ANISOTROPIC NAVIER KIRCHHOFF PROBLEMS WITH CONVECTION AND LAPLACIAN DEPENDENCE

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Abstract. We consider the Navier problem

$$-\Delta_{k,p}^2 u(x) = f(x, u(x), \nabla u(x), \Delta u(x)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = 0,$$

driven by the sign-changing (degenerate) Kirchhoff type $p(x)$-biharmonic operator, and involving a $(\nabla u, \Delta u)$-dependent nonlinearity f. We prove the existence of solutions, in weak sense, defining an appropriate Nemitsky map for the nonlinearity. Then, the Brouwer fixed point theorem assested for a Galerkin basis of the Banach space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, leads to the existence result. The case of non-degenerate Kirchhoff type $p(x)$-biharmonic operator is also considered with respect to the theory of pseudo-monotone operators, and an asymptotic analysis is derived.

1. Introduction

In this article we study equations whose main operator is a degenerate (sign-changing) Kirchhoff type $p(x)$-biharmonic operator, namely $u \to -\Delta_{k,p}^2 u$, for a function u given on a bounded domain $\Omega \subseteq \mathbb{R}^N$ with smooth boundary $\partial\Omega$. The appropriate setting to develop this study, is the Banach space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, where $W^{r,p(x)}(\Omega)$ $(r \geq 1)$ means the generalized variable exponent Sobolev space, and $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ (see Section 2 for the precise notion). Here, the exponent p leaves in $C(\overline{\Omega})$ and possesses sufficient regularities. The new operator $\Delta_{k,p}^2$ is constructed over the $p(x)$-biharmonic operator $\Delta_{p(x)}^2$ (of fourth order) whose formula links to the $p(x)$-Laplace operator. Indeed, for a $p(x)$-Laplace operator $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ for all $u \in W_0^{1,p(x)}(\Omega)$, we have the corresponding $p(x)$-biharmonic operator $\Delta_{p(x)}^2 u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$. Starting from the sign-changing Kirchhoff type weight defined by

$$K(p, \Delta u) = a - b \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx, \quad \text{with } a, b > 0, \tag{1}$$

we introduce the operator

$$\Delta_{k,p}^2 u = K(p, \Delta u)\Delta_{p(x)}^2 u = \left(a - b \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\right) \Delta_{p(x)}^2 u,$$

for all $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, and consequently we define the Navier problem

$$(P) \qquad -\Delta_{k,p}^2 u(x) = f(x, u(x), \nabla u(x), \Delta u(x)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = 0.$$

We remark that the $p(x)$-Laplace operator, in contrast to the isotropic p-Laplacian (that is, the case $p(x) = p = \text{constant}$), is not homogeneous and this is a source of difficulties in the analysis of anisotropic problems. According to the relevant literature on the variable

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exponents Sobolev spaces (see the book of Rădulescu & Repovš [21]), we assume that $p \in C(\overline{\Omega})$ satisfies the bound condition

$$1 < p^- = \inf_{x\in\overline{\Omega}} p(x) \leq p(x) \leq p^+ = \sup_{x\in\overline{\Omega}} p(x) < +\infty.$$

To complete the presentation of the problem, we point out that the nonlinearity is assumed of Carathéodory type (that is, for all $(z, y, v) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, $x \to f(x, z, y, v)$ is measurable and for almost all $x \in \Omega$, $(z, y, v) \to f(x, z, y, v)$ is continuous). The ∇u-dependence is appropriate to cover the physical situations where convective phenomena of fluid dynamics cannot be neglected (and hence, there is energy transfer accomplished by particles motion). Moreover, we recall a classical direction of research aimed to analyze situations when f depends on the derivatives of u (see for example Carrião et al. [4], and the references therein). These situations motivate our choice to consider a ∇u-dependent nonlinearity. About equation (1), we recall that the Kirchhoff weight is a useful way to represent (in a physical model setting) how transverse vibrations imply changes in length of a string/beam. In details, Kirchhoff [12] provided a generalization of the D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

