LANDESMAN-LAZER TYPE (p,q)-EQUATIONS WITH NEUMANN CONDITION

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ABSTRACT. We consider a Neumann problem driven by the (p,q)-Laplacian under the Landesman-Lazer type condition. Using the classical saddle point theorem and other classical results of calculus of variations, we show that the problem has at least one nontrivial weak solution.

1. INTRODUCTION

Let Ω be a bounded regular domain in the Euclidean space $(\mathbb{R}^N, |\cdot|)$. In this paper we study the following Neumann problem

$$\begin{cases} -\triangle_p u(z) - \triangle_q u(z) = f(z,u(z)) & \text{in } \Omega,\ 1<q<p<+\infty, \\ \dfrac{\partial u}{\partial \nu_{pq}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

By Δ_p and Δ_q we denote the p-Laplace and q-Laplace differential operators defined by

$$\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W^{1,p},\ p\in(1,+\infty),$$
$$\triangle_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u) \quad \text{for all } u \in W^{1,q},\ q\in(1,+\infty).$$

In (1) we assume that $p \geq 2$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function (that is, for all $z \in \mathbb{R}$, $z \to f(z,u)$ is measurable and for a.a. $z \in \Omega$, $u \to f(z,u)$ is continuous, and for any $s > 0$ there exists $l_s \in L^1(\Omega)$ with $|f(z,u)| \leq l_s(u)$, for a.a. $z \in \Omega$ and for all $|u| \leq s$). In the boundary condition, we mention that $\dfrac{\partial u}{\partial \nu_{pq}}$ is the conormal derivative, which means

$$\frac{\partial u}{\partial \nu_{pq}} = (|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u)\nu := (|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u, \nu)_{\mathbb{R}^N}, \quad u \in C^1(\overline{\Omega}),$$

where ν is the outer unit normal to $\partial\Omega$ (see Gasiński-Papageorgiou [3], p. 210).

We mention that (p,q)-equations attracted considerable interest and there have been various existence and multiplicity results for such equations (for instance, such a kind of two phase equation is used to model various physical processes). We recall the recent works of Papageorgiou-Vetro [10] ($(p(z),q(z))$-equations), Papageorgiou-Vetro-Vetro [11, 12, 13] ($(p,2)$-equations), Papageorgiou-Rădulescu-Repovš [7] and Tanaka [14] ((p,q)-equations). For problems satisfying the Landesman-Lazer type condition, see Jiang-Ma-Paşca [4], Tang [15, 16], Wu-Tan [17].

Key words and phrases. Neumann problem, (p,q)-Laplacian, critical point, saddle point theorem, Landesman-Lazer type condition.

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The structure of problem (1) is variational. So, we conbine the saddle point theorem and other critical point tools to prove three results concerning the existence of at least one and two weak solutions.

Existence results for different types of double phase equations can be found in Bahrouni-Rădulescu-Repovš [1], Cencelj-Rădulescu-Repovš [2], Papageorgiou-Rădulescu-Repovš [8], Zhang-Rădulescu [18].

2. Mathematical Preliminaries

In the study of problem (1), we need the Lebesgue space $L^r(\Omega)$ and the Sobolev space $W^{1,r}(\Omega)$, with $1 \leq r < +\infty$ (see also the book of Papageorgiou-Rădulescu-Repovš [9]). By $\|\cdot\|_r$ we denote the norm of the Lebesgue space $L^r(\Omega)$ and by $\|\cdot\|$ the norm of the Sobolev space $W^{1,r}(\Omega)$, given as

$$\|u\| = \left[\|u\|_r^r + \|\nabla u\|_r^r\right]^{\frac{1}{r}} \quad \text{for all } u \in W^{1,r}(\Omega).$$

If we set

$$\overline{u} = \frac{1}{|\Omega|}\int_\Omega u(z)dz \quad \text{and} \quad \mathbb{W}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \overline{u} = 0\},$$

we get the representation

$$W^{1,p}(\Omega) = \mathbb{R} \oplus \mathbb{W}^{1,p}(\Omega).$$

We recall some facts related to $\mathbb{W}^{1,p}(\Omega)$, and other preliminaries.

