Proposition 3.1*] *Let X be a T_1 space. The following affirmations are equivalent*

- *X is set SSL*
- *$e(X) = \aleph_0$*

Proof. Let X be a T_1 space having countable extent. Assume, by contradiction, that X is not set SSL. Then there exist a non empty subset A of X and a family $\mathcal{U}$ of open sets of X such that $\overline{A} \subseteq \bigcup\mathcal{U}$ and for every countable subset C of $\overline{A}$, $A \not\subseteq st(C,\mathcal{U})$. Then, by induction, for each $\alpha < \omega_1$ choose a point $a_\alpha \in A$ such that $a_\alpha \notin st(\{a_\gamma : \gamma < \alpha\},\mathcal{U})$. Let $B = \{a_\alpha : \alpha < \omega_1\} \subseteq \overline{A}$. Since $e(X) = \aleph_0$, there exists a limit point a^* for B in $\overline{A}$. Then there exist $U^* \in \mathcal{U}$ and $\alpha_1, \alpha_2 \in \omega_1$ with $\alpha_1 \neq \alpha_2$ such that $a^* \in U^*$ and $\{a_{\alpha_1}, a_{\alpha_2}\} \subset U^*$; a contradiction. Viceversa, assume that X is set SSL and let D be a closed and discrete subset of X. For every $x \in D$ there exists an open neighbourhood U_x of x such that $U_x \cap D = \{x\}$. Put $\mathcal{U} = \{U_x : x \in D\}$. Then, $\overline{D} = D \subset \bigcup\mathcal{U}$. Since X is set SSL, there exists a countable subset C of $\overline{D} = D$ such that $D \subset st(C,\mathcal{U})$. Then $C = D$; hence D is countable. □

It can be proved the following generalization of Proposition 2.5.

Proposition 2.1.18. [*Bonanzinga, M., 2022, Proposition 3.2*] *If X is a collectionwise Hausdorff set SL space, then $e(X) = \aleph_0$ (equivalently, X is set SSL).*

Proof. Suppose that X is a collectionwise Hausdorff set SL space and let D be a closed and discrete subspace of X. For each $x \in D$, there exists an open neighbourhood U_x of x such that $U_x \cap D = \{x\}$. Since X is collectionwise Hausdorff, assume that U_x are pairwise disjoint. Put $\mathcal{U} = \{U_x : x \in D\}$. Then, $\bigcup\mathcal{U} \supseteq D = \overline{D}$ and, by hypothesis there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $D \subseteq st(\bigcup\mathcal{V},\mathcal{U})$. Then D is countable. □

Corollary 2.1.19. [*Bonanzinga, M., 2022, Corollary 3.1*] *If X is a collectionwise Hausdorff space, then the following properties are equivalent:*

1. *$e(X) = \aleph_0$*
2. *X is set SSL*
3. *X is set SL*
4. *X is SSL.*

Note that SL property cannot be added to the list of equivalences of the previous corollary even in the class of Tychonoff spaces, as the following example shows.

Example 2.1.20. [*Bonanzinga, M., 2022, Example 3.1*] *A Tychonoff collectionwise Hausdorff $\mathcal{K}$-SC (hence SL) space which is not SSL.*

Let D be a discrete space of cardinality $\mathfrak{c}$. Consider the space $X = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$. The space X is collectionwise Hausdorff and $e(X) > \omega$; then by the previous result it is not SSL. It is enough to prove that X is $\mathcal{K}$-SC (hence SL). Let $\mathcal{U}$ be an open cover of X. For each $\alpha \in \mathfrak{c}$, there exist $n_\alpha \in \omega$ and $U_n \in \mathcal{U}$ such that $\{\alpha\} \times (n_\alpha, \omega] \subseteq U_\alpha$. Let $\gamma = \sup\{n_\alpha : \alpha \in \mathfrak{c}\}$. The set $K_1 = \beta D \times (\gamma + 1)$ is compact and $D \times \{\omega\} \subseteq st(K_1,\mathcal{U})$; indeed $(\alpha, \omega) \in U_\alpha \in \mathcal{U}$ and $U_\alpha \cap K_1 \neq \emptyset$, for every $\alpha \in D$. The subspace $\beta D \times \omega$ is CC, hence there exists a finite subset F of

$\beta D \times \omega$ such that $\beta D \times \omega \subset st(F, \mathcal{U})$. Then $K = K_1 \cup F$ is a compact subset of X such that $st(K, \mathcal{U}) = X$. △

However the following holds.

Corollary 2.1.21. [*Bonanzinga, M.*, 2022, *Corollary 3.2*] *If X is a normal collectionwise Hausdorff space, then the following properties are equivalent:*

1. $e(X) = \aleph_0$
2. *X is set SSL*
3. *X is set SL*
4. *X is SSL*
5. *X is SL.*

Proof. It is enough to prove 5. $\Rightarrow$ 1.. Suppose by contradiction that X contains an uncountable closed discrete subset D. By the collectionwise Hausdorff property of X, for every $x \in D$, it is possible to find an open neighbourhood U_x of x such that $\{U_x : x \in D\}$ is a disjoint family. The set $F = X \setminus \bigcup\{U_x : x \in D\}$ is closed, and by the SL property of X and the fact that $\{U_x : x \in D\}$ is an uncountable disjoint family, F is non-empty. So by normality of X choose disjoint open sets U and V such that $D \subset U$ and $F \subset V$. Now $\mathcal{U} = \{U_x : x \in D\} \cup \{V\}$ is an open cover of X and thus, by the SL property of X there is a countable subcollection $\mathcal{V} \subset \mathcal{U}$ such that $st(\bigcup \mathcal{V}, \mathcal{U}) = X$. Without loss of generality it is possible to assume that there is a countable set $C \subset D$ such that $\mathcal{V} = \{U_x : x \in C\} \cup \{V\}$. But then $(D \setminus C) \cap st(\bigcup \mathcal{V}, \mathcal{U}) = \emptyset$, which is a contradiction. □

The following fact will be used in Example 2.1.23 to produce a SSL not set SL space.

Proposition 2.1.22. [*Bonanzinga, M.*, 2022, *Proposition 3.4*] *Let X be a space. If there exist a closed and discrete subspace D of X having uncountable cardinality and a disjoint family $\mathcal{U} = \{O_a : a \in D\}$ of open neighbourhoods of points $a \in D$, then X is not set SL.*

Proof. Let D and $\mathcal{U}$ like in the hypothesis. For every countable subfamily $\mathcal{V} \subset \mathcal{U}$, there exists $b \in D \setminus \bigcup \mathcal{V}$. Since O_b is the only element of $\mathcal{U}$ containing b, $st(\bigcup \mathcal{V}, \mathcal{U}) \not\supset D$. □

Example 2.1.23. [*Bonanzinga, M.*, 2022, *Example 3.2*] *A Tychonoff SSL non set SL space.*

Let D be a discrete space of cardinality $\mathfrak{c}$. Consider the space $X = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$ and the Isbell-Mrówka space $Y = \omega \cup \mathcal{A}$ where $\mathcal{A}$ is a maximal almost disjoint family of infinite subsetes of ω such that $|\mathcal{A}| = \mathfrak{c}$; of course, assume $X \cap Y = \emptyset$. D is homeomorphic to the subspace $(\beta D \times \{\omega\}) \setminus ((\beta D \setminus D) \times \{\omega\})$. Let $f : D \to \mathcal{A}$ be a bijection and consider the quotient mapping $h : X \oplus Y \to Z$ which identifies x with $f(x)$, for every $x \in D$, where $X \oplus Y$ is the discrete sum of X and Y. Consider Z as the union of the compact spaces $X_n = \beta D \times \{n\}$, $n \in \omega$ and the SSL space Y. Since the SSL property is preserved by countable unions, Z is SSL. Since $A = (\beta D \times \{\omega\}) \setminus ((\beta D \setminus D) \times \{\omega\})$ is a closed and discrete subset of Z of cardinality $\mathfrak{c}$ and $\mathcal{U} = \{\{d\} \times (\omega + 1) : d \in D\}$ is a disjoint family of open neighbourhoods of points $\langle d, \omega \rangle \in A$, by Proposition 2.1.22, Z is not set SL. △

Example 2.1.24. [*Bonanzinga, M.,* 2022, *Example 3.3*] *A Hausdorff $\mathcal{K}$-SC (hence SC and SL) non set SL (hence non set SC nor set $\mathcal{K}$-SC) space.*

Let $A = [0, \mathfrak{c}), B = [0, \omega), Y = A \times B$. Put $X = Y \cup A \cup \{a\}$ where $a \notin Y \cup A$. Topologize X as follows: every point of Y is isolated; a basic neighborhood of $\alpha \in A$ takes the form:

$$U_\alpha(n) = \{\alpha\} \cup \{\langle \alpha, m \rangle : n < m\}, \text{for } n \in \omega$$

and a basic neighborhood of the point a takes the form:

$$U_\alpha(F) = \{a\} \cup \left(\bigcup \{\langle \alpha, n \rangle : \alpha \in A \setminus F, n \in B\} \right), \text{for a finite subset } F \text{ of } A.$$

The space X is clearly T_2, but not T_3 (in fact, the point a cannot be separated from the closed set A by disjoint open sets). In [Song, 2015] it is proved that the space X is $\mathcal{K}$-SC. It is enough to show that the space X is not set SL. Note that A is a closed and discrete subspace of X of cardinality $\mathfrak{c}$ and the family $\mathcal{U} = \{U_\alpha : \alpha \in A\}$, where for every $\alpha \in A$

$$U_\alpha = \{\alpha\} \cup \{\langle \alpha, n \rangle : n \in B\}$$

of pairwise disjoint open set such that $\bigcup \mathcal{U} \supset A = \overline{A}$. Then, by Proposition 2.1.22, the space X is not set SL. △

In the next paragraph an example of Tychonoff SC non set SC will be given (see Example 2.2.10).

Corollary 2.1.28 below will show that "every c.c.c. space is set SL". This will allow to show that a set SL space need not be set SSL, or, equivalently, need not having countable extent . First recall the following :

Definition 2.1.25. A space X is

- *weakly Lindelöf*, briefly wL , if for every open cover $\mathcal{U}$ of X there exists a countable family $\mathcal{V} \subset \mathcal{U}$ such that $\overline{\bigcup \mathcal{V}} = X$ [Hodel, 1984];
- *weakly Lindelöf with respect to closed sets* , briefly wL_c, if for every closed set $F \subseteq X$ and for every family $\mathcal{U}$ of open sets such that $F \subset \bigcup \mathcal{U}$ there esists a countable family $\mathcal{V} \subset \mathcal{U}$ such that $\overline{\bigcup \mathcal{V}} \supseteq F$ [Alas, 1993, Definition 1].

Of course, every Lindelöf space is wL_c and every wL_c space is wL.

Proposition 2.1.26. [*Bonanzinga, M.,* 2022, *PRoposition 3.5*] *Every wL_c space is set SL.*

Proof. Let X be a wL_c space, $A \subseteq X$ and $\mathcal{U}$ be a family of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}$. By hypothesis, there exists a countable family $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\overline{\bigcup \mathcal{U}_0} \supseteq \overline{A}$. It is sufficient to prove that $A \subseteq St(\bigcup \mathcal{U}_0, \mathcal{U})$. Fix $x \in A$. Then, there exists $U \in \mathcal{U}$ such that $x \in U$. Since $\overline{\bigcup \mathcal{U}_0} \supseteq \overline{A}$, $U \cap \bigcup \mathcal{U}_0 \neq \emptyset$ and then $x \in St(\bigcup \mathcal{U}_0, \mathcal{U})$. □

Note that the converse of the previous proposition is not true as the following example shows.

Example 2.1.27. [*Bonanzinga, M.,* 2022, *Example 3.4*] *A T_5 set SL space X which is not wL (hence not wL_c).*

Consider the ordinal space ω_1. This space is not wL because the open cover $\mathcal{U} = \{[0, \alpha) : \alpha < \omega_1\}$ has no countable subfamily the union of which is dense in ω_1. Since ω_1 is CC, it is set SL. △

Corollary 2.1.28. [*Bonanzinga, M., 2022, Corollary 3.3*] *Every c.c.c. space is set SL.*

Proof. By [Alas, 1993, Inequality 3)] every c.c.c. space is wL_c. □

Using Corollary 2.1.28 and Proposition 2.1.17, the following example can be exposed.

Example 2.1.29. [*Bonanzinga, M., 2022, Example 3.5*] *A Tychonoff separable (hence set SL) non set SSL space.*

Consider the Isbell-Mrówka space $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$, where $\mathcal{A}$ is the maximal almost disjoint family of cardinality $\mathfrak{c}$. $\Psi(\mathcal{A})$ is separable hence c.c.c. and, by the previous result, set SL. Since $\mathcal{A}$ is a closed and discrete subspace of $\Psi(\mathcal{A})$ of uncountable cardinality, $e(\Psi(\mathcal{A})) > \omega$ and, equivalentely, $\Psi(\mathcal{A})$ is not set SSL. △

Note that the space of the previous example is SSL because it is separable. In [van Douwen, Reed, Roscoe, and Tree, 1991, Example 3.2.3.2] it is proved that the Pixley-Roy hyperspace of $\mathbb{R}$ is a c.c.c. (hence set SL) space which is not SSL (hence not set SSL).

In [Bonanzinga, 1998] the author introduced the following star-covering property and its hereditary, with respect to closed sets, version. Below it is shown that these properties are related to set SL property.

Definition 2.1.30. A space X is

- *absolutely star-Lindelöf* , briefly a-st-L, if for every open cover $\mathcal{U}$ of X and every dense subspace $D \subseteq X$ there exists a countable set $C \subset D$ such that $st(C, \mathcal{U}) = X$ [Bonanzinga, 1998, Definition 1.5];
- *hereditarely closed absolutely star-Lindelöf* , briefly h-cl-a-st-L, provided that every its closed subspace is a-st-L [Bonanzinga, 1998, p. 83].

Recall also that absolute star-Lindelöfness is the countable version of the following property introduced by M. Matveev:

Definition 2.1.31. [*Matveev, 1994, Definition 1.1*]A space X is *absolutely countably compact* , briefly acc, if for every open cover $\mathcal{U}$ of X and every dense subspace $D \subset X$ there exists a finite subset $F \subset D$ such that $st(F, \mathcal{U}) = X$.

The h-cl-a-st-L and wL_c properties are indipendent as the following two examples show.

Example 2.1.32. [*Bonanzinga, M., 2022, Example 3.6*] *A Tychonoff wL_c space which is not h-cl-a-st-L.*

The Isbell-Mrówka space $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$, where $\mathcal{A}$ is a maximal almost disjoint family of cardinality $\mathfrak{c}$, is separable hence wL_c. However $e(\Psi(\mathcal{A})) > \omega$ and then $\Psi(\mathcal{A})$ is not h-cl-a-star-Lindelöf. △

Example 2.1.33. [*Bonanzinga, M., 2022, Example 3.7*] *A T_5 h-cl-a-st-L which is not wL_c.*

The ordinal space ω_1 is not wL_c (see Example 2.1.27). By [Matveev, 1994, Theorem 1.8], every CC space having countable tightness is acc. Since CC is hereditary with respect to closed subspace and tightness is a hereditary cardinal function, every closed subspace of ω_1 is acc hence ω_1 is h-cl-a-st-L. △

The set Lindelöf-type covering properties are related to absolute Lindelöf properties, as the following result shows.

Proposition 2.1.34. [Bonanzinga, M., 2022, Proposition 3.6] *Every h-cl-a-st-L T_1 space is set SSL (or equivalently, has countable extent).*

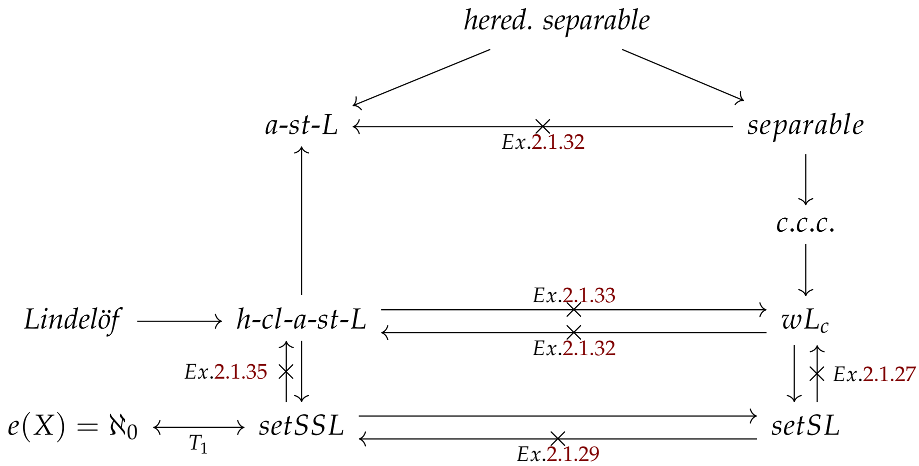
Proof. Let X be a h-cl-a-st-L space, A be a subset of X and $\mathcal{U}$ be a family of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}$. By the hypothesis, there exists a countable subset C of A such that $st(C, \mathcal{U}) \supseteq \overline{A} \supseteq A$. □

The converse of the previous result is not true as the following example shows.

Example 2.1.35. [Bonanzinga, M., 2022, Example 3.8] *A Tychonoff set SSL space which is not h-cl-a-st-L.*

The space $\omega_1 \times (\omega_1 + 1)$, where $\omega_1 + 1$ is considered with the order topology, is CC, hence it has countable extent. However this space is not a-st-L, since it is CC non acc, hence it is not h-cl-a-st-L. △

The following diagram sums up the implications obtained up to now.



Consider the following relative version of acc property:

Definition 2.1.36. [Bonanzinga, M., 2022]A subset A of a space X is *relatively acc in* X if for every collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subset \bigcup \mathcal{U}$, and for every dense subspace $D \subset X$ there exists a finite subset $F \subset D$ such that $st(F, \mathcal{U}) \supset A$.

In [Kočinac and Singh, 2020] the authors introduced the following definition (with different terminology).

Definition 2.1.37. [Kočinac and Singh, 2020, Definition 7]A space X is *set absolutely countably compact* , briefly set acc, if every subset A of X is relatively acc in X.

In the same article the authors pose the following question:

Question 2.1.38. [Kočinac and Singh, 2020, Problem 1] *Does an acc space which is not set-acc exist?*

The Proposition 2.1.39 below answers in the negative to the previous question.

Proposition 2.1.39. [Bonanzinga, M., 2022, Proposition 4.1] *Let X be a space. The following affirmations are equivalent.*

- *X is set acc*

- *X is acc*

Proof. Of course every set acc space is acc. Then, suppose X is an acc space. Let $A \subseteq X$ be a non-empty subset, $\mathcal{U}$ be a family of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}$ and D be a dense subset of X. Define

$$\mathcal{U}' = \mathcal{U} \cup \{X \setminus \overline{A}\}.$$

Clearly, $\mathcal{U}'$ is an open cover for X. Since the space X is acc, there is a finite subset F of D such that $st(F, \mathcal{U}') = X$. Fix $x \in A$. Thus

$$x \in st(F, \mathcal{U}') \Leftrightarrow st(x, \mathcal{U}') \cap F \neq \emptyset$$

and then

$$st(x, \mathcal{U}') = st(x, \mathcal{U}).$$

So, $A \subset st(F, \mathcal{U})$. □

In [Aurichi, 2013] the following definition is introduced:

Definition 2.1.40. [*Aurichi, 2013, Definition 2.1*]A space X is *selectively c.c.c.* if for every sequence $(\mathcal{A}_n : n \in \omega)$ of maximal cellular open families in X, there is a sequence $(A_n : n \in \omega)$ such that for each $n \in \omega$, $A_n \in \mathcal{A}_n$, and $\bigcup_{n\in\omega} A_n$ is dense in X.

In [Bal and Kočinac, 2020] the following star version of the previous property is introduced:

Definition 2.1.41. [*Bal and Kočinac, 2020*] A space X is *selectively star-c.c.c.* if for each open cover $\mathcal{U}$ of X and every sequence $(\mathcal{A}_n : n \in \omega)$ of maximal cellular open families in X, there is a sequence $(A_n : n \in \omega)$ such that for each $n \in \omega$, $A_n \in \mathcal{A}_n$, and $X = st(\bigcup_{n\in\omega} A_n, \mathcal{U})$.

For further developement about selectively star-c.c.c. space, see [Song and Xuan, 2019] and [Xuan and Song, 2020a].

Finally in [Kočinac and Singh, 2020], using the following relative version of selectively star-c.c.c. property:

Definition 2.1.42. [*Bonanzinga, M., 2022*]A subset A of a space X is *relatively selectively star-c.c.c. in* X if for each collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subset \bigcup \mathcal{U}$, and for each sequence $(\mathcal{A}_n : n \in \omega)$ of maximal cellular open families in X, there is a sequence $(A_n : n \in \omega)$ such that for each $n \in \omega$, $A_n \in \mathcal{A}_n$, and $A \subset st(\bigcup_{n\in\omega} A_n, \mathcal{U})$.

The following definition was introduced.

Definition 2.1.43. [*Kočinac and Singh, 2020, Definition 8*] A space X is *set selectively star-c.c.c.* if every subset A of X is relatively selectively star-c.c.c. in X.

By Proposition 2.1.39 and [Kočinac and Singh, 2020, Theorem 5] the following result is obvious.

Proposition 2.1.44. [*Bonanzinga, M., 2022, Proposition 4.2*] *Every acc space is set selectively star-c.c.c., hence selectively star-c.c.c..*

Proposition 2.1.46 below give a negative answer to the question:

Question 2.1.45. [*Kočinac and Singh, 2020, Problem 2*] *Does there exists a Tychonoff selectively star-c.c.c. space which is not set selectively star-c.c.c.?*

Proposition 2.1.46. [*Bonanzinga, M., 2022, Proposition 4.3*] *Let X be a space. The following affirmations are equivalent*

- *X is set selectively star-c.c.c.*
- *X is selectively star-c.c.c..*

Proof. Of course, every set selectively star-c.c.c. space is selectively star-c.c.c.. Then, suppose X is a selectively star-c.c.c. space. Let $A \subseteq X$ be a non-empty subset, $\mathcal{U}$ be a family of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}$ and $(\mathcal{A}_n : n \in \omega)$ be a sequence of maximal cellular open families in X. Define

$$\mathcal{U}' = \mathcal{U} \cup \{X \setminus \overline{A}\}.$$

Clearly, $\mathcal{U}'$ is an open cover for X. Since the space X is selectively star-c.c.c., there is a sequence $(A_n : n \in \omega)$ such that for each $n \in \omega$, $A_n \in \mathcal{A}_n$, and $X = st(\bigcup_{n\in\omega} A_n, \mathcal{U})$. Fix $x \in A$. Thus

$$x \in st(\textstyle\bigcup_{n\in\omega} A_n, \mathcal{U}') \Leftrightarrow st(x, \mathcal{U}') \cap (\bigcup_{n\in\omega} A_n) \neq \emptyset$$

and then

$$st(x, \mathcal{U}') = st(x, \mathcal{U}).$$

So, $A \subset st(\bigcup_{n\in\omega} A_n, \mathcal{U})$. □

Replacing "st" with "st^m", $m \in \omega$, in the definitions of selective star-c.c.c. and set selective star-c.c.c. properties, the notions of *selective m-star-c.c.c. property* ([Bal and Kočinac, 2020], see also [Xuan and Song, 2020b]) and *set selective m-star-c.c.c. property* ([Kočinac and Singh, 2020]) were defined. A selectively 1-star-c.c.c. (set selectively 1-star-c.c.c.) space is exactly a selectively star-c.c.c. (resp., set selectively star-c.c.c.) space. Of course, every selectively m-star-c.c.c. (set selectively m-star-c.c.c.) space is selectively $(m+1)$-star-c.c.c. (resp., set selectively $(m+1)$-star-c.c.c.).

