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Fuzzy Nilpotent Lie Algebras: Bases, Isomorphisms, and a Measure of Nilpotency

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Abstract

This paper investigates the structure of fuzzy Lie subalgebras, with particular emphasis on isomorphisms and nilpotency. Building on two prior conference contributions, one of which established foundational results on fuzzy bases of Lie algebras, we develop here a more complete and unified treatment of these themes. We introduce a notion of isomorphism between fuzzy Lie subalgebras based on the transfer principle via t-cut sets, and we prove that isomorphic fuzzy Lie subalgebras necessarily share the same nilpotency measure. The central contribution of the paper is a fuzzy measure of nilpotency $N(\mu) \in [0,1]$, defined for any non-constant fuzzy Lie subalgebra μ of a Lie algebra $\mathfrak{g}$. This invariant equals 1 precisely when μ is fuzzy nilpotent, and decreases as the subalgebra departs from nilpotency. We show that nilpotency of the underlying Lie algebra implies $N(\mu) = 1$, but that the converse fails in general, as witnessed by an explicit counterexample.

Keywords: fuzzy Lie algebra; nilpotent Lie algebra; isomorphisms

MSC: 03E72 (primary); 17B99; 17B30; 17B40

1. Introduction

The interplay between fuzzy set theory and algebraic structures has been a fertile area of research since Zadeh's foundational work [1]. The idea of endowing classical algebraic objects with a graded membership function—rather than a sharp, crisp boundary—opens up a richer landscape in which structural properties can be studied not just as binary predicates, but as matters of degree. After Zadeh, Rosenfeld started researching how the fuzzy set theory could be used in abstract algebra, and wrote about this in a well-known work [2]. He was inspired by a paper [3], in which appears the first occurrence of fuzzy set theory used in topological spaces. For more about the history of this topic the reader is referred to [4]. Lie algebra is a well-known algebraic structure that is widely used for its complex bracket operations and its links to geometry and physics. It is a great subject to use in this type of fuzzy enrichment.

The concept of a fuzzy Lie subalgebra, first discussed in [5], has since been extensively investigated, with a particularly detailed analysis being offered in [6]. The majority of

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the current research concentrates on the definitional and categorical dimensions of fuzzy Lie subalgebras, encompassing their substructures, homomorphisms, and ideals. The question of nilpotency, however, has received comparatively little attention in the fuzzy setting, despite being one of the most central notions in classical Lie theory. This is the main motivation of the present work. We want to understand how nilpotency can be formulated in the fuzzy setting and, at the same time, how one can measure the extent to which a fuzzy Lie subalgebra behaves as a nilpotent one.

The present paper grows out of two conference contributions [7,8]. The first [7] laid the groundwork by investigating the foundational properties of fuzzy Lie algebras as fuzzy vector spaces, with particular attention to the notion of a fuzzy basis and the conditions under which a classical basis retains its fuzzy character. The second [8] introduced a definition of nilpotency tailored to the fuzzy setting and undertook a comparative analysis with both the classical notion and existing alternatives in the literature. Here, we consolidate and significantly extend those results, providing a more complete and unified treatment.

Specifically, we introduce a definition of nilpotency for fuzzy Lie subalgebras via an ascending central series adapted to the fuzzy context, inspired by the one previously introduced in [9] for fuzzy groups. We also show that while nilpotency of the underlying crisp Lie algebra implies fuzzy nilpotency, the converse fails—a phenomenon that is both natural and, we believe, geometrically meaningful. To capture this asymmetry, we propose a numerical invariant $N(\mu) \in [0,1]$, which we call the *fuzzy nilpotency measure* of a fuzzy Lie subalgebra μ. This measure equals 1 precisely when μ is fuzzy nilpotent, and takes smaller values the further μ is from nilpotency.

We also study isomorphisms between fuzzy Lie subalgebras, proposing a definition based on the transfer principle through t-cut sets. We compare this with the categorical notion developed in [10], and we prove that isomorphic fuzzy Lie subalgebras necessarily share the same nilpotency measure. Finally, we interpret N as a fuzzy measure in the sense of Sugeno, and establish some of its basic properties, including subadditivity.

The paper is organised as follows. Section 2 recalls the necessary background on Lie algebras and fuzzy Lie subalgebras, including fuzzy bases and t-cut sets. In Section 3, we study isomorphisms of fuzzy Lie subalgebras and their connection with flags of subalgebras. Section 4 introduces nilpotent fuzzy Lie subalgebras and the fuzzy nilpotency measure N, along with its basic properties.

2. Preliminaries

A vector space $\mathfrak{g}$ over a field $\mathbb{F}$ is a Lie algebra if it is equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the Lie bracket, satisfying the following properties:

$$[x, x] = 0_{\mathfrak{g}} \text{ for all } x \in \mathfrak{g}, \tag{1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0_{\mathfrak{g}}, \tag{2}$$

for every $x, y, z \in \mathfrak{g}$.

This study focuses on finite-dimensional Lie algebras, meaning those with a finite basis as vector spaces. As illustrated by Equation (1), this bilinear map is alternating, and it is crucial to recognise that this implies the map is skew-symmetric. Indeed, we have

$$[x + y, x + y] = 0_{\mathfrak{g}}.$$

Thus, since the Lie bracket is bilinear, the previous equation and Equation (1) imply that

$$[x, y] + [y, x] = 0_{\mathfrak{g}},$$

and then
$$[x, y] = -[y, x]. \tag{3}$$
If the field $\mathbb{F}$ has a characteristic that is not 2, then the last condition on the Lie bracket is equivalent to requiring that it is alternating, since

$$[x, x] = -[x, x] \Rightarrow 2[x, x] = 0_{\mathfrak{g}} \Rightarrow [x, x] = 0_{\mathfrak{g}}.$$

Given the assumptions on the underlying field, Equation (3) can replace the initial formulation. Throughout this work, we consider Lie brackets defined over real vector spaces, that is, over the field $\mathbb{R}$. Unless stated otherwise, the Lie bracket is assumed to be skew-symmetric by definition.

The identity in Equation (2), known as the Jacobi identity, plays a central role in Lie algebra theory. It ensures the coherent behaviour of the Lie bracket operation. A detailed treatment of this identity is beyond the scope of this paper; for further reading, see [11,12].

As is common in abstract algebra, once a structure is defined, its substructures are also introduced. These are subsets that themselves satisfy the axioms of the original structure (e.g., subgroups, subrings, linear subspaces). Lie subalgebras follow this pattern: they are vector subspaces of a Lie algebra that also form Lie algebras under the same bracket operation.

The notion of a fuzzy Lie subalgebra was first introduced by Yehia in [5]. The reader will find the majority of the definitions of fuzzy Lie algebra that are to be recalled in the same paper. For our purposes, we adopt the formalism and notation from [6].

Definition 1 ([6] Definitions 1.16–1.17)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. A fuzzy set $\mu : \mathfrak{g} \to [0, 1]$ is called a fuzzy Lie subalgebra of $\mathfrak{g}$ over $\mathbb{F}$ if:*

1. $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$;
2. $\mu(ax) \geq \mu(x)$;
3. $\mu([x, y]) \geq \min\{\mu(x), \mu(y)\}$,

for all $x, y \in \mathfrak{g}$ and $a \in \mathbb{F}$.

As a consequence of the second condition, we obtain $\mu(-x) \geq \mu(x)$ and $\mu(0) \geq \mu(x)$ for all $x \in \mathfrak{g}$.