Proposition 1 ([17], Proposition 1)**.** *One can find $\tau > 0$ satisfying*

$$\|\nabla u\|_p^p \geq \tau \|u\|_p^p, \quad \textit{for all } u \in \mathbb{W}^{1,p}(\Omega),\ p \in (1, +\infty). \tag{2}$$

Remark 1. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathbb{W}^{1,p}(\Omega)$ is a sequence satisfying $\|u_n\| \to +\infty$ as $n \to +\infty$, then (2) leads to $\|\nabla u_n\|_p \to +\infty$ as $n \to +\infty$.

From the Sobolev embedding theorem (see Motreanu-Motreanu-Papageorgiou [6], p. 11) we get:

Proposition 2. *Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ such that $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ with $p > N$. Then $u_n \to u$ uniformly in $\overline{\Omega}$.*

By a weak solution of problem (1) we mean a function $u \in W^{1,p}(\Omega)$ satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dz + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dz = \int_\Omega f(z,u)v dz,$$

for each $v \in W^{1,p}(\Omega)$.

Let $F(z,t) = \int_0^t f(z,s)ds$. The energy functional $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ associated to problem (1) is given by

$$\varphi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega F(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

It is well known that the weak solutions of problem (1) correspond to the critical points of φ in $W^{1,p}(\Omega)$. Also, φ is continuously differentiable in $W^{1,p}(\Omega)$ with

$$\langle \varphi'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dz + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dz - \int_\Omega f(z,u)v dz$$

for all $u, v \in W^{1,p}(\Omega)$.

Remark 2 ([6], p. 25). One can find $c_r > 0$ satisfyng

$$(|y|^{r-2}y - |h|^{r-2}h, y-h)_{\mathbb{R}^N} \geq c_r|y-h|^r \quad \text{for all } y,h \in \mathbb{R}^N,\ r \in [2,+\infty).$$

Whenever $r \in (1,2)$, then $(|y|^{r-2}y - |h|^{r-2}h, y-h)_{\mathbb{R}^N} \geq 0$ for all $y,h \in \mathbb{R}^N$.

As a consequence of Proposition 2, we prove the following convergence result.

Proposition 3. *If $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$, with $p \in (N,+\infty)$, is a bounded sequence such that $\varphi'(u_n) \to 0$ in $W^{1,p}(\Omega)^*$ (that is, the topological dual of $W^{1,p}(\Omega)$), then $\{u_n\}_{n\in\mathbb{N}}$ converges in $W^{1,p}(\Omega)$.*

Proof. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$, by the reflexivity of $W^{1,p}(\Omega)$ we can find $u \in W^{1,p}(\Omega)$ and a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (for reader convenience say still $\{u_n\}_{n\in\mathbb{N}}$) satisfying

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega).$$

So, we have

$$\lim_{n\to+\infty} \|u_n - u\|_\infty = 0 \qquad \text{(by Proposition 2).}$$

Now, we can find $\rho > 0$ with

$$\|u_n\|_\infty \leq \rho \quad \text{for all } n \in \mathbb{N}.$$

From

$$\begin{aligned}
&\langle \varphi'(u_n) - \varphi'(u_m), u_n - u_m\rangle \\
&= \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)(\nabla u_n - \nabla u_m)dz \\
&+ \int_\Omega (|\nabla u_n|^{q-2}\nabla u_n - |\nabla u_m|^{q-2}\nabla u_m)(\nabla u_n - \nabla u_m)dz \\
&- \int_\Omega [f(z,u_n) - f(z,u_m)](u_n - u_m)dz \\
&\geq c_p \int_\Omega |\nabla(u_n - u_m)|^p dz - \int_\Omega [f(z,u_n) - f(z,u_m)](u_n-u_m)dz, \\
&\qquad\qquad \text{for some } c_p > 0 \text{ with } p \geq 2 \text{ (by Remark 2),}
\end{aligned}$$