In [Bal and Kočinac, 2020] the following question is posed.

Question 2.1.47. [*Bal and Kočinac, 2020, Problem 4.9*] *Does there exists a space which is selectively 2-star-c.c.c. but not SSL?*

The following example gives a positive answer to the previous question.

Example 2.1.48. [*Bonanzinga, M., 2022, Example 4.1*] *There exists a T_2 selectively star-c.c.c. (hence selectively 2-star c.c.c.) space which is not SSL.*

In [Aurichi, 2013, Corollary 2.9] it is proved that for every Tychonoff space X, the function space $C_p(X)$ is selectively c.c.c. and hence selectively star-c.c.c. Consider the space $C_p(\omega_1 + 1)$, where $\omega_1 + 1$ is endowed with the order topology. Then $C_p(\omega_1 + 1)$ is selectively star-c.c.c.. In [Bonanzinga and Matveev, 2000, p.117] it is proved that $C_p(\omega_1 + 1)$ is not SSL. △

Recall that

Theorem 2.1.49. [*Bal and Kočinac, 2020, Theorem 3.2*] *Every Lindelöf space is selectively star-ccc.*

Theorem 2.1.50. [*Kočinac and Singh, 2020*] *Every set SSL space is set selectively* 2 *star-ccc.*

Since every set selectively star-ccc space is set selectively 2 star-ccc and recalling that the set version of selective star-ccc property is equivalent to the original property (Proposition 2.1.46) and that set SSL is equivalent to countable extent (Proposition 2.1.17), the following result improves both the previous theorems.

Theorem 2.1.51. [*Bonanzinga, M., 2022, Theorem 4.3*] *Every space with countable extent is selectively star-c.c.c. .*

Proof. Let X be a space such that $e(X) = \aleph_0$. It is enough to prove that X is set selectively star-ccc; by Proposition 2.1.46, this will prove the theorem. Let $B \subseteq X$ be a non-empty subset, $\mathcal{U}$ be a family of open sets of X such that $\overline{B} \subseteq \bigcup \mathcal{U}$ and $(\mathcal{A}_n : n \in \omega)$ be a sequence of maximal cellular open families in X. By Proposition 2.1.17, there exists a countable subset of B, say $Y = \{y_1, ..., y_n\}$, such that $B \subset St(Y, \mathcal{U})$.

Fix $b \in B$. Then

$$b \in st(Y, \mathcal{U}) \Leftrightarrow \text{ there exists } n \in \omega \text{ such that } y_n \in st(b, \mathcal{U}).$$

For every $n \in \omega$, put $U_n = \bigcup_{b \in B} \{st(b, \mathcal{U}) : y_n \in st(b, \mathcal{U})\}$. Then, U_n is an open subset of X and, by maximality of $\mathcal{A}_n$, there exists $A_n \in \mathcal{A}_n$ such that $U_n \cap A_n \neq \emptyset$. Then $B \subset \bigcup_{n \in \omega} U_n \subset st(\bigcup_{n \in \omega} A_n, \mathcal{U})$. Therefore, X is set selectively star-ccc. □

The Theorem 2.1.53 will show that a space X is selectively m-star-ccc if and only if it is set selectively m-star-ccc, for every $m \in \mathbb{N}$.

Lemma 2.1.52. [*Bonanzinga, M., 2022, Lemma 4.1*] *Let X be a topological space, $A \subseteq X$ be a nonempty subset, and $\mathcal{U}$ be a family of open sets of X such that $\overline{A} \subset \bigcup \mathcal{U}$. Consider the family $\mathcal{U}' = \mathcal{U} \cup \{X \setminus \overline{A}\}$. Then for every $m \in \mathbb{N}$, $st^m(x, \mathcal{U}) = st^m(x, \mathcal{U}')$ for all $x \in A$.*

Proof. Fix $x \in A$. For the basis of the induction, see the proof of Proposition 2.1.4. Now, assume that $st^m(x, \mathcal{U}) = st^m(x, \mathcal{U}')$ is true for a fixed $k \in \mathbb{N}$. Then for every $V \in \mathcal{U}$,

$$V \cap st^m(x, \mathcal{U}') \neq \emptyset \Leftrightarrow V \cap st^m(x, \mathcal{U}) \neq \emptyset.$$

So $st^{m+1}(x, \mathcal{U}') = st^{m+1}(x, \mathcal{U})$ as required. □

Using the previous lemma and proceeding as in the proof of Proposition 2.1.46, it is possible to obtain the following result.

Theorem 2.1.53. [*Bonanzinga, M., 2022, Theorem 4.4*] *Let $m \in \mathbb{N}$. A space X is selectively m-star-ccc iff it is set selectively m-star-ccc.*

Lastly, it is consdered the behaviour of the productivity of set SSL and set SL spaces. Note that, by Proposition 2.1.10 (resp. Corollary 2.1.12), set SSC (resp. set SC) property is not productive and it is preserved in the Hausdorff (resp. regular) product with compact spaces. Recall that a map is *perfect* if it is continuous, closed, onto and each fiber is compact.

Proposition 2.1.54. [*Bonanzinga, Giacopello, M., 2023, Proposition 3.3*] *If $f : X \to Y$ is a perfect map and A is an uncountable closed and discrete subspace of X, then $f(A)$ is an uncountable closed and discrete subspace of Y.*

Proof. Let f and A be as in the hypothesis. Clearly $f(A)$ is closed in Y. Note that, for every $y \in f(A)$, $f^{-1}(y) \cap A$ is a closed subset of the compact subspace $f^{-1}(y)$ and then, since A is discrete, it is finite. Now, fix $y \in f(A)$ and say $f^{-1}(y) \cap A = \{x_1, ..., x_n\}$. For every $i = 1, ..., n$ fix an open subset U_i of X such that $A \cap U_i = \{x_i\}$ and put $U = \bigcup_{i=1}^{n} U_i$. Since $A \setminus U$ is a closed subset of X, $f(A \setminus U) = f(A) \setminus \{y\}$ is a closed subset of Y, and then $\{y\}$ is open in $f(A)$ with the topology inherited from Y. □

Corollary 2.1.55. [*Bonanzinga, Giacopello, M.*, 2023, *Corollary 3.4*] *The product of a space having countable extent with a compact space has countable extent.*

Proof. Let X be a space with countable extent and Y be a compact space. The projection from $X \times Y$ onto X is a perfect map. Then, by Proposition 2.1.54, $e(X \times Y) = \omega$. □

By Proposition 2.1.17, the previous result can be restated as follows.

Proposition 2.1.56. [*Bonanzinga, Giacopello, M.*, 2023, *Proposition 3.5*] *The product of a T_1 set SSL space with a compact space is set SSL.*

Proposition 2.1.57. [*Bonanzinga, Giacopello, M.*, 2023, *Proposition 3.9*] *If X and Y are T_1 spaces with $e(X) > \aleph_0$ and $c(Y) > \aleph_0$ then $X \times Y$ is not set SL.*

Proof. Let $S = \{s_\alpha : \alpha < \omega_1\}$ be a closed and discrete subset of X, $\mathcal{O} = \{O_\alpha : \alpha < \omega_1\}$ be a pairwise disjoint family of nonempty open subsets of Y. For every $\alpha < \omega_1$, fix $t_\alpha \in O_\alpha$. Put $A = \{(s_\alpha, t_\alpha) : \alpha < \omega_1\}$. It is obvious that A is an uncountable discrete subspace of $X \times Y$. It is enough to show that A is closed. For every $\alpha < \omega_1$ there exists an open set, say N_α, such that $N_\alpha \cap S = \{s_\alpha\}$. Then $(X \times Y) \setminus A = ((X \setminus S) \times Y) \cup \bigcup_{\alpha<\omega_1} (N_\alpha \times (Y \setminus \{t_\alpha\}))$. Then, by Proposition 2.1.22, $X \times Y$ is not set SL. □

Example 2.1.58. [*Bonanzinga, Giacopello, M.*, 2023, *Example 3.10*] *There exists a T_1 set SC (hence set SL) space X and a compact space Y with $c(Y) > \aleph_0$ such that $X \times Y$ is not set SL (hence not set SC).*

Consider the set $X = \omega_1 \cup A$, where $A = \{a_\alpha : \alpha \in \omega_1\}$ is a set of cardinality ω_1, topologized as follows: ω_1 has the usual order topology and is an open subspace of X; a basic neighborhood of a point $a_\alpha \in A$ takes the form

$$O_\beta(a_\alpha) = \{a_\alpha\} \cup (\beta, \omega_1), \text{ where } \beta < \omega_1.$$

In 2.1.15 it is said that X is set SC, hence X is set SM. Moreover $e(X) > \aleph_0$. If Y is any compact space with $c(Y) > \aleph_0$, by Proposition 2.1.57, $X \times Y$ is not set SL. △

2.2 Some relative Menger-type and Hurewicz-type properties defined by stars

The following result gives a characterization of the SSM property in terms of a relative version of it.

Proposition 2.2.1. [*Bonanzinga, Giacopello, M.*, 2023, *Proposition 2.2*] *The followings are equivalent for a space X:*

1. *X is SSM;*

2. *for each nonempty subset A of X and each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets of X such that $\overline{A} \subset \bigcup \mathcal{U}_n$ for every $n \in \omega$, there exists a sequence $(F_n : n \in \omega)$ such that F_n, $n \in \omega$, is a finite subset of X and $A \subset \bigcup_{n\in\omega} st(F_n, \mathcal{U}_n)$.*

Proof. 2. $\Rightarrow$ 1. is obvious. 1. $\Rightarrow$ 2. Let $A \subseteq X$ be a nonempty subset and $(\mathcal{U}_n : n \in \omega)$ be a sequence of families of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}_n$ for every $n \in \omega$. Define

$$\mathcal{U}_n' = \mathcal{U}_n \cup \{X \setminus \overline{A}\}$$

for all $n \in \omega$. Clearly, each $\mathcal{U}_n'$ is an open cover for X. Since X is SSM, there is a sequence $(F_n : n \in \omega)$ of finite subsets of X such that $X = \bigcup_{n\in\omega} st(F_n, \mathcal{U}_n)$. Fix $x \in A$. Then there exists $n \in \omega$ such that $x \in st(F_n, \mathcal{U}_n')$. Observe that

$$x \in st(F_n, \mathcal{U}_n') \Leftrightarrow st(x, \mathcal{U}_n') \cap F_n \neq \emptyset.$$

Also

$$st(x, \mathcal{U}_n') = \bigcup\{U \in \mathcal{U}_n' : x \in U\} = \bigcup\{U \in \mathcal{U}_n : x \in U\} = st(x, \mathcal{U}_n).$$

So

$$x \in st(F_n, \mathcal{U}_n') \Leftrightarrow st(x, \mathcal{U}_n') \cap F_n \neq \emptyset \Leftrightarrow st(x, \mathcal{U}_n) \cap F_n \neq \emptyset \Leftrightarrow x \in st(F_n, \mathcal{U}_n).$$

Since x is an arbitrary point of A, $A \subset \bigcup_{n\in\omega} st(F_n, \mathcal{U}_n)$. □

The following result is a characterization of SSH property in terms of a relative version of it. The proof is similar to the previous one.

Proposition 2.2.2. [*Bonanzinga, Giacopello, M.*, 2023, *Proposition 4.2*] *The following affirmations are equivalent for a space X:*

1. *X is SSH;*

2. *for each nonempty subset A of X and for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of collection of open sets of X such that $\overline{A} \subset \bigcup \mathcal{U}_n$ for every $n \in \omega$, there exists a sequence $(F_n : n \in \mathbb{N})$ such that F_n, $n \in \omega$, is a finite subset of X and $\forall x \in A$, $x \in st(F_n, \mathcal{U}_n)$ for all but finitely many $n \in \omega$.*

Notice that in the propositions above, the affirmation 2. consents to choose the finite sets are required to be subsets of X. Asking that the finite sets must be subsets of the closure of A, the authors in [Kočinac, Konca, and Singh, 2022] defined the following relative version of the SM and SSM properties were considered.

Definition 2.2.3. A space X is

- *set strongly star-Menger*, briefly set SSM, if for each nonempty subset A of X and for each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets of X such that $\overline{A} \subset \bigcup \mathcal{U}_n$ for every $n \in \omega$, there exists a sequence $(F_n : n \in \omega)$ such that F_n, $n \in \omega$, is a finite subset of $\overline{A}$ and $A \subset \bigcup_{n\in\omega} st(F_n, \mathcal{U}_n)$ [Kočinac, Konca, and Singh, 2022, Definition 4];

- *set star-Menger*, briefly set SM, if for each nonempty subset A of X and for each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets of X such that $\overline{A} \subset \bigcup \mathcal{U}_n$ for every $n \in \omega$, there exists a sequence $(\mathcal{V}_n : n \in \omega)$ such that $\mathcal{V}_n$, $n \in \omega$, is a finite subset of $\mathcal{U}_n$ and $A \subset \bigcup_{n\in\omega} st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ [Kočinac, Konca, and Singh, 2022, Definition 4];

- *set strongly star-Hurewicz*, briefly set SSH, if for each nonempty subset $A \subseteq X$ and for each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}_n$ for every $n \in \omega$, there exists a sequence $(F_n : n \in \omega)$ such that F_n, $n \in \omega$, is a finite subset of $\overline{A}$ and $\forall x \in A$, $x \in st(F_n, \mathcal{U}_n)$ for all but finitely many $n \in \omega$ [Kočinac, Konca, and Singh, 2022, Definition 5];
- *set star-Hurewicz*, briefly set SH, if for each nonempty subset $A \subseteq X$ and for each sequence $(\mathcal{U}_n : n \in \omega)$ of collection of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}_n$ for every $n \in \omega$, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that $\mathcal{V}_n$, $n \in \omega$, is a finite subset of $\mathcal{U}_n$ and $\forall x \in A$, $x \in st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many $n \in \omega$ [Kočinac, Konca, and Singh, 2022, Definition 5].

The set SSH and set SSM properties lie between CC and the property having countable extent.

Theorem 2.2.4. [*Bonanzinga, Giacopello, M., 2023*] *Let X be a space. The following relations hold*

$$X\ CC \longrightarrow X\ set\ SSH \longrightarrow X\ set\ SSM \longrightarrow e(X) = \aleph_0$$

The following examples show that the previous implications cannot be reversed.

Example 2.2.5. [*Bonanzinga, Giacopello, M., 2023*] *A metrizable set SSH space which is not CC*

Consider ω with the discrete topology. This is a H (hence set SSH) non CC space. △

Example 2.2.6. [*Bonanzinga, Giacopello, M., 2023, Example 4.4*] *($\mathfrak{b} < \mathfrak{d}$) There is a Tychonoff set SSM space which is not set SSH.*

Consider an unbounded subset X of the Baire space ω^ω of cardinality $\mathfrak{b}$. Then $e(X) = \aleph_0$, and X is not Hurewicz, hence not set SSH by [Bonanzinga, Cammaroto, and Kočinac, 2004, Proposition 4.1]. △

Example 2.2.7. [*Bonanzinga, Giacopello, M., 2023, Example 2.17*] *A Tychonoff space having countable extent which is not set SSM.*

Let $\mathbb{I}$ be the space of irrationals and $X \subset \mathbb{I}$ any non-Menger subspace of cardinality $\mathfrak{d}$ (for instance, consider the Baire space ω^ω which is homeomorphic to $\mathbb{I}$ and take a cofinal subset of cardinality $\mathfrak{d}$). It is well known that any cofinal subset of ω^ω is not Menger. Of course, X is a paracompact space having countable extent. By [Kočinac, 1999, Theorem 2.8], in the class of paracompact Hausdorff spaces Menger property is equivalent to SM, hence X is not set SSM. △

Recall that [Matveev, 2002, Theorem 1] shows that the extent of a Tychonoff SSL space can be arbitrarily big. In [Sakai, 2014, Example 3.1] the same space was used to prove that the extent of a Tychonoff SM (in fact SC) space can be arbitrarily large. The following result shows that the extent of a regular set SM space cannot exceed $\mathfrak{c}$.

Theorem 2.2.8. [*Bonanzinga, Giacopello, M., 2023, Theorem 2.8*] *If X is a regular set SM space, then every closed and discrete subspace of X has cardinality less than $\mathfrak{c}$. Hence, $e(X) \leq \mathfrak{c}$.*

Proof. Fix a closed and discrete subspace Y of X and assume $|Y| = \mathfrak{c}$. Consider a family $\mathcal{B}$ of open subsets of X such that for every $y \in Y$ there exists $B \in \mathcal{B}$ such that $y \in B$ and $\overline{B} \cap Y = \{y\}$ and suppose that $|\mathcal{B}| = \mathfrak{c}$. Denote by $[\mathcal{B}]^{<\omega}$ the family of all finite subsets of $\mathcal{B}$, by $\mathbb{P} = ([\mathcal{B}]^{<\omega})^{\omega}$ the family of all the sequences of elements of $[\mathcal{B}]^{<\omega}$ and introduce on $\mathbb{P}$ the partial order "$\leq$" defined as follows: if $(\mathcal{B}'_n)_{n\in\omega}, (\mathcal{B}''_n)_{n\in\omega} \in \mathbb{P}$ then $(\mathcal{B}'_n)_{n\in\omega} \leq (\mathcal{B}''_n)_{n\in\omega}$ means $\mathcal{B}'_n \subseteq \mathcal{B}''_n$ for every $n \in \omega$. Let $\{(\mathcal{B}_{\alpha,n})_{n\in\omega} : \alpha < \mathfrak{c}\}$ be a cofinal family in $(\mathbb{P}, \leq)$. Take $Z = \{y_\alpha : \alpha < \mathfrak{c}\}$ by choosing for every $\alpha < \mathfrak{c}$ a point $y_\alpha \in Y \setminus \bigcup_{n\in\omega} \overline{\bigcup \mathcal{B}_{\alpha,n}}$ and $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$. For every $\alpha < \mathfrak{c}$ let $\{V_n(y_\alpha) : n \in \omega\}$ be a sequence of open neighbourhoods of y_α such that $V_n(y_\alpha) \subseteq B$ for some $B \in \mathcal{B}$ and every $n \in \omega$ and $V_n(y_\alpha) \cap \bigcup \mathcal{B}_{\alpha,n} = \emptyset$ for every $n \in \omega$. For every $n \in \omega$ put $\mathcal{U}_n = \{V_n(y_\alpha) : \alpha < \mathfrak{c}\}$. Clearly $Z = \overline{Z} \subseteq \bigcup \mathcal{U}_n$ for every $n \in \omega$. It is enough to show that the subset Z and the sequence $(\mathcal{U}_n : n \in \omega)$ do not satisfy the set SM property. Let $(\mathcal{V}_n : n \in \omega)$ be any sequence of finite subsets of $\mathcal{U}_n$ for every $n \in \omega$. Let $(\mathcal{B}'_n : n \in \omega) \in \mathbb{P}$ such that every member of $\mathcal{V}_n$ is contained in a member of $\mathcal{B}'_n$. Since $\{(\mathcal{B}_{\alpha,n})_{n\in\omega} : \alpha < \mathfrak{c}\}$ is a cofinal family in $\mathbb{P}$, there exists $\gamma < \mathfrak{c}$ such that $\mathcal{B}'_n \subseteq \mathcal{B}_{\gamma,n}$ for every $n \in \omega$. Then $V_n(y_\gamma) \cap \bigcup \mathcal{V}_n \subseteq V_n(y_\gamma) \cap \bigcup \mathcal{B}_{\gamma,n} = \emptyset$ for every $n \in \omega$. Since $V_n(y_\gamma)$ is the only member of $\mathcal{U}_n$ containing y_γ, $y_\gamma \notin \bigcup_{n\in\omega} st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$. □

Example 2.2.9 below gives a consistent example of a SSM space which is not set SSM. In fact, such an example was already described in [Kočinac, Konca, and Singh, 2022]; here it is given a shorter proof.

Example 2.2.9. [*Kočinac, Konca, and Singh, 2022*] *($\omega_1 < \mathfrak{d}$) There exists a Tychonoff SSM space which is not set SSM.*

Assume $\omega_1 < \mathfrak{d}$ and consider $\Psi(\mathcal{A})$ with $|\mathcal{A}| = \omega_1$. By Theorem 2.10 and since $e(\Psi(\mathcal{A})) > \omega$, $\Psi(\mathcal{A})$ is a SSM not set SSM space. △

Using Theorem 2.2.8 it is possible to construct a Tychonoff space distinguishing SM and set SM properties. In fact, the following example distinguishes SC and set SC properties.

Example 2.2.10. [*Bonanzinga, Giacopello, M., 2023, Example 2.12*] *A Tychonoff SC (hence SM) space which is not set SM (hence not set SC).*

In [Matveev, 2002], for each infinite cardinal τ the following space $X(\tau)$ was considered. Let $Z = \{f_\alpha : \alpha < \tau\}$ where f_α denotes the points in 2^τ with only the αth coordinate equal to 1. Consider the set

$$X(\tau) = (2^\tau \times (\tau^+ + 1)) \setminus ((2^\tau \setminus Z) \times \{\tau^+\})$$

with the topology inherited from the product topology on $2^\tau \times (\tau^+ + 1)$. Denote $X_0 = 2^\tau \times \tau^+$ and $X_1 = Z \times \{\tau^+\}$. Then $X(\tau) = X_0 \cup X_1$. X_1 is a closed and discrete subspace of $X(\tau)$ of cardinality τ. So the extent of $X(\tau)$ is τ. In [Sakai, 2014] it is proved that the space $X(\mathfrak{c})$ is SC (hence SM). By Theorem 2.2.8, $X(\mathfrak{c})$ it is not set SM. △

The following are the set versions of Proposition 2.12.