The first two conditions in Definition 1 characterise the fuzzy closure of vector addition and scalar multiplication, respectively. Together, they define a fuzzy subspace. The third condition is particularly significant, ensuring the fuzzy closure under the Lie bracket. From now on, we write $\mu \leq \mathfrak{g}$ to indicate that μ is a fuzzy Lie subalgebra of $\mathfrak{g}$. Here, we recall some basic properties about fuzzy Lie subalgebras that will be utilised in our subsequent investigations.

Proposition 1. *Let μ be a fuzzy subspace of a vector space V. Then, for all $x, y \in V$, we have:*

1. $\mu(x) = \mu(-x)$;
2. *If* $\mu(x - y) = \mu(0)$, *then* $\mu(x) = \mu(y)$;
3. *If* $\mu(x) < \mu(y)$, *then* $\mu(x - y) = \mu(x) = \mu(y - x)$.

We now recall that, if $\mu : \mathfrak{g} \to [0, 1]$ is a fuzzy subset of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$, the (crisp) set
$$U(\mu, t) = \{x \in \mathfrak{g} \mid \mu(x) \geq t\}$$
is the *t-cut set* of μ for every $t \in [0, 1]$.

The final part of this section introduces definitions and results that are essential for our study. We begin with a key concept from [13].

Definition 2 ([13] Definition 3.1). *Let V be a fuzzy vector space over a field $\mathbb{F}$. A set of vectors $\mathcal{B} = \{e_1, \dots, e_n\}$ is said to be fuzzy linearly independent if:*

1. $\{e_1, \dots, e_n\}$ *is linearly independent;*
2. $\mu(\sum_{i=1}^{n} a_i e_i) = \bigwedge_{i=1}^{n} \mu(a_i e_i)$,

for all $a_i \in \mathbb{F}$, $1 \leq i \leq n$.

Definition 3 ([13] Definition 4.1). *A fuzzy basis of a fuzzy vector space V is a fuzzy linearly independent set that spans V.*

In the classical setting, a basis of a vector space is a maximal set of linearly independent vectors that spans the space. Analogously, a fuzzy basis is a spanning set of fuzzy linearly independent vectors.

Any finite basis of a vector space V is trivially a fuzzy basis of V. Indeed, if $\{e_1, \dots, e_n\}$ is a crisp basis, then for any scalars $a_i \in \mathbb{F}$, we have $a_i e_i \in V$ and $\sum_{i=1}^{n} a_i e_i \in V$. Therefore, $\mu(a_i e_i) = 1$ and $\mu(\sum_{i=1}^{n} a_i e_i) = 1$, satisfying condition (2) in Definition 2.

We now recall some properties of fuzzy Lie algebras from [7].

Proposition 2 ([7] Proposition 2). *Let $\mathfrak{g}$ be a real Lie algebra and let $\mu \leq \mathfrak{g}$. Then:*

1. $\mu(0_{\mathfrak{g}}) = \sup\{\mu(x) \mid x \in \mathfrak{g}\}$;
2. $\mu(ax) = \mu(x)$, *for all $a \in \mathbb{R}^*$ and $x \in \mathfrak{g}$;*
3. *If $\mu(x) \neq \mu(y)$ for $x, y \in \mathfrak{g}$, then*

$$\mu(x + y) = \mu(x) \wedge \mu(y).$$

We now extend the concept of fuzzy bases to Lie algebras. In classical Lie theory, the basis of a Lie algebra is simply a basis of the underlying vector space. The same idea applies in the fuzzy context.

In this fuzzy setting, the notion of idealof a Lie algebra, i.e., a subalgebra I of $\mathfrak{g}$ such that $[I, \mathfrak{g}] \subseteq I$ has a fuzzy counterpart.

Definition 4 ([6] Definition 1.18). *A fuzzy set $\mu : \mathfrak{g} \to [0, 1]$ is called a fuzzy Lie ideal of $\mathfrak{g}$ if:*

1. $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$;
2. $\mu(\alpha x) \geq \mu(x)$;
3. $\mu([x, y]) \geq \mu(x)$

for all $x, y \in \mathfrak{g}$, $\alpha \in \mathbb{F}$.

After this definition, we have to remember this important tool in the theory of fuzzy Lie algebra.

The following theorem is a particular case of Proposition 1 of [10].

Theorem 1. *A fuzzy set μ of a Lie algebra $\mathfrak{g}$ is a fuzzy Lie subalgebra of $\mathfrak{g}$ if and only if each nonempty set $U(\mu, t)$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof. Let μ be a fuzzy subalgebra of a Lie algebra $\mathfrak{g}$ and let $x, y \in U(\mu, t)$, with $t \in [0, 1]$. Then $\mu(x), \mu(y) \geq t$,

$$\mu(x + y) \geq \min\{\mu(x), \mu(y)\} \geq \min\{t, t\} = t,$$

$$\mu(\alpha x) \geq \mu(x) \geq t$$

and

$$\mu([x,y]) \geq \min\{\mu(x),\mu(y)\} \geq \min\{t,t\} = t$$

for every $\alpha \in \mathbb{F}$. As a consequence, $x+y, \alpha x, [x,y] \in U(\mu,t)$. Moreover,

$$\mu(0) \geq \mu(x) \geq t$$

for every $x \in U(\mu,t)$ and hence $0 \in U(\mu,t)$ and $U(\mu,t)$ is a Lie subalgebra of $\mathfrak{g}$.

Conversely, suppose that $U(\mu,t)$ is a Lie subalgebra of $\mathfrak{g}$ for every $t \in [0,1]$, let $x,y \in \mathfrak{g}$ and consider $t = \min\{\mu(x),\mu(y)\}$. Then, since $\mu(x) \geq t$ and $\mu(y) \geq t$, we have $x,y \in U(\mu,t)$ and hence $x+y, [x,y] \in U(\mu,t)$. As a consequence,

$$\mu(x+y) \geq t = \min\{\mu(x),\mu(y)\}$$

and

$$\mu([x,y]) \geq t = \min\{\mu(x),\mu(y)\}.$$

Moreover, since $x \in U(\mu,\mu(x))$, we have $\alpha x \in U(\mu,\mu(x))$ and hence $\mu(\alpha x) \geq \mu(x)$. □

More precisely, this result extends to a broad class of algebras. Statements of the form "a subset A has property P" are classified by Kondo and Dudek [14] as type 0. They established a transfer principle that allows one to lift such properties from classical algebraic structures to their fuzzy counterparts, and vice versa. As they observed, this principle also suggests that genuinely new results in fuzzy algebra should go beyond what can be directly inferred from it. For further details, we refer the reader to the paper cited above and the references therein.

We conclude this section with a result from [7] that provides a practical criterion for deciding when, under a particular choice of the membership function, a basis of a Lie algebra, is also a fuzzy basis. Here we recall that a set $\mathcal{B} = \{e_1,\ldots,e_n\}$ is a fuzzy basis for a real vector space V if the elements of $\mathcal{B}$ are fuzzy linearly independent elements, i.e.,

$$\mu\left(\sum_{i=1}^{n} a_i e_i\right) = \bigwedge_{i=1}^{n} \mu(a_i e_i)$$

for all $a_1,\ldots,a_n \in \mathbb{R}$.

Theorem 2. *Let $\mu \leq \mathfrak{g}$ and $\mathcal{B} = \{e_1,\ldots,e_n\}$ be a basis of L with $\mu(e_1) = a$ and $\mu(e_i) = b$, with $1 < i \leq n$ and $a \geq b$. The set $\mathcal{B}$ is a fuzzy basis of μ if and only if*

$$\mu(x) = \min_{i\in\{1,\ldots,n\}} \mu(e_i) = b, \tag{4}$$

for every $x \in \mathfrak{g} \setminus \operatorname{span}\{e_1\}$.