we get

$$\begin{aligned}
&c_p \int_\Omega |\nabla(u_n - u_m)|^p dz \\
&\leq \langle \varphi'(u_n) - \varphi'(u_m), u_n - u_m\rangle + \int_\Omega [f(z,u_n) - f(z,u_m)](u_n - u_m)dz \\
&\leq \|\varphi'(u_n) - \varphi'(u_m)\|\,\|u_n - u_m\| + 2\|u_n - u_m\|_\infty \int_\Omega \sup_{|t|\leq\rho} |f(z,t)|dz, \\
&\to 0 \quad \text{as } n,m \to +\infty \text{ (recall that } f \text{ is } L^1\text{-Carathéodory).}
\end{aligned}$$

So, we have

$$\begin{aligned}
&\|u_n - u_m\|^p = \int_\Omega |\nabla(u_n - u_m)|^p dz + \int_\Omega |u_n - u_m|^p dx \to 0 \text{ as } n,m \to +\infty, \\
\Rightarrow\quad & \{u_n\}_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } W^{1,p}(\Omega), \\
\Rightarrow\quad & u_n \to u \text{ in } W^{1,p}(\Omega), \text{ as } n \to +\infty \text{ (since } W^{1,p}(\Omega) \text{ is complete).}
\end{aligned}$$

∎

3. Main Results

We start with a theorem producing the existence of at least one weak solution of problem (1) in $W^{1,p}(\Omega)$ with $p \in (N, +\infty)$. First we recall the following saddle-point theorem (Theorem 5.41 of Motreanu-Motreanu-Papageorgiou [6], p. 119).

Theorem 1. *Let X be a Banach space. If $X = X_1 \oplus X_2$, with $\dim X_1 < +\infty$, $\varphi \in C^1(X, \mathbb{R})$, there exists $r > 0$ such that*

$$\max\{\varphi(z) : z \in \partial B_r(0) \cap X_1\} \leq \inf\{\varphi(z) : z \in X_2\} = b,$$
$$\Gamma = \{h \in C(\overline{B_r(0)} \cap X_1, X) : h\Big|_{\partial B_r(0)\cap X_1} = \mathrm{id}_{\partial B_r(0)\cap X_1}\},$$
$$c = \inf_{h\in\Gamma} \sup_{z\in\overline{B_r(0)}\cap X_1} \varphi(h(z)),$$

and φ satisfies the $(C)_c$-condition, then $c \geq b$, c is a critical value of φ, and if $c = b$, then $K_\varphi^c \cap X_2 \neq \emptyset$.

Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition at the level $c \in \mathbb{R}$" (the "$(C)_c$-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\varphi(u_n) \to c$ in $\mathbb{R}$ and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* (that is, the topological dual of X) as $n \to +\infty$, admits a strongly convergent subsequence".

Now we give the hypotheses on the data of the problem:

(H1) If $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is a sequence with $\lim_{n\to+\infty} \|u_n\| = +\infty$ and $\lim_{n\to+\infty} \frac{|\overline{u}_n||\Omega|^{1/p}}{\|u_n\|} = 1$, then
$$\limsup_{n\to+\infty} \int_\Omega f(z, u_n)\frac{\overline{u}_n}{|\overline{u}_n|}dz > 0.$$

(H2) $\limsup_{|t|\to+\infty} \dfrac{F(z,t)}{|t|^p} < \dfrac{\tau^*}{p}$ uniformly for a.a. $z \in \Omega$, where $\tau^* > 0$ is the biggest constant such that (2) holds true.

We mention that (H1) is known as the Landesman-Lazer type condition.