Proposition 2.2.11. [*Bonanzinga, Giacopello, M., 2023, Proposition 2.14*] *Every set SL space of cardinality less than $\mathfrak{d}$ is set SM.*

Proof. Let X be a set SL space of cardinality less than $\mathfrak{d}$. Let $A \subseteq X$ and $(\mathcal{U}_n : n \in \omega)$ be a sequence of families of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}_n$ for every $n \in \omega$. For every $n \in \omega$ there is a countable subfamily $\mathcal{V}_n = \{V_{n,m} : m \in \omega\}$ of $\mathcal{U}_n$ such that $A \subseteq st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$. For every $x \in A$ choose a function $f_x \in \omega^\omega$ such that $st(x, \mathcal{U}_n) \cap V_{n,f_x(n)} \neq \emptyset$ for all $n \in \omega$. Since $\{f_x : x \in A\}$ is not a cofinal family in $(\omega^\omega, \leq)$, there are some $g \in \omega^\omega$ and $n_x \in \omega$ for $x \in A$ such that $f_x(n_x) < g(n_x)$. Let $\mathcal{W}_n = \{V_{n,j} : j \leq g(n)\}$. Then $A \subseteq \bigcup_{n \in \omega} st(\bigcup \mathcal{W}_n, \mathcal{U}_n)$. □

Proposition 2.2.12. [*Bonanzinga, Giacopello, M.,* 2023*, Proposition 2.15*] *Every set SSL space of cardinality less than $\mathfrak{d}$ is set SSM.*

Proof. Similar to the previous one. □

It is thus obtained the following.

Corollary 2.2.13. [*Bonanzinga, Giacopello, M.,* 2023*, Corollary 2.16*] *Let X be a T_1 space of cardinality less than $\mathfrak{d}$. The following affirmations are equivalent:*

1. *X is set SSM*
2. *$e(X) = \aleph_0$.*

In the previous corollary the hypothesis on the cardinality of the space is mandatory: see Example 2.2.7 where the space X has cardinality $\mathfrak{d}$.

Corollary 2.2.14. [*Bonanzinga, Giacopello, M.,* 2023*, Corollary 2.18*] *The following statements are equivalent:*

1. *$\omega_1 < \mathfrak{d}$;*
2. *every T_1 space of cardinality ω_1 having countable extent is set SSM.*

Proof. It follows by Corollary 2.2.13 and Example 2.2.7. □

The following result is easy to check.

Proposition 2.2.15. [*Bonanzinga, Giacopello, M.,* 2023*, Proposition 2.4*] *A space X is set SSM if and only if every closed subspace of X is SSM.*

The previous result is not true for set SM spaces: the space of Example 2.1.58 below contains a discrete subspace of uncountable cardinality, hence not SM.

Using Theorem 2.15, proved by Sakai, it is obtained the following list of equivalent conditions.

Theorem 2.2.16. [*Bonanzinga, Giacopello, M.,* 2023*, Theorem 2.20*] *The following statements are equivalent*

1. *$\omega_1 = \mathfrak{d}$;*
2. *if X is a regular SSM space, then $e(X) \leq \omega$;*
3. *for regular spaces of cardinality less than $< \mathfrak{d}$, set SSM and SSM are equivalent properties.*
4. *for regular spaces of cardinality $< \mathfrak{d}$, set SSL and SSL are equivalent properties.*
5. *every closed subspace of a regular (set) SSM space X such that $|X| < \mathfrak{d}$ is (set) SSM.*

Proof. 1. $\Leftrightarrow$ 2. holds by Theorem 2.15. 2. $\Rightarrow$ 3. Let X be a space of cardinality less than $\mathfrak{d}$. By Corollary 2.2.13, X is SSM iff X is set SSM. 3. $\Rightarrow$ 1. Assume $\omega_1 < \mathfrak{d}$. Consider the space $\Psi(\mathcal{A})$ with $|\mathcal{A}| = \omega_1$. By Theorem 2.10, $\Psi(\mathcal{A})$ is SSM, and since $e(\Psi(\mathcal{A})) > \omega$, $\Psi(\mathcal{A})$ is not set SSM. 3. $\Leftrightarrow$ 4. is obvious. 3. $\Leftrightarrow$ 5. follows from Proposition 2.2.15. □

Of course, countable spaces are Menger, then set SSM and SSM.

Corollary 2.2.17. [*Bonanzinga, Giacopello, M.*, 2023, *Corollary 2.21*] *For regular spaces X such that $\omega < |X| < \mathfrak{d}$, SSM and set SSM are not equivalent properties.*

Corollary 2.2.18. [*Bonanzinga, Giacopello, M.*, 2023, *Corollary 2.22*] *Uncountable regular spaces in which SSM and set SSM are equivalent properties have cardinality $\geq \mathfrak{d}$.*

In [Kočinac, Konca, and Singh, 2022, Example 5] the authors constructed a T_1 set SM space which is not set SSM and posed the following question.

Question 2.2.19. [*Kočinac, Konca, and Singh*, 2022, *Problem 1*] *Does there exist a Tychonoff set SM space which is not set SSM?*

Now it is possible to give a consistent positive answer to Question 2.2.19.

Example 2.2.20. [*Bonanzinga, Giacopello, M.*, 2023, *Example 2.24*] *($\omega_1 < \mathfrak{d}$) A Tychonoff set SM space which is not set SSM.*

Assume $\omega_1 < \mathfrak{d}$ and consider $\Psi(\mathcal{A})$ with $|\mathcal{A}| = \omega_1$. Since $\Psi(\mathcal{A})$ is separable, it is set SL hence, by Proposition 2.2.11, it is set SM. Since $e(\Psi(\mathcal{A})) > \omega$, $\Psi(\mathcal{A})$ is not set SSM. △

All the result obtained for Menger-type properties can be extended to Hurewicz-type properties.

Example 2.2.21. [*Bonanzinga, Giacopello, M.*, 2023, *Example 4.6*] *($\omega_1 < \mathfrak{b}$) There exists a Tychonoff SSH not set SSH space.*

Assume $\omega_1 < \mathfrak{b}$ and consider $\Psi(\mathcal{A})$ with $|\mathcal{A}| = \omega_1$. Then, by Theorem 2.11 and since $e(\Psi(\mathcal{A})) > \omega$, $\Psi(\mathcal{A})$ is SSH not a set SSH space. △

Following Theorem 2.2.8 step by step with little modifications it is possible to prove the following result.

Theorem 2.2.22. [*Bonanzinga, Giacopello, M.*, 2023, *Theorem 4.8*] *If X is a regular set SH space, then every closed and discrete subspace of X has cardinality less than $\mathfrak{c}$. Hence $e(X) \leq \mathfrak{c}$.*

In [Singh and Kočinac, 2021, Example 2.4] it is given a Hausdorff SH space which is not set SH. The following example improves the latter with a higher separation axiom.

Example 2.2.23. [*Bonanzinga, Giacopello, M.*, 2023, *Example 4.9*] *A Tychonoff SC (hence SH) space which is not set SH.*

Consider the space $X(\mathfrak{c})$ of Example 2.2.10. $X(\mathfrak{c})$ is SC (hence SH) and, by Theorem 2.2.22, it is not set SH. △

In analogy to Proposition 2.2.11 and Proposition 2.2.12, it is possible prove the following.

Proposition 2.2.24. [*Bonanzinga, Giacopello, M., 2023, Proposition 4.11*] *Every set SL (set SSL) space of cardinality less than $\mathfrak{b}$ is set SH (set SSH).*

Proof. Let X be a set SL space of cardinality less than $\mathfrak{b}$ (the proof is similar if X is set SSL). Let $A \subseteq X$ and $(\mathcal{U}_n : n \in \omega)$ be a sequence of families of open sets of X such that $\overline{A} \subseteq \bigcup \mathcal{U}_n$ for every $n \in \omega$. For every $n \in \omega$ there is a countable subfamily $\mathcal{V}_n = \{V_{n,m} : m \in \omega\}$ of $\mathcal{U}_n$ such that $A \subseteq st(\bigcup \mathcal{V}_n, \mathcal{U}_n)$. For every $x \in A$ choose a function $f_x \in \omega^\omega$ such that $st(x, \mathcal{U}_n) \cap V_{n,f_x(n)} \neq \emptyset$ for all $n \in \omega$. Since $\{f_x : x \in A\}$ is a bounded family in $(\omega^\omega, \leq^*)$, there exists $g \in \omega^\omega$ such that for every $x \in A$, $f_x(n) \leq g(n)$ for all but finitely many $n \in \omega$. Let $\mathcal{W}_n = \{V_{n,j} : j \leq g(n)\}$. Then for every $x \in A$, $x \in st(\bigcup \mathcal{W}_n, \mathcal{U}_n)$ for all but finitely many $n \in \omega$. □

Corollary 2.2.25. [*Bonanzinga, Giacopello, M., 2023, Corollary 4.12*] *Let X be a T_1 space X of cardinality less than $\mathfrak{b}$. The following affirmations are equivalent:*

1. *X is set SSH*
2. *$e(X) = \aleph_0$.*

Corollary 2.2.26. [*Bonanzinga, Giacopello, M., 2023, Corollary 4.13*] *For spaces X such that $|X| < \mathfrak{b}$, the following affirmations are equivalent:*

1. *X is set SSM*
2. *X is set SSH*
3. *$e(X) = \aleph_0$.*

In [Singh and Kočinac, 2021] the authors give a T_1 set SH space which is not set SSH. The next example gives a consistent improvement.

Example 2.2.27. [*Bonanzinga, Giacopello, M., 2023, Example 4.14*] *($\omega_1 < \mathfrak{b}$) A Tychonoff set SH space which is not set SSH.*

Assume $\omega_1 < \mathfrak{b}$ and consider $\Psi(\mathcal{A})$ with $|\mathcal{A}| = \omega_1$. Since $\Psi(\mathcal{A})$ is separable, it is set SL hence, by Proposition 2.2.24, it is set SH. Since $e(\Psi(\mathcal{A})) > \omega$, $\Psi(\mathcal{A})$ is not set SSH. △

In what follows a partial answer to Question 2.18 (Proposition 2.2.29), and a negative answer to Question 2.19 (Example 2.2.30) are exposed; subsequently, Proposition 2.2.32 and Example 2.2.33 prove the same assertions for Hurewicz-type properties.

Proposition 2.2.28. [*Bonanzinga, Giacopello, M., 2023, Corollary 3.6*] *The product of a T_1 set SSM space with a compact space has countable extent.*

Proof. It follows from Proposition 2.1.56. □

Corollary 2.2.29. [*Bonanzinga, Giacopello, M., 2023, Corollary 3.7*] *The T_1 product of a set SSM space with cardinality less than $\mathfrak{d}$ and a compact space is set SSM.*

Proof. By Corollary 2.2.13. □

Example 2.2.30. [*Bonanzinga, Giacopello, M., 2023, Example 3.10*] *There exists a T_1 set SM space X and a compact space Y with $c(Y) > \aleph_0$ such that $X \times Y$ is not set SM.*

Consider Example 2.1.58. △

Proposition 2.2.31. [*Bonanzinga, Giacopello, M., 2023, Proposition 4.16*] *The product of a set SSH space with a compact space has countable extent.*

Proof. By Corollary 2.1.55. □

Proposition 2.2.32. [*Bonanzinga, Giacopello, M., 2023, Proposition 4.17*] *The product of a T_1 set SSH space having cardinality less than $\mathfrak{b}$ with a compact space is set SSH.*

Proof. By Corollary 2.2.26 and Proposition 2.1.56. □

Proposition 2.2.33. [*Bonanzinga, Giacopello, M., 2023, Proposition 4.15*] *Set SH property is not preserved in the product with compact spaces.*

Proof. Consider Example 2.1.58. □

The following diagram sums up implications and counterexamples for all the compact-type, Lindelöf-type, Menger-type and Hurewicz-type properties. A disconnected crossed arrow means that the counterexample is a consistent one.

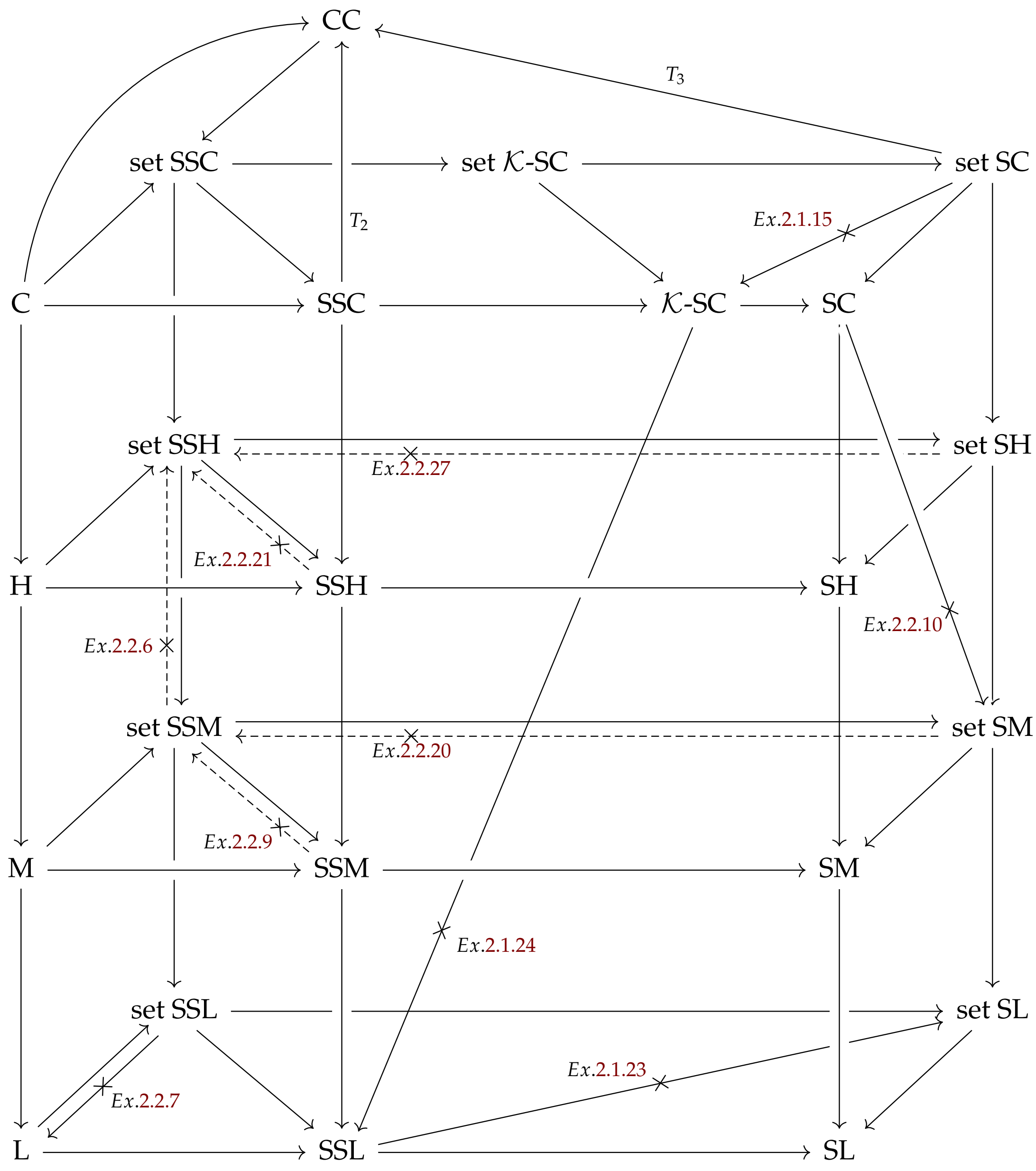


FIGURE 2.1: Summing diagram

CHAPTER 3

FRÉCHET-URYSOHN PROPERTY AND SELECTIVE SEPARABILITY

All the new contributions obtained by the author of the thesis included in this chapter are contained in the article [Bardyla, Maesano and Zdomskyy, 2023].

Definition 3.1. [*Fréchet*, *1906*]A space X is said to be *Fréchet-Urysohn*, briefly FU, if for every $A \subset X$ and $x \in \overline{A} \setminus A$ there exists a sequence $S \subset A \setminus \{x\}$ converging to x, i.e., $S \in [A \setminus \{x\}]^\omega$ and $|U \setminus A| < \omega$ for every open neighbourhood U of x.

Definition 3.2. [*Scheepers*, *1999*, *with different terminology*]A space X is said to be

- *M-separable* (or *selectively separable*) if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there are finite sets $F_n \subset D_n$, $n \in \omega$, such that $\bigcup\{F_n : n \in \omega\}$ is dense in X [Bella, Bonanzinga, Matveev, and Tkachuk, 2008, Definition 2.1];
- *H-separable* if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there are finite sets $F_n \subset D_n$, $n \in \omega$, such that every nonempty open set of X meets all but finitely many F_n [Bella, Bonanzinga, and Matveev, 2009, Definition 28];
- *R-separable* if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there points $x_n \in D_n$, $n \in \omega$, such that $\{x_n : n \in \omega\}$ is dense in X [Bella, Bonanzinga, and Matveev, 2009, Definition 47].

By [Bella, Bonanzinga, Matveev, and Tkachuk, 2008, Proposition 2.3(1)], every space with a countable π-base is M-separable; in [Gruenhage and Sakai, 2011] it is pointed out how a space with countable π-weight is also R-separable and H-separable.

It is in fact possible to distnguish all the mentioned principles.

Corollary 3.3. [*Bella, Bonanzinga, and Matveev*, *2009*] *The existence of a countable M-separable space which is not H-separable is consistent with ZFC.*

In the same article the authors prove informally the existence of a H-separable (hence M-separable) non R-separable space (see [Bella, Bonanzinga, and Matveev, 2009, p. 1247]).

It was proved by Barman and Dow that FU property implies selective separablity in the class of separable spaces.

Theorem 3.4. [*Barman and Dow, 2011, Theorem 2.9*] *Each separable FU space is M-separable.*

This result was improved later in the same year.

Corollary 3.5. [*Gruenhage and Sakai, 2011, Corollary 4.2*] *Each separable FU space is R-separable.*

Recall the following weaker versions of selective separability properties.

Definition 3.6. A space X is said to be

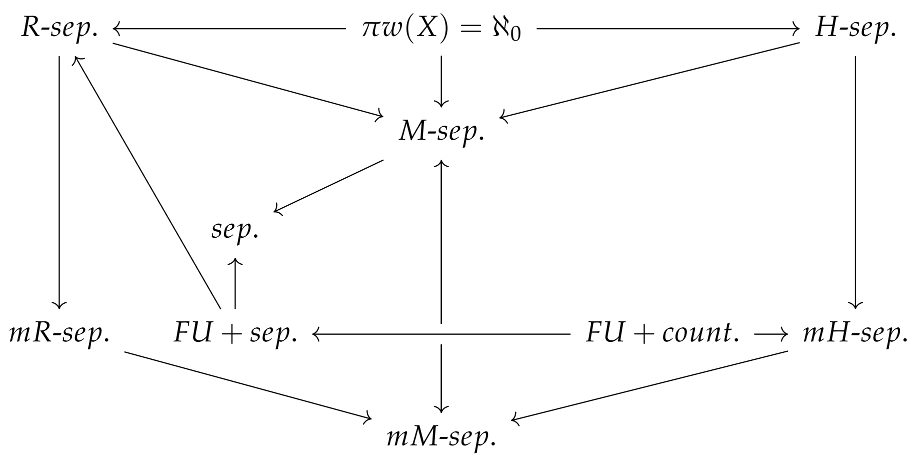
- *mM-separable* if for every decreasing sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there are finite sets $F_n \subset D_n$, $n \in \omega$, such that $\bigcup\{F_n : n \in \omega\}$ is dense in X;
- *mH-separable* if for every decreasing sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there are finite sets $F_n \subset D_n$, $n \in \omega$, such that every nonempty open set of X meets all but finitely many F_n [Gruenhage and Sakai, 2011, Definition 2.6];
- *mR-separable* if for every decreasing sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there points $x_n \in D_n$, $n \in \omega$, such that $\{x_n : n \in \omega\}$ is dense in X [Gruenhage and Sakai, 2011, Definition 2.6].

The previous weaker versions of selective separability were introduced by the authors in the light of the following result.

Lemma 3.7. [*Gruenhage and Sakai, 2011, Lemma 2.1*] *A space is M-separable if and only if it is mM-separable.*

Notice also that, by [Gruenhage and Sakai, 2011, Lemma 2.7(2) and Corollary 4.2], every countable FU space is mH-separable.

The following diagram sums up the implications between the cited selective properties.



In this chapter the spaces considered will be mainly countable and without isolated points in the light of the following results.

Proposition 3.8. [*Bella, Bonanzinga, Matveev, and Tkachuk, 2008, Proposition 2.3 (3)*] *Given a space X, if X has a dense open M-separable subspace, then it is M-separable.*

Lemma 3.9. [*Barman and Dow, 2011, Lemma 2.3*] *A space X is M-separable if and only if the set I of isolated points is countable and* $X \setminus \overline{I}$ *is M-separable.*

In the proof of Theorem 3.2.3, the following result will be used, originally proved with different terminology

Theorem 3.10. [*Blass, 1973, Theorem 2*] $(\mathfrak{p} = \mathfrak{c})$ *There exist* $2^{\mathfrak{c}}$ $P_{\mathfrak{c}}$*-point ultrafilters.*

In fact the statement of the theorem assume MA instead of $(\mathfrak{p} = \mathfrak{c})$, but the proof takes advantage only of the equality.

Recall also the following proposition

Proposition 3.11. [*Bella, Bonanzinga, and Matveev, 2013, Proposition 3.2*] *Let A be a countable subset of a space X. If* $x \in \overline{A}$ *and* $\chi(x, X) < \mathfrak{p}$*, then there exists a sequence* $S \subseteq A$ *which converges to x.*

The following result is crucial for what studied in paragraph 3.3.

Theorem 3.12. [*Barman and Dow, 2012, Theorem 3.3*] *(PFA) The product of two countable FU spaces is M-separable.*

Notice that in a previous article the authors proved the converse of the theorem above using an opposite set-theoretical assumption.

Theorem 3.13. [*Barman and Dow, 2011, Theorem 3.3*] *(CH) There exist two countable FU spaces whose product is not M-separable.*

3.1 A Hausdorff Fréchet-Urysohn space which is not H-separable in ZFC

Theorem 3.1.1. [*Bardyla, M., Zdomskyy, 2023, Theorem 2.1*] *There exists a countable Hausdorff FU space X without isolated points which is not H-separable.*

Proof. The underlying set of X will be ω, and let τ_0 be a topology on ω such that $\langle \omega, \tau_0 \rangle$ is homeomorphic to the rationals $\mathbb{Q}$ with the topology induced from the euclidean one.