with ρ, P_0, h, E, L denoting physical parameters (that is, mass density, initial tension, area of the cross-section, Young modulus of the material, length of the string), to describe the changes in string length subject to free vibrations. Referring to the main equation in (P) in this setting, then u means the displacement, the coefficients a and b mean, respectively, the intrinsic features and initial tension of the string, finally $f(x, u, \nabla u, \Delta u)$ represents the external force acting on it. On the other hand, we recall that fourth-order elliptic type equations are useful to describe physical phenomena as diffusion on solids, phase field models of multiphasic systems and others (see also Kefi & Rădulescu [11], Section 1). As we will say in the sequel, there is an active literature on establishing the existence and nonexistence of solutions to this type of problems under general conditions for the nonlinearity (see again [11]) and adopting the techniques of the Calculus of Variations (we remark that the nonexistence of a priori estimates, with respect to the norms of the gradient and the Laplacian of solution, is the main difficulty in using variational techniques). Turning to the mathematical content of our manuscript, we aim to obtain existence results of weak solutions to (P) (see equation (5)). Since the $(\nabla u, \Delta u)$-dependent nonlinearity cannot be considered using variational methods, we adopt topological tools. Precisely, we center the proof on fixed-point arguments, and the preparation work is made from two perspective: the introduction of a suitable Nemitsky map linked to the nonlinearity $f(x, u, \nabla u, \Delta u)$, and a discretization of the Banach space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ via the definition of a Galerkin basis.

Some recent references supporting our strategy can be considered in respect of three categories:

(A) problems with gradient and Laplacian dependent nonlinearities;
(B) problems with weighted Kirchhoff terms;
(C) problems with biharmonic operators.

For the category (A), we first mention the work of Carrião et al. [4] dealing with nonlinear biharmonic equations under Navier boundary conditions. Using an iterative scheme of the mountain pass approximated solutions together with useful truncations, the authors establish the existence of at least one solution. In [4], the nonlinearity f depends on both the gradient and the Laplacian of u, and this is the first paper where we find the similar representation of f as in our manuscript. Usually in the literature are considered the dependence by first and second order derivatives, and [4] is the first case where we found the Laplacian dependence explicitly stated (at the best of our knowledge). When f does not depend on the Laplacian Δu, we recall the works of Bai et al. [2] (nonhomogeneous partial differential operator with Robin boundary condition) and of Papageorgiou et al. [19] (constant exponent p-Laplacian operator with Neumann boundary condition), where the authors use Leray-Schauder alternative principle, together with truncation and comparison techniques. Both these works established the existence of smooth positive solutions, without imposing any global growth condition on the reaction term. Finally, we mentione the work of Ourraoui [17] where the Galerkin's approach jointly with useful a priori estimates, is adopted to conclude the existence of solutions to a class of elliptic problems. This time, the toy problem is driven by a p-Kirchhoff type operator with constant exponent p, Dirichlet boundary condition and a convection term.

For the category (B), we can mention the work of Vetro [24] dealing with the variable exponents Lebesgue and Sobolev spaces, in the case of a single $p(x)$-Kirchhoff type operator and a Dirichlet boundary condition. We point out that the Kirchhoff weight in (1) was previously considered by Hamdani et al. [10], and the related differential problem was approached by variational methods, since the reaction therein is neither gradient dependent nor Laplacian dependent. At the basis of the recent interest for boundary value problems with a Kirchhoff weight, there is the monography [14] by Lions. However, we usually find in the literature a positive restriction to the values of the Kirchhoff term (that is, the form $a+b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx > 0$, with $a, b > 0$), which means a non-degenerate term. To enlarge the discussion over the sign of the Kirchhoff weight, we mention the recent works of Figueiredo & Nascimento [7], Santos Júnior & Siciliano [22], where the involved Kirchhoff terms can vanish in many different points. In all of them, existence and non-existence of solutions are established via fixed point theorems. Finally, we mention the work of Maia [15] where the author studies a class of $p(x)$-Choquard equations with a nonlocal and non-degenerate Kirchhof term, establishing a multiplicity of solutions, combining truncation arguments with Krasnoselskii's genus.