Example 1. *The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ given by $f(z,t) = a|t|^{p-2}t$ with $a \in (0, \tau^*)$ satisfies hypothesis (H1). Clearly, $F(z,t) = \int_0^t f(z,s)ds$ satisfies hypothesis (H2).*

Theorem 2. *If (H1) and (H2) hold, then problem* (1) *has at least one weak solution in $W^{1,p}(\Omega)$ with $p \in (N, +\infty)$.*

Proof. We claim that $\inf\{\varphi(u) : u \in \mathbb{W}^{1,p}(\Omega)\} > -\infty$. So, by (H2) we can find $0 < \varepsilon < \tau^* p^{-1}$ satisfying

$$F(z,t) < \left(\frac{\tau^*}{p} - \varepsilon\right)|t|^p \quad \text{for all } t \in \mathbb{R} \text{ such that } c_1 < |t|, \text{ for some } c_1 > 0.$$

If we put $\beta_1(z) = \sup_{|t|\leq c_1} F(z,t)$, then we have

$$F(z,t) < \left(\frac{\tau^*}{p} - \varepsilon\right)|t|^p + \beta_1(z) \quad \text{for all } t \in \mathbb{R} \text{ and a.a. } z \in \Omega. \tag{3}$$

Now, for all $u \in \mathbb{W}^{1,p}(\Omega)$, by (3) we get

$$\varphi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega F(z,u)dz$$

$$
\begin{aligned}
&> \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \left(\frac{\tau^*}{p} - \varepsilon\right)\|u\|_p^p - \int_\Omega \beta_1(z)dz \\
&= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \left(\frac{1}{p} - \frac{\varepsilon}{\tau^*}\right)\tau^*\|u\|_p^p - \int_\Omega \beta_1(z)dz \\
&\geq \frac{\varepsilon}{\tau^*}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \beta_1(z)dz \\
&\geq -\int_\Omega \beta_1(z)dz \\
\Rightarrow \quad & \inf\{\varphi(u) : u \in \mathbb{W}^{1,p}(\Omega)\} > -\infty,
\end{aligned}
$$

which proves the claim.

Now we show that there is $r > 0$ such that $\max\{\varphi(z) : z \in \partial B_r(0) \cap \mathbb{R}\} \leq \inf\{\varphi(z) : z \in \mathbb{W}^{1,p}(\Omega)\}$. To this aim, we show that $\varphi(u) \to -\infty$ as $|u| \to +\infty$, $u \in \mathbb{R}$. So, we suppose that there are $c_2 > 0$ and $\gamma > 0$ with

$$\int_\Omega f(z,t)tdz \geq \gamma|t| \quad \text{for } t \in \mathbb{R} \text{ with } |t| \geq c_2. \tag{4}$$

Arguing indirectly, assume that at least for a sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $|t_n| \to +\infty$ as $n \to +\infty$, we have

$$\int_\Omega f(z,t_n)\frac{t_n}{|t_n|}dz < \frac{1}{n} \quad \text{for every } n \geq 1.$$

So, we have a contradiction to (H1). Thus, for all $u \in X_1 = \mathbb{R}$ with $c_2 < |u|$, we get

$$
\begin{aligned}
\varphi(u) &= -\int_\Omega F(z,u)dz = -\int_\Omega \left[\int_0^u f(z,s)ds\right]dz \\
&= -\int_\Omega \left[\int_0^1 f(z,us)uds\right]dz \\
&= -\int_\Omega \left[\int_0^{\frac{c_2}{|u|}} f(z,us)uds + \int_{\frac{c_2}{|u|}}^1 f(z,us)uds\right]dz.
\end{aligned}
$$