Proceeding recursively over ordinals $\alpha \in \mathfrak{c}$, a topology $\tau = \tau_{\mathfrak{c}}$ turning ω into a space with the needed properties will be constructed as an increasing union $\tau_{\mathfrak{c}} = \bigcup_{\alpha<\mathfrak{c}} \tau_\alpha$ of intermediate topologies.

Let $\langle E_n : n \in \omega \rangle$ be a sequence of dense subsets of $\langle \omega, \tau_0 \rangle$; without loss of generality it is possible to assume that those are mutually disjoint. Consider an enumeration $\{\langle S_\alpha, x_\alpha \rangle : \alpha < \mathfrak{c}\}$ of $[\omega]^\omega \times \omega$ such that for each $\langle S, x \rangle \in [\omega]^\omega \times \omega$ there are cofinally many ordinals α for which $\langle S, x \rangle = \langle S_\alpha, x_\alpha \rangle$, and an enumeration $\{\langle F^\alpha_n : n \in \omega \rangle : \alpha < \mathfrak{c}\}$ of $\prod_{n\in\omega} [E_n]^{<\omega}$.

Let $\mathcal{A} \subset [\omega]^\omega$ be a compact almost disjoint family of size $\mathfrak{c}$ (it can easily be constructed by, e.g., considering the family of all branches through $2^{<\omega}$, and then copying it via any bijection between ω and $2^{<\omega}$) and $\mathcal{C}$ be any mad family extending $\mathcal{A}$. Note that $\mathcal{C} \neq \mathcal{A}$ since there are no analytic mad families by [Mathias, 1977, Corollary 4.7].

Suppose that for some $\alpha < \mathfrak{c}$ and all $\delta \in \alpha$ a topology τ_δ on ω, a family $\mathcal{Y}_\delta \subset [\omega]^\omega$ were contructed, and for every $Y \in \mathcal{Y}_\delta$ either $n(Y) \in \omega$ (such Y will be called *vertical*) or $C(Y) \in \mathcal{C}$ (such Y will be called *horizontal*) such that the following conditions are satisfied:

(1) $\tau_\beta \subset \tau_\delta$ for all $\beta \leq \delta$;

(2) $\mathcal{Y}_\beta \subset \mathcal{Y}_\delta$ for all $\beta \leq \delta$ and $\mathcal{Y}_\delta$ consists of sequences convergent in $\langle \omega, \tau_\delta \rangle$;

(3) For every $\beta < \delta$, if x_β is a limit point of S_β (with respect to τ_δ), then there exists $Y \in \mathcal{Y}_\delta$ converging to x_β with $|Y \cap S_\beta| = \omega$;

(4) Every vertical $Y \in \mathcal{Y}_\delta$ is contained in $E_{n(Y)}$, and each horizontal $Y \in \mathcal{Y}_\delta$ is a subset of $\bigcup_{n \in C(Y)} E_n$; Moreover, in the latter case $|Y \cap E_n| \leq 1$ for all $n \in C(Y)$;

(5) For every $\beta < \delta$ there exists $U_\beta \in \tau_\delta$ and $A_\beta \in \mathcal{A}$ such that:

- $U_\beta = \omega \setminus \bigcup_{n \in A_\beta} F_n^\beta$,
- $\{A_\beta : \beta < \delta\} \cap \{C(Y) : Y \in \mathcal{Y}_\delta\} = \emptyset$
- $A_\beta \neq A_{\beta'}$ for any $\beta \neq \beta'$ with $\beta' < \delta$

(6) τ_δ is generated by $\tau_0 \cup \{U_\beta : \beta < \delta\}$ as a subbase;

(7) E_n is dense in $\langle \omega, \tau_\delta \rangle$ for all $n \in \omega$.

Observe that (5) and (6) imply (7), as well as the fact that for every $\mathcal{C}' \in [\mathcal{C}]^{<\omega}$ and $K \in [\omega]^{<\omega}$, the topology $\tau_\delta \upharpoonright \bigcup_{n \in K \cup (\bigcup \mathcal{C}')} E_n$ has a countable base. Moreover, if $\mathcal{C}' \in [\mathcal{C} \setminus \{A_\beta : \beta < \delta\}]^{<\omega}$, then

$$\tau_\delta \upharpoonright \bigcup_{n \in K \cup (\bigcup \mathcal{C}')} E_n = \tau_0 \upharpoonright \bigcup_{n \in K \cup (\bigcup \mathcal{C}')} E_n.$$

Several cases are now to be considered, depending on α:

I. α is limit.
It is easily checked that the topology τ_α generated by $\bigcup_{\delta < \alpha} \tau_\delta$ as a base, along with $\mathcal{Y}_\alpha = \bigcup_{\delta < \alpha} \mathcal{Y}_\delta$ satisfies (1)-(7) for $\delta = \alpha$.

II. $\alpha = \delta + 1$ and x_δ is a limit point of S_δ in $\langle \omega, \tau_\delta \rangle$. Two sub-cases are considered:

(i) If $|Y \cap S_\delta| = \omega$ for some $Y \in \mathcal{Y}_\delta$ converging to x_δ, then define $\mathcal{Y}_\alpha = \mathcal{Y}_\delta$ and pick any

$$A_\delta \in \mathcal{A} \setminus \{C(Y) : Y \in \mathcal{Y}_\delta\},$$

let U_δ be such as in item (6), denote by τ_α the topology generated by $\tau_\delta \cup \{U_\delta\}$ as a subbase, and note that all the conditions (1)-(7) are satisfied, e.g., each $Y \in \mathcal{Y}_\alpha = \mathcal{Y}_\delta$ is convergent also in $\langle \omega, \tau_\alpha \rangle$ since U_δ almost contains all such Y.

(ii) Suppose $|Y \cap S_\delta| < \omega$ for all $Y \in \mathcal{Y}_\delta$ converging to x_δ. If there exists $n \in \omega$ such that $x_\delta \in \overline{S_\delta \cap E_n}^{\tau_\delta}$, let $Y_\delta \in [S_\delta \cap E_n]^\omega$ be a sequence converging to x_δ (remember that all the τ_β's restricted to E_n are the same, and thus turn E_n into a copy of the rationals), and in this case set $n(Y_\delta) = n$. Conversely, assume that there is no such n.

Claim 1. *There exists a sequence $Y \in [S_\delta]^\omega$ converging to x_δ in $\langle \omega, \tau_\delta \rangle$.*

Proof of the Claim 1. Let $\mathcal{C}'$ be the family of all those $C \in \mathcal{C}$ such that there exists a sequence $Y_C \in [S_\delta]^\omega$ convergent to x_δ in $\langle \omega, \tau_0 \rangle$ such that $Y_C \subset \bigcup_{n \in C} E_n$ and $|Y_C \cap E_n| \leq 1$ for all $n \in C$. Two cases are possible:

a) $\mathcal{C}'$ is finite. It follows that x_δ is not in the closure of $S'_\delta = S_\delta \setminus \bigcup\{E_n : n \in \cup\mathcal{C}'\}$ with respect to τ_0. Indeed, otherwise there exists a sequence $Y \in [S'_\delta]^\omega$ converging to x_δ in $\langle \omega, \tau_0 \rangle$, and since it cannot have infinite intersection with any E_n, it is possible to conclude that there exists $L \in [\omega]^\omega$ with $|Y \cap E_n| \neq \emptyset$ for all $n \in L$. Then $L \cap (\cup\mathcal{C}') = \emptyset$, but on the other hand any $C \in \mathcal{C}$ with $|C \cap L| = \omega$ must be in C', a contradiction. Consequently, x_δ is also not in the closure of S'_δ with respect to τ_δ, and hence it must be in the closure of $S''_\delta = S_\delta \cap \bigcup\{E_n : n \in \cup\mathcal{A}'\}$ with respect to τ_δ. However, $\tau_\delta \restriction S''_\delta \cup \{x_\delta\}$ has countable base, so there exists a sequence $Y_\delta \in [S''_\delta]^\omega$ convergent to x_δ with respect to τ_δ. By shrinking Y_δ if necessary, it is possible to assume to find $C(Y_\delta) \in \mathcal{C}'$ satisfying (4) along with Y_δ. It remains to set $\mathcal{Y}_\alpha = \mathcal{Y}_\delta \cup \{Y_\delta\}$, pick any $A_\delta \in \mathcal{A} \setminus (\{C(Y) : Y \in \mathcal{Y}_\alpha\} \cup \{A_\beta : \beta < \delta\})$, and let U_δ and τ_α be such as required in (5), (6) for α.

b) $\mathcal{C}' \supset \{C_k : k \in \omega\}$, where $C_k \neq C_m$ for $k \neq m$. For every $k \in \omega$ let Y_{C_k} be witnessing for $C_k \in \mathcal{C}'$. By shrinking Y_{C_k}'s, if necessary, assume that $|\bigcup_{k\in\omega} Y_{C_k} \cap E_n| \leq 1$ for all $n \in \omega$. Let $Y \subset \bigcup_{k\in\omega} Y_{C_k}$ be such that $Y_{C_k} \subset^* Y$ for all $k \in \omega$ and Y converges to x_δ with respect to τ_0. Such Y obviously exists since τ_0 has a countable base. Set $L = \{n \in \omega : Y \cap E_n \neq \emptyset\}$ and note that L cannot be covered by finitely many elements of $\mathcal{C}$ because $|L \cap C_k| = \omega$ for all $k \in \omega$. Thus

$$\{A \cap L \,:\, A \in \mathcal{A}, |A \cap L| = \omega\}$$

cannot be a mad family of infinite subsets of L since it is analytic, and hence there exists $L' \in [L]^\omega$ which is almost disjoint from all $A \in \mathcal{A}$. Let $C \in \mathcal{C}$ be such that $|C \cap L'| = \omega$, denote by Y_δ the sequence $Y \cap \bigcup_{n \in C\cap L'} E_n$, set $C(Y_\delta) = C$, and $Y_\alpha = \mathcal{Y}_\delta \cup \{Y_\delta\}$. Finally, pick any $A_\delta \in A \setminus (\{C(Y) : Y \in \mathcal{Y}_\alpha\} \cup \{A_\beta : \beta < \delta\})$ and let U_δ and τ_α be such as required in (5), (6) for α. Since $A_\beta \cap C(Y)$ is finite for all $Y \in \mathcal{Y}_\alpha$ and $\beta < \alpha$, $Y \subset^* U_\beta$ for all $Y \in \mathcal{Y}_\alpha$ and $\beta < \alpha$, and hence all $Y \in \mathcal{Y}_\alpha$ remain convergent with respect to τ_α.

□

III. $\alpha = \delta + 1$ and x_δ is not a limit point of S_δ in $\langle \omega, \tau_\delta \rangle$. In this case set $x'_\delta = 0$ (in fact, any $i \in \omega$ instead of 0 would work), $S'_\delta = \omega \setminus \{0\}$, and repeat the same argument of *II*, with $\langle S'_\delta, x'_\delta \rangle$ instead of $\langle S_\delta, x_\delta \rangle$. Note that x'_δ must be in the closure of S'_δ in $\langle \omega, \tau_\delta \rangle$ since this topological space has no isolated points, because it has disjoint dense subsets by (7).

This completes the recursive construction of the objects mentioned in (1)-(7) so that these conditions are satisfied. The space $\langle \omega, \tau_\mathfrak{c} \rangle$ is as required: (2), (3) imply the FU property, and (5) entails the failure of the H-separability. □

3.2 A Fréchet-Urysohn α_4 space which is not H-separable under $\mathfrak{p} = \mathfrak{c}$

Recall the following local properties of points of a space.

Definition 3.2.1. [*Arhangel'skii, 1972, p. 267*]Let X be a space, and denote by Γ_x the set of all $A \in [X \setminus \{x\}]^\omega$ which converge to $x \in X$. A point $x \in X$ has the property:

(α_1) if for each $\langle S_n : n \in \omega \rangle \in [\Gamma_x]^\omega$, there is $S \in \Gamma_x$ such that $S_n \subset^* S$ for all $n \in \omega$;

(α_2) if for each $\langle S_n : n \in \omega \rangle \in [\Gamma_x]^\omega$, there is $S \in \Gamma_x$ such that $S_n \cap S$ is infinite for all $n \in \omega$;

(α_3) if for each $\langle S_n : n \in \omega \rangle \in [\Gamma_x]^\omega$, there is $S \in \Gamma_x$ such that $S_n \cap S$ is infinite for infinitely many $n \in \omega$;

(α_4) if for each $\langle S_n : n \in \omega \rangle \in [\Gamma_x]^\omega$, there is $S \in \Gamma_x$ such that $S_n \cap S \neq \emptyset$ for infinitely many $n \in \omega$.

A space X is an α_i space, where $i \in \{1,2,3,4\}$, if it is an α_i space at each $x \in X$.

Each of these α_i-properties obviously imply the next one.

The main result of this section endowes the space of Theorem 3.1.1 of zero-dimensionality and α_4 property, but at the cost of an additional set-theoretic assumption.

As a preliminary, the next easy statement is proved: it can be considered as a folklore and it is possible to refer to it as "there are no $(\omega, < \mathfrak{b})$-gaps". Its proof is similar to the argument of [Jech, 2003, Theorem 29.8].

Lemma 3.2.2. [*Bardyla, M., Zdomskyy, 2023, Lemma 3.1*] *Suppose that* $\mathcal{A} \subset [\omega]^\omega$, $|\mathcal{A}| < \mathfrak{b}$, $\{B_i : i \in \omega\} \subset [\omega]^\omega$, *and* $|A \cap B_i| < \omega$ *for all* $A \in \mathcal{A}$ *and* $i \in \omega$. *Then there exists* $X \subset \omega$ *such that* $A \subset^* \omega \setminus X$ *and* $B_i \subset^* X$ *for any* $A \in \mathcal{A}$ *and* $i \in \omega$.

Proof. For each $A \in \mathcal{A}$ find an increasing function $f_A \in \omega^\omega$ with $A \cap B_i \subset f_A(i)$ (for example, consider $f_A(i) = max(A \cap B_i) + 1$) for all $i \in \omega$, and let $f \in \omega^\omega$ be such that $f_A \leq^* f$ for all $A \in \mathcal{A}$. Then $X = \bigcup_{i \in \omega}(B_i \setminus f(i))$ is as required. □

Theorem 3.2.3. [*Bardyla, M., Zdomskyy, 2023, Theorem 3.2*] ($\mathfrak{p} = \mathfrak{c}$) *There exists a countable zero-dimensional* α_4 *FU space* X *without isolated points which is not H-separable.*

Proof. The underlying set of X will be ω, and let τ_0 be a topology on ω such that $\langle \omega, \tau_0 \rangle$ is homeomorphic to the rationals $\mathbb{Q}$ with the topology induced from the euclidean one.

By Theorem 3.10, it is possible to fix a $P_\mathfrak{c}$-point $\mathcal{G}$.

The final topology $\tau = \tau_\mathfrak{c}$ turning ω into a space with the needed properties will be constructed recursively over ordinals $\alpha \in \mathfrak{c}$ as an increasing union $\tau_\mathfrak{c} = \bigcup_{\alpha < \mathfrak{c}} \tau_\alpha$ of intermediate zero-dimensional topologies.

Let $\langle E_n : n \in \omega \rangle$ be a sequence of mutually disjoint dense subsets of $\langle \omega, \tau_0 \rangle$ and $\{\langle S_\alpha, \mathcal{A}_\alpha, x_\alpha \rangle : \alpha < \mathfrak{c}\}$ be an enumeration of $[\omega]^\omega \times [[\omega]^\omega]^\omega \times \omega$ such that for each $\langle S, \mathcal{A}, x \rangle \in [\omega]^\omega \times [[\omega]^\omega]^\omega \times \omega$ there are cofinally many ordinals α such that $\langle S, \mathcal{A}, x \rangle = \langle S_\alpha, \mathcal{A}_\alpha, x_\alpha \rangle$. Fix also an enumeration $\{\langle F_n^\alpha : n \in \omega \rangle : \alpha < \mathfrak{c}\}$ of $\prod_{n \in \omega}[E_n]^{<\omega}$.

Suppose to have already constructed a zero-dimensional topology τ_δ on ω, an almost disjoint family $\mathcal{Y}_\delta \subset [\omega]^\omega$, and for every $Y \in Y_\delta$ and element $I_Y \in \mathcal{G}^* = \mathcal{P}(\omega) \setminus \mathcal{G}$ for some $\alpha < \mathfrak{c}$ and all $\delta \in \alpha$ such that the following conditions are satisfied:

(1) The weight of τ_δ is $< \mathfrak{c}$;

(2) For every $n, x \in \omega$ there exists $Y \in \mathcal{Y}_0$ such that $Y \in [E_n]^\omega$ and Y converges to x;

(3) $\mathcal{Y}_\beta \subset \mathcal{Y}_\delta$ for all $\beta \leq \delta < \alpha$ and $\mathcal{Y}_\delta$ consists of sequences convergent in $\langle \omega, \tau_\delta \rangle$;

(4) For every $\beta < \delta$, if x_β is a limit point of S_β and every $A \in \mathcal{A}_\beta$ is a sequence converging to x_β in $\langle \omega, \tau_\delta \rangle$, then

- there exists $Y \in \mathcal{Y}_\delta$ converging to x_β with $|Y \cap S_\beta| = \omega$;
- there exists $Y \in \mathcal{Y}_\delta$ such that $Y \cap (A \setminus \{x_\beta\}) \neq \emptyset$ for infinitely many $A \in \mathcal{A}_\beta$;

(5) Every $Y \in \mathcal{Y}_\delta$ is contained in $\bigcup_{n \in I_Y} E_n$; Moreover, either $|Y \cap E_n| = 1$ for all $n \in I_Y$, or $|I_Y| = 1$;

(6) For every $\beta < \delta$ there exists $U_\beta \in \tau_\delta$ and $G_\beta \in \mathcal{G}$ such that $U_\beta \cap \bigcup_{n \in G_\beta} F_n^\beta = \emptyset$;

(7) E_n is dense in $\langle \omega, \tau_\delta \rangle$ for all $n \in \omega$.

Several cases are now to be considered, depensing on α:

I. α is limit.
It is easily checked that the topology τ_α generated by $\bigcup_{\delta<\alpha} \tau_\delta$ as a base, along with $\mathcal{Y}_\alpha = \bigcup_{\delta<\alpha} \mathcal{Y}_\delta$, satisfies (1)-(7) for $\delta = \alpha$.

II. $\alpha = \delta + 1$, x_δ is a limit point of S_δ in $\langle \omega, \tau_\delta \rangle$, and $\mathcal{A}_\delta$ consists of mutually disjoint sequences convergent to x_δ.
If $|Y \cap S_\delta| = \omega$ for some $Y \in \mathcal{Y}_\delta$ converging to x_δ, then denote by $Y_{0,\delta}$ one of these Y's. Similarly, if $\{A \in \mathcal{A}_\beta : Y \cap (A \setminus \{x_\beta\}) \neq \emptyset\}$ is infinite for some $Y \in \mathcal{Y}_\delta$, denote by $Y_{1,\delta}$ one of these Y's.

So consider the case $|Y \cap S_\delta| < \omega$ for all $Y \in \mathcal{Y}_\delta$ converging to x_δ. Since the weight of $\langle \omega, \tau_\delta \rangle$ is $< \mathfrak{c} = \mathfrak{p}$, by Proposition 3.11 there exists a sequence $Y_{0,\delta} \in [S_\delta]^\omega$ convergent to x_δ. Passing to an infinite subset of $Y_{0,\delta}$, if necessary, it is possible to assume that either there exists $n \in \omega$ such that $Y_{0,\delta} \subset E_n$, in which case set $I_{Y_{0,\delta}} = \{n\}$, or there exists an infinite $I_{Y_{0,\delta}} \in \mathcal{G}^*$ such that $Y_{0,\delta} \subset \bigcup_{n \in I_{Y_{0,\delta}}} E_n$ and $|Y_{0,\delta} \cap E_n| = 1$ for all $n \in I_{Y_{0,\delta}}$. Note that $Y_{0,\delta}$ is almost disjoint from any $Y \in \mathcal{Y}_\delta$.

Now suppose that $\{A \in \mathcal{A}_\beta : Y \cap (A \setminus \{x_\beta\}) \neq \emptyset\}$ is finite for all $Y \in \mathcal{A}$. Replacing each $A \in \mathcal{A}_\delta$ with an infinite subset thereof, it is possible to assume that for each $A \in \mathcal{A}_\delta$, either there exists $n(A) \in \omega$ such that $A \subset E_{n(A)}$ (such A will be called *vertical*), or $|A \cap E_n| \leq 1$ for all $n \in \omega$ (such A will be called *horizontal*). Since every space of character $< \mathfrak{b}$ is α_1, there exists a sequence $Y \subset \cup \mathcal{A}_\delta$ convergent to x_δ such that $|Y \cap A| = 1$ for all $A \in \mathcal{A}_\delta$. The fact that no element of $\mathcal{Y}_\delta$ converging to x_δ intersects infinitely many $A \in \mathcal{A}_\delta$ yields that Y is almost disjoint from all elements of $\mathcal{Y}_\delta$. Note also that $\mathcal{A}_\delta$ is a disjoint family, and hence each infinite subset of Y intersects infinitely many elements of $\mathcal{A}_\delta$.

If there are infinitely many horizontal sequences, then $|Y \cap E_n| \neq \emptyset$ for infinitely many n, and hence by shrinking Y to some infinite $Y_{1,\delta}$ it is possible to assume that $|Y_{1,\delta} \cap E_n| \leq 1$ for all n and $|Y_{1,\delta} \cap E_n| = 1$ if and only if $n \in I$ for some $I \in \mathcal{G}^*$. In this case set $I_{Y_{1,\delta}} = I$.

Now suppose that all but finitely many $A \in \mathcal{A}_\delta$ are vertical. If $\{n(A) : A \in \mathcal{A}_\delta, A \text{ is vertical}\}$ is infinite, then as before by shrinking Y to some infinite $Y_{1,\delta}$ it is possible to assume that $|Y_{1,\delta} \cap E_n| \leq 1$ for all n and $|Y_{1,\delta} \cap E_n| = 1$ if and only if $n \in I$ for some $I \in \mathcal{G}^*$. Also in this case set $I_{Y_{1,\delta}} = I$.

In the remaining case there exists $n \in \omega$ with $\{A \in \mathcal{A}_\delta : n(A) = n\}$ infinite. Then set $Y_{1,\delta} = Y \cap E_n$, $I_{Y_{1,\delta}} = \{n\}$, and note that $Y_{1,\delta}$ is infinite.

Shrinking $Y_{0,\delta}$ and $Y_{1,\delta}$, if necessary, it is possible to assume that they are disjoint. Finally, set $\mathcal{Y}_\alpha = \mathcal{Y}_{\delta+1} = \mathcal{Y}_\delta \cup \{Y_{0,\delta}, Y_{1,\delta}\}$ and note that (3), (4), and (5) are satisfied for $\delta + 1$ instead of δ.