Proof. Let $\mathcal{B} = \{e_1,\ldots,e_n\}$ be a basis of L with $\mu(e_1) = a$ and $\mu(e_i) = b$, with $1 < i \leq n$ and $a \geq b$. For every $x \in \mathfrak{g} \setminus \operatorname{span}\{e_1\}$, we put $x = \sum_{i=1}^{n} x_i e_i$, with $x_i \in \mathbb{F}$ and with $x_j \in \mathbb{F}^*$ for some $j \in \{2,\ldots,n\}$. By Theorem 4 of [7], we have that $\mathcal{B}$ is a fuzzy basis of μ if and only if $\min_{i\in\{1,\ldots,n\}} \mu(e_i) \geq \mu(x)$, for any $x \in \mathfrak{g} \setminus \{0_{\mathfrak{g}}\}$. Moreover, by Definition 1, we have $\min_{i\in\{1,\ldots,n\}} \mu(e_i) \geq \mu(x)$ for any $x \in \mathfrak{g} \setminus \{0_{\mathfrak{g}}\}$ if and only if

$$b = \min_{i\in\{1,\ldots,n\}} \mu(e_i) \geq \mu(x) = \mu(\sum_{i=1}^{n} x_i e_i) \geq \min_{i=1,\ldots,n} \mu(x_i e_i) \geq \min_{i=1,\ldots,n} \mu(e_i) = \min(a,b) = b,$$

where the equality $\min_{i=1,\dots,n} \mu(e_i) = \min(a,b)$ follows from the fact that $x \in \mathfrak{g} \setminus \text{span}\{e_1\}$. □

3. Isomorphisms of Fuzzy Lie Subalgebras

In this section, we propose a definition of isomorphism between fuzzy Lie subalgebras based on the transfer principle via t-cut sets, and we compare it with the categorical notion introduced in [10].

For our purposes, we only need the notion of a filtration on an object in a category and, more importantly, that of a morphism of filtered objects, which will serve as the foundation for our definition of isomorphism between fuzzy Lie subalgebras. These concepts were originally introduced by Deligne [15] in the setting of abelian categories, but since we do not require any of the additional structure that such categories provide, the definitions make sense in any category, and in particular in the category Lie of Lie algebras.

Let C be a category and let A be an object of C. A filtration $\mathcal{F}$ on A is a family of subobjects $(\mathcal{F}^n A)_{n\in\mathbb{Z}}$ such that

$$n \leq m \ \Rightarrow \ \mathcal{F}^n A \subset \mathcal{F}^m A.$$

The pair $(A, \mathcal{F})$ is called a filtered object, and the filtration is said to be finite if there exist integers $n, m \in \mathbb{Z}$ such that $\mathcal{F}^n A = A$ and $\mathcal{F}^m A = 0$. A morphism of filtered objects $f : (A, \mathcal{F}) \to (B, \mathcal{F})$ is a morphism $f : A \to B$ in C such that

$$f(\mathcal{F}^n A) \subset \mathcal{F}^n B \quad \text{for all } n \in \mathbb{Z}.$$

Let now $\mathfrak{g}$ be a Lie algebra and let μ be a fuzzy Lie subalgebra of $\mathfrak{g}$. Clearly, if $t_1 < t_2$, then $U(\mu, t_1) \supset U(\mu, t_2)$. Hence, given a partition of the interval $[0,1]$, the corresponding family of t-cut sets defines an increasing filtration of $\mathfrak{g}$ (for more results on t-cuts set and partitions, we refer the reader to [7]). Under these assumptions, and in view of the definitions above, we introduce the notion of isomorphism of fuzzy Lie subalgebras as follows.

Definition 5. *Let μ, ν be two fuzzy Lie subalgebras of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We say that μ is isomorphic to ν, denoted by $\mu \cong \nu$, if for every $t \in [0,1]$ there exists an isomorphism of Lie algebras $f_t : U(\mu, t) \to U(\nu, t)$.*

In the previous definition, it follows straightforwardly that the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic via f_0. In fact, since $U(\mu, 0) = \mathfrak{g}$ and $U(\nu, 0) = \mathfrak{h}$, if $\mu \cong \nu$ then $f_0 : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism of Lie algebras. We point this out in the following corollary.

Corollary 1. *If $\mu \cong \nu$, then $\mathfrak{g} \cong \mathfrak{h}$.*

Thanks to the previous result, from now on, we may, without loss of generality, restrict our attention to isomorphisms of fuzzy Lie subalgebras of the same Lie algebra.

Regarding Definition 5, one might expect that it is sufficient to consider an automorphism f of $\mathfrak{g}$ and define $f_t := f_{|U(\mu,t)}$ for every $t \in [0,1]$. However, this is not the case. Indeed, in the example below, we show that there exists an automorphism f of a certain Lie algebra $\mathfrak{g}$ whose restriction $f_{|U(\mu,0.8)}$ is such that $f_{|U(\mu,0.8)}(U(\mu, 0.8)) \neq U(\nu, 0.8)$.

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Example 1. *Let $\mathfrak{g} = \langle E_1, E_2, E_3, E_4 \rangle$ be the 4-dimensional Lie algebra generated by the matrices*

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The only nontrivial bracket between the generators is $[E_1, E_2] = E_1$. Consider the ideal $I = \langle E_1, E_2 \rangle$. This ideal is not characteristic (see [16]), meaning that there exists an automorphism f of $\mathfrak{g}$ such that $f(I) \neq I$. For instance, define f by $f(E_2) = E_2 + E_3$ and $f(E_i) = E_i$ for $i \in \{1, 3, 4\}$. Then $f(I) \neq I$. Now let μ be a fuzzy Lie subalgebra of $\mathfrak{g}$ defined by $\mu(0_{\mathfrak{g}}) = 0.9$, $\mu(E_1) = \mu(E_2) = 0.8$, and $\mu(E_3) = \mu(E_4) = 0.6$, and let $\nu = \mu$. Then, for $t = 0.8$, we have $U(\mu, 0.8) = I$, but $f(U(\mu, 0.8)) = f(I) \neq I = U(\nu, 0.8)$. Hence, the restriction $f_{|U(\mu,0.8)} : U(\mu, 0.8) \to \mathfrak{g}$ is not an isomorphism of Lie algebras between the 0.8-sets $U(\mu, 0.8)$ and $U(\nu, 0.8)$.

Proposition 3. *Let μ, ν be two fuzzy Lie subalgebras of $\mathfrak{g}$ and let f be an automorphism of $\mathfrak{g}$ such that $\mu(x) = \nu(f(x))$. Then, μ is isomorphic to ν.*

Proof. We have $f(U(\mu, t)) \subseteq U(\nu, t)$ for every $t \in [0, 1]$. Indeed, suppose that $x \in U(\mu, t)$. Then $\mu(x) \geq t$ and hence

$$\nu(f(x)) = \mu(x) \geq t.$$

Moreover, let $y \in U(\nu, t)$. Then $f^{-1}(y) \in U(\mu, t)$, indeed

$$\mu(f^{-1}(y)) = \nu(f(f^{-1}(y))) = \nu(y) \geq t$$

and $f(f^{-1}(y)) = y$. □

We recall now the notion of isomorphism between fuzzy Lie subalgebras of a Lie algebra $\mathfrak{g}$ presented in [10]. Let Lie^F be the category whose objects are the pairs $(\mathfrak{g}, \mu)$, where $\mathfrak{g}$ is a Lie algebra, and μ is a fuzzy subalgebra of $\mathfrak{g}$, and whose morphisms $f : (\mathfrak{g}, \mu) \to (\mathfrak{h}, \nu)$ are homomorphisms of Lie algebras $f \colon \mathfrak{g} \to \mathfrak{h}$ such that $\mu(x) \leq \nu(f(x))$. We observe that an isomorphism $f \colon (\mathfrak{g}, h) \to (\mathfrak{h}, \nu)$ in Lie^F is an isomorphism of Lie algebras $f \colon \mathfrak{g} \to \mathfrak{h}$ such that $\mu(x) = \nu(f(x))$ and $\nu(y) = \mu(f^{-1}(y))$ for any $x \in X$ and $y \in Y$.