Set $\beta_2(z) = \sup_{|su|\leq c_2}|f(z,us)|$, so that we have

$$
\begin{aligned}
\left|\int_\Omega \int_0^{\frac{c_2}{|u|}} f(z,us)udsdz\right| &\leq \int_\Omega \int_0^{\frac{c_2}{|u|}} |f(z,us)|\,|u|dsdz \\
&\leq \int_\Omega \int_0^{\frac{c_2}{|u|}} \beta_2(z)|\,|u|dsdz \\
&= c_2\int_\Omega \beta_2(z)dz \in \mathbb{R} \quad (\beta_2 \in L^1(\Omega), \text{ as } f \text{ is } L^1\text{-Carathéodory}).
\end{aligned}
$$

Next by (4) we deduce that

$$
\begin{aligned}
\int_\Omega \int_{\frac{c_2}{|u|}}^1 f(z,us)udsdz &= \int_{\frac{c_2}{|u|}}^1 \frac{1}{s}\left[\int_\Omega f(z,us)us\,dz\right]ds \\
&\geq \int_{\frac{c_2}{|u|}}^1 \frac{1}{s}(\gamma|us|)ds
\end{aligned}
$$

$$= \gamma|u|\left(1 - \frac{c_2}{|u|}\right) = \gamma|u| - \gamma c_2.$$

Therefore, we conclude

$$\varphi(u) \le c_2 \int_\Omega \beta_2(z)dz - \gamma|u| + \gamma c_2,$$
$$\Rightarrow \quad \lim_{|u|\to+\infty} \varphi(u) = -\infty, \qquad u \in \mathbb{R}.$$

Next step is to show that the $(C)_c$-condition holds. Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ such that $\varphi(u_n) \to c$ in $\mathbb{R}$ and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ as $n \to +\infty$. We will show that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. Arguing indirectly, assume that at least for a subsequence (say still $\{u_n\}_{n\in\mathbb{N}}$) we have

$$\|u_n\| \to +\infty \quad \text{as} \quad n \to +\infty. \tag{5}$$

We set $v_n = \eta\frac{u_n}{\|u_n\|}$ with $\eta = \left(\frac{1}{1+\tau^*}\right)^{\frac{1}{p}}$. Note that $\|v_n\| = \eta$ and the sequence $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$. So, we can find $v \in W^{1,p}(\Omega)$ and a subsequence of $\{v_n\}_{n\in\mathbb{N}}$ (for reader convenience, say $\{v_n\}_{n\in\mathbb{N}}$) with

$$v_n \xrightarrow{w} v \text{ in } W^{1,p}(\Omega) \text{ and } v_n \to v \text{ in } L^p(\Omega).$$

By (3), we get

$$\begin{aligned}
\frac{\varphi(u_n)}{\|u_n\|^p} &= \frac{1}{p}\int_\Omega \frac{|\nabla u_n|^p}{\|u_n\|^p}dz + \frac{1}{q}\int_\Omega \frac{|\nabla u_n|^q}{\|u_n\|^p}dz - \frac{1}{\|u_n\|^p}\int_\Omega F(z,u)dz \\
&> \frac{1}{p\eta^p}\int_\Omega |\nabla v_n|^p dz - \left(\frac{1}{p} - \frac{\varepsilon}{\tau^*}\right)\frac{\tau^*}{\eta^p}\int_\Omega |v_n|^p dz - \frac{1}{\|u_n\|^p}\int_\Omega \beta_1(z)dz \\
&\ge \frac{\varepsilon}{p\tau^*\eta^p}\|\nabla v_n\|_p^p - \frac{1}{\|u_n\|^p}\int_\Omega \beta_1(z)dz.
\end{aligned}$$