Now consider the following construction of U_δ, G_δ satisfying (6). This will require that if $x \in U_\delta$ and $Y \in \mathcal{Y}_\delta$ converges to x, then $Y \subset^* U_\delta$. Let $G_\delta \in \mathcal{G}$ be such that $|G_\delta \cap I_Y| < \omega$ for all $Y \in \mathcal{Y}_\alpha$. Then $F_\delta := \bigcup_{n \in G_\delta} F^\delta_n$ is almost disjoint from any $Y \in \mathcal{Y}_\alpha$.

Claim 2. *For every $n \in \omega$ there exists $C_n \subset \omega$ such that*

(8) *$Y \subset^* C_n$ for any $Y \in \mathcal{Y}_\alpha$ converging to n in $\langle \omega, \tau_\delta \rangle$;*

(9) *$C_n \cap C_m = \emptyset$ for all $n \neq m$;*

(10) *$C_n \cap F_\delta = \emptyset$ for all $n \neq m$.*

Proof. For every $n \in \omega$ denote by $\mathcal{Y}_{\alpha,n}$ the family of all $Y \in \mathcal{Y}_\alpha$ converging to n, and fix a family $\{O^n_k : k \in \omega \setminus \{n\}\} \subset \tau_0$ such that O^n_k is a clopen neighbourhood of k not containing n. Then $|Y \cap O^n_k| < \omega$ for all $Y \in \mathcal{Y}_{\alpha,n}$ and $k \neq n$, and hence Lemma 3.2.2 implies that there exists $C^0_n \subset \omega$ such that $Y \subset^* C^0_n$ and $|C^0_n \cap O^n_k| < \omega$ for all Y, k as above. Notice that $|C^0_n \cap Y| < \omega$ for all $Y \in \mathcal{Y}_\alpha \setminus \mathcal{Y}_{\alpha,n}$ because $Y \subset^* O^n_k$ for $k \neq n$ being the limit point of Y. Thus letting $C_0 = C^0_0 \setminus F_\delta$ and $C_n = C^0_n \setminus (\bigcup_{n' < n} C_{n'} \cup F_\delta)$ it is obtained that $\{C_n : n \in \omega\}$ is a disjoint family satisfying $(8) - (10)$. □

The construction of $V_\delta := \omega \setminus U_\delta$ will be done recursively over $k \in \omega$, namely it will be constructed as an increasing union $\bigcup_{k \in \omega} V^\delta_k$. Set $V^\delta_{-1} = \emptyset$, $V^\delta_0 = F_\delta$ and assuming that V^δ_k is constructed, let

$$V^\delta_{k+1} = V^\delta_k \cup \bigcup \{C_n \setminus \{\min C_n\} : n \in V^\delta_k \setminus V^\delta_{k-1}\}.$$

In the sequel a subset B of ω it is said to be *saturated* if $n \in B$ implies $C_n \subset^* B$. It follows that V_δ defined above is saturated. It is sufficient to prove that $U_\delta := \omega \setminus V_\delta$ is saturated as well. Indeed, otherwise there exists $n \in U_\delta$ such that $C_n \cap V_\delta \neq \emptyset$. Let $k \in \omega$ be the minimal such that $C_n \cap V^\delta_{k+1} \neq \emptyset$ (note that $C_n \cap V^\delta_0 = C_n \cap F_\delta = \emptyset$ by (9)). Then there exists $m \in V^\delta_k \setminus V^\delta_{k-1}$ with $C_n \cap C_m \neq \emptyset$, which is impossible since $n \neq m$ (because $n \in U_\delta$ and $m \in V_\delta$).

Finally, $U_\delta \neq \emptyset$ since $\min C_n \in U_\delta$ for all $n \in V_\delta$.

Let τ_α be the topology generated by $\tau_\delta \cup \{U_\delta, V_\delta\}$ as a base. This way a 0-dimensional topology satisfying (6) is obtained. Since both U_α, V_α are saturated, it is easy to see that all $Y \in \mathcal{Y}_\alpha$ are convergent also in $\langle \omega, \tau_\alpha \rangle$. Indeed, suppose that y is the limit of $Y \in \mathcal{Y}_\alpha$ and $y \in U \cap U_\delta$, where $U \in \tau_\delta$. Then $Y \subset^* U$ (because Y converges in $\langle \omega, \tau_\delta \rangle$) and $Y \subset^* C_y \subset^* U_\delta$, and therefore $Y \subset^* U \cap U_\delta$. The same argument works also for V_δ instead of U_δ. Since $U \in \tau_\delta$ was arbitrary, Y converges to y also in $\langle \omega, \tau_\alpha \rangle$.

To see that also (7) holds, fix $n \in \omega$ and $y \in U$, where $U \in \tau_\alpha$. Let $Y \in \mathcal{Y}_0 \cap [E_n]^\omega \subset \mathcal{Y}_\alpha$ be convergent to y. Then $Y \subset^* U$, and hence $Y \subset^* U \cap E_n$, which yields $U \cap E_n \neq \emptyset$.

This completes the construction of the objects mentioned in (1)-(7) for all $\delta < \mathfrak{c}$, so that these conditions are satisfied. Condition (4) implies that $X = \langle \omega, \tau_\mathfrak{c} \rangle$ is FU and α_4. Indeed, if $\mathcal{A}$ is a countable family of mutually disjoint sequences convergent to some $x \in \omega$ in $X = \langle \omega, \tau_\mathfrak{c} \rangle$, then there exists $\delta < \mathfrak{c}$ such that $\langle \omega \setminus \{x\}, \mathcal{A}, x \rangle = \langle S_\delta, \mathcal{A}_\delta, x_\delta \rangle$. Then $Y_{1,\delta} \in \mathcal{Y}_{\delta+1}$ is convergent to x_α in $\langle \omega, \tau_\mathfrak{c} \rangle$, and it intersects infinitely many elements of $\mathcal{A}_\delta = \mathcal{A}$. Similarly one can check the FU property. Finally, (6) implies the failure of the H-separability, which completes the proof. □

The assumption of being α_4 cannot be much improved in Theorem 3.2.3 as the following fact shows.

Proposition 3.2.4. [*Bardyla, M., Zdomskyy, 2023, Proposition 3.3*] *Every separable FU α_2 space X is H-separable.*

Proof. Let $\langle E_n : n \in \omega \rangle$ be a sequence of dense subsets of X, and D be a countable dense subset of X. Let $\{d_i : i \in \omega\}$ be an enumeration of all non-isolated elements of D, and $\{d'_i : i \in \omega\} = D \setminus \{d_i : i \in \omega\}$. For every $i, n \in \omega$ fix an injective sequence S^i_n of elements of $E_n \setminus \{d_i\}$ convergent to d_i. Applying α_2 it is possible to find a sequence $\langle s^i_n : n \in \omega \rangle$ convergent to d_i such that $s^i_n \in S^i_n \subset E_n$. Note that $\{d'_i : i \in \omega\}$ consists of isolated points of X and hence it is a subset of E_n for all $n \in \omega$. It follows that the sequence

$$\langle F_n := \{d'_i : i \leq n\} \cup \{s^i_n : i \leq n\} \ : \ n \in \omega \rangle$$

is a witness for the H-separability of X. Indeed, given any open $U \subset X$, either there exists $i \in \omega$ such that $d'_i \in U$, and thus $U \cap F_n \neq \emptyset$ for all $n \geq i$; or there exists $i \in \omega$ with $d_i \in U$, and then $U \cap F_n \neq \emptyset$ as soon as $s^i_n \in U$ and $n \geq i$, and $s^i_n \in U$ for all but finitely many n since $\langle s^i_n : n \in \omega \rangle$ convergent to d_i. □

Question 3.2.5. *Does there exists a zero-dimensional FU zero-dimensional α_3 space which is not H-separable?*

3.3 Products of Fréchet-Urysohn spaces and M-separability

Definition 3.3.1. [*Bardyla, M., Zdomskyy, 2023*] Let τ_0 be a topology on ω turning it into a space homeomorphic to the rationals $\mathbb{Q}$. In what follows $(*_\mathbb{Q})$ stands for the following statement:

MA + there exists a mad family $\mathcal{A}$ on ω such that

- every disjoint pair $\mathcal{A}', \mathcal{A}'' \in [\mathcal{A}]^{<\mathfrak{c}}$ is *separated*, i.e. there exists $S \subset \omega$ such that $A' \subset^* S$ and $A'' \subset^* \omega \setminus S$ for any $A' \in \mathcal{A}'$ and $A'' \in \mathcal{A}''$;
- Every $A \in \mathcal{A}$ is either closed discrete or a convergent sequence in $\langle \omega, \tau_0 \rangle$.

The formally weaker statement obtained from $(*_\mathbb{Q})$ by dropping the second item is known to be consistent. More precisely, the following result was proved by Dow and Shelah.

Theorem 3.3.2. [*Dow and Shelah, 2012*] *It is consistent with MA and $\mathfrak{c} = \omega_2$ that there is a maximal almost disjoint family such that any disjoint pair of its $\leq \aleph_1$-sized subsets are separated.*

Osservation 3.3.3. Consider a topology τ_0 on ω such that $\langle \omega, \tau_0 \rangle$ is homeomorphic to the rationals in $[0,1]$, and let $\theta : \omega \to \mathbb{Q} \cap [0,1]$ be the corresponding homeomorphism.

In what follows, all the topological properties of subsets of ω, (e.g., convergence, being closed discrete, etc.) are considered with respect to the topology τ_0.

Fix an almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ consisting of convergent sequences as well as closed discrete subsets. Given any $n \in \omega$, denote by $\mathcal{F}_n(\mathcal{A})$ the filter generated by $\mathcal{F}(\mathcal{A}) \cup \{O \in \tau_0 : n \in O\}$ and note that if G is $\mathbb{M}_{\mathcal{F}_n(A)}$-generic, then $X_G := \bigcup_{\langle s,F \rangle \in G} s$ is a sequence convergent to n, and $|X_G \cap Y| = \omega$ for any sequence $Y \in V$ convergent to n, provided that $Y \not\subset \bigcup \mathcal{A}'$ for any $\mathcal{A}' \in [\mathcal{A}]^\omega$. Indeed, any such Y belongs to $\mathcal{F}_n(\mathcal{A})^+$, and X_G has infinite intersection with any element of $\mathcal{F}_n(\mathcal{A})^+$ by genericity.

Fix $r \in [0,1] \setminus \mathbb{Q}$ and let $\mathcal{F}_r(\mathcal{A})$ be the filter generated by

$$\mathcal{F}(\mathcal{A}) \cup \{\theta^{-1}[O \cap \mathbb{Q}] : r \in O \text{ is an open subset of } \mathbb{R}\}.$$

By genericity, if G is $\mathbb{M}_{\mathcal{F}_r(A)}$-generic, then $X_G := \bigcup_{\langle s,F \rangle \in G} s \subset \omega$ is closed and discrete because $\theta[X_G]$ converges to an irrational number $r \in [0,1]$, and $|X_G \cap Y| = \omega$ for any closed discrete subset Y of ω such that r is a limit point of $\theta[Y]$ and $Y \not\subset \bigcup \mathcal{A}'$ for any $\mathcal{A}' \in [\mathcal{A}]^{<\omega}$. Indeed, similarly as in the case of $\mathcal{F}_n(\mathcal{A})$, any such Y belongs to $\mathcal{F}_n(\mathcal{A})^+$, and X_G has infinite intersection with any ground model element of $\mathcal{F}_n(\mathcal{A})^+$ by the genericity.

Definition 3.3.4. [*Dow and Shelah, 2012, definitions 1.3 and 1.4*]An almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ is called *special*, if there exists a function $c : [\mathcal{A}]^{<\omega} \to \omega$ and a linear ordering $<$ of $\mathcal{A}$ such that for each $n \in \omega$ and sequences

$$\langle B_0, B_1, \ldots, B_{n-1} \rangle, \langle C_0, C_1, \ldots, C_{n-1} \rangle \in \mathcal{A}^n$$

increasing with respect to $<$, if

$$c\langle B_0, B_1, \ldots, B_{n-1} \rangle = c\langle C_0, C_1, \ldots, C_{n-1} \rangle = k,$$

then for all $i \neq j$, $i, j \in n$, $B_i \cap C_j \subset k$. An almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ is called ω_1-*special*, if any $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$ is special.

Observation 3.3.5. There is a natural poset introduced in [Dow and Shelah, 2012, Definition 2.4] and denoted by $\mathbb{Q}_{\mathcal{A},<}$, which for an almost disjoint family $\mathcal{A}$ and a linear order $<$ thereof, introduces a function c having the above property: A condition in this poset is a finite function $d : \mathcal{P}(\mathcal{A}_d) \to \omega$, where $\mathcal{A}_d \in [\mathcal{A}]^{<\omega}$, such that for each $n \in \omega$ and $<$-increasing sequences

$$\langle B_0, B_1, \ldots, B_{n-1} \rangle, \langle C_0, C_1, \ldots, C_{n-1} \rangle \in \mathcal{A}_d^n,$$

if $d\langle B_0, B_1, \ldots, B_{n-1} \rangle = d\langle C_0, C_1, \ldots, C_{n-1} \rangle = k$, then for all $i \neq j$, $i, j \in n$ we have $B_i \cap C_j \subset k$. It is clear that $\mathcal{A}$ is special in $V^{\mathbb{Q}_{\mathcal{A},<}}$. However, this poset may collapse cardinals: In ZFC one can construct an almost disjoint family $\mathcal{A}$ of size ω_1 which cannot be made special by any forcing which preserves ω_1, see [Dow and Shelah, 2012, p. 108] and references therein.

Lemma 3.3.6. *[Dow and Shelah, 2012, Prop. 1.5] ($MA_{\aleph_1}$) Let $\mathcal{A} \in [[\omega]^\omega]^{\omega_1}$ be a ω_1-special almost disjoint family. Then $\mathcal{A}$ is separated.*

Theorem 3.3.7. [*Bardyla, M., Zdomskyy, 2023, Theorem 4.4*] *($*_{\mathbb{Q}}$) is consistent.*

Proof. Denote by Λ the set of all limit ordinals below ω_2. Let V ba a model of GCH. Construct a finitely supported iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of c.c.c. posets of size ω_1, along with a sequence $\langle \dot{\mathcal{A}}_\beta : \beta < \omega_2, \beta \in \Lambda \rangle$, where $\dot{\mathcal{A}}_\beta = \{\dot{X}_\xi : \xi \in (\beta+1) \cap \Lambda\}$ is a $\mathbb{P}_{\beta+1}$-name for an almost disjoint family, as follows:

- If β is a successor ordinal, then $\dot{\mathbb{Q}}_\beta$ is ($\mathbb{P}_\beta$-name for) a ccc poset of size ω_1;
- For $\beta \in \Lambda$ the poset $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for the product $\dot{\mathbb{Q}}^0_\beta \times \dot{\mathbb{Q}}^1_\beta$, where
  - (a) Letting $\dot{\mathcal{A}}^-_\beta$ be a $\mathbb{P}_\beta$-name for $\bigcup_{\xi \in \beta \cap \Lambda} \dot{\mathcal{A}}_\xi = \{\dot{X}_\xi : \xi \in \beta \cap \Lambda\}$, $\dot{\mathbb{Q}}^0_\beta$ is either $\mathbb{M}_{\mathcal{F}_{\dot{r}_\beta}(\dot{\mathcal{A}}^-_\beta)}$ for some $\mathbb{P}_\beta$-name $\dot{r}_\beta$ for an irrational number in $[0,1]$, or $\mathbb{M}_{\mathcal{F}_{n_\beta}(\dot{\mathcal{A}}^-_\beta)}$ for some $n_\beta \in \omega$, decided by a bookkeeping function as described below; $\dot{\mathbb{Q}}^0_\beta$ produces a generic subset $\dot{X}_\beta$ almost disjoint from all elements of $\dot{\mathcal{A}}^-_\beta$, and set $\dot{\mathcal{A}}_\beta = \dot{\mathcal{A}}^-_\beta \cup \{\dot{X}_\beta\}$;
  - (b) $\dot{\mathbb{Q}}^1_\beta = \dot{\mathbb{Q}}_{\dot{\mathcal{A}}^-_\beta, <}$, where $<$ is the linear order (actually wellorder) of $\dot{\mathcal{A}}^-_\beta$ generated by the indices of its elements, i.e., $\dot{X}_\xi < \dot{X}_\zeta$ iff $\xi < \zeta$, $\xi, \zeta \in \beta \cap \Lambda$.

Almost literally following the argument in the proof of [Dow and Shelah, 2012, Theorem 2.6], it is possible to show that $\mathbb{P}_{\omega_2}$ is c.c.c., the only non-trivial ingredient being that $\dot{\mathbb{Q}}^1_\beta$ is forced to be c.c.c. by $1_{\mathbb{P}_\beta}$. This, roughly speaking, happens because elements of $\dot{\mathcal{A}}^-_\beta$ are added as generic reals for corresponding Mathias posets. Now a standard choice of a suitable bookkeeping function "delivering" the names for posets $\dot{\mathbb{Q}}_\beta$ at successor steps β, and n_β or $\dot{r}_\beta$ for limit β, guarantees that $(*_\mathbb{Q})$ holds in $V^{\mathbb{P}_{\omega_2}}$. More precisely, MA is guaranteed at successor stages, and $\mathcal{A} = \bigcup_{\beta \in \Lambda} \mathcal{A}_\beta$ is a mad family consisting of convergent sequences (added at stages $\beta \in \Lambda$ for which the bookkeeping function gives $n_\beta \in \omega$) and closed discrete sets (added at stages $\beta \in \Lambda$ for which the bookkeeping function gives $\dot{r}_\beta$); The presence of $\dot{\mathbb{Q}}^1_\beta$'s for $\beta \in \Lambda$ implies that any $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$ is special, which together with Lemma 3.3.6 implies that any such $\mathcal{A}'$ is separated, and thus completes the proof of the theorem. □

Observation 3.3.8. Even though the main part of the proof of Theorem 3.3.7, namely that $\dot{\mathbb{Q}}^1_\beta$ is forced to be c.c.c. by $1_{\mathbb{P}_\beta}$, can be done in exactly the same way as the corresponding step in the proof of [Dow and Shelah, 2012, Theorem 2.6], the overall scheme of the proof of Theorem 3.3.7 is very different from that of [Dow and Shelah, 2012, Theorem 2.1]. More precisely, the proof of the latter gives a so-called *tight* mad family, i.e., a mad family $\mathcal{A}$ such that for every countable $\mathcal{X} \subset \mathcal{I}(\mathcal{A})^+$ (recall that $X \in \mathcal{I}(\mathcal{A})^+$ iff X has infinite intersection with infinitely many elements of $\mathcal{A}$) there exists $A \in \mathcal{A}$ such that $|A \cap X| = \omega$ for all $X \in \mathcal{X}$. On the other hand, no mad family $\mathcal{A}$ of subsets of $\mathbb{Q}$ consisting of convergent sequences and closed discrete sets can be tight. Indeed, it easy to prove that for every $0 < \delta < \epsilon$ there exists a closed discrete $D \subset \mathbb{Q}$ such that $D \in \mathcal{I}(\mathcal{A})^+$ and $\delta < |x| < \epsilon$ for all $x \in D$. Now let $\langle D_n : n \in \omega \rangle$ be a sequence of closed discrete subsets of $\mathbb{Q}$ which lie in $\mathcal{I}(\mathcal{A})^+$ and such that $\frac{3}{5 \cdot 2^n} < |x| < \frac{4}{5 \cdot 2^n}$ for all $x \in D_n$. Then it is clear that there is neither a closed discrete subset of $\mathbb{Q}$ nor a convergent sequence there which has infinite intersection with each (even infinitely many) of D_n's.

The overall scheme of the proof of the next theorem is patterned after that of [Barman and Dow, 2011, Theorem 2.24]. However, there are also essential differences because this construction lasts $> \omega_1$-many steps and hence it is necessary to make sure to not face a kind of Hausdorff gaps consisting of convergent sequences in this construction, which is done with the help of a mad family witnessing $(*_\mathbb{Q})$.

Theorem 3.3.9. [*Bardyla, M., Zdomskyy, 2023, Theorem 4.5*] $(*_{\mathbb{Q}})$ *There exist two countable regular (hence zero-dimensional) FU spaces* X_0, X_1 *without isolated points whose product is not M-separable.*

Proof. Let the underlying sets of X_0, X_1 be ω and $\tau_0 = \sigma_0$ be a topology on ω such that $\langle \omega, \tau_0 \rangle$ is homeomorphic to the rationals. Let $\mathcal{A}$ be a mad family on ω whose existence is guaranteed by $(*_{\mathbb{Q}})$.

The final topologies $\tau = \tau_{\mathfrak{c}}$ and $\sigma = \sigma_{\mathfrak{c}}$ turning ω into Frechet-Urysohn-spaces with non-M-separable product will be constructed recursively over ordinals $\alpha \in \mathfrak{c}$ as increasing unions $\tau_{\mathfrak{c}} = \bigcup_{\alpha<\mathfrak{c}} \tau_\alpha$ and $\sigma_{\mathfrak{c}} = \bigcup_{\alpha<\mathfrak{c}} \sigma_\alpha$ of intermediate zero-dimensional topologies. Let $\langle E_n : n \in \omega \rangle$ be a sequence of mutually disjoint dense subsets of $\langle \omega \times \omega, \tau_0 \otimes \sigma_0 \rangle$ such that $E := \bigcup_{n\in\omega} E_n$ is (the graph of) a permutation of ω. It will be needed that the sets E_n remain dense in $\langle \omega \times \omega, \tau_\alpha \otimes \sigma_\alpha \rangle$ for all $\alpha \le \mathfrak{c}$. Let $\{\langle S_\alpha, x_\alpha \rangle : \alpha < \mathfrak{c}\}$ be an enumeration of $[\omega]^\omega \times \omega$ such that for each $\langle S, x \rangle \in [\omega]^\omega \times \omega$ there are cofinally many even as well as odd[1] ordinals α such that $\langle S, x \rangle = \langle S_\alpha, x_\alpha \rangle$. Fix also an enumeration $\{\langle F^\alpha_n : n \in \omega \rangle : \alpha < \mathfrak{c}\}$ of $\prod_{n\in\omega} [E_n]^{<\omega}$. For the convenience of those readers familiar with the proof of [Barman and Dow, 2011, Theorem 2.24], for a subset A of ω, denote $E[A]$ and $E^{-1}[A]$ (here E is seen as a map from ω to ω) by $E(A, 1)$ and $E(A, 0)$, respectively.

