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Flag manifolds without one codimensional strata

In this paper, we propose a classification of finite-dimensional real Lie algebras that admit subalgebras of codimension two, but no subalgebra of codimension one, which leads to a description of homogeneous spaces without isotropy subgroups of codimension one. This study expands upon the existing body of research on codimension-one subalgebras, aiming to extend the analysis to a new setting. The result of this expansion is a comprehensive description of the relevant families. It is demonstrated that a such Lie algebra $\mathfrak{g}$ contains an ideal $\mathfrak{u}$, containing the radical of $\mathfrak{g}$, such that $\mathfrak{g}/\mathfrak{u}$ belongs to a list $\mathcal{L}$, consisting of one compact and two non compact Lie algebras.

Keywords: homogeneous spaces, Lie algebras, codimension two subalgebras, simple Lie algebras, compact real forms

1. Introduction

Homogeneous geometries G/H with minimal isotropy H of codimension 2 are considered via the study of real Lie algebras containing a subalgebra of codimension two. Indeed, in the study of Lie groups and their actions on manifolds, understanding the structure of subalgebras of small codimension in their Lie algebras plays a central role. The case of Lie algebras admitting 1-codimensional subalgebras has been comprehensively investigated and is

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well understood (see, e. g., Hofmann [5]). The latter's applications range from the classification of transitive group actions with 1-dimensional orbits to the analysis of boundaries of certain topological semigroups. Recently, there has been a renewed interest in the lattice structure of subalgebras of simple Lie algebras [1—3], motivated by applications in representation theory, geometry, and mathematical physics. This has led to a more detailed classification of subalgebras with specific codimensions, inspiring the present study focused on Lie algebras admitting subalgebras of codimension two but not of codimension one.

Building on previous work on the classification of Lie algebras by subalgebra codimension, we focus on the real finite-dimensional case where subalgebras of codimension 2 exist, while none of codimension 1 do. We provide a detailed analysis and classification of such Lie algebras, highlighting the algebraic constraints imposed by the absence of codimension one subalgebras, and describing the possible structures that arise in this intermediate setting. This study fills a gap in the literature between the classical codimension-one case and the broader theory of maximal subalgebras: whereas Hofmann's Theorem 1 [5, p. 638] establishes that if a real Lie algebra $\mathfrak{g}$ has a subalgebra $\mathfrak{h}$ of codimension 1, then either $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, or $\mathfrak{h}$ contains an ideal $\mathfrak{i}$ of $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{i}$ is isomorphic to the 2-dimensional non-abelian Lie algebra, we show (Theorem 3.2) that if $\mathfrak{g}$ admits a subalgebra of codimension 2 but none of codimension 1, then it contains an ideal $\mathfrak{u}$, containing the radical of $\mathfrak{g}$ and lying inside the given subalgebra, such that $\mathfrak{g}/\mathfrak{u}$ belongs to the list $\mathcal{L} = \{\mathfrak{su}(2), \mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}, \mathfrak{sl}(3, \mathbb{R})\}$. This work consists essentially of two sections, in addition to this introduction. In Section 2 we recall some definitions and results to provide autonomy and self-consistency to the present work, since the statements referenced in this section will be used throughout the proofs of our new results in Section 3. In the latter, we present intermediate results, as well as the complete family of Lie algebras under consideration, culminating in the main theorem of this work.

The results obtained in this study contribute to a deeper understanding of the lattice of subalgebras in low codimension, with potential applications in geometry and representation theory.

2. Preliminaries

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. We denote by $\mathfrak{g}^{\mathbb{C}}$ the *complexification* of $\mathfrak{g}$, that is, $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. The complexification is obtained by extending the scalar field from $\mathbb{R}$ to $\mathbb{C}$, resulting in a complex vector space endowed with the naturally extended Lie bracket.

Now, let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. A *real form* of a complex Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus i\, \mathfrak{g}_0$$

as a direct sum of real vector spaces. Here, the *realification* $\mathfrak{g}^{\mathbb{R}}$ is defined by considering the same underlying vector space of $\mathfrak{g}$ but viewed as a vector space over $\mathbb{R}$, with the Lie bracket extended by $\mathbb{R}$-bilinearity. In other words, $\mathfrak{g}^{\mathbb{R}}$ is the Lie algebra over $\mathbb{R}$ obtained by forgetting the complex scalar multiplication on $\mathfrak{g}$.