Passing to the limit as $n \to +\infty$ in the previous inequality, we get

$$\|\nabla v\|_p^p = 0,$$

that is, $|\nabla v(z)| = 0$ for all $z \in \Omega$ and so v is a constant function. It follows that $|v|^p = \eta^p|\Omega|^{-1}$, and hence

$$\begin{aligned}
\lim_{n\to+\infty} \frac{|\overline{u}_n|^p}{\|u_n\|^p} &= \lim_{n\to+\infty}\left|\frac{1}{|\Omega|}\int_\Omega \frac{u_n}{\|u_n\|}dz\right|^p = \lim_{n\to+\infty}\left|\frac{1}{|\Omega|\eta}\int_\Omega v_n dz\right|^p \\
&= \left|\frac{1}{|\Omega|\eta}\int_\Omega v dz\right|^p \\
&= \left(\frac{1}{|\Omega|\eta}\int_\Omega |v| dz\right)^p = \frac{1}{|\Omega|}, \\
\Rightarrow \quad &\lim_{n\to+\infty} \frac{|\overline{u}_n||\Omega|^{\frac{1}{p}}}{\|u_n\|^p} = 1, \\
\Rightarrow \quad &\limsup_{n\to+\infty} \int_\Omega f(z,u_n)\frac{\overline{u}_n}{|\overline{u}_n|}dz > 0 \quad \text{(by (5) and hypothesis (H1))}.
\end{aligned} \tag{6}$$

From $(1+\|u_n\|)\varphi'(u_n)\to 0$ as $n\to+\infty$, we deduce that $\varphi'(u_n)\to 0$ as $n\to+\infty$. So, we have

$$\begin{aligned}\int_\Omega f(z,u_n)\frac{\overline{u}_n}{|\overline{u}_n|}dz &= -\langle\varphi'(u_n),\frac{\overline{u}_n}{|\overline{u}_n|}\rangle+\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\frac{\overline{u}_n}{|\overline{u}_n|}dz\\ &\qquad+\int_\Omega|\nabla u_n|^{q-2}\nabla u_n\nabla\frac{\overline{u}_n}{|\overline{u}_n|}dz\\ &= -\langle\varphi'(u_n),\frac{\overline{u}_n}{|\overline{u}_n|}\rangle\to 0 \text{ as } n\to+\infty,\end{aligned}$$

a contradiction to (6). So, we conclude that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$ and, since $\varphi'(u_n)\to 0$ as $n\to+\infty$, we deduce that the $(C)_c$-condition holds true for every $c\in\mathbb{R}$ (by Proposition 3). So, by Theorem 1 the functional φ has a critical value, which means that problem (1) has at least one weak solution. □

To obtain our second result, we use the following key-result (Theorem 5.51 of Motreanu-Motreanu-Papageorgiou [6], p. 122).

Theorem 3. *Let X be a Banach space with a direct sum decomposition $X=X_1\oplus X_2$, with $\dim X_1<+\infty$ and let $\varphi\in C^1(X,\mathbb{R})$ be bounded below with $\inf_X\varphi<0$ and satisfy the (PS)-condition. Assume that φ has a local linking to zero with respect to the pair (X_1,X_2) (that is, there is $r>0$ such that $\varphi(z)\le 0$ if $z\in X_1$ with $\|z\|\le r$, and $\varphi(z)\ge 0$ if $z\in X_2$ with $\|z\|\le r$). Then φ has at least two nonzero critical points.*

We consider the following hypotheses on the data of the problem:

(H3) If $\{u_n\}_{n\in\mathbb{N}}\subseteq W^{1,p}(\Omega)$ is a sequence with $\lim\limits_{n\to+\infty}\|u_n\|=+\infty$ and $\lim\limits_{n\to+\infty}\frac{|\overline{u}_n||\Omega|^{1/p}}{\|u_n\|}=1$, then
$$\lim_{n\to+\infty}\int_\Omega F(z,u_n)dz=-\infty.$$

(H4) $\limsup\limits_{|t|\to+\infty}\dfrac{F(z,t)}{|t|^p}<0$ uniformly for a.a. $z\in\Omega$.

(H5) We can find $r^*>0$ satisfying
$$0\le F(z,t)\le\frac{\tau^*}{p}|t|^p,\qquad \text{for all } |t|\le r^* \text{ and a.a. } z\in\Omega.$$

Note that (H3) is also a Landesman-Lazer type condition.