Suppose that for some $\alpha < \mathfrak{c}$ and all $\delta \in \alpha$ topologies $\tau_\delta, \sigma_\delta$ on ω and almost disjoint families $\mathcal{Y}_\delta, \mathcal{Z}_\delta \subset [\omega]^\omega$, such that the following conditions are satisfied:

(1) The weight of $\tau_\delta, \sigma_\delta$ is $< \mathfrak{c}$;

(2) (*a*) $\mathcal{Y}_\beta \subset \mathcal{Y}_\delta$ for all $\beta \le \delta$ and $\mathcal{Y}_\delta$ consists of sequences convergent in $\langle \omega, \tau_\delta \rangle$;

(*b*) $\mathcal{Z}_\beta \subset \mathcal{Z}_\delta$ for all $\beta \le \delta$ and $\mathcal{Z}_\delta$ consists of sequences convergent in $\langle \omega, \sigma_\delta \rangle$;

(3) (*a*) For every $\beta < \delta$, if β is even and x_β is a limit point of S_β in $\langle \omega, \tau_\delta \rangle$, then there exists $Y \in \mathcal{Y}_\delta$ converging to x_β in $\langle \omega, \tau_\delta \rangle$ with $|Y \cap S_\beta| = \omega$;

(*b*) For every $\beta < \delta$, if β is odd and x_β is a limit point of S_β in $\langle \omega, \sigma_\delta \rangle$, then there exists $Z \in \mathcal{Z}_\delta$ converging to x_β in $\langle \omega, \sigma_\delta \rangle$ and such that $|Z \cap S_\beta| = \omega$;

(4) (*a*) There are injective maps $\phi^\delta_0, \phi^\delta_1 : \mathcal{Y}_\delta \to \mathcal{A}$ such that $Y \subset \phi^\delta_0(Y)$ and $E(Y, 1) \subset \phi^\delta_1(Y)$ for all $Y \in \mathcal{Y}_\delta$;

(*b*) There are injective maps $\psi^\delta_1, \psi^\delta_0 : \mathcal{Z}_\delta \to \mathcal{A}$ such that $Z \subset \psi^\delta_1(Y)$ and $E(Z, 0) \subset \psi^\delta_0(Z)$ for all $Z \in \mathcal{Z}_\delta$;

(*c*) $\psi^\delta_1[\mathcal{Z}_\delta] \cap \phi^\delta_1[\mathcal{Y}_\delta] = \emptyset$ and $\psi^\delta_0[\mathcal{Z}_\delta] \cap \phi^\delta_0[\mathcal{Y}_\delta] = \emptyset$;

d) $\phi^\beta_i \subset \phi^\delta_i$ and $\psi^\beta_i \subset \psi^\delta_i$ for any $\beta < \delta$ and $i \in 0, 1$;

(5) (*a*) $E(\phi^\delta_0(Y), 1)$ and $\phi^\delta_1(Y)$ are closed and discrete in $\langle \omega, \sigma_\delta \rangle$ for all $Y \in \mathcal{Y}_\delta$;

(*b*) $E(\psi^\delta_1(Z), 0)$ and $\psi^\delta_0(Z)$ are closed and discrete in $\langle \omega, \tau_\delta \rangle$ for all $Z \in \mathcal{Z}_\delta$;

(6) For every $\beta < \delta$ there are $U_\beta \in \tau_\delta$ and $V_\beta \in \sigma_\delta$ such that[2]

$$(U_\beta \times V_\beta) \cap \bigcup_{n\in\omega} F^\beta_n = \emptyset.$$

(7) E_n is dense in $\langle \omega \times \omega, \tau_\delta \otimes \sigma_\delta \rangle$ for all $n \in \omega$.

[1] An ordinal is *even* (resp. *odd*) if it can be written in the form $\omega \cdot \alpha + i$ for some even (resp. odd) $i \in \omega$.

[2] In particular, $U_\beta, V_\beta, I_\beta$ depend only on β and not on δ.

were already contructed. By (7) the space $(\omega\times\omega,\tau_\delta\otimes\sigma_\delta)$ is crowded, since no space with isolated points can contain disjoint dense subsets. Several cases are now to be considered, depending on α:

I. α is limit.
It is easily checked that topologies τ_α and σ_α generated by $\bigcup_{\delta<\alpha}\tau_\delta$ and $\bigcup_{\delta<\alpha}\sigma_\delta$ as a base, respectively, along with $\mathcal{Y}_\alpha=\bigcup_{\delta<\alpha}\mathcal{Y}_\delta$, $\mathcal{Z}_\alpha=\bigcup_{\delta<\alpha}\mathcal{Z}_\delta$, $\phi^\alpha_i=\bigcup_{\delta<\alpha}\phi^\delta_i$ and $\psi^\alpha_i=\bigcup_{\delta<\alpha}\psi^\delta_i$ satisfy (1)-(7) for $\delta=\alpha$ and $i\in\{0,1\}$.

II. $\alpha=\delta+1$, δ is even, x_δ is a limit point of S_δ in $\langle\omega,\tau_\delta\rangle$, and $|Y\cap S_\delta|<\omega$ for all $Y\in\mathcal{Y}_\delta$ converging to x_δ.
Since the weight of $\langle\omega,\tau_\delta\rangle$ is $<\mathfrak{c}=\mathfrak{p}$, there exists a sequence $T\in[S_\delta]^\omega$ convergent to x_δ. Let $A^\delta_0\in\mathcal{A}$ be such that $|A^\delta_0\cap T|=\omega$.

Analogously, since $|\psi^\delta_0(Z)\cap T|<\omega$ for all $Z\in\mathcal{Z}_\delta$ (because $\psi^\delta_0(Z)$ is closed discrete in $\langle\omega,\tau_\delta\rangle$), then $\psi^\delta_0(Z)\neq A^\delta_0$ for all $Z\in\mathcal{Z}_\delta$. For the same reason $|E(\psi^\delta_1(Z),0)\cap T|<\omega$ for all $Z\in\mathcal{Z}_\delta$. Since E is a permutation of ω, $\mathcal{A}_{1\downarrow 0}:=\{E(A,0):A\in\mathcal{A}\}$ is a mad family, and hence there exists $A^\delta_1\in\mathcal{A}$ such that $|E(A^\delta_1,0)\cap A^\delta_0\cap T|=\omega$. It follows from the above that $E(A^\delta_1,0)\neq E(\psi^\delta_1(Z),0)$ for all $Z\in\mathcal{Z}_\delta$, and hence also $A^\delta_1\neq\psi^\delta_1(Z)$.

Finally, set $Y_\delta=T\cap A^\delta_0\cap E(A^\delta_1,0)$, $\mathcal{Y}_\alpha:=\mathcal{Y}_\delta\cup\{Y_\delta\}$, $\mathcal{Z}_\alpha=\mathcal{Z}_\delta$, $\phi^\alpha_0(Y_\delta)=A^\delta_0$, $\phi^\alpha_1(Y_\delta)=A^\delta_1$, $\phi^\alpha_0\restriction\mathcal{Y}_\delta=\phi^\delta_0$, and $\phi^\alpha_1\restriction\mathcal{Y}_\delta=\phi^\delta_1$. It follows from the construction that item (4) is satisfied. To proceed further we need the following direct consequence of [Barman and Dow, 2011, Lemma 2.23].

Claim 3. Suppose that X is a countable crowded space of weight less than $\mathfrak{p}$, $\mathcal{T}\subset\mathcal{P}(X)$ is a family of almost disjoint converging sequences in X, $|\mathcal{T}|<\mathfrak{p}$, and $\mathcal{R}\subset\mathcal{P}(X)$ is a countable family of subsets of X such that each $R\in\mathcal{R}$ has dense complement and is almost disjoint from each member of $\mathcal{T}$, with all but possibly one elements of $\mathcal{R}$ being discrete. Then there is an expansion of the topology of X to a larger crowded one, obtained by adding countably many sets along with their complements, in which each $R\in\mathcal{R}$ is a closed nowhere dense set, and each member of $\mathcal{T}$ is again a converging sequence.

For every $\langle x,y\rangle\in\omega^2$ and $n\in\omega$ find an injective sequence

$$\big\langle\langle s^{\langle x,y\rangle,n}_i,t^{\langle x,y\rangle,n}_i\rangle:i\in\omega\big\rangle\in\big(E_n\cap\big((\omega\setminus A^\delta_0)\times(\omega\setminus A^\delta_1)\big)\big)^\omega$$

convergent to $\langle x,y\rangle$. This is possible because all elements of $\mathcal{A}$ are either closed discrete or convergent sequences with respect to $\tau_0=\sigma_0$, and hence they remain nowhere dense in any stronger topology without isolated points, e.g., τ_δ and σ_δ. It follows that $T^{x,y}:=\{t^{\langle x,y\rangle,n}_i:i\in\omega\}$ is almost disjoint from each $Z\in\mathcal{Z}_\alpha$. Indeed, $E(T^{\langle x,y\rangle,n},0)=\{s^{\langle x,y\rangle,n}_i:i\in\omega\}$ is a convergent sequence in $\langle\omega,\tau_\delta\rangle$, whereas $E(Z,0)$ is closed discrete in this space for all $Z\in\mathcal{Z}_\delta$. Also, $E(T^{\langle x,y\rangle,n},0)\cap A^\delta_0=\emptyset$ by the choice of $T^{\langle x,y\rangle,n}$, and hence $T^{\langle x,y\rangle,n}\cap E(A^\delta_0,1)=\emptyset$ for all x,y as above. In particular, since $y\in\omega$ is arbitrary, it shows that $\omega\setminus E(A^\delta_0,1)$ is dense[3] in $\langle\omega,\sigma_\delta\rangle$. Also, $T^{\langle x,y\rangle,n}$ is almost disjoint from A^δ_1 for all x,y by construction. Applying Lemma 3 to $X=\langle\omega,\sigma_\delta\rangle$, $\mathcal{T}=\mathcal{Z}_\delta\cup\{T^{\langle x,y\rangle,n}:x,y,n\in\omega\}$, $\mathcal{R}=\{E(A^\delta_0,1),A^\delta_1\}$, a 0-dimensional topology $\sigma^-_\alpha\supset\sigma_\delta$ on ω in which elements of $\mathcal{T}$ are converging is obtained, and those

[3]This is of course easy and follows directly from the fact that A^δ_0 is nowhere dense in $\langle\omega,\tau_\delta\rangle$.

of $\mathcal{R}$ are closed discrete. By the choice of $\mathcal{T}$, E_n is dense in $\langle \omega \times \omega, \tau_\delta \otimes \sigma_\alpha^- \rangle$ for all n, because there is a sequence in E_n converging to each $\langle x, y \rangle$. Thus condition (5) will be satisfied because the topology σ_α is going to be crowded and stronger than σ_α^-. For convenience set $\tau_\alpha^- = \tau_\delta$.

Next, consider the following construction of U_δ, V_δ satisfying (6). This condition requires that if $x \in U_\delta$ and $Y \in \mathcal{Y}_\delta$ converges to x, then $Y \subset^* U_\delta$.

Claim 4. *There are families $\{C_n : n \in \omega\}$ and $\{D_n : n \in \omega\}$ such that*

(8) *$Y \subset^* C_n$ for any $Y \in \mathcal{Y}_\alpha$ converging to n in $\langle \omega, \tau_\alpha^- \rangle$;*

(9) *$Z \subset^* D_n$ for any $Z \in \mathcal{Z}_\alpha$ converging to n in $\langle \omega, \sigma_\alpha^- \rangle$;*

(10) *$C_n \cap C_m = \emptyset$ and $D_n \cap D_m = \emptyset$ for all $n \neq m$;*

(11) *$E(C_n, 1) \cap D_m = \emptyset$ (or equivalently, $E(D_m, 0) \cap C_n = \emptyset$) for any $n, m \in \omega$.*

Proof. For every $n \in \omega$ denote by $\mathcal{Y}_{\alpha,n}$ the family of all $Y \in \mathcal{Y}_\alpha$ converging to n, and fix a family $\{O_k^n : k \in \omega \setminus \{n\}\} \subset \tau_0$ such that O_k^n is a clopen neighbourhood of k not containing n. Then $|Y \cap O_k^n| < \omega$ for all $Y \in \mathcal{Y}_{\alpha,n}$ and $k \neq n$, and hence Lemma 3.2.2 implies that there exists $C_n^0 \subset \omega$ such that $Y \subset^* C_n^0$ and $|C_n^0 \cap O_k^n| < \omega$ for all Y, k as above. Notice that $|C_n^0 \cap Y| < \omega$ for all $Y \in \mathcal{Y}_\alpha \setminus \mathcal{Y}_{\alpha,n}$ because $Y \subset^* O_k^n$ for k being the limit point of Y. Thus letting $C_0^1 = C_0^0$ and $C_n^1 = C_n^0 \setminus \bigcup_{n'<n} C_{n'}^0$, we get that $\{C_n^1 : n \in \omega\}$ is a disjoint family such that $Y \subset^* C_n^1$ for all $Y \in \mathcal{Y}_{\alpha,n}$. Similarly it is possible to get a disjoint family $\{D_n^1 : n \in \omega\}$ such that $Z \subset^* D_n^1$ for all $Z \in \mathcal{Z}_{\alpha,n}$, where $\mathcal{Z}_{\alpha,n} = \{Z \in \mathcal{Z}_\alpha : Z$ converges to $n\}$. Finally, by item (4) and the property of $\mathcal{A}$ stated in $(*_\mathbb{Q})$, it is possible to find $C, D \subset \omega$ such that $\phi_0^\alpha(Y) \subset^* C$ and $\phi_1^\alpha(Y) \subset^* D$ for all $Y \in \mathcal{Y}_\alpha$, and $|\psi_1^\alpha(Z) \cap D| < \omega$ and $|\psi_0^\alpha(Z) \cap C| < \omega$ for all $Z \in \mathcal{Z}_\alpha$.

Set $C_n = C_n^1 \cap C \cap E(D, 0)$ and $D_n = D_n^1 \cap (\omega \setminus D) \cap (\omega \setminus E(C, 1))$. The families $\{C_n : n \in \omega\}$ and $\{D_n : n \in \omega\}$ are as required. Indeed, by (8), for all $Y \in \mathcal{Y}_{\alpha,n}$ it is true that $Y \subset^* C_n^1$, $Y \subset \phi_0^\delta(Y) \subset^* C$, and $E(Y, 1) \subset \phi_1^\alpha(Y) \subset^* D$, the latter implying $Y \subset^* E(D, 0)$, and hence $Y \subset^* C_n$. (9) is analogous: for all $Z \in \mathcal{Z}_{\alpha,n}$, $Z \subset^* D_n^1$, $Z \subset \psi_1^\delta(Z) \subset^* \omega \setminus D$, and $E(Z, 0) \subset \psi_0^\alpha(Z) \subset^* \omega \setminus C$, the latter implying $Z \subset^* \omega \setminus E(C, 1)$, and hence $Z \subset^* D_n$. Condition (10) follows from $C_n \subset C_n^1$ and $D_n \subset D_n^1$. And finally,

$$E(C_n, 1) \cap D_m \subset E\big(E(D, 0), 1\big) \cap (\omega \setminus D) = D \cap (\omega \setminus D) = \emptyset$$

for all $n, m \in \omega$, which yields (11). □

For every $\langle x, y \rangle \in \omega^2$ and $n \in \omega$ fix an injective sequence

$$\big\langle \langle s_k^{\langle x,y \rangle, n}, t_k^{\langle x,y \rangle, n} \rangle : k \in \omega \big\rangle \in (E_n \setminus F_n^\delta)^\omega$$

converging to $\langle x, y \rangle$ in $\langle \omega \times \omega, \tau_\alpha^- \otimes \sigma_\alpha^- \rangle$. Assume in addition that these sequences are mutually disjoint, i.e., $s_k^{\langle x,y \rangle, n} = s_{k'}^{\langle x',y' \rangle, n'}$ (resp. $t_k^{\langle x,y \rangle, n} = t_{k'}^{\langle x',y' \rangle, n'}$) iff $x = x'$, $y = y'$, $n = n'$, and $k = k'$. Set $S^{\langle x,y \rangle, n} = \{s_k^{\langle x,y \rangle, n} : k \in \omega\}$ and $T^{\langle x,y \rangle, n} = \{t_k^{\langle x,y \rangle, n} : k \in \omega\}$ for all $\langle x, y \rangle$ and n as above.

Replacing the D_m's and C_m's with smaller sets, if necessary, it is additionally possible to assume that

$$|S^{\langle x,y \rangle, n} \cap C_m| < \omega \text{ and } |T^{\langle x,y \rangle, n} \cap D_m| < \omega$$

for all $\langle x,y\rangle$, n, and m. Indeed, since $E(Y,1)$ is closed and discrete in $\langle \omega, \sigma_\alpha^-\rangle$ for every $Y \in \mathcal{Y}_\alpha$ and $E(S^{\langle x,y\rangle,n},1) = T^{\langle x,y\rangle,n}$ is a convergent sequence in this topology, then $|S^{\langle x,y\rangle,n} \cap Y| < \omega$ for all $\langle x,y\rangle, n$ as above. Since there are no $(\omega, < \mathfrak{b})$ gaps, there exists $S \subset \omega$ such that $S^{\langle x,y\rangle,n} \subset^* S$ and $|Y \cap S| < \omega$ for all $\langle x,y\rangle, n$. Thus replacing C_m with $C_m \setminus S$ for all $m \in \omega$, if necessary, it is possible to assume that $|S^{\langle x,y\rangle,n} \cap C_m| < \omega$ for all $\langle x,y\rangle, n$, in addition to all properties of C_m stated above. Analogously with $|T^{\langle x,y\rangle,n} \cap D_m| < \omega$ for all $\langle x,y\rangle, n$.

Finally, consider the following construction of U_δ and V_δ. This will be done recursively over $k \in \omega$, namely they will be constructed as increasing unions $\bigcup_{k\in\omega} U_k^{\delta,1}$ and $\bigcup_{k\in\omega} V_k^{\delta,1}$, respectively. Moreover adding their complements to the corresponding topologies can also be done without any harm by constructing these as increasing unions $\bigcup_{k\in\omega} U_k^{\delta,0}$ and $\bigcup_{k\in\omega} V_k^{\delta,0}$, respectively. These objects are constructed along with

- non-decreasing sequences $\langle H_k^j : k \in \omega\rangle$ and $\langle G_k^j : k \in \omega\rangle$ of finite subsets of ω, where $j \in 2$; and
- for each $x \in \bigcup_{k\in\omega} H_k^1 \cup \bigcup_{k\in\omega} H_k^0$, $y \in \bigcup_{k\in\omega} G_k^1 \cup \bigcup_{k\in\omega} G_k^0$ and $n \in \omega$ cofinal subsets $\tilde{C}_x$, $\tilde{D}_y$, $\tilde{S}^{\langle x,y\rangle,n}$ and $\tilde{T}^{\langle x,y\rangle,n}$ of C_x, D_y, $S^{\langle x,y\rangle,n}$ and $T^{\langle x,y\rangle,n}$, respectively,

such that $H_0^1 = \{x_0^1\}$ and $G_0^1 = \{y_0^1\}$ for some $\langle x_0^1, y_0^1\rangle \notin \bigcup_{n\in\omega} F_n^\delta$, $H_0^0 = G_0^0 = \emptyset$, and the following conditions are satisfied for all $k \in \omega$ and $i \in 2$:

(i) $H_k^1 \cap H_k^0 = G_k^1 \cap G_k^0 = \emptyset$;

(ii) $U_k^{\delta,i} = \big[H_k^i \cup \bigcup\{\tilde{C}_x : x \in H_k^i\} \cup$
$\cup \bigcup\{\tilde{S}^{\langle x,y\rangle,n} : n \le k, \langle x,y\rangle \in H_k^i \times (G_k^1 \cup G_k^0)\}\big] \cup \Delta_k^i$,
$V_k^{\delta,i} = \big[G_k^i \cup \bigcup\{\tilde{D}_y : y \in G_k^i\} \cup$
$\cup \bigcup\{\tilde{T}^{\langle x,y\rangle,n} : n \le k, \langle x,y\rangle \in (H_k^1 \cup H_k^0) \times G_k^i\}\big] \cup \Sigma_k^i$;

(iii) $\Delta_k^1 = \emptyset$ for all k and $\Delta_k^0 = \{k\}$ if $k \notin U_k^{\delta,1}$, otherwise $\Delta_k^0 = \emptyset$; $\Sigma_k^1 = \emptyset$ for all k and $\Sigma_k^0 = \{k\}$ if $k \notin V_k^{\delta,1}$, otherwise $\Sigma_k^0 = \emptyset$;

(iv) $U_k^{\delta,1} \cap U_k^{\delta,0} = V_k^{\delta,1} \cap V_k^{\delta,0} = \emptyset$;

(v) If $\langle x,y\rangle \in (H_k^1 \cup H_k^0) \times (G_k^1 \cup G_k^0)$, then $\tilde{T}^{\langle x,y\rangle,n} = E(\tilde{S}^{\langle x,y\rangle,n},1)$ for all $n \le k$;

(vi) $(U_k^{\delta,1} \times V_k^{\delta,1}) \cap \bigcup_{n\in\omega} F_n^\delta = \emptyset$;

(vii) $H_{k+1}^i = H_k^i \cup \{\min(U_k^{\delta,i} \setminus H_k^i)\}$ provided that $U_k^{\delta,i} \ne \emptyset$, and $H_{k+1}^i = H_k^i = \emptyset$ otherwise; $G_{k+1}^i = G_k^i \cup \{\min(V_k^{\delta,i} \setminus G_k^i)\}$ provided that $V_k^{\delta,i} \ne \emptyset$, and $G_{k+1}^i = G_k^i = \emptyset$ otherwise.