Notice that the real dimension of $\mathfrak{g}^{\mathbb{R}}$ is twice the complex dimension of $\mathfrak{g}$, that is,

$$\dim_{\mathbb{R}}(\mathfrak{g}^{\mathbb{R}}) = 2\dim_{\mathbb{C}}(\mathfrak{g}).$$

Moreover, it is useful for this work to recall this result (as a reference, see, for example, [8]).

Theorem 2.1 [8, Theorem 1, p. 9]. *Let* $\mathfrak{g}$ *be a real Lie algebra.* $\mathfrak{g}$ *is abelian (resp. nilpotent, solvable, semisimple) if and only if* $\mathfrak{g}^{\mathbb{C}}$ *is.*

Furthermore, we also have the following.

Theorem 2.2. $\mathfrak{g}^{\mathbb{C}}$ *is simple if and only if either* $\mathfrak{g}$ *is simple, or* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{s} \oplus \overline{\mathfrak{s}}$*, where* $\mathfrak{s}$ *and* $\overline{\mathfrak{s}}$ *are simple complex Lie algebras that are conjugate to each other.*

Proof. See [8, discursive remark following Theorem 9, p. 9].

In this work, as well as in all the references we will cite and/or refer to, a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is said to be *compact* if there exists a positive definite scalar product on $\mathfrak{g}$ which is invariant under the adjoint representation. Any subalgebra of a compact Lie algebra is itself compact.

Proposition 2.1 [7, Proposition 1.3, p. 131]. *The Killing form of a compact Lie algebra is nonpositive definite. Moreover, a real Lie algebra is semisimple and compact if and only if its Killing form is negative definite.*

The following result states a sufficient condition for a complex Lie algebra to admit a compact real form.

Theorem 2.3 [6, Theorem 6.11, p. 353]. *If $\mathfrak{g}$ is a complex semisimple Lie algebra, then $\mathfrak{g}$ has a real compact form $\mathfrak{g}_0$.*

After briefly summarising the necessary definitions and recalling some results on real forms and compact Lie algebras, what remains is to state a lemma from the 1965 work of K. H. Hofmann. This result will be invoked to prove certain statements in the next section.

Lemma 2.2 [5, Lemma 1]. *If $\mathfrak{g}$ is a n-dimensional real compact simple Lie algebra, containing a d-codimensional subalgebra $\mathfrak{h} \neq 0$, then $n \leq \frac{d(d+1)}{2}$.*

3. Main Results

In this section we present the new results of this work, providing a complete list of Lie algebras that admit a subalgebra of codimension 2 but no subalgebra of codimension 1. Furthermore, the final theorem describes the precise structure of such Lie algebras. Throughout the paper, when writing dim without a subscript, the underlying field will be clear from the context. When the field needs to be specified, we shall write $\dim_{\mathbb{F}}$, where $\mathbb{F}$ denotes the field over which the dimension is taken.

Lemma 3.1. *If $\mathfrak{g}$ is a n-dimensional real simple Lie algebra, containing a 2-codimensional subalgebra $\mathfrak{h} \neq 0$, then $n \leq 10$.*

Proof. If $\mathfrak{g}$ is compact, there is nothing more to prove, as Lemma 2.2 ensures that $n \leq 3$. If $\mathfrak{g}$ is not compact, then two cases arise.

• Case 1: $\mathfrak{g}$ is not a realification of a complex Lie algebra. Then by Theorem 2.2, $\mathfrak{g}^{\mathbb{C}}$ is simple, and then semisimple. Hence, by Theorem 2.3 there exists a real compact form, namely $\mathfrak{g}_c$. This real compact form is also simple (see point b) of Theorem 3.1) and $\dim\mathfrak{g}_c = n$. Let us now consider $\mathfrak{h}^{\mathbb{C}}$ and the subalgebra $\mathfrak{k} := \mathfrak{h}^{\mathbb{C}} \cap \mathfrak{g}_c$. We first note that $\mathfrak{k}$ is a proper subalgebra of $\mathfrak{g}_c$. Indeed, if this were not the case, i. e., if $\mathfrak{h}^{\mathbb{C}} \cap \mathfrak{g}_c = \mathfrak{g}_c$, then it would follow that $\mathfrak{g}_c \subseteq \mathfrak{h}^{\mathbb{C}}$. This would imply that $\mathfrak{g}_c$ is the complex vector space generated by its compact real form, which is the whole $\mathfrak{g}^{\mathbb{C}}$. However, this leads to a contradiction due to dimension considerations. The dimension of $\mathfrak{k}$ is given by