For the category (C), we can mention the work of Guo et al. [9], where the Kirchhoff type $p(x)$-biharmonic problem is approached via mountain pass theorem and Ekeland's variational principle. The involved problem is not gradient dependent in the reaction term. The similar problem (but without the Kirchhoff weight) and the same technique of proofs are adopted by Mbarki in [16]. We also mention the work of Boureanu et al. [3], where the authors consider a no-flux boundary condition (useful to cover the cases of surfaces being impermeable to certain contaminants). Finally, we cite the paper of Zhou [26] where the author establishes existence, multiplicity and nonexistence results for a Navier $p(x)$-biharmonic problem with a parametric reaction, involving variational methods too; see also [11] for a Navier $p(x)$-biharmonic problem with singular weights.

Inspired by the above-mentioned works, we consider problem (P) under the combined effects of a sign-changing Kirchhoff weight (that is, we deal with the degenerate case) and a principal $p(x)$-biharmonic operator, in the case of a gradient and Laplacian dependent nonlinearity. The manuscript is organized as follows. In Section 2, we collect the basic facts on variable exponent Lebesgue and Sobolev spaces, useful norm inequalities, properties of Banach spaces and a Brouwer type fixed point result. In Section 3 we give the main theorems and their proofs are shown in Section 4. In Section 5 we briefly discuss the case of a non-degenerate Kirchhoff weight (that is, we deal with a positive constant sign weight), with respect to the theory of pseudo-monotone operators, and establish an asymptotic result assuming that the coefficient b in the Kirchhoff term works as a parameter. A short Section 6 concludes the manuscript.

2. Preliminaries

For a comprehensive coverage of the variable exponent Lebesgue and Sobolev spaces (which are special cases of generalized Orlicz spaces) we refer to the monographs of Diening et al. [5] and of Rădulescu & Repovš [21]. In the sequel we assume that $p(x) > 1$ for all $x \in \overline{\Omega}$, even when it is not explicitly stated. Given a bounded domain $\Omega \subseteq \mathbb{R}^N$ with smooth boundary $\partial\Omega$, our study consider the Banach space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Thus, we start recalling the definition of the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ as follows

$$L^{p(x)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\},$$

with $M(\Omega)$ being the space of all measurable functions $u : \Omega \to \mathbb{R}$. Thus, we define the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \rho_p \left(\frac{u}{\lambda} \right) \leq 1 \right\},$$

for the modular

$$\rho_p(u) = \int_\Omega |u(x)|^{p(x)} dx \quad \text{for all } u \in L^{p(x)}(\Omega).$$

Different from the constant exponent $L^p(\Omega)$ (that is, the case $p(x) = p =$ constant), the variable exponent space is useful in the analysis of boundary value problems with nonstandard growth conditions. However, even if the passage from the constant setting to the variable one is natural, it is not trivial as some sources of difficulties occur (for example, we recall that $L^{p(x)}(\Omega)$ is not invariant to translations and the convolution is not in general continuous; see Kováčik & Rákosník [13]). However, under additional hypotheses on the exponent $p(\cdot)$ we can recover the situation and obtain boundedness and other properties useful to conclude the study. For reader convenience, we recall that $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$ is a separable, reflexive and uniformly convex Banach space. Moreover $\|\cdot\|_{L^{p(x)}(\Omega)}$ and $\rho_p(\cdot)$ meet the following theorem.