Notice that, by (ii) and the choices of H_0^1 and G_0^1, the equalities $U_k^{\delta,i} = \emptyset$ and $V_k^{\delta,i} = \emptyset$ are possible only for $i = 0$. To start off the inductive construction, set $\tilde{C}_{x_0^1} = C_{x_0^1} \setminus E(\{y_0^1\},0)$, $\tilde{D}_{y_0^1} = D_{y_0^1} \setminus E(\{x_0^1\},1)$,

$$\hat{T}^{\langle x_0^1,y_0^1\rangle,0} = T^{\langle x_0^1,y_0^1\rangle,0} \setminus \big(E(C_{x_0^1},1) \cup E(\{x_0^1\},1)\big),$$

$$\hat{S}^{\langle x_0^1,y_0^1\rangle,0} = S^{\langle x_0^1,y_0^1\rangle,0} \setminus \big(E(D_{y_0^1},0) \cup E(\{y_0^1\},0)\big)$$

and then

$$\tilde{T}^{\langle x_0^1,y_0^1\rangle,0} = \hat{T}^{\langle x_0^1,y_0^1\rangle,0} \cap E(\hat{S}^{\langle x_0^1,y_0^1\rangle,0},1)$$

and

$$\tilde{S}^{\langle x_0^1,y_0^1\rangle,0} = \hat{S}^{\langle x_0^1,y_0^1\rangle,0} \cap E(\hat{T}^{\langle x_0^1,y_0^1\rangle,0},0).$$

The equation

$$E(\tilde{S}^{\langle x_0^1,y_0^1\rangle,0},1) = E\big(\hat{S}^{\langle x_0^1,y_0^1\rangle,0} \cap E(\hat{T}^{\langle x_0^1,y_0^1\rangle,0},0),1\big) = \\ E(\hat{S}^{\langle x_0^1,y_0^1\rangle,0},1) \cap E\big(E(\hat{T}^{\langle x_0^1,y_0^1\rangle,0},0),1\big) = \\ = E(\hat{S}^{\langle x_0^1,y_0^1\rangle,0},1) \cap \hat{T}^{\langle x_0^1,y_0^1\rangle,0} = \tilde{T}^{\langle x_0^1,y_0^1\rangle,0}$$

proves (v) for $k=0$. To check (vi) note that

$$U_0^{0,1} \times V_0^{0,1} = \big(\{x_0^1\} \cup \tilde{C}_{x_0^1} \cup \tilde{S}^{\langle x_0^1,y_0^1\rangle,0}\big) \times \big(\{y_0^1\} \cup \tilde{D}_{y_0^1} \cup \tilde{T}^{\langle x_0^1,y_0^1\rangle,0}\big) = \\ \{\langle x_0^1,y_0^1\rangle\} \cup (\{x_0^1\} \times \tilde{D}_{y_0^1}) \cup (\{x_0^1\} \times \tilde{T}^{\langle x_0^1,y_0^1\rangle,0}) \cup \\ \cup(\tilde{C}_{x_0^1} \times \{y_0^1\}) \cup (\tilde{C}_{x_0^1} \times \tilde{D}_{y_0^1}) \cup (\tilde{C}_{x_0^1} \times \tilde{T}^{\langle x_0^1,y_0^1\rangle,0}) \cup \\ \cup(\tilde{S}^{\langle x_0^1,y_0^1\rangle,0} \times \{y_0^1\}) \cup (\tilde{S}^{\langle x_0^1,y_0^1\rangle,0} \times \tilde{D}_{y_0^1}) \cup (\tilde{S}^{\langle x_0^1,y_0^1\rangle,0} \times \tilde{T}^{\langle x_0^1,y_0^1\rangle,0}),$$

i.e., $U_0^{0,1} \times V_0^{0,1}$ is a union of 9 sets, of which 7 ones "in the middle" have empty intersection with $\bigcup_{n\in\omega} E_n$ because $A \times B \cap \bigcup_{n\in\omega} E_n = \emptyset$ provided that $A \cap E(B,0) = \emptyset$, which is the case for these 7 products by the choice or definition of the corresponding sets. Also, $\langle x_0^1, y_0^1\rangle \notin \bigcup_{n\in\omega} F_n^\delta$ by the choice, whereas

$$\tilde{S}^{\langle x_0^1,y_0^1\rangle,0} \times \tilde{T}^{\langle x_0^1,y_0^1\rangle,0} \cap \bigcup_{n\in\omega} E_n = \{\langle s_k^{\langle x_0^1,y_0^1\rangle,0}, t_k^{\langle x_0^1,y_0^1\rangle,0}\rangle : k\in\omega\} \subset E_0 \setminus F_0^\delta,$$

and hence $\tilde{S}^{\langle x_0^1,y_0^1\rangle,0} \times \tilde{T}^{\langle x_0^1,y_0^1\rangle,0}$ also has empty intersection with $\bigcup_{n\in\omega} F_n^\delta$. This completes the case $k=0$ since all other items are easily checked.

Suppose now that items $(i)-(vii)$ hold for k and note that (vii) gives the unique way to define H_{k+1}^i and G_{k+1}^i for $i\in\{0,1\}$, and the points (iv) (vii) for k yields (i) for $k+1$. Given $i\in 2$ and $x\in H_{k+1}^i \setminus H_k^i$ and $y \in G_{k+1}^i \setminus G_k^i$, set

$$\tilde{C}_x = C_x \setminus \Big(E(G_{k+1}^1,0) \cup \bigcup\{E(T^{\langle x',y'\rangle,n'},0) : \\ \langle x',y'\rangle \in (H_{k+1}^1 \cup H_{k+1}^0) \times (G_{k+1}^1 \cup G_{k+1}^0), n' \le k\} \cup H_{k+1}^0\Big) = \\ = C_x \setminus \Big(E(G_{k+1}^1,0) \cup \bigcup\{S^{\langle x',y'\rangle,n'} : \\ \langle x',y'\rangle \in (H_{k+1}^1 \cup H_{k+1}^0) \times (G_{k+1}^1 \cup G_{k+1}^0), n' \le k\} \cup H_{k+1}^0\Big)$$

and

$$\tilde{D}_y = D_y \setminus \Big(E(H_{k+1}^1,1) \cup \bigcup\{E(S^{\langle x',y'\rangle,n'},1) : \\ \langle x',y'\rangle \in (H_{k+1}^1 \cup H_{k+1}^0) \times (G_{k+1}^1 \cup G_{k+1}^0), n' \le k\} \cup G_{k+1}^0\Big) = \\ = D_y \setminus \Big(E(H_{k+1}^1,1) \cup \bigcup\{T^{\langle x',y'\rangle,n'} : \\ \langle x',y'\rangle \in (H_{k+1}^1 \cup H_{k+1}^0) \times (G_{k+1}^1 \cup G_{k+1}^0), n' \le k\} \cup G_{k+1}^0\Big).$$

For every

$$\langle\langle x',y'\rangle,n'\rangle \in \big((H^1_{k+1}\cup H^0_{k+1})\times(G^1_{k+1}\cup G^0_{k+1})\times(k+2)\big)\setminus \\ \setminus\big((H^1_k\cup H^0_k)\times(G^1_k\cup G^0_k)\times(k+1)\big)$$

set

$$\hat{T}^{\langle x',y'\rangle,n'} = T^{\langle x',y'\rangle,n'}\setminus\big[E\big(\bigcup\{C_x : x\in H^1_{k+1}\}\cup H^1_{k+1},1\big)\cup G^0_{k+1}\cup \\ \cup\bigcup\{D_y : y\in G^1_{k+1}\cup G^0_{k+1}\}\big],$$
$$\hat{S}^{\langle x',y'\rangle,n'} = S^{\langle x',y'\rangle,n'}\setminus\big[E\big(\bigcup\{D_y : y\in G^1_{k+1}\}\cup G^1_{k+1},0\big)\cup H^0_{k+1}\cup \\ \cup\bigcup\{C_x : x\in H^1_{k+1}\cup H^0_{k+1}\}\big],$$

and finally

$$\tilde{T}^{\langle x',y'\rangle,n'} := \hat{T}^{\langle x',y'\rangle,n'}\cap E(\hat{S}^{\langle x',y'\rangle,n'},1),$$
$$\tilde{S}^{\langle x',y'\rangle,n'} := \hat{S}^{\langle x',y'\rangle,n'}\cap E(\hat{T}^{\langle x',y'\rangle,n'},0).$$

The equality

$$E(\tilde{S}^{\langle x',y'\rangle,n'},1) = E\big(\hat{S}^{\langle x',y'\rangle,n'}\cap E(\hat{T}^{\langle x',y'\rangle,n'},0),1\big) = \\ E(\hat{S}^{\langle x',y'\rangle,n'},1)\cap E\big(E(\hat{T}^{\langle x',y'\rangle,n'},0),1\big) = \\ = E(\hat{S}^{\langle x',y'\rangle,n'},1)\cap\hat{T}^{\langle x',y'\rangle,n'} = \tilde{T}^{\langle x',y'\rangle,n'}$$

for all $\langle\langle x',y'\rangle,n'\rangle$ as above yields (v) for $k+1$. Let $U^{\delta,i}_{k+1}$ and $V^{\delta,i}_{k+1}$ be defined according to (ii) and (iii) for $k+1$. To check (vi) for $(k+1)$, consider the product

$$\begin{array}{ll}
 & U^{\delta,1}_{k+1}\times V^{\delta,1}_{k+1} = \\
(\alpha) & = \big(H^1_{k+1}\times G^1_{k+1}\big)\cup\big(H^1_{k+1}\times\bigcup\{\tilde{D}_y : y\in G^1_{k+1}\}\big)\cup \\
(\beta) & \cup\big(H^1_{k+1}\times\bigcup\{\tilde{T}^{\langle x,y\rangle,n} : n\le k+1, \langle x,y\rangle\in \\
 & \in (H^1_{k+1}\cup H^0_{k+1})\times G^1_{k+1}\}\big)\cup \\
(\gamma) & \cup\big(\bigcup\{\tilde{C}_x : x\in H^1_{k+1}\}\times G^1_{k+1}\big)\cup \\
(\zeta) & \cup\big(\bigcup\{\tilde{C}_x : x\in H^1_{k+1}\}\times\bigcup\{\tilde{D}_y : y\in G^1_{k+1}\}\big)\cup \\
(\varepsilon) & \cup\big(\bigcup\{\tilde{C}_x : x\in H^1_{k+1}\}\times \\
 & \times\bigcup\{\tilde{T}^{\langle x,y\rangle,n} : n\le k+1, \langle x,y\rangle\in(H^1_{k+1}\cup H^0_{k+1})\times G^1_{k+1}\}\big)\cup \\
(\eta) & \cup\big(\bigcup\{\tilde{S}^{\langle x,y\rangle,n} : n\le k+1, \langle x,y\rangle\in \\
 & \in H^1_{k+1}\times(G^1_{k+1}\cup G^0_{k+1})\}\times G^1_{k+1}\big)\cup \\
(\theta) & \cup\big(\bigcup\{\tilde{S}^{\langle x,y\rangle,n} : n\le k+1, \langle x,y\rangle\in H^1_{k+1}\times(G^1_{k+1}\cup G^0_{k+1})\}\times \\
 & \times\bigcup\{\tilde{D}_y : y\in G^1_{k+1}\}\big)\cup \\
(\lambda) & \cup\big(\bigcup\{\tilde{S}^{\langle x,y\rangle,n} : n\le k+1, \langle x,y\rangle\in H^1_{k+1}\times(G^1_{k+1}\cup G^0_{k+1})\}\times \\
 & \times\bigcup\{\tilde{T}^{\langle x,y\rangle,n} : n\le k+1, \langle x,y\rangle\in(H^1_{k+1}\cup H^0_{k+1})\times G^1_{k+1}\}\big).
\end{array}$$

To show that this product is disjoint from $F=\bigcup_{n\in\omega}F^\delta_n$ it is needed to analyze sets appearing in the above long formula one by one. $H^1_{k+1}\times G^1_{k+1}$ is disjoint from F by the inductive assumption, namely (vi) for k, since $H^1_{k+1}\subset U^{\delta,1}_k$ and $G^1_{k+1}\subset V^{\delta,1}_k$. The second product in (α) is disjoint from F because it was explicitly made $E(H^1_{k+1},1)$ disjoint from $\tilde{D}_y$ if $y\in G^1_{k+1}\setminus G^1_k$, and hence $H^1_{k+1}\times\tilde{D}_y$ is disjoint even from E in this case. And if $y\in G^1_k$ then it is possible to use

(vi) for k since $H^1_{k+1} \subset U^{\delta,1}_k$ and $\tilde{D}_y \subset V^{\delta,1}_k$. Analogously it is possible to show that also the product in item (γ) is disjoint from F.

The product in (β) is disjoint from F because $E(H^1_{k+1}, 1)$ was explicitly made disjoint from $\tilde{T}^{\langle x,y\rangle,n}$ if $\langle\langle x, y\rangle, n\rangle$ lies in the difference $N :=$

$$\big(((H^1_{k+1} \cup H^0_{k+1}) \times G^1_{k+1}) \times (k+2)\big) \setminus \big(((H^1_k \cup H^0_k) \times G^1_k) \times (k+1)\big),$$

and if

$$\langle\langle x, y\rangle, n\rangle \in ((H^1_k \cup H^0_k) \times G^1_k) \times (k+1)$$

then it is possible to use (vi) for k since $H^1_{k+1} \subset U^{\delta,1}_k$ and $\tilde{T}^{\langle x,y\rangle,n} \subset V^{\delta,1}_k$. Analogously it is possible to show that also the product in item (η) is disjoint from F.

The product in (ζ) is disjoint even from E since $E(C_x, 1) \cap D_y = \emptyset$ for any $x, y \in \omega$ by Claim 4(11).

The product in item (ε) is disjoint from F because $E(\tilde{C}_x, 1)$ were explicitly made disjoint from $\tilde{T}^{\langle x',y'\rangle,n}$ if $x \in H^1_{k+1} \setminus H^1_k$ or $\langle\langle x', y'\rangle, n\rangle \in N$, and if both $x \in H^1_k$ and

$$\langle\langle x', y'\rangle, n\rangle \in (H^1_k \cup H^0_k) \times (G^1_k \cup G^0_k) \times (k+1)$$

then it is possible to use (vi) for k since $\tilde{C}_x \subset U^{\delta,1}_k$ and $\tilde{T}^{\langle x',y'\rangle,n} \subset V^{\delta,1}_k$. Analogously it is possible to show that also the product in item (θ) is disjoint from F.

Finally, consider the product in (λ). Given $\langle\langle x, y\rangle, n\rangle$ and $\langle\langle x', y'\rangle, n'\rangle$ in

$$((H^1_{k+1} \cup H^0_{k+1}) \times G^1_{k+1}) \times (k+2),$$

two cases are possible. If $\langle\langle x, y\rangle, n\rangle = \langle\langle x', y'\rangle, n'\rangle$, then $\tilde{T}^{\langle x,y\rangle,n} = E(\tilde{S}^{\langle x,y\rangle,n}, 1)$, and hence

$$(\tilde{S}^{\langle x,y\rangle,n} \times \tilde{T}^{\langle x,y\rangle,n}) \bigcap E \subset \{\langle s^{\langle x,y\rangle,n}_k, t^{\langle x,y\rangle,n}_k\rangle : k \in \omega\} \subset E_n \setminus F^\delta_n,$$

i.e., $\tilde{S}^{\langle x,y\rangle,n} \times \tilde{T}^{\langle x,y\rangle,n}$ is disjoint from F. If $\langle\langle x, y\rangle, n\rangle \neq \langle\langle x', y'\rangle, n'\rangle$, then

$$E(\tilde{S}^{\langle x,y\rangle,n}, 1) = \tilde{T}^{\langle x,y\rangle,n} \subset T^{\langle x,y\rangle,n},$$

and hence

$$E(\tilde{S}^{\langle x,y\rangle,n}, 1) \cap \tilde{T}^{\langle x',y'\rangle,n'} \subset T^{\langle x,y\rangle,n} \cap T^{\langle x',y'\rangle,n'} = \emptyset,$$

which implies that $(\tilde{S}^{\langle x,y\rangle,n} \cap \tilde{T}^{\langle x',y'\rangle,n'}) \cap E = \emptyset$. This completes the proof of (vi) for $(k+1)$.

Finally, it remains to check (iv) for $k+1$. Write $U^{\delta,1}_{k+1} \cap U^{\delta,0}_{k+1}$ as follows:

$$\begin{array}{ll}
 & U^{\delta,1}_{k+1} \cap U^{\delta,0}_{k+1} = \\
(\alpha_\cap) & = \big(H^1_{k+1} \cap H^0_{k+1}\big) \cup \big(H^1_{k+1} \cap \bigcup\{\tilde{C}_x : x \in H^0_{k+1}\}\big) \cup \\
(\beta_\cap) & \cup \big(H^1_{k+1} \cap \bigcup\{\tilde{S}^{\langle x,y\rangle,n} : n \le k+1, \langle x,y\rangle \in \\
 & \in H^0_{k+1} \times (G^1_{k+1} \cup G^0_{k+1})\}\big) \cup \\
(\gamma_\cap) & \cup \big(\bigcup\{\tilde{C}_x : x \in H^1_{k+1}\} \cap H^0_{k+1}\big) \cup \\
(\zeta_\cap) & \cup \big(\bigcup\{\tilde{C}_x : x \in H^1_{k+1}\} \cap \bigcup\{\tilde{C}_x : x \in H^0_{k+1}\}\big) \cup \\
(\varepsilon_\cap) & \cup \big(\bigcup\{\tilde{C}_x : x \in H^1_{k+1}\} \cap \\
 & \cap \bigcup\{\tilde{S}^{\langle x,y\rangle,n} : n \le k+1, \langle x,y\rangle \in H^0_{k+1} \times (G^1_{k+1} \cup G^0_{k+1})\}\big) \cup \\
(\eta_\cap) & \cup \big(\bigcup\{\tilde{S}^{\langle x,y\rangle,n} : n \le k+1, \langle x,y\rangle \in \\
 & \in H^1_{k+1} \times (G^1_{k+1} \cup G^0_{k+1})\} \cap H^0_{k+1}\big) \cup \\
(\theta_\cap) & \cup \big(\bigcup\{\tilde{S}^{\langle x,y\rangle,n} : n \le k+1, \langle x,y\rangle \in H^1_{k+1} \times (G^1_{k+1} \cup G^0_{k+1})\} \cap \\
 & \cap \bigcup\{\tilde{C}_x : x \in H^0_{k+1}\}\big) \cup \\
(\lambda_\cap) & \cup \big(\bigcup\{\tilde{S}^{\langle x,y\rangle,n} : n \le k+1, \langle x,y\rangle \in H^1_{k+1} \times (G^1_{k+1} \cup G^0_{k+1})\} \cap \\
 & \cap \bigcup\{\tilde{S}^{\langle x,y\rangle,n} : n \le k+1, \langle x,y\rangle \in H^0_{k+1} \times (G^1_{k+1} \cup G^0_{k+1})\}\big).
\end{array}$$

Similarly (but easier) to the proof of (vi) it is possible to check that all the intersections in the formula displayed above are empty. For instance, consider the intersection in item $(\theta_\cap)$. Fix any

$$\langle\langle x,y\rangle,n\rangle \in \big(H^1_{k+1} \times (G^1_{k+1} \cup G^0_{k+1})\big) \times (k+2)$$

and $x' \in H^0_{k+1}$. If $x' \notin H^0_k$, then, since $S^{\langle x,y\rangle,n}$ was explicitly subtracted from $C_{x'}$ in the definition of $\tilde{C}_{x'}$, so $\tilde{C}_{x'} \cap \tilde{S}^{\langle x,y\rangle,n} = \emptyset$. If $\langle\langle x,y\rangle,n\rangle \in N$ (see the proof of (vi) for the definition thereof), then, since $C_{x'}$ was explicitly subtracted from $S^{\langle x,y\rangle,n}$ in the definition of $\tilde{S}^{\langle x,y\rangle,n}$, so again $\tilde{C}_{x'} \cap \tilde{S}^{\langle x,y\rangle,n} = \emptyset$. Otherwise $\tilde{S}^{\langle x,y\rangle,n} \subset U^{\delta,1}_k$ and $\tilde{C}_{x'} \subset U^{\delta,0}_k$ and therefore it is possible to use (iv) for k in order to get $\tilde{C}_{x'} \cap \tilde{S}^{\langle x,y\rangle,n} = \emptyset$.

Thus the inductive construction of all the objects needed for defining U_δ and V_δ, so that conditions $(i)-(vii)$ are satisfied, is finished.

Set $U_\delta = \bigcup_{k\in\omega} U^{\delta,1}_k$, $V_\delta = \bigcup_{k\in\omega} V^{\delta,1}_k$, and note that $\omega \setminus U_\delta = \bigcup_{k\in\omega} U^{\delta,0}_k$ and $\omega \setminus V_\delta = \bigcup_{k\in\omega} V^{\delta,0}_k$ by (iii) and (iv). Let τ_α and σ_α be the topologies generated by $\tau^-_\alpha \cup \{U_\delta, \omega \setminus U_\delta\}$ and $\sigma^-_\alpha \cup \{V_\delta, \omega \setminus V_\delta\}$ as a base, respectively. This way 0-dimensional topologies are obtained and (vi) implies that (6) is satisfied. Moreover, (vii) gives that $U_\delta = \bigcup_{k\in\omega} H^1_k$, $\omega \setminus U_\delta = \bigcup_{k\in\omega} H^0_k$, $V_\delta = \bigcup_{k\in\omega} G^1_k$, and $\omega \setminus V_\delta = \bigcup_{k\in\omega} G^0_k$. Thus by (ii) for every $x \in U_\delta$ the set C_x is almost contained in U_δ, and hence also $Y \subset^* U_\delta$ for all $Y \in \mathcal{Y}_\alpha$ converging to x. The same holds for $\omega \setminus U_\delta$. It follows that all $Y \in \mathcal{Y}_\alpha$ remain convergent sequences in $\langle\omega, \tau_\alpha\rangle$. Analogously, all $Z \in \mathcal{Z}_\alpha$ remain convergent sequences in $\langle\omega, \sigma_\alpha\rangle$, and hence (3) holds for α. Finally, whenever $n \in \omega$, $\ell \in 2^2$, and $\langle x,y\rangle \in U^{(\ell(0))}_\delta \times V^{(\ell(1))}_\delta$, then[4], then $\{\langle s^{\langle x,y\rangle,n}_k, s^{\langle x,y\rangle,n}_k\rangle : k \in \omega\} \subset^* U^{(\ell(0))}_\delta \times V^{(\ell(1))}_\delta$. Indeed, by (vii) there exists $k \in \omega$ such that $x \in U^{\delta,\ell(0)}_k$ and $y \in V^{\delta,\ell(1)}_k$, and therefore (ii) implies

$$\tilde{S}^{\langle x,y\rangle,n} \subset^* U^{\delta,\ell(0)}_k \subset U^{\ell(0)}_\delta \quad \text{and} \quad \tilde{T}^{\langle x,y\rangle,n} \subset^* V^{\delta,\ell(1)}_k \subset V^{\ell(1)}_\delta.$$

[4] For $I \subset \omega$ set $I^{(1)} = I$ and $I^{(0)} = \omega \setminus I$.

As a result E_n remains dense in $\langle \omega \times \omega, \tau_\alpha \otimes \sigma_\alpha \rangle$ for all n, because there is an injective sequence in E_n converging to each $\langle x, y \rangle$. This also proves that there are no isolated points in $\langle \omega \times \omega, \tau_\alpha \otimes \sigma_\alpha \rangle$. This completes the verification of $(1)-(7)$ for $\delta = \alpha$.

III. $\alpha = \delta + 1$, δ is odd, x_δ is a limit point of S_δ in $\langle \omega, \sigma_\delta \rangle$, and $|Z \cap S_\delta| < \omega$ for all $Z \in \mathcal{Z}_\delta$ converging to x_δ.
In this case it is possible to repeat the argument from Case II, with the roles of $\mathcal{Z}_\delta$ and $\mathcal{Y}_\delta$ interchanged, so that again (1)-(7) are satisfied for $\delta = \alpha$.