$$\dim\mathfrak{k} = \dim\mathfrak{h}^{\mathbb{C}} + \dim\mathfrak{g}_c - \dim(\mathfrak{h}^{\mathbb{C}} + \mathfrak{g}_c) = (2n-4) + n - t,$$

where

$$n := \dim\mathfrak{g}_c = \dim\mathfrak{g},$$

and

$$t := \dim(\mathfrak{h}^{\mathbb{C}} + \mathfrak{g}_c) \geq \max\{2n-4, n\}.$$

Thus, if $n > 4$, then $\dim\mathfrak{k} \leq n - 4$, and again by Lemma 2.2, we obtain

$$\dim\mathfrak{g} = \dim\mathfrak{g}_c \leq \frac{4 \cdot 5}{2} = 10.$$

• Case 2: $\mathfrak{g}$ is the realification of a complex Lie algebra $\mathfrak{G}$, that is $\mathfrak{g} = \mathfrak{G}^{\mathbb{R}}$, with $\dim_{\mathbb{C}}\mathfrak{G} = n$. Clearly, since $\mathfrak{g}$ is simple, it follows that $\mathfrak{G}$ is also simple. Hence, $\mathfrak{G}$ is semisimple and, by Theorem 2.3, it contains a compact real form, which we denote by $\mathfrak{G}_c$ (so that $\mathfrak{G}_c^{\mathbb{C}} = \mathfrak{G}$). Hence we have $\dim_{\mathbb{R}}\mathfrak{G}_c = n$, whereas $\dim_{\mathbb{R}}\mathfrak{h} = 2n-2$, and $\dim_{\mathbb{R}}\mathfrak{g} = 2\dim_{\mathbb{C}}\mathfrak{G} = 2n$. Let now $\mathfrak{k} = \mathfrak{h} \cap \mathfrak{G}_c$. Similarly to what was done in Case 1, one can show that $\mathfrak{k}$ is a proper subalgebra of $\mathfrak{G}_c$. Indeed, if by contradiction we assume that $\mathfrak{k} = \mathfrak{G}_c$, then it would follow that $\mathfrak{G}_c \subseteq \mathfrak{h}$. Considering these vector spaces over $\mathbb{C}$, this would imply $\mathfrak{G} \subseteq \mathfrak{h}^{\mathbb{C}}$, which is impossible since $\dim_{\mathbb{R}}\mathfrak{h} = 2n - 2$, so $\dim_{\mathbb{C}}\mathfrak{h}^{\mathbb{C}} = n - 1$, whereas $\dim_{\mathbb{C}}\mathfrak{G} = n$. Proceeding with similar arguments as above and applying again Lemma 2.2, we conclude that $\dim_{\mathbb{R}}\mathfrak{g} \leq 6$.

The following result gives, up to isomorphism, the complete list of real simple Lie algebras.

Theorem 3.1 [6, Theorem 6.105, p. 421]. *Up to isomorphism, every simple real Lie algebra is contained in the following list, and every Lie algebra in the list is simple:*

(a) The Lie algebra $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{g}$ is a complex simple Lie algebra of type A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4, or G_2.

(b) The compact real form of any $\mathfrak{g}$ as in (a).

(c) The following classical matrix Lie algebras:

$$\mathfrak{su}(p,q) \text{with} p \geq q > 0 \text{and} p + q \geq 2,$$

$$\mathfrak{so}(p,q) \ \begin{cases} \text{with } p > q > 0, & p + q \text{ odd, and } p + q \geq 5, \\ \text{or with } p \geq q > 0, & p + q \text{ even, and } p + q \geq 8, \end{cases}$$

$$\mathfrak{sp}(p,q) \text{ with } p \geq q > 0 \text{ and} p \ + q \geq 3,$$

$$\mathfrak{sp}(n,\mathbb{R}) \text{ with } n \geq 3,$$

$$\mathfrak{so}^*(2n) \text{ with } n \geq 4,$$

$$\mathfrak{sl}(n,\mathbb{R}) \text{ with } \ n \geq 3,$$

$$\mathfrak{sl}(n,\mathbb{H}) \text{ with } \ n \geq 2.$$

(d) The 12 exceptional, non-complex, non-compact, simple Lie algebras listed in Figures 6.2 and 6.3 of [6].