Theorem 1 (Fan & Zhao [6], Theorem 1.3)**.** *For $u \in L^{p(x)}(\Omega)$ we get:*

(i) $\|u\|_{L^{p(x)}(\Omega)} < 1 \ (= 1, \ > 1) \Leftrightarrow \rho_p(u) < 1 \ (= 1, \ > 1)$;

(ii) *if* $\|u\|_{L^{p(x)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho_p(u) \leq \|u\|^{p^+}_{L^{p(x)}(\Omega)}$;

(iii) *if* $\|u\|_{L^{p(x)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho_p(u) \leq \|u\|^{p^-}_{L^{p(x)}(\Omega)}$.

Let $p' \in C(\overline{\Omega})$ be the conjugate variable exponent to $p(\cdot)$, that is, the following formula holds:

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Consequently we denote the cojugate of $L^{p(x)}(\Omega)$ by $L^{p(x)}(\Omega)^* = L^{p'(x)}(\Omega)$, and in the case $p^- > 1$ we get the Hölder inequality

$$\int_\Omega uw dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{L^{p(x)}(\Omega)} \|w\|_{L^{p'(x)}(\Omega)} \leq 2 \|u\|_{L^{p(x)}(\Omega)} \|w\|_{L^{p'(x)}(\Omega)},$$

for $u \in L^{p(x)}(\Omega)$, $w \in L^{p'(x)}(\Omega)$. This inequality leads us to the existence of embedding results. For example, [6, Theorem 1.11] ensures the continuity of the embedding $L^{p_1(x)}(\Omega) \hookrightarrow L^{p_2(x)}(\Omega)$, whenever $p_1, p_2 \in C(\overline{\Omega})$ with $p_1(x) \geq p_2(x) > 1$ for all $x \in \overline{\Omega}$. Using the variable exponent Lebesgue space, for every integer $r > 0$ and fixed multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$, we can define the variable exponent generalized Sobolev space

$$W^{r,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), \, 1 \leq |\alpha| \leq r \}, \quad p \in C(\overline{\Omega}),$$

where $|\alpha| = \sum_{i=1}^n \alpha_i$ (that is, the order) and $D^\alpha u = \partial^{|\alpha|} u / \partial^{\alpha_1} x_1 \cdots \partial^{\alpha_N} x_N$. As already mentioned in the Introduction, by $W_0^{r,p(x)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{r,p(x)}(\Omega)$. Also, we consider the norm

$$\|u\|_{W^{r,p(x)}(\Omega)} = \sum_{|\alpha| \leq r} \|D^\alpha u\|_{L^{p(x)}(\Omega)}.$$

From [6] and [13], we know that $(W^{r,p(x)}(\Omega), \|\cdot\|_{W^{r,p(x)}(\Omega)})$ and $(W_0^{r,p(x)}(\Omega), \|\cdot\|_{W^{r,p(x)}(\Omega)})$ are separable and uniformly convex (hence reflexive) Banach spaces. Now, taking in mind the Poincaré inequality (for a reference, consider [5, Theorem 8.2.18])

$$\|u\|_{L^{p(x)}(\Omega)} \leq c_1 \|\nabla u\|_{L^{p(x)}(\Omega)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega), \text{ some } c_1 > 0,$$

we recall that the norms $\|u\|_{W^{1,p(x)}(\Omega)}$ and $\|\nabla u\|_{L^{p(x)}(\Omega)}$ are equivalent on $W_0^{1,p(x)}(\Omega)$. According to [25, Definition 4.3], we recall that for a couple of Banach spaces, namely X_1 and X_2, we define the norm on the space $X = X_1 \cap X_2$ by

$$\|u\|_X = \|u\|_{X_1} + \|u\|_{X_2}.$$

This remark is useful to our discussion, as we are interested to work on the Banach space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Hence, we consider as ingredients, the two norms:

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)} \quad (\text{where } \|\nabla u\|_{L^{p(x)}(\Omega)} = \||\nabla u|\|_{L^{p(x)}(\Omega)}),$$

and

$$\|u\|_{W^{2,p(x)}(\Omega)} = \sum_{|\alpha|=2} \|D^\alpha u\|_{L^{p(x)}(\Omega)}.$$