Proposition 4. *Let $f : \mathfrak{g} \to \mathfrak{h}$ be an isomorphism of Lie algebras and let μ be a fuzzy Lie subalgebra of $\mathfrak{g}$. Then there exists a fuzzy Lie subalgebra ν of $\mathfrak{h}$ such that $(\mathfrak{g}, \mu)$ and $(\mathfrak{h}, \nu)$ are isomorphic in Lie^F.*

Proof. Since every $y \in \mathfrak{h}$ is given by $y = f(x)$, with $x \in \mathfrak{g}$, hence the statement is proven by defining $\nu(y) := \mu(x)$. □

Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{B}$ be a basis of $\mathfrak{g}$. Now, consider $\mathcal{B}'$ another basis of $\mathfrak{g}$. Thus, by Theorem 2 and Proposition 4, it follows that $\mathcal{B}'$ is a fuzzy basis of $\mathfrak{g}$ as in Theorem 2.

Definition 6. *Let $\mathfrak{g}$ be a Lie algebra. We say that $\mathfrak{g}$ has a flag of subalgebras if there is a chain*

$$0 = \mathfrak{g}_n < \mathfrak{g}_{n-1} < \ldots < \mathfrak{g}_0 = \mathfrak{g}$$

where $\mathfrak{g}_i$ is a $(n - i)$-dimensional subalgebra of $\mathfrak{g}$ for $0 \leq i \leq n$.

In other words, a flag of subalgebras is a finite filtered object in the category Lie of Lie algebras. A huge class of Lie algebras that admit a flag of subalgebras is the class of solvable Lie algebras [17] (Theorem 2.7). The following result shows that, given a flag of subalgebras, it is always possible to construct a fuzzy Lie subalgebra whose associated family of t-cut sets recovers exactly the original flag.

Proposition 5. *Let $\mathfrak{g}$ be a Lie algebra that has a flag of subalgebras*

$$0 = \mathfrak{g}_n < \mathfrak{g}_{n-1} < \ldots < \mathfrak{g}_0 = \mathfrak{g}.$$

Then there exists a fuzzy subalgebras $\mu : \mathfrak{g} \to [0,1]$ of $\mathfrak{g}$ such that there exists $t_0, t_1, \ldots, t_n \in [0,1]$ such that $U(\mu, t_i) = \mathfrak{g}_i$.

Proof. Let $\mu : \mathfrak{g} \to [0,1]$ be the fuzzy subset defined as

$$\mu(x) = \begin{cases} \frac{i}{n}, & \text{if } x \in \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}, \\ 1, & \text{if } x \in \mathfrak{g}_n, \end{cases}$$

for every $x \in \mathfrak{g}$ and $i \in \{0, \ldots, n-1\}$. For every $i \in \{1, \ldots, n-1\}$, let $t_i \in [0,1]$ be such that $\frac{i-1}{n} < t_i \leq \frac{i}{n}$. Then

$$\begin{aligned} U(\mu, t_i) &= \{x \in \mathfrak{g} \mid \mu(x) \geq t_i\} \\ &= \{x \in \mathfrak{g} \mid \mu(x) \geq \frac{i}{n}\} \\ &= \mathfrak{g}_i. \end{aligned}$$

Moreover, if $t_0 = 0$, we have $U(\mu, t_0) = \{x \in \mathfrak{g} \mid \mu(x) \geq 0\} = \mathfrak{g}$. As a consequence, for every $t \in [0,1]$, $U(\mu, t)$ is a subalgebra of $\mathfrak{g}$, and hence $\mu \colon \mathfrak{g} \to [0,1]$ is a fuzzy subalgebra of $\mathfrak{g}$. □

Remark 1. *Let μ be a fuzzy subalgebra of a Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}'$ be a subalgebra of $\mathfrak{g}$. Then the fuzzy subset $\mu' \colon \mathfrak{g}' \to [0,1]$ given by the restriction of μ to $\mathfrak{g}'$ is a fuzzy subalgebra of $\mathfrak{g}'$. Indeed, for every $x, y \in \mathfrak{g}'$,*

$$\mu'(x+y) = \mu(x+y) \geq \mu(x) \wedge \mu(y) = \mu'(x) \wedge \mu'(y),$$

$$\mu'(\alpha x) = \mu(\alpha x) = \mu(x) = \mu'(x)$$

and

$$\mu'([x,y]) = \mu([x,y]) \geq \mu(x) \wedge \mu(y) = \mu'(x) \wedge \mu'(y).$$

Moreover, the inclusion $\iota \colon \mathfrak{g}' \to \mathfrak{g}$ of $\mathfrak{g}'$ in $\mathfrak{g}$ is such that $\mu(x) \leq \mu'(\iota(x))$.

Consequently, given a Lie algebra $\mathfrak{g}$ that has a flag of subalgebras $0 = \mathfrak{g}_n < \mathfrak{g}_{n-1} < \ldots < \mathfrak{g}_0 = \mathfrak{g}$, there exists, for every $0 \leq i \leq n$, a fuzzy subalgebra $\mu_i : \mathfrak{g}_i \to [0,1]$ of $\mathfrak{g}_i$. Moreover, the inclusion $\iota \colon \mathfrak{g}_i \to \mathfrak{g}_{i-1}$ is such that $\mu_i(x) \leq \mu_{i-1}(\iota(x))$, for every $x \in \mathfrak{g}_i$.

Then, for every $i \in \{0, \ldots, n\}$, the map $\iota \colon (\mathfrak{g}_i, \mu_i) \to (\mathfrak{g}_{i-1}, \mu_{i-1})$ given by the inclusion $\iota \colon \mathfrak{g}_i \to \mathfrak{g}_{i-1}$ of $\mathfrak{g}_i$ in $\mathfrak{g}_{i-1}$ is a morphism in Lie^F.