IV. $\alpha = \delta + 1$, δ is even, and either x_δ is *not* a limit point of S_δ in $\langle \omega, \tau_\delta \rangle$, or it is and there exists $Y \in \mathcal{Y}_\delta$ converging to x_δ in $\langle \omega, \tau_\delta \rangle$ with $|Y \cap S_\delta| = \omega$. Then set $\mathcal{Y}_\alpha = \mathcal{Y}_\delta$, $\mathcal{Z}_\alpha = \mathcal{Z}_\delta$, $\tau_\alpha^- = \tau_\delta$, $\sigma_\alpha^- = \sigma_\delta$, and repeat the argument from Case II starting from Claim 4.

V. $\alpha = \delta + 1$, δ is odd, and either x_δ is *not* a limit point of S_δ in $\langle \omega, \sigma_\delta \rangle$, or it is and there exists $Z \in \mathcal{Z}_\delta$ converging to x_δ in $\langle \omega, \sigma_\delta \rangle$ with $|Z \cap S_\delta| = \omega$. Then set $\mathcal{Y}_\alpha = \mathcal{Y}_\delta$, $\mathcal{Z}_\alpha = \mathcal{Z}_\delta$, $\tau_\alpha^- = \tau_\delta$, $\sigma_\alpha^- = \sigma_\delta$, and repeat the argument from Case III starting from Claim 4.

All in all, this completes the construction of the objects mentioned in (1)-(7) for all $\delta < \mathfrak{c}$, so that these conditions are satisfied. Conditions (3) and (2) imply that $X = \langle \omega, \tau_\mathfrak{c} \rangle$ and $Y = \langle \omega, \sigma_\mathfrak{c} \rangle$ are FU, and $X \times Y$ is not M-separable by (6). Moreover, both spaces are zero-dimensional since they started from zero-dimensional topologies and they were always enlarged by adding new sets along with their complements. This completes the proof of Theorem 3.3.9. □

Osservation 3.3.10. *If X_0 and X_1 are the space constructed in the previous Theorem, then $X_0 \sqcup \omega$ and $X_1 \sqcup \omega$ are countable regular FU spaces and not crowded whose product is not M-separable; however it is easy to see that, if X_0 is M-separable and X_1 contains a dense set of isolated points, then $X_0 \times X_1$ is M-separable.*

Corollary 3.3.11. [*Bardyla, M., Zdomskyy, 2023*] *It is consistent with (MA+¬CH) that the product of two countable regular FU spaces is not M-separable.*

3.4 Products of H-separable spaces in the Laver model

Definition 3.4.1. [*Bardyla, M., Zdomskyy, 2023, Definition 5.1*]A topological space $\langle X, \tau \rangle$ is said to be *bounded box-separable* (briefly, *b.b.-separable*) if for every function R assigning to each countable family $\mathcal{U}$ of non-empty open subsets of X a sequence $R(\mathcal{U}) = \langle F_n : n \in \omega \rangle$ of finite non-empty subsets of X such that $\{n : F_n \subset U\}$ is infinite for every $U \in \mathcal{U}$, there exists $\mathbb{U} \subset [\tau \setminus \{\emptyset\}]^\omega$ of size $|\mathbb{U}| = \omega_1$ and a sequence $\langle l_i : i \in \omega \rangle \in \omega^\omega$ such that for all $U \in \tau \setminus \{\emptyset\}$ there exists $\mathcal{U} \in \mathbb{U}$ such that for all but finite $i \in \omega$ there is $n \in [l_i, l_{i+1})$ such that $R(\mathcal{U})(n) \subseteq U$.

The proof of the following statement is close to that of [Repovš and Zdomskyy, 2018, Lemma 2.2], the only difference being a more careful analysis of sets of the form $\{n : R(\mathcal{U})(n) \subset U\}$.

Proposition 3.4.2. [*Bardyla, M., Zdomskyy, 2023, Proposition 5.2*] *In the Laver model every countable H-separable space is b.b.-separable*

Proof. Consider $V[G_{\omega_2}]$, where G_{ω_2} is $\mathbb{P}_{\omega_2}$-generic and $\mathbb{P}_{\omega_2}$ is the iteration of length ω_2 with countable supports of the Laver forcing. Fix an H-separable space of the form $\langle \omega, \tau \rangle$ and a function R such as in the definition of b.b-separability. By a standard argument (see, e.g., the proof of [Blass and Shelah, 1987, Lemma 5.10]) there exists an ω_1-club $C \subseteq \omega_2$ such that for every $\alpha \in C$ the following conditions hold:

(i) $\tau \cap V[G_\alpha] \in V[G_\alpha]$ and for every sequence $\langle D_n : n \in \omega \rangle \in V[G_\alpha]$ of dense subsets of $\langle \omega, \tau \rangle$ there exists a sequence $\langle K_n : n \in \omega \rangle \in V[G_\alpha]$ such that $K_n \in [D_n]^{<\omega}$ and for every $U \in \tau \setminus \{\emptyset\}$ the intersection $U \cap K_n$ is non-empty for all but finitely many $n \in \omega$;

(ii) $R(\mathcal{U}) \in V[G_\alpha]$ for any $\mathcal{U} \in [\tau \setminus \{\emptyset\}]^\omega \cap V[G_\alpha]$; and

(iii) For every $A \in \mathcal{P}(\omega) \cap V[G_\alpha]$ the interior $Int(A)$ also belongs to $V[G_\alpha]$.

By [Laver, 1976, Lemma 11] there is no loss of generality in assuming that $0 \in C$. Set $\mathbb{U} := [\tau \setminus \{\emptyset\}]^\omega \cap V$. It suffices to prove the following auxiliary

Claim 5. *For every $A \in \tau \setminus \{\emptyset\}$ there are $\mathcal{U} \in \mathbb{U}, J \in [\omega]^\omega \cap V[G_1]$ such that for every consecutive $j, j' \in J$ there exists $n \in [j, j')$ with the property $R(\mathcal{U})(n) \subset A$.*

Proof. Equivalent, the following statement will be proved:

> For every $A \in \tau \setminus \{\emptyset\}$ there are $\mathcal{U} \in \mathbb{U}, J \in [\omega]^\omega \cap V[G_1]$ such that for every $m \in \omega$ there exists $n \in [m, J(m))$ with the property $R(\mathcal{U})(n) \subset A$, where $J(m)$ is the m-th element of J with respect to its increasing enumeration.

Suppose that there exists $A \in \tau \setminus \{\emptyset\}$ for which the lemma is false. Let $\dot{A}$ be a $\mathbb{P}_{\omega_2}$-name for A and $p \in \mathbb{P}_{\omega_2}$ a condition forcing the negation of the statement quoted above. Applying [Laver, 1976, Lemma 14] to the sequence $\langle \dot{a}_i : i \in \omega \rangle$ such that $\dot{a}_i = \dot{A}$ for all $i \in \omega$, it is possible to consider a condition $p' \leq p$ such that $p'(0) \leq^0 p(0)$, and a finite set $\mathcal{U}_s \subset \mathcal{P}(\omega)$ for every $s \in p'(0)$ with $p'(0)\langle 0 \rangle \leq s$, such that for each $n \in \omega$, $s \in p'(0)$ with $p'(0)\langle 0 \rangle \leq s$, and for all but finitely many immediate successors t of s in $p'(0)$,

$$p'(0)^\frown_t p' \restriction [1, \omega_2) \Vdash \exists U \in \mathcal{U}_s (\dot{A} \cap n = U \cap n).$$

Subclaim 5.1. *There exists $p'' \leq p'$ s.t. for every $s \in p''(0), p''(0)\langle 0 \rangle \leq s$, $n \in \omega$ and every immediate successor t of s in $p''(0)$,*

$$p''(0)^\frown_t p'' \restriction [1, \omega_2) \Vdash \exists U \in \mathcal{U}_s (Int(U) \neq \emptyset \wedge \dot{A} \cap n = U \cap n).$$

Proof. Suppose by contradiction that

($**$) For every $p'' \leq p'$ it is possible to find $s \in p''(0)$ with $p''(0)\langle 0 \rangle \leq s$, $n \in \omega$, infinitely many immediate successors t of s in $p''(0)$, for which there exists $r = r(t) \leq p''(0)^\frown_t p'' \restriction [1, \omega_2)$ forcing
$\forall U \in \mathcal{U}_s \, (U \cap n = \dot{A} \cap n \Rightarrow Int(U) = \emptyset)$.

Fix $p'' \leq p'$ and let s be as in ($**$). Note that it is possible to replace n with any other bigger number and ($**$) will be still satisfied, so it is possible to assume that $U \cap n \neq U' \cap n$ for any distinct $U, U' \in \mathcal{U}_s$. Let $p^{(3)} \in \mathbb{P}_{\omega_2}$ be the condition obtained by strengthening $p''(0)$ by leaving only those infinitely many immediate successors of s like in ($**$), and $p^{(3)}(\alpha) = p''(\alpha)$ for all $\alpha > 0$. Next, removing yet another finite collection of immediate successors of s in $p^{(3)}(0)$ and keeping the other coordinates the same, it is possible to assume that for every immediate successor t of s in $p^{(3)}(0)$ it is true that

$$p^{(3)}(0)_t^\frown p'' \upharpoonright [1,\omega_2) \Vdash \exists U \in \mathcal{U}_s\, (U \cap n = \dot{A} \cap n),$$

and hence by strengthening $r(t)$, if necessary, it is additionally possible to assume that, for some $U(t) \in \mathcal{U}_s$ with $Int(U(t)) = \emptyset$, $r(t) \Vdash \dot{A} \cap n = U(t) \cap n$. By the choice of n, such an $U(t)$ is unique. Furthermore, strengthening $p^{(3)}$ again in the way described above and using the finiteness of $\mathcal{U}_s$, it is possible to assume that there exists $U_s \in \mathcal{U}_s$ with $U(t) = U_s$ for all t as above. Summarizing all the modifications of p'' mentioned above, it is obtained the following:

$(**)'$ For every $p'' \leq p'$ it is possible to find $p^{(3)} \leq p''$, $s \in p^{(3)}(0)$ with $p^{(3)}(0)\langle 0\rangle \leq s$, $n \in \omega$ with $U \cap n \neq U' \cap n$ for any distinct $U, U' \in \mathcal{U}_s$, $U_s \in \mathcal{U}_s$ with $Int(U_s) = \emptyset$, and for every immediate successors t of s in $p^{(3)}(0)$ a condition $r(t) \leq p^{(3)}(0)_t^\frown p'' \upharpoonright [1,\omega_2)$ forcing $U_s \cap n = \dot{A} \cap n$.

Let $\langle D_k : k \in \omega\rangle \in V$ be a sequence of dense subsets of $\langle \omega, \tau\rangle$ such that for every $U \in \bigcup\{\mathcal{U}_s : s \in p'(0), p'(0)\langle 0\rangle \leq s\}$ with $Int(U) = \emptyset$ there are infinitely many $k \in \omega$ such that $D_k = \omega \setminus U$. Let $\langle F_k : k \in \omega\rangle \in V$ be a witness of the H-separability of X for $\langle D_k : k \in \omega\rangle$. So it is possible to choose $k_0 \in \omega$ and $p'' \leq p'$ such that $p'' \Vdash (\forall k \geq k_0)(\dot{A} \cap F_k \neq \emptyset)$.

Fix $p^{(3)}, s, n, U_s$, and $p(t)$'s such as in $(**)'$. Let $k_1 \geq k_0$ be such that $D_{k_1} = \omega \setminus U_s$, and thus $F_{k_1} \cap U_s = \emptyset$. Choose now $n_1 > n, \max F_{k_1}$ and an immediate successor t of s in $p^{(3)}(0)$ with

$$p^{(3)}(0)_t^\frown p'' \upharpoonright [1,\omega_2) \Vdash \exists U \in \mathcal{U}_s (\dot{A} \cap n_1 = U \cap n_1).$$

Thus $r(t)$ also forces the above property (being stronger than $p^{(3)}(0)_t^\frown p'' \upharpoonright [1,\omega_2)$) as well as $U_s \cap n = \dot{A} \cap n$. Since all the differences between elements of $\mathcal{U}_s$ show up below n, $r(t)$ forces $U_s \cap n_1 = \dot{A} \cap n_1$. Since

$$F_{k_1} \subset D_{k_1} \cap n_1 = (\omega \setminus U_s) \cap n_1 = n_1 \setminus U_s,$$

then $r(t)$ forces $F_{k_1} \cap \dot{A} = \emptyset$. On the other hand $k_1 \geq k_0$, $r(t) \leq p''$, and the latter forces $\dot{A} \cap F_k \neq \emptyset$ for all $k \geq k_0$, a contradiction. □

It is now possible to assume that every $U \in \mathcal{U}_s$ with $s \in p'(0)$ and $p'(0)\langle 0\rangle \leq s$ has nonempty interior. Let $\mathcal{U} = \{Int(U) : U \in \bigcup\{\mathcal{U}_s : s \in p''(0), p''(0)\langle 0\rangle \leq s\}$ (or any countable family containing it if this family of interiors is finite) and $R(\mathcal{U})(n) = \langle F_n : n \in \omega\rangle$. Replace $p'(0)$ with a tree T in the Laver forcing $\mathcal{L}$ such that $T\langle 0\rangle = p'(0)\langle 0\rangle$ and for all $s \in T$ with $s \geq T\langle 0\rangle$ there exists $N_s \in \omega$ so that for every immediate successor t of s in T the following two properties hold:

(a) $\forall U \in \mathcal{U}_s\ \exists n = n(s,U) > |s|\ (F_n \subset Int(U) \cap N_s)$

(b) $T_t{}^\frown p' \upharpoonright [1,\omega_2) \Vdash \exists U \in \mathcal{U}_s\ (\dot{A} \cap N_s = U \cap N_s)$

For any $s \in T$ set

$$j_s = \max\{n(s,U) : U \in \mathcal{U}_s\} + 1$$

Let G_1 be $\mathcal{L}$-generic over V with $T \in G_1$ and $\ell \in \omega^{\uparrow\omega}$ be the Laver real generated by G_1. Finally put

$$J = \{j_{\ell\upharpoonright m} : m \geq |T\langle 0\rangle|\}$$

and note that $J \in V[G_1]$.

Subclaim 5.2. $T^\frown p' \upharpoonright [1, \omega_2) \Vdash \forall m \geq |T\langle 0\rangle| \, \exists n \in [m, j_{\ell \upharpoonright m}) \, (F_n \subset \dot{A})$

Proof. Suppose the statement is false and pick $r \leq T^\frown p' \upharpoonright [1, \omega_2)$ and $m \geq |T\langle 0\rangle|$ such that

$$r \Vdash \forall n \in [m, j_{\ell \upharpoonright m})(F_n \not\subset \dot{A}).$$

Observe that $r = R^\frown r \upharpoonright [1, \omega_2)$ and without loss of generality assume $|R\langle 0\rangle| \geq m+1$. Setting $\{s\} = R \cap \omega^m$ and $\{t\} = R \cap \omega^{m+1}$, two elements of T which contradict the disjunction of (a) and (b) above are obtained. □

It follows that

$$T^\frown p' \upharpoonright [1, \omega_2) \leq p' \leq p$$

and $T^\frown p' \upharpoonright [1, \omega_2)$ forces for $\dot{A}$ the quoted statement after the formulation of Claim 5, contradicting the assumption that p forces for $\dot{A}$ the negation of that statement. This contradiction completes the proof of Claim 5 and thus also of Proposition 3.4.2. □

□

Lemma 3.4.3. *[Bardyla, M., Zdomskyy, 2023, Lemma 5.4] Suppose $\mathfrak{b} > \omega_1$, X is a b.b.-separable space and Y is an H-separable space. Then $X \times Y$ is mH-separable, provided it is separable.*

Proof. Let $\langle D_n : n \in \omega\rangle$ be a decreasing sequence of countable dense subsets of $X \times Y$. Fix a countable family $\mathcal{U}$ of open non-empty subsets of X and a partition $\{\Omega_U : U \in \mathcal{U}\}$ of ω into infinite pieces. For every $U \in \mathcal{U}$ and $n \in \Omega_U$ set

$$D_n^{\mathcal{U}} = \{y \in Y : \exists x \in U(\langle x, y\rangle \in D_n)\}.$$

Note that every $D_n^{\mathcal{U}}$ defined in this way is dense in Y. Since Y is H-separable there exists a sequence of sets $\langle L_n^{\mathcal{U}} : n \in \omega\rangle$ such that $L_n^{\mathcal{U}} \in [D_n^{\mathcal{U}}]^{<\omega}$ for all $n \in \omega$ and every open subset of Y intersects all but finite $L_n^{\mathcal{U}}$'s. For every $U \in \mathcal{U}$ and $n \in \Omega_U$ find a set $K_n^{\mathcal{U}} \in [U]^{<\omega}$ such that for every $y \in L_n^{\mathcal{U}}$ there exists $x \in K_n^{\mathcal{U}}$ such that $\langle x, y\rangle \in D_n$, and set $R(\mathcal{U}) = \langle K_n^{\mathcal{U}} : n \in \omega\rangle$. Note that $K_n^{\mathcal{U}} \subset U$ for all $n \in \Omega_U$, so R is as in the definition of b.b.-separability. Therefore there exist a family $\mathbb{U}$ of countable collections of open non-empty subsets of X of cardinality ω_1 and a sequence $\langle l_i : i \in \omega\rangle \in \omega^\omega$ that witness the b.b.-separability. By the hypothesis $|\mathbb{U}| < \mathfrak{b}$, so it is possible to select a sequence $\langle F_n : n \in \omega\rangle$ such that $F_n \in [D_n]^{<\omega}$ and $K_n^{\mathcal{U}} \times L_n^{\mathcal{U}} \subset F_n$ for all $\mathcal{U} \in \mathbb{U}$ and all but finitely many $n \in \omega$. It is sufficient to prove that the sequence

$$\langle F_i' := \bigcup_{n \in [l_i, l_{i+1})} F_n \, : \, i \in \omega\rangle$$

witnesses the mH-separability of $X \times Y$. First of all, note that $F_i' \subset D_{l_i} \subset D_i$ by the monotonicity of the sequence $\langle D_n : n \in \omega\rangle$. Now fix an open non-empty subset $U \times V \subseteq X \times Y$ and find $\mathcal{U} \in \mathbb{U}$ and $i_0 \in \omega$ such that for all $i \geq i_0$ there exists $n \in [l_i, l_{i+1})$ with $R(\mathcal{U})(n) = K_n^{\mathcal{U}} \subset U$. Suppose that $i \geq i_0$, and both of the conditions $K_n^{\mathcal{U}} \times L_n^{\mathcal{U}} \subset F_n$ and $L_n^{\mathcal{U}} \cap V \neq \emptyset$ hold true for all $n \geq l_i$. Given $n \in [l_i, l_{i+1})$ with $R(\mathcal{U})(n) = K_n^{\mathcal{U}} \subset U$, it follows from the above that $F_n \cap (U \times V) \neq \emptyset$, which combined with $F_n \subset F_i'$ proves the mH-separability of $X \times Y$. □

Proposition 3.4.2, Lemma 3.4.3, and the fact that $\mathfrak{b} > \omega_1$ is true in the Laver model all together imply the following

Theorem 3.4.4. [*Bardyla, M., Zdomskyy,* 2023*, Theorem 5.5*] *In the Laver model, the product of two H-separable spaces is mH-separable provided that it is separable. In particular, in this model the product of two countable H-separable spaces is mH-separable.*

LIST OF SYMBOLS

ZF	Zermelo-Fraenkel's axioms	1
AC	the Axiom of Choice	1
ZFC	Zermelo-Fraenkel's axioms togheter with the Axiom of Choice	1
λ, κ	cardinal numbers	1
κ^+	the least cardinal greater than the cardinal κ	1
$\alpha, \beta, \gamma, \delta$	ordinal numbers	1
$\aleph_0$	the first infinite cardinal	1
$\aleph_1$	the first uncountable cardinal	1
$\aleph_{n+1}$	the successor cardinal of $\aleph_n$	1
ω	the first infinite ordinal	1
ω_1	the first uncountable ordinal	1
n, m, k	integers	1
$\mathbb{N}$	the set of all integers	1
$^{<\omega}\omega$	the set of all finite sequences of integers	1
$\lvert A \rvert$	the cardinality of a set A	2
$[A]^\kappa$	the set of all subsets of a set A of cardinality κ	2
$[A]^{\leq\kappa}([A]^{<\kappa}, [A]^{\geq\kappa}, [A]^{>\kappa})$	set of all subsets of A of cardinality $\leq \kappa$ ($< \kappa, \geq \kappa, > \kappa$)	2
$\mathfrak{c}$,$2^{\aleph_0}$	the cardinality of the set of the real numbers	2
$cf(\kappa)$	the cofinality of a cardinal κ	2
$\subseteq^*$	almost contained relation	2
$\mathfrak{b}$	the minimal cardinality of a unbounded family in ω^ω	2
$\mathfrak{d}$	the minimal cardinality of a cofinal family in ω^ω	2
$\mathfrak{p}$	the minimal cardinality of a family in $[\omega]^\omega$ with SFIP but with no pseudointersection	2
CH	the Continuum Hypothesis	2
X	a topological space	2
τ	a topology	2
$\mathcal{O}$	the family of all the covers of a space	3
Γ	the family of all the γ-covers of a space	3
Ω	the family of all the ω-covers of a space	3
$d(X)$	the density of a space X	4
$\pi\chi(x, X)$	the π-character of a point $x \in X$	4
$\chi(x, X)$	the character of a point $x \in X$	4
$\pi\chi(X)$	the π-character of a space X	4
$\chi(X)$	the character of a space X	4

$e(X)$	the extent of a space X	5
$L(X)$	the Lindelöf degree of X	5
ω^ω	the Baire space	6
$\mathcal{A}$	an almost disjoint or a maximal almost disjoint family	6
$\Psi(\mathcal{A})$	the Isbell-Mröwka space on an almost disjoint family $\mathcal{A}$	6
$C_p(X)$	the space of continuous functions from X to the real line $\mathbb{R}$	6
$\mathcal{F}$	a filter or an ultrafilter	6
$\mathcal{F}(\mathcal{A})$	the dual filter on a family $\mathcal{A}$	7
$\mathbb{P}$	a forcing notion	7
H_θ	the set of all sets whose transitive closure has cardinality less than θ	7
M, V	models	7
MA	the Martin's Axiom	8
PFA	the Proper Forcing Axiom	9
$(\mathcal{T}, \leq)$	a tree	10
$\mathbb{M}_\mathcal{F}$	the Mathias forcing on the filter $\mathcal{F}$	10
$H(X)$	the Hausdorff number of a space X	12
$st(A, \mathcal{V})$	the star of the set A with respect the family of sets $\mathcal{V}$	22
$U(X)$	the Urysohn number of a space X	12

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