Example 2. *The function $F:\Omega\times\mathbb{R}\to\mathbb{R}$ given by $F(z,t)=\dfrac{\tau^*}{p}|t|^p-\lambda|t|^{p+1}$ with $\lambda\in(0,\tau^*/p)$ satisfies hypotheses (H3)-(H5).*

Theorem 4. *If (H3)-(H5) hold, then problem* (1) *has at least two weak solutions in $W^{1,p}(\Omega)$ with $p\in(N,+\infty)$.*

Proof. We prove that $\varphi(u)\to+\infty$ as $\|u\|\to+\infty$ for $u\in W^{1,p}(\Omega)$ (that is, φ is coercive). Arguing indirectly, assume that for a sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq W^{1,p}(\Omega)$ we have

$$\|u_n\|\to+\infty\quad\text{ as } n\to+\infty, \text{ and } \varphi(u_n)\le c_3 \text{ (for some } c_3>0). \tag{7}$$

Set $v_n=\frac{u_n}{\|u_n\|}$, so that $\|v_n\|=1$ and $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$. Now, we can find $v\in W^{1,p}(\Omega)$ and a subsequence of $\{v_n\}_{n\in\mathbb{N}}$ (say $\{v_n\}_{n\in\mathbb{N}}$) with

$$v_n\xrightarrow{w}v \text{ in } W^{1,p}(\Omega) \text{ and } v_n\to v \text{ in } L^p(\Omega).$$

Fixed $\varepsilon > 0$, by (H4) we can find $c_4 > 0$ with

$$F(z,t) < \frac{\varepsilon}{p}|t|^p, \quad \text{for all } t \in \mathbb{R} \text{ with } |t| > c_4.$$

Set $\beta_3(z) = \sup_{|t|\leq c_4} F(z,t)$ so that we get

$$F(z,t) < \frac{\varepsilon}{p}|t|^p + \beta_3(z), \quad \text{for all } t \in \mathbb{R} \text{ and a.a. } z \in \Omega.$$

Also, we have

$$\begin{aligned}
\frac{c_3}{\|u_n\|^p} \geq \frac{\varphi(u_n)}{\|u_n\|^p} &= \frac{1}{p}\int_\Omega \frac{|\nabla u_n|^p}{\|u_n\|^p}dz + \frac{1}{q\|u_n\|^{p-q}}\int_\Omega \frac{|\nabla u_n|^q}{\|u_n\|^q}dz - \frac{1}{\|u_n\|}\int_\Omega F(z,u_n)dz \\
&> \frac{1}{p}\|\nabla v_n\|_p^p + \frac{1}{q\|u_n\|^{p-q}}\|\nabla v_n\|_q^q - \frac{\varepsilon}{p}\|v_n\|_p^p - \frac{1}{\|u_n\|}\int_\Omega \beta_3(z)dz \\
&\geq \frac{1}{p} - \frac{1}{p}\|v_n\|_p^p - \frac{\varepsilon}{p}\|v_n\|_p^p - \frac{1}{\|u_n\|}\int_\Omega \beta_3(z)dz. \qquad (8)
\end{aligned}$$

Now, we get

$$\begin{aligned}
& 0 \geq \frac{1}{p} - \frac{1}{p}(1+\varepsilon)\|v\|_p^p \quad \text{(by (7) and (8) as } n \to +\infty), \\
\Rightarrow \quad & \|v\|_p^p \geq 1 \quad \text{as } \varepsilon \to 0. \qquad (9)
\end{aligned}$$

Since $\|\cdot\|$ is weakly lower semi-continuous, we have

$$\begin{aligned}
& \|v\| \leq \liminf_{n\to+\infty} \|v_n\| = 1, \\
\Rightarrow \quad & \|\nabla v\|_p^p + \|v\|_p^p = \|v\|^p \leq 1, \\
\Rightarrow \quad & \|\nabla v\|_p^p = 0 \quad \text{(by (9))}.
\end{aligned}$$

So,we deduce that v is a constant function and $\lim_{n\to+\infty} \frac{|\overline{u}_n||\Omega|^{\frac{1}{p}}}{\|u_n\|^p} = 1$.