Proposition 6. *If a Lie algebra $\mathfrak{g}$ has a flag of subalgebras, then every epimorphic image of $\mathfrak{g}$ also has a flag of subalgebras.*

In particular, if $\mathfrak{g}$ is a Lie algebra with a flag of subalgebras

$$0 = \mathfrak{g}_n < \mathfrak{g}_{n-1} < \ldots < \mathfrak{g}_0 = \mathfrak{g}$$

and $f\colon \mathfrak{g} \to \mathfrak{h}$ is a surjective homomorphism of Lie algebras, then the chain

$$0 \subseteq \frac{\mathfrak{g}_{n-1} + K}{K} \subseteq \frac{\mathfrak{g}_{n-2} + K}{K} \subseteq \cdots \subseteq \frac{\mathfrak{g} + K}{K} \cong \mathfrak{h} \tag{5}$$

is a flag of subalgebras of $\mathfrak{h}$, where $K = \ker(f)$. Now, let $\mu\colon \mathfrak{g} \to [0,1]$ the fuzzy subalgebra associated with $\mathfrak{g}$ by Proposition 5 and let $\nu\colon \frac{\mathfrak{g}}{K} \to [0,1]$ be the fuzzy subset of $\mathfrak{h}$ defined as

$$\nu(x + K) = \mu(x)$$

for every $x \in \mathfrak{g}$. Then, ν is the fuzzy subalgebra associated with the flag (5) by Proposition 5. Indeed, let $x + K \in \mathfrak{h}_i \setminus \mathfrak{h}_{i+1} = \frac{\mathfrak{g}_i + K}{K} \setminus \frac{\mathfrak{g}_{i+1} + K}{K}$. Then

$$\nu(x + K) = \mu(x) = \frac{i}{n}.$$

4. Nilpotent Fuzzy Lie Subalgebras

In this section, we address the following question: Can nilpotency be formulated in the context of fuzzy algebra? We focus on nilpotency as a first step toward introducing a suitable measure of the degree to which a given property is satisfied by a class of fuzzy algebraic structures (for instance, nilpotency in the case of fuzzy Lie subalgebras). We provide two answers to this question. The first is more classical: we introduce a central series associated with a fuzzy subalgebra of a Lie algebra and define nilpotency by requiring that this series eventually stabilizes. This approach is inspired by the work of [9,18], where nilpotent subgroups were extended to the fuzzy setting; we propose here a similar generalisation for Lie algebras by introducing the notion of *nilpotent fuzzy Lie subalgebras*. The second approach takes into account the intrinsic vagueness of the fuzzy framework and proposes a way to measure the degree of nilpotency of a fuzzy Lie subalgebra. We note that a substantial part of this section is based on [8], which is here extended and complemented by the study of isomorphisms introduced in Section 3.

For any $x, y_1, y_2, \ldots, y_k \in \mathfrak{g}$, we denote by

$$[y_1, y_2, \ldots, y_k, x] := [y_1, [y_2, [\ldots, [y_k, x]] \ldots]]$$

the iterated Lie bracket. Recall that a Lie algebra $\mathfrak{g}$ is said to be nilpotent if there exists $k \in \mathbb{N}$ such that

$$[y_1, y_2, \ldots, y_k, x] = 0 \quad \text{for all } x, y_1, \ldots, y_k \in \mathfrak{g}. \tag{6}$$

This condition is equivalent to the vanishing of the lower central series of $\mathfrak{g}$, and hence it characterises the nilpotency of $\mathfrak{g}$. Here, we recall that the lower central series of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^i \supseteq \mathfrak{g}^{i+1} \supseteq \ldots,$$

where $\mathfrak{g}^i$ denotes $[\mathfrak{g}, \mathfrak{g}^{i-1}]$, for every $i > 1$. We have that $\mathfrak{g}$ is nilpotent if and only if there exists a natural number k such that $\mathfrak{g}^k = \{0_{\mathfrak{g}}\}$. The smallest such integer is called the nil-index of $\mathfrak{g}$.

Let $\mu \leq \mathfrak{g}$ be a fuzzy Lie subalgebra. We define the ascending central series associated to μ inductively as follows: we put

$$Z_0(\mu) = \{0_{\mathfrak{g}}\},$$
$$Z_1(\mu) = \{x \in \mathfrak{g} \mid \ \mu([y,x]) = \mu(0_{\mathfrak{g}}) \text{ for any } y \in \mathfrak{g}\}$$

and, inductively, for every $k \geq 2$,

$$Z_k(\mu) = \{x \in \mathfrak{g} \mid \mu([y_1, y_2, \dots, y_k, x]) = \mu(0_{\mathfrak{g}}), \text{for any } y_1, \dots, y_k \in \mathfrak{g}\}.$$

It is straightforward to verify that the sequence $(Z_k(\mu))_{k\geq 0}$ is increasing, i.e., $Z_k(\mu) \subseteq Z_{k+1}(\mu)$, for all $k \geq 0$.

Definition 7. *A fuzzy Lie subalgebra μ of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is called a nilpotent fuzzy Lie subalgebra of $\mathfrak{g}$ if there exists $k \geq 0$ such that $Z_k(\mu) = \mathfrak{g}$. The smallest k such that $Z_k(\mu) = \mathfrak{g}$ is called fuzzy nil-index of μ.*

As one would expect, for a nilpotent Lie algebra, fuzzy nilpotency is trivially necessary, as the following result demonstrates.

Proposition 7. *If μ is a fuzzy Lie subalgebra of a nilpotent Lie algebra $\mathfrak{g}$, then μ is a fuzzy nilpotent Lie subalgebra. Moreover, the fuzzy nil-index of μ is at most the nil-index of $\mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ is nilpotent, there exists an integer $k \geq 1$ such that $\mathfrak{g}^k = \{0_{\mathfrak{g}}\}$, with $\mathfrak{g}^{k-1} \neq \{0_{\mathfrak{g}}\}$. Hence, for every $y_1, \dots, y_k \in \mathfrak{g}$, $[y_1, \dots, y_k] = 0_{\mathfrak{g}}$. Then, for every $x \in \mathfrak{g}$, $[y_1, \dots, y_{k-1}, x] = 0_{\mathfrak{g}}$. This implies that $\mu([y_1, \dots, y_{k-1}, x]) = \mu(0_{\mathfrak{g}})$, and statement is proved because $Z_k(\mu) = \mathfrak{g}$, whereas it may happen $Z_j(\mu) = \mathfrak{g}$, with $j < k$. □

Example 2. *Let $\mathfrak{g}$ be the real Lie algebra of dimension 6 with basis $\{e_1, \dots, e_6\}$, and brackets*

$$[e_1, e_2] = e_4, \ [e_1, e_3] = e_5, \ [e_2, e_4] = e_6, \ [e_3, e_5] = -e_6.$$

This algebra is nilpotent with nil-index 3, indeed $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \langle e_4, e_5, e_6\rangle$, $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] = \langle e_6\rangle$ and $\mathfrak{g}^3 = [\mathfrak{g}, \mathfrak{g}^2] = \langle 0_{\mathfrak{g}}\rangle$. However, the fuzzy subalgebra μ defined by

$$\mu(x) = \begin{cases} 0.6 & \text{if } x = 0_{\mathfrak{g}}, \ e_6, \\ 0.5 & \text{if } x = e_i, \text{ for } i = 1, \dots, 5, \end{cases}$$

has fuzzy nil-index 2. To see this, let us find $Z_1(\mu)$ and $Z_2(\mu)$. Since $\mu([e_1, e_2]) = \mu([e_1, e_3]) = 0.5$, we obtain that $e_1, e_2, e_3 \notin Z_1(\mathfrak{g})$. Moreover, we have that $e_4, e_5, e_6 \in Z_1(\mathfrak{g})$ as the only non-zero brackets $[y, e_i]$ with $i = 4, 5, 6$ are $[e_2, e_4] = e_6$ and $[e_3, e_5] = -e_6$ but $\mu(e_6) = 0.5 = \mu(0_{\mathfrak{g}})$. Hence, $Z_1(\mu) = \langle e_4, e_5, e_6\rangle$. Now we find

$$Z_2(\mu) = \{x \in \mathfrak{g} \mid \mu([y_1, y_2, x]) = \mu(0_{\mathfrak{g}}), \text{ for any } y_1, y_2 \in \mathfrak{g}\}.$$