Consequently we introduce the norm

$$\|u\| = \|u\|_{W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)} = \|u\|_{W^{2,p(x)}(\Omega)} + \|u\|_{W_0^{1,p(x)}(\Omega)}$$

for all $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

Moreover, from Zang & Fu [25] we know that the norm $\|u\|$ is equivalent to $\|\Delta u\|_{L^{p(x)}(\Omega)}$ in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Indeed in [25], we have the inequality

$$\|\Delta u\|_{L^{p(x)}(\Omega)} \leq \|u\| \leq c_2 \|\Delta u\|_{L^{p(x)}(\Omega)}, \tag{2}$$

for all $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, where $c_2 > 0$ is independent of u.

Next, for $p \in C(\overline{\Omega})$ we recall the formula of the critical Sobolev exponent $p_r^*(\cdot)$ given as

$$p_r^*(x) = \begin{cases} \frac{Np(x)}{N - rp(x)} & \text{if } rp(x) < N, \\ +\infty & \text{if } N \leq rp(x), \end{cases} \quad \text{for all } x \in \overline{\Omega}. \tag{3}$$

From [11] we recall the following Sobolev embeddings properties related to the Banach space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

Proposition 1. *Given $p, \alpha \in C(\overline{\Omega})$ with $p(x) > 1$ and $1 < \alpha(x) < p_r^*(x)$ for all $x \in \overline{\Omega}$, we have that $(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)) \hookrightarrow L^{\alpha(x)}(\Omega)$ is a continuous and compact embedding.*

On the other hand, we recall a general embedding result for Banach spaces (refer to Gasiński & Papageorgiou [8, Lemma 2.2.27]).

Proposition 2. *Let (X_1, X_2) be a couple of Banach spaces satisfying $X_1 \subseteq X_2$. Then, if X_1 is dense in X_2 and the embedding is continuous, also the embedding $X_2^* \subseteq X_1^*$ is continuous. Moreover, if X_1 is reflexive then X_2^* is dense in X_1^*.*

To develop our arguments of proofs, we use the features of appropriate operators of monotone type. Precisely, we are interested to psedo-monotonicity. For reader convenience, we recall some well-know facts about the class of pseudo-monotone operators.

Definition 1. Let $\langle \cdot, \cdot \rangle$ be the duality pairing in Banach spaces, and consider a reflexive Banach space X, with dual space X^*. Thus, $A : X \to X^*$

(i) satisfies the (S_+)-property if $u_n \xrightarrow{w} u$ in X and $\limsup_{n \to +\infty} \langle A(u_n), u_n - u \rangle \leq 0$ imply $u_n \to u$ in X;

(ii) is pseudo-monotone if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0$ imply

$$\liminf_{n\to+\infty}\langle A(u_n), u_n - v\rangle \ge \langle A(u), u - v\rangle \quad \text{for all } v \in X;$$

(iii) is coercive if

$$\lim_{\|u\|_X\to+\infty}\frac{\langle A(u), u\rangle}{\|u\|_X} = +\infty.$$

Remark 1. For a bounded operator $A : X \to X^*$, Definition 1(ii) is equivalent to the following implication: $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0$ imply $A(u_n) \xrightarrow{w} A(u)$ and $\langle A(u_n), u_n\rangle \to \langle A(u), u\rangle$. We will use these convergences in the sequel.

Pseudo-monotone operators lead to useful conditions for the existence of solutions to certain operator equations. This depends on their surjectivity properties, as stated in the following theorem (see also Papageorgiou & Winkert [20, Theorem 6.1.57]); see also Papageorgiou et al. [18] for related abstract results.

Theorem 2. *If $A\colon X \to X^*$ is a pseudo-monotone, bounded, and coercive operator, defined on a real and reflexive Banach space X. Then, the equation $Au = b$ with $b \in X^*$, admits a solution.*

Remaining focused on the problem of solutions to operator equations, we note the following byproduct of the Brouwer fixed point theorem.