The only isomorphism among Lie algebras in the list above is $\mathfrak{so}^*(8) \cong \mathfrak{so}(6,2)$.

Recalling that

$$\dim \mathfrak{su}(p,q) = (p+q)^2 - 1, \quad \text{with } \ p \geq q > 0,$$

$$\dim \mathfrak{so}(p,q) = \frac{(p+q)(p+q-1)}{2}, \ \text{ with } \ p \geq q > 0,$$

$$\dim \mathfrak{sp}(p,q) = (p+q)(2p+2q+1), \ \text{ with } \ p \geq q > 0,$$

$$\dim \mathfrak{sp}(n,\mathbb{R}) = n(2n+1), \ \text{ with } \ n \geq 3,$$
$$\dim \mathfrak{so}^*(2n) = n(2n-1), \ \text{ with } \ n \geq 4,$$
$$\dim \mathfrak{sl}(n,\mathbb{R}) = n^2 - 1, \ \text{ with } \ n \geq 3,$$
$$\dim \mathfrak{sl}(n,\mathbb{H}) = 4n^2 - 1, \ \text{ with } \ n \geq 2,$$

we deduce that, given Lemma 3.1, the Lie algebras that may arise in our investigation are precisely those appearing in Table.

List of simple real Lie algebras of dimension less than or equal to 10, relevant for our analysis. Isomorphisms are indicated when applicable

Lie Algebra	Dimension	Isomorphisms
$\mathfrak{su}(2)$	3	$\mathfrak{sp}(1) \cong \mathfrak{so}(3)$
$\mathfrak{su}(3)$	8	–
$\mathfrak{so}(5)$	10	$\mathfrak{sp}(2)$
$\mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}}$	6	$\mathfrak{so}(3,1)$
$\mathfrak{su}(2,1)$	8	–
$\mathfrak{so}(3,2)$	10	–
$\mathfrak{so}(4,1)$	10	–
$\mathfrak{sl}(2,\mathbb{R})$	3	–
$\mathfrak{sl}(3,\mathbb{R})$	8	–

Lemma 3.2. *If the real simple Lie algebra* $\mathfrak{g}$ *contains a 2-codimensional subalgebra* $\mathfrak{h} \neq 0$*, but does not contain any 1-codimensional subalgebra, then it is isomorphic to one of the list*

$$\mathcal{L} = \{\mathfrak{su}(2), \mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}}, \mathfrak{sl}(3,\mathbb{R})\}.$$

Proof. If $\mathfrak{g}$ is compact, then by Lemma 2.2 it has dimension 3, hence it must be isomorphic to $\mathfrak{su}(2)$. It is well known that $\mathfrak{su}(2)$ is isomorphic to the 3-dimensional Lie algebra on $\mathbb{R}^3$ with Lie bracket given by the vector cross product. Since the cross product of two nonzero vectors in $\mathbb{R}^3$ is always orthogonal to the plane spanned by those vectors, it follows that $\mathfrak{su}(2)$ cannot have a subalgebra of codimension 1. Indeed, the product of any two independent vectors yields a third vector independent of the first two, preventing the existence of a 2-dimensional subalgebra. Moreover, for dimensional reasons, we must also exclude in the compact case both $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$ from Table.

From now on, let us assume that $\mathfrak{g}$ is not compact. The Lie algebra $\mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}}$, which is isomorphic to $\mathfrak{so}(3,1)$, appears in our list $\mathcal{L}$ by [4] (see Sections 6 and 7). In particular, the authors construct in Section 6 a 4-dimensional Borel subalgebra of $\mathfrak{so}(3,1)$, and prove in Section 7 that no 5-dimensional subalgebras exist.

The Lie algebra $\mathfrak{su}(2,1)$ does not belong to $\mathcal{L}$, as it admits neither subalgebras of codimension 1 nor subalgebras of codimension 2 (see [3]).