Next, (7) and hypothesis (H3) imply

$$\begin{aligned}
& \lim_{n\to+\infty} \int_\Omega F(z,u_n)dz = -\infty, \\
\Rightarrow \quad & c_3 \geq \lim_{n\to+\infty} \varphi(u_n) \geq -\lim_{n\to+\infty}\int_\Omega F(z,u_n)dz = +\infty,
\end{aligned}$$

which leads to a contradiction with the same (7), and so we conclude that $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$ for $u \in W^{1,p}(\Omega)$. Also, φ is weakly lower semi-continuous, and hence it is bounded from below.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a (PS)-sequence for the functional φ (that is, $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$). Since φ is coercive, then $\{u_n\}_{n\in\mathbb{N}}$ is bounded. By Proposition 3, $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in $W^{1,p}(\Omega)$ and so φ satisfies the (PS)-condition. By (H5) we have

$$\varphi(u) = -\int_\Omega F(z,u)dz \leq 0 \quad \text{for all } u \in \mathbb{R} \text{ with } |u| \leq r^*. \qquad (10)$$

Also, for all $u \in \mathbb{W}^{1,p}(\Omega)$ with $\|u\|_\infty \leq r^*$, by (H5) we get

$$\varphi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega F(z,u)dz$$

$$\geq \frac{1}{p}[\|\nabla u\|_p^p - \tau^*\|u\|_p^p] + \frac{1}{q}\|\nabla u\|_q^q$$
$$\geq \frac{1}{q}\|\nabla u\|_q^q \geq 0 \quad \text{(by Proposition 1)}.$$

As $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}$ compactly for all $\alpha \in (0, 1 - \frac{N}{p})$, then we can find $\rho_0 > 0$ such that

$$\|u\|_\infty \leq \rho_0\|u\|, \quad u \in W^{1,p}(\Omega).$$

If we put $r = \min\{r^*\rho_0^{-1}, r^*|\Omega|^{1/p}\}$, then we have $\varphi(u) \leq 0$ for every $u \in \mathbb{R}$ with $\|u\| \leq r$ and $\varphi(u) \geq 0$ for $u \in \mathbb{W}^{1,p}(\Omega)$ with $\|u\| \leq r$.

Now, if $\inf_{W^{1,p}(\Omega)} \varphi < 0$, we conclude by Theorem 3. On the other hand, if $\inf_{W^{1,p}(\Omega)} \varphi \geq 0$, by (10) we get

$$\varphi(u) = \inf_{W^{1,p}(\Omega)} \varphi = 0 \quad \text{for all } u \in \mathbb{R} \text{ with } \|u\| \leq r,$$

that is, all $u \in \mathbb{R}$ with $\|u\| \leq r$ are solutions of problem (1). □

Here, we recall another key-result (which is an immediate consequence of Theorem 1.1 of Mawhin-Willem [5], p. 3).

Theorem 5. *Let X be a reflexive Banach space. If a functional $\varphi \in C^1(X, \mathbb{R})$ is weakly lower semi-continuous and coercive, then there exists $\widetilde{z} \in X$ such that $\inf_{z\in X} \varphi(z) = \varphi(\widetilde{z})$ and $\widetilde{z}$ is also a critical point of φ, that is $\varphi'(\widetilde{z}) = 0$.*

Theorem 6. *If (H3) and (H4) hold, then problem* (1) *has at least one weak solution in $W^{1,p}(\Omega)$.*

Proof. By the proof of Theorem 4, we know that hypotheses (H3) and (H4) imply that φ is coercive. Since φ is weakly lower semi-continuous, by Theorem 5 we deduce that problem (1) has at least one weak solution. □

Remark 3. In the last theorem, we do not need any restriction on the range of p.

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