Proposition 3. *Given a continuous map $A : X \to X^*$, with $(X, \|\cdot\|_X)$ being a normed finite-dimensional space, then*

If there exists some $R > 0$ such that

$$\langle A(w), w\rangle \ge 0 \quad \textit{for all } w \in X \textit{ with } \|w\|_X = R,$$

then $A(w) = 0$ has a solution $\widehat{w} \in X$ satisfying the upper bound condition $R \ge \|\widehat{w}\|_X$.

3. Assumptions and results

In this section, we discuss the assumptions used in developing our study. Then, we present the obtained results. About the exponent $p \in C(\overline{\Omega})$, we require the following condition involving the finite values p^- and p^+:

$(A1)$ $p \in C(\overline{\Omega})$ is finite with $p^- > p^+/2$.

The relevance in adopting such a condition, can be easily clarified referring to [10, Theorem 1.1] where the authors provide sufficient conditions to obtain the existence of a weak solution to a degenerate (sign-changing) Kirchhoff equation without convection term. Moreover, $(A1)$ is adopted in [24] in the case of convection. We complete the set of assumptions, controlling the growth of the Carathéodory nonlinearity $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, as follows:

$(A2)$ there exist $\sigma \in L^{\alpha'(x)}(\Omega)$, $\alpha \in C(\overline{\Omega})$ with $1 < \alpha(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$ and $c > 0$ such that

$$|f(x, z, y, v)| \le c(\sigma(x) + |z|^{\alpha(x)-1} + |y|^{\frac{p(x)}{\alpha'(x)}} + |v|^{\frac{p(x)}{\alpha'(x)}})$$

for a.a. $x \in \Omega$, all $z, v \in \mathbb{R}$, all $y \in \mathbb{R}^N$;

$(A3)$ there exist $\sigma_0 \in L^1(\Omega)$ and $b_1, b_2, b_3 \ge 0$ such that

$$|f(x, z, y, v)z| \le \sigma_0(x) + b_1|z|^{p(x)} + b_2|y|^{p(x)} + b_3|v|^{p(x)}$$

for a.a. $x \in \Omega$, all $z, v \in \mathbb{R}$, all $y \in \mathbb{R}^N$.

Assumptions $(A2)$ and $(A3)$ are dictated by the technical needs of our proofs and are useful to establish a priori bounds to integral terms, and sign constraints to the involved operators (see Section 4). On the other hand, we note that in dealing with practical situations (remaining into a physical context, we think to evolution systems and related problems), it is natural to impose control constraints on the growth of involved terms. Using $(A3)$ we can obtain the inequality

$$\begin{aligned} (4) \qquad \int_\Omega |f(x,u,\nabla u,\Delta u)u|dx &\leq \lambda^* \|\Delta u\|^{p^+}_{L^{p(x)}(\Omega)} + \|\sigma_0\|_{L^1(\Omega)}, \\ &\leq \lambda^* \|u\|^{p^+} + \|\sigma_0\|_{L^1(\Omega)} \quad (\|u\| \geq \|\Delta u\|_{L^{p(x)}(\Omega)}, \text{ by } (2)), \end{aligned}$$

for all $u \in W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$ with $\|\Delta u\|_{L^{p(x)}(\Omega)} \geq 1$, where we set $\lambda^* = (b_1 + b_2)c_3 + b_3$, for some constant $c_3 > 0$. The relevance in getting such an estimate, follows immediately from the definition of weak solution to (P). We note that $u \in W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$ satisfying $u = \Delta u = 0$ on $\partial\Omega$ is a weak solution to (P) if

$$(5) \qquad \langle -\Delta^2_{k,p} u, w\rangle = \int_\Omega f(x,u,\nabla u,\Delta u) w dx$$

for all $w \in W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$. Of course, $u \equiv 0$ such that (5) holds true, is a trivial weak solution of (P).