Let x be any element in $\mathfrak{g}$. Then, $[y_1, y_2, x] = [y_1, [y_2, x]] = [y_1, \alpha e_4 + \beta e_5 + \gamma e_6]$, for some $\alpha, \beta, \gamma \in \mathbb{R}$, since $[y_2, x] \in \mathfrak{g}^1$. However, $[y_1, \alpha e_4 + \beta e_5 + \gamma e_6] = \alpha[y_1, e_4] + \beta[y_1, e_5] + \gamma[y_1, e_6] = \delta e_6$, for some $\delta \in \mathbb{R}$, for the same reason as before. Hence $\mu([y_1, y_2, x]) = \mu(\delta e_6) \geq \mu(e_6) = \mu(0_{\mathfrak{g}})$, but $\mu(0_{\mathfrak{g}}) = \sup\{\mu(y) \mid y \in \mathfrak{g}\}$ (Proposition 2), so $\mu([y_1, y_2, x]) = \mu(0_{\mathfrak{g}})$ for any $y_1, y_2 \in \mathfrak{g}$. Therefore, $Z_2(\mu) = \mathfrak{g}$.

In conclusion, we have constructed an example of a fuzzy nilpotent Lie subalgebra μ with fuzzy nil-index 2 on a nilpotent Lie algebra $\mathfrak{g}$ of nil-index 3.

We show by the following example that the converse does not hold. More precisely, the example shows that it is possible to define a fuzzy nilpotent subalgebra μ on a non-nilpotent Lie algebra $\mathfrak{g}$.

Example 3. *Let $\mathfrak{g}$ be the unique 2-dimensional non-abelian Lie algebra over the field $\mathbb{R}$, with basis the set $\{e_1, e_2\}$, and brackets given by $[e_1, e_2] = e_1$. It is well-known that $\mathfrak{g}$ is not nilpotent. Indeed, $\mathfrak{g}^1 = \langle e_1 \rangle$ and*

$$\mathfrak{g}^k = \langle e_1 \rangle \neq \{0_{\mathfrak{g}}\}$$

for every $k \geq 2$. Let us define a fuzzy subalgebra $\mu \leq \mathfrak{g}$ with the following membership function:

$$\mu(x) = \begin{cases} 0.5 & \text{if } x = 0_{\mathfrak{g}} \vee x = \alpha e_1,\ \alpha \in \mathbb{R}, \\ 0.1 & \text{otherwise.} \end{cases}$$

We want to check whether μ is fuzzy nilpotent and, if it is the case, to compute the fuzzy nil-index of μ. In order to do this, we need to find the ascending central series associated with μ, and see if and when $Z_k(\mu) = \mathfrak{g}$. Of course, by definition, $Z_0(\mu) = \{0_{\mathfrak{g}}\}$. The first non-trivial set of the series is the subset

$$Z_1(\mu) = \{x \in \mathfrak{g} \mid \ \mu([y, x]) = \mu(0_{\mathfrak{g}}) \text{ for any } y \in \mathfrak{g}\}.$$

Let $x = x_1 e_1 + x_2 e_2$ and $y = y_1 e_1 + y_2 e_2$ be two generic elements of $\mathfrak{g}$. Let us compute $[y, x]$. We follows that

$$\begin{aligned} [y, x] &= [y_1 e_1 + y_2 e_2, x_1 e_1 + x_2 e_2] \\ &= y_1 x_2 [e_1, e_2] + y_2 x_1 [e_2, e_1] \\ &= y_1 x_2 [e_1, e_2] - y_2 x_1 [e_1, e_2] \\ &= (y_1 x_2 - y_2 x_1)[e_1, e_2] \\ &= (y_1 x_2 - y_2 x_1) e_1. \end{aligned}$$

Hence, for every $x, y \in \mathfrak{g}$, it holds that $\mu([y, x]) = \mu(\alpha e_1) = \mu(0_{\mathfrak{g}})$. This means that $Z_1(\mu) = \mathfrak{g}$, thus μ is fuzzy nilpotent with fuzzy nil-index equal to 1 whereas $\mathfrak{g}$ is not even nilpotent.

We now want to introduce a sort of fuzzy measure for the nilpotency of a fuzzy Lie algebra. In particular, we are looking for a function $N : \mathcal{V}(\mu, \mathfrak{g}) \to [0, 1]$, where $\mathcal{V}(\mu, \mathfrak{g}) = \{\mu : \mathfrak{g} \to [0, 1] \mid \mu \text{ non-constant fuzzy Lie subalgebra of } \mathfrak{g}\}$, such that $N(\mu) = 1$ when μ is a nilpotent fuzzy Lie subalgebra of $\mathfrak{g}$, and $N(\mu) \in [0, 1]$ otherwise.
To this end, we set $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$. By definition, the following inequality holds for every $x, y \in \mathfrak{g}$

$$\mu([x, y]) \geq \mu(x) \wedge \mu(y).$$

In the next term of the series, $\mathfrak{g}^2 := [\mathfrak{g}, \mathfrak{g}^1]$ this inequality extends as follows:

$$\begin{aligned} \mu([x, [y, z]]) &\geq \mu(x) \wedge \mu([y, z]) \\ &= \mu(x) \wedge (\mu(y) \wedge \mu(z)) \\ &= \mu(x) \wedge \mu(y) \wedge \mu(z). \end{aligned}$$

More generally, for each for $k \geq 3$, we define recursively $\mathfrak{g}^k := [\mathfrak{g}, \mathfrak{g}^{k-1}]$. Then, for any $x_1, \ldots, x_k \in \mathfrak{g}$, we have

$$\mu([x_1, \ldots, x_{k-1}, x_k]) \geq \bigwedge_{i=1}^{k} \mu(x_i).$$

To emphasize this, we consider the minimum of the possible membership degrees for a k-tuple of elements, that is $\bigwedge_{x_i \in \mathfrak{g}} \mu([x_1, x_2, ..., x_k])$. If $\mathfrak{g}$ is nilpotent, then there exists $k \geq 1$, called *the nil-index* of $\mathfrak{g}$, such that $\mathfrak{g}^{k-1} \neq \{0_{\mathfrak{g}}\}$ and $\mathfrak{g}^k = \{0_{\mathfrak{g}}\}$. We observe that if $\mathfrak{g}$ is nilpotent with nil-index k, then every fuzzy Lie subalgebra μ of $\mathfrak{g}$ is nilpotent with nil-index k. To quantify the extent to which a fuzzy Lie subalgebra approaches nilpotency, we define a fuzzy measure of nilpotency. This measure reflects how the membership values of higher-order commutators behave in the fuzzy setting. Formally, we put

$$N(\mu) = \lim_{k \to \infty} \frac{\bigwedge_{x_i \in \mathfrak{g}} \mu([x_1, x_2, ..., x_k])}{\mu(0_{\mathfrak{g}})}. \tag{7}$$

It is clear that, since $\mu(x) \leq \mu(0_{\mathfrak{g}})$ for every $x \in \mathfrak{g}$, then $N(\mu) \in [0, 1]$ and, in particular, if $\mathfrak{g}$ is nilpotent then $N(\mu) = 1$ for every fuzzy Lie subalgebra μ of $\mathfrak{g}$. In this formula, the numerator takes the minimum membership degree over all iterated Lie brackets of length k for two reasons. First, since $\mu(0_{\mathfrak{g}}) \geq \mu(x)$ for all $x \in \mathfrak{g}$, taking the maximum would trivially yield $N(\mu) = 1$, which would defeat the purpose. Second, we want to capture the worst-case behaviour of iterated brackets, that is, how far they can be from the zero vector in terms of membership degree. This is then normalised by $\mu(0_{\mathfrak{g}})$, the membership degree of the zero vector, which serves as a natural reference value. The only reason for considering the limit in Equation (7) as $k \to \infty$ is to theoretically extend this measure to infinite-dimensional Lie algebras, and moreover, because the ascending central series does not necessarily terminate a priori.