Regarding the Lie algebra $\mathfrak{so}(3,2)$, the authors in [1] proved that its maximal subalgebras fall into three categories: maximal parabolic subalgebras, the reducible subalgebra $\mathfrak{so}(4)$, and a simple irreducible subalgebra $\mathfrak{sl}_2^5(\mathbb{C})$ defined by the natural action on $S^4(\mathbb{C}^2)$. However, none of the subalgebras listed in their work has dimension 8 or 9. Therefore, $\mathfrak{so}(3,2)$ does not belong to the family $\mathcal{L}$.

We now prove that $\mathfrak{so}(4,1) \notin \mathcal{L}$ by showing that it does not possess any subalgebras of codimension 1 or codimension 2.

Suppose $\mathfrak{h} \subseteq \mathfrak{so}(4,1)$ is a subalgebra with $\dim\mathfrak{h} = t$, where $t \in \{8,9\}$. Consider the subalgebra

$$\mathfrak{m} = \left\{\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} | A \in \mathfrak{so}(4)\right\}.$$

Recall here that the matrices in $\mathfrak{so}(4,1)$ are of the form

$$\begin{pmatrix} X & Y \\ Y^t & 0 \end{pmatrix},$$

with $X \in \mathfrak{so}(4)$. Obviously, $\mathfrak{m} \cong \mathfrak{so}(4)$, so $\mathfrak{m}$ is compact and $\dim\mathfrak{m} = 6$. Then, by Lemma 2.2, $\mathfrak{m}$ contains neither subalgebras of codimension 1 nor codimension 2. If $\mathfrak{m} \subseteq \mathfrak{h}$, then for every $A \in \mathfrak{so}(4)$ and every

$$\begin{pmatrix} X & Y \\ Y^t & 0 \end{pmatrix} \in \mathfrak{h}$$

with $Y \neq 0$, we have

$$\left[\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X & Y \\ Y^t & 0 \end{pmatrix}\right] = \begin{pmatrix} [A,X] & AY \\ -Y^tA & 0 \end{pmatrix} \in \mathfrak{h},$$

which would imply that $\mathfrak{h} = \mathfrak{so}(4,1)$, a contradiction.

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is not included in the family $\mathcal{L}$ because it contains a subalgebra of codimension 1, for example

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle.$$

Finally, the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ belongs to the family $\mathcal{L}$ since, according to [2, Theorem 20, p. 18]: it has subalgebras of dimension 6 but no subalgebras of dimension 7.

We are now in a position to prove the main theorem of this work.

Theorem 3.2. *If* $\mathfrak{h} \neq 0$ *has codimension two in* $\mathfrak{g}$ *and there is no subalgebra* $\mathfrak{k}$ *of codimension one in* $\mathfrak{g}$*, then* $\mathfrak{g}$ *has an ideal* $\mathfrak{u}$*, contained in* $\mathfrak{h}$*, such that* $\mathfrak{g}/\mathfrak{u}$ *is isomorphic to one of the list* $\mathcal{L}$*. Moreover,* $\mathfrak{u}$ *contains the radical* $\mathfrak{r}$ *of* $\mathfrak{g}$.

Proof. By contradiction, let $\mathfrak{g}$ be a counterexample of minimal dimension. First, observe that $\mathfrak{h}$ cannot contain any non-zero ideal $\mathfrak{v}$ of $\mathfrak{g}$, for otherwise the quotient $\mathfrak{g}/\mathfrak{v}$ would yield a counterexample of smaller dimension. In particular, $\mathfrak{g}$ cannot have a one-dimensional ideal $\mathfrak{u}$, since in that case $\mathfrak{h} + \mathfrak{u}$ would be a subalgebra of $\mathfrak{g}$ of codimension one.