Remark 2. *We briefly explain why, at least in the present manuscript, we have not introduced an analogue of the measure N for fuzzy subgroups. In the Lie algebra setting, nilpotency admits two equivalent characterisations: via the ascending central series terminating at the trivial subalgebra, and via the condition expressed in Equation (6). The same equivalence does not hold in the group-theoretic setting. Indeed, for groups, the condition in Equation (6), where the bracket $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$, characterises the class of k-Engel groups, which coincides with the class of nilpotent groups only in the finite case. As soon as one moves to finitely generated groups, this equivalence breaks down. Since our measure N is built precisely on the condition in Equation (6), extending it to the group setting would capture Engel-type behaviour rather than nilpotency proper, making the analogy less meaningful. For a thorough treatment of nilpotent groups and Engel groups, we refer the reader to [19–21].*

Conversely, even if $\mathfrak{g}$ is not nilpotent, a fuzzy Lie subalgebra μ may still satisfy $N(\mu) = 1$. In particular, we require that every $\mu \in \mathcal{V}(\mu, \mathfrak{g})$ has to be not constant since if $\mu(x) = m \in]0, 1[$ for every $x \in \mathfrak{g}$, then $N(\mu) = 1$. Moreover, if μ is a nilpotent fuzzy Lie subalgebra of $\mathfrak{g}$, then $N(\mu) = 1$.

Now we present an example of a non-nilpotent Lie algebra endowed with a fuzzy non-nilpotent Lie subalgebra μ, and we show that $N(\mu) < 1$.

Example 4. *This example is Example 1.3 in [6]. Let $\mathfrak{g} = \mathbb{R}^3$ the 3–dimensional real vector space with Lie bracket $[x, y] = x \times y$, where $x, y \in \mathbb{R}^3$, that is the classical cross product. Then $\mathfrak{g} = (\mathbb{R}^3, \times)$ is a real Lie algebra. We define a fuzzy set μ on $\mathfrak{g}$ by*

$$\mu(x) = \begin{cases} 0.9 & \text{if } x = (0, 0, 0), \\ 0.6 & \text{if } x = (a, 0, 0), a \neq 0, \\ 0.2 & \text{otherwise}. \end{cases}$$

One can easily check that μ is a fuzzy Lie subalgebra. With the values above, we have $\mu(e_1) = 0.6$ and $\mu(e_2) = \mu(e_3) = 0.2$, then $\min_{e_i \in \mathcal{B}} = 0.2 = \min \mu(x)$, for every $x \in \mathbb{R}^3 \setminus \{0_{\mathbb{R}^3}\}$. Moreover, using Theorem 2, it is straightforward to see that $\mathcal{B} = \{e_1, e_2, e_3\}$ is a fuzzy basis of μ.

Now, $\mathfrak{g}$ is not a nilpotent Lie algebra. Indeed, the non-zero brackets are

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad \textit{and} \quad [e_2, e_3] = e_1.$$

Hence, the k-th term of ascending central series is $\mathfrak{g}$, for every $k \geq 1$. Then $\bigwedge_{x_i \in \mathfrak{g}} \mu([x_1, \ldots, x_k]) = 0.2$, for every $k \geq 1$, and this implies that $N(\mu) = \frac{0.2}{0.9} = \frac{2}{9}$.

The fuzzy nilpotent measure N induces an order relation on the set of fuzzy Lie subalgebras of a fixed fuzzy Lie algebra $\mathfrak{g}$. Specifically, for any $\mu_1, \mu_2 \leq \mathfrak{g}$ we define

$$\mu_1 \leq_N \mu_2 \quad \text{if, and only if,} \quad N(\mu_1) \leq N(\mu_2).$$

Then, if $\mu_1 \leq_N \mu_2$ holds, we say that μ_1 is *less* nilpotent than μ_2. We recall that a preorder on a set X is a binary relation $\leq$ that is reflexive and transitive; that is, for all $x, y, z \in X$,

$$x \leq x, \qquad x \leq y \text{ and } y \leq z \implies x \leq z.$$

A total order on a set X is a binary relation $\leq$ that is a preorder on X, and is moreover antisymmetric and total; that is, for all $x, y \in X$,

$$x \leq y \text{ and } y \leq x \implies x = y,$$

and

$$x \leq y \text{ or } y \leq x.$$

More precisely, we have the following.

Theorem 3. *The relation $\leq_N$ is a total preorder on $\mathcal{V}(\mu, \mathfrak{g})$.*

Proof. For every $\mu \in \mathcal{V}(\mu, \mathfrak{g})$, we clearly have

$$N(\mu) \leq N(\mu),$$

hence $\leq_N$ is reflexive.

Moreover, if $\mu_1 \leq_N \mu_2$ and $\mu_2 \leq_N \mu_3$, then

$$N(\mu_1) \leq N(\mu_2) \quad \text{and} \quad N(\mu_2) \leq N(\mu_3),$$

which implies

$$N(\mu_1) \leq N(\mu_3).$$

Therefore, $\mu_1 \leq_N \mu_3$, and $\leq_N$ is transitive.

Finally, since $[0, 1]$ is totally ordered, for every $\mu_1, \mu_2 \in \mathcal{V}(\mu, \mathfrak{g})$, either

$$N(\mu_1) \leq N(\mu_2)$$

or

$$N(\mu_2) \leq N(\mu_1).$$

Hence, either $\mu_1 \leq_N \mu_2$ or $\mu_2 \leq_N \mu_1$, proving that $\leq_N$ is total. □

In the following example, we show that $\leq_N$ cannot be a total order on $\mathcal{V}(\mu, \mathfrak{g})$ because the anti-symmetry condition does not hold in general.

Example 5. *Let $\mathfrak{g}$ and μ be as in Example 4. Now let ν be the following fuzzy Lie subalgebra of $\mathfrak{g}$:*

$$\nu(x) = \begin{cases} 0.9 & \text{if } x = (0,0,0), \\ 0.7 & \text{if } x = (a,0,0), a \neq 0, \\ 0.2 & \text{otherwise}. \end{cases}$$

Here $N(\nu) = N(\mu) = \frac{2}{9}$, but clearly $\nu \neq \mu$.