Note that $\mathfrak{g}$ must be perfect; otherwise, any codimension-1 subalgebra of the abelian quotient $\mathfrak{g}/\mathfrak{g}'$ would correspond to a codimension-1 subalgebra of $\mathfrak{g}$. However, by Lemma 3.2, $\mathfrak{g}$ is not simple. Moreover, $\mathfrak{g}$ is not semisimple. Indeed, if

$$\mathfrak{g} = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_s$$

is semisimple (with $\mathfrak{s}_j$ simple and $s > 1$), then each $\mathfrak{s}_j$ is not contained in $\mathfrak{h}$, admits no subalgebra of codimension 1, and $\mathfrak{h} \cap \mathfrak{s}_j$ has codimension 2 in $\mathfrak{s}_j$, since $\mathfrak{g} = \mathfrak{h} + \mathfrak{s}_j$. Hence, $\mathfrak{s}_j \in \mathcal{L}$. Now we show that one of the ideals $\mathfrak{s}_j$ is contained in $\mathfrak{h}$, leading to a contradiction. Indeed, let

$$B_{\mathfrak{h}} = \{x_{11}, \dots, x_{t_1 1};\ \ x_{12}, \dots, x_{t_2 2};\ \ \dots;\ \ x_{1s}, \dots, x_{t_s s};\ \ y_1, \dots, y_{2s}\}$$

be a basis of $\mathfrak{h}$ such that, for each j, the set $\{x_{1j}, \dots, x_{t_j j}\}$ is a basis of $\mathfrak{h} \cap \mathfrak{s}_j$. Write $y_1 = y_{11} + \ \dots + y_{1s}$ with $y_{1j} \in \mathfrak{s}_j$. Then, for any $x_{ij} \in \mathfrak{h} \cap \mathfrak{s}_j$, we have

$$[y_1, x_{ij}] = [y_{1j}, x_{ij}] \in \mathfrak{h} \cap \mathfrak{s}_j.$$

There exists at least one index j such that $y_{1j} \notin \mathfrak{h}$; otherwise y_1 would be a linear combination of the vectors in $\{x_{11}, \dots, x_{t_1 1};\ \ x_{12}, \dots, x_{t_2 2};\ \ \dots;\ \ x_{1s}, \dots, x_{t_s s}\}$, which is impossible. But then $\langle y_{1j}\rangle + (\mathfrak{h} \cap \mathfrak{s}_j)$ would be a subalgebra of $\mathfrak{s}_j$ of codimension one, contradicting $\mathfrak{s}_j \in \mathcal{L}$. In particular, the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is nonzero. Setting $\mathfrak{v} = \mathfrak{h} \cap \mathfrak{n}$, we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$, and hence $\mathfrak{v}$ has codimension 2 in $\mathfrak{n}$.

Let $\mathfrak{v} \neq 0$. If $\mathfrak{v}$ is maximal in $\mathfrak{n}$, then it is an ideal of the nilpotent algebra $\mathfrak{n}$, and hence also an ideal of $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$, which is impossible since $\mathfrak{v} \subseteq \mathfrak{h}$. Therefore there exists $\mathfrak{v} \subsetneq \mathfrak{u} \subsetneq \mathfrak{n}$, and again $\mathfrak{g} = \mathfrak{h} + \mathfrak{u}$. Since $\mathfrak{v}$ is maximal in $\mathfrak{u}$, it is an ideal of $\mathfrak{u}$, and thus also of $\mathfrak{g} = \mathfrak{h} + \mathfrak{u}$, which is again a contradiction. Thus $\mathfrak{v} = 0$, and $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$ with $\dim \mathfrak{n} = 2$. Hence $\mathfrak{n}$ is abelian and $\mathfrak{h} \cong \mathfrak{g}/\mathfrak{n}$ is reductive. Moreover, since $\mathfrak{g}$ is perfect and $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$, the subalgebra $\mathfrak{h}$ is perfect as well, and thus semisimple. At least one simple component $\mathfrak{s}_i$ of $\mathfrak{h}$ must act non-trivially on the 2-dimensional abelian ideal $\mathfrak{n}$, which forces $\mathfrak{s}_i \cong \mathfrak{sl}_2$. But $\mathfrak{sl}_2$ contains the 1-codimensional subalgebra of traceless triangular matrices; hence $\mathfrak{h}$ contains a 1-codimensional subalgebra $\mathfrak{w}$, and $\mathfrak{g}$ contains the 1-codimensional subalgebra $\mathfrak{w} \ltimes \mathfrak{n}$ — a final contradiction.

Finally, to prove that the ideal $\mathfrak{u}$ contains the radical $\mathfrak{r}$ of $\mathfrak{g}$, it suffices to observe that $\mathfrak{g}/\mathfrak{u}$ is simple; therefore the solvable ideal $(\mathfrak{r} + \mathfrak{u})/\mathfrak{u}$ must vanish.

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