Properties of N

Let μ and ν be two fuzzy subalgebras of a Lie algebra $\mathfrak{g}$. We define

$$\begin{aligned} \mu \wedge \nu \colon \mathfrak{g} &\to [0,1] \\ x &\mapsto \mu(x) \wedge \nu(x) \end{aligned}$$

and

$$\begin{aligned} \mu + \nu \colon \mathfrak{g} &\to [0,1] \\ x &\mapsto \mu(x) + \nu(x) - \mu(x) \cdot \nu(x). \end{aligned}$$

Proposition 8 ([6] Theorem 1.2)**.** *The fuzzy sets $\mu \wedge \nu$ and $\mu + \nu$ are fuzzy ideals of $\mathfrak{g}$.*

Proposition 9. *Let μ, ν be fuzzy ideals of $\mathfrak{g}$, with μ less nilpotent than ν (i.e., $N(\mu) \leq N(\nu)$). Then*

$$N(\mu \cap \nu) \begin{cases} = N(\mu) & \text{if } \mu(0_{\mathfrak{g}}) = \nu(0_{\mathfrak{g}}) \\ \leq N\Big(\arg_{\mu,\nu} \min\{\mu(0_{\mathfrak{g}}), \nu(0_{\mathfrak{g}})\}\Big) & \text{otherwise}. \end{cases}$$

Proof. We have

$$\begin{aligned} N(\mu \wedge \nu) &= \lim_{k \to \infty} \bigwedge_{x_i \in \mathfrak{g}} \frac{(\mu \wedge \nu)([x_1, \dots, x_k])}{(\mu \wedge \nu)(0_{\mathfrak{g}})} \\ &= \lim_{k \to \infty} \bigwedge_{x_i \in \mathfrak{g}} \frac{\mu([x_1, \dots, x_k]) \wedge \nu([x_1, \dots, x_k])}{\mu(0_{\mathfrak{g}}) \wedge \nu(0_{\mathfrak{g}})}. \end{aligned}$$

If $\mu(0_{\mathfrak{g}}) = \nu(0_{\mathfrak{g}})$, then

$$\bigwedge_{x_i \in \mathfrak{g}} \mu([x_1, \dots, x_k]) \leq \bigwedge_{x_i \in \mathfrak{g}} \nu([x_1, \dots, x_k])$$

since $N(\mu) \leq N(\nu)$. As a consequence,

$$\begin{aligned} N(\mu \wedge \nu) &= \lim_{k \to \infty} \bigwedge_{x_i \in \mathfrak{g}} \frac{\mu([x_1, \dots, x_k]) \wedge \nu([x_1, \dots, x_k])}{\mu(0_{\mathfrak{g}}) \wedge \nu(0_{\mathfrak{g}})} \\ &= \lim_{k \to \infty} \bigwedge_{x_i \in \mathfrak{g}} \frac{\mu([x_1, \dots, x_k])}{\mu(0_{\mathfrak{g}})} \\ &= N(\mu). \end{aligned}$$

If $\mu(0_{\mathfrak{g}}) \leq \nu(0_{\mathfrak{g}})$, then

$$\begin{aligned} N(\mu \wedge \nu) &= \lim_{k\to\infty} \bigwedge_{x_i \in \mathfrak{g}} \frac{\mu([x_1, ..., x_k]) \wedge \nu([x_1, ..., x_k])}{\mu(0_{\mathfrak{g}}) \wedge \nu(0_{\mathfrak{g}})} \\ &= \lim_{k\to\infty} \bigwedge_{x_i \in \mathfrak{g}} \frac{\mu([x_1, ..., x_k]) \wedge \nu([x_1, ..., x_k])}{\mu(0_{\mathfrak{g}})} \\ &\leq \lim_{k\to\infty} \bigwedge_{x_i \in \mathfrak{g}} \frac{\mu([x_1, ..., x_k])}{\mu(0_{\mathfrak{g}})} \\ &= N(\mu). \end{aligned}$$

□

The following theorem establishes that, if two objects $(\mathfrak{g}, \mu), (\mathfrak{h}, \nu)$ in Lie^F are isomorphic, then μ and ν have the same fuzzy measure of nilpotency.

Theorem 4. *Let $(\mathfrak{g}, \mu), (\mathfrak{h}, \nu) \in \mathsf{Lie}^F$ such that $\mathfrak{g}$ and $\mathfrak{h}$ are finite dimensional. Let $f : (\mathfrak{g}, \mu) \to (\mathfrak{h}, \nu)$ an isomorphism in Lie^F. Hence, $N(\mu) = N(\nu)$.*

Proof. Let $(\mathfrak{g}, \mu), (\mathfrak{h}, \nu) \in \mathsf{Lie}^F$ such that $\mathfrak{g}$ and $\mathfrak{g}$ are finite dimensional and such that there exists an isomorphism $f \colon (\mathfrak{g}, \mu) \to (\mathfrak{h}, \nu)$ in Lie^F . Then we have $\mu(0_{\mathfrak{g}}) = \nu(0_{\mathfrak{h}})$. Moreover, let $x \in \mathfrak{g}$ be such that $\mu(x) = \bigwedge_{x_i \in \mathfrak{g}} \mu([x_1, \ldots, x_k])$. Then

$$\mu(x) = \nu(f(x)) = \bigwedge_{y_i \in \mathfrak{h}} \nu([y_1, \ldots, y_k]).$$

Indeed, if $\nu(f(x)) > \bigwedge_{y_i \in \mathfrak{h}} \nu([y_1, \ldots y_k])$, then there exists $y \in \mathfrak{h}$ such that $\nu(y) < \nu(f(x))$ and hence $\mu(f^{-1}(y)) = \nu(y) < \nu(f(x)) = \mu(x)$, which is a contradiction. Therefore, for every k,

$$\bigwedge_{x_i \in \mathfrak{g}} \mu([x_1, \ldots, x_k]) = \bigwedge_{y_i \in \mathfrak{h}} \nu([y_1, \ldots, y_k]),$$

and since $\mu(0_{\mathfrak{g}}) = \nu(0_{\mathfrak{h}})$, taking limits in the definition of N gives

$$N(\mu) = N(\nu).$$

□

5. Conclusions and Future Work

In this paper, we have developed a framework for the study of fuzzy Lie subalgebras with particular emphasis on isomorphisms and nilpotency. By adopting the transfer principle via t-cut sets, we introduced a natural notion of isomorphism that is well aligned with the underlying crisp structure while preserving the graded nature of fuzziness. Within this framework, we established that if two fuzzy Lie subalgebras are isomorphic in the categorical sense, then they share the same nilpotency measure, thereby confirming the robustness of the invariant $N(\mu)$ under structural equivalence.

Future Work

The results obtained in this paper open several promising directions for further investigation. First, it would be natural to extend the approach developed here to other structural properties of Lie algebras. In particular, one may ask whether analogous fuzzy measures can be defined for properties such as commutativity, solvability or semisimplicity. Such measures could provide a unified quantitative framework for comparing different algebraic behaviours in the fuzzy setting. Second, the theory could be generalised beyond Lie algebras to other algebraic structures. A particularly interesting candidate is given by Leibniz algebras. Another direction worth exploring, suggested by the broader landscape of non-associative algebra, is the extension of the present framework to Jordan algebras and, more generally, to algebraic structures governed by operads. Operadic methods offer a unified language for treating a wide range of algebraic structures, from associative and Lie algebras to commutative and other non-associative ones [22], and it would be natural to ask whether fuzzy versions of operadic identities can be studied through suitable families of level sets, in the spirit of the t-cut approach used here. A further possible direction concerns tropical algebra and semiring theory (see, for instance, [23]). Since fuzzy membership degrees interact naturally with order-theoretic and idempotent operations such as minima and maxima, semiring-based methods may offer a useful language for studying fuzzy algebraic structures, and exploring these connections could help place fuzzy Lie theory within a richer algebraic framework. Lastly, from a categorical perspective, it would be worthwhile to investigate whether other classes of maps admit a meaningful fuzzy extension compatible with the t-cut approach. In this direction, derivations constitute a natural object of study. Developing a theory of fuzzy derivations could provide deeper insight into the internal symmetries of fuzzy algebraic structures.

We believe that the perspective adopted in this work contributes to bridging the gap between classical Lie theory and fuzzy logic, and that the tools introduced here will prove useful in the continued development of the field.

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