Before presenting the results of this manuscript, we introduce the last ingredient of our strategy, namely the discrete Galerkin approximation (i.e., Galerkin basis) of the separable Banach space $W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$. That is, we introduce a sequence $\{V_n\}_{n\in\mathbb{N}}$ of vector subspaces of $W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$ with the following properties:

(i) each subspace is finite dimensional, that is, $\dim(V_n) < +\infty$ for all $n \in \mathbb{N}$;
(ii) each previous subspace of the sequence is contained in the subsequent one, that is, $V_n \subseteq V_{n+1}$ for all $n \in \mathbb{N}$;
(iii) the closure of the union of all subspaces is the vector space, that is, $\overline{\cup_{n\in\mathbb{N}} V_n} = W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$.

The Galerkin basis above, means an approximation sequence of a given Banach space, via finite-dimensional subspaces. It is strongly related to the well-known Galerkin method for numerical approximation of solutions to continuous problems by discrete finite-dimensional problems. This approach works well for operator equations in weak form as equation (5), and hence we establish the following result.

Proposition 4. *Let $\{V_n\}_{n\in\mathbb{N}}$ be a Galerkin basis of $W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$. If assumptions $(A1) - (A3)$ hold, then for all $n \in \mathbb{N}$ we can find $u_n \in V_n$ satisfying*

$$(6) \qquad \langle -\Delta^2_{k,p} u_n, w\rangle = \int_\Omega f(x,u_n,\nabla u_n,\Delta u_n) w dx \quad \text{for all } w \in V_n.$$

Turning to the idea behind the Galerkin approximation method, our next step is to ensure suitable properties of the approximation sequence of solutions originated in Proposition 4, namely the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^\infty V_n$. Thus, the boundedness in $W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$ of $\{u_n\}_{n\in\mathbb{N}}$ is established in the following proposition.

Proposition 5. *Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^\infty V_n$ be the Galerkin sequence originated in Proposition 4. If assumptions $(A1) - (A3)$ hold, then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega)$.*

We note that we will use the Galerkin sequence $\{u_n\}_{n\in\mathbb{N}}$ in working with a special class of maps, namely the Nemitsky maps. Thus, for the Carathéodory function f, we introduce the Nemitsky map $N^*_f : W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega) \subset L^{\alpha(x)}(\Omega) \to L^{\alpha'(x)}(\Omega)$ given as

$$(7) \qquad N^*_f(u)(\cdot) = f(\cdot, u(\cdot), \nabla u(\cdot), \Delta u(\cdot)) \quad \text{for all } u \in W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega).$$

Referring to the works of Fan & Zhao [6] and Kováčik & Rákosník [13], one can show that this map is well-defined, bounded and continuous. We note that this characterization of $N_f^*(\cdot)$ follows directly by assumption $(A2)$. Additionally denote the dual space $\mathbb{W}(\Omega) = (W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega))^*$, we have that $i^* : L^{\alpha'(x)}(\Omega) \to \mathbb{W}(\Omega)$ is a continuous embedding (recall Proposition 2). Consequently, the operator $N_f : W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \to \mathbb{W}(\Omega)$ defined by $N_f = i^* \circ N_f^*$ is bounded and continuous. Using this operator and referring to Proposition 3, we establish the following existence theorem.

Theorem 3. *Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^{\infty} V_n$ be the Galerkin sequence originated in Proposition 4. If*

$$\liminf_{n\to+\infty} |K(p, \Delta u_n)| > 0,$$

and the assumptions $(A1)$–$(A3)$ hold, then problem (P) admits a weak solution $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

We note that the additional assumption in Theorem 3 (that is, the fact that the sequence $\{|K(p, \Delta u_n)|\}_{n\in\mathbb{N}}$ admits a positive inferior limit as n goes to infinity) is not so restrictive. Indeed, it means that

$$\int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} dx \not\to \frac{a}{b} \quad \text{for } n \to +\infty,$$
$$\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^{\infty} V_n.$$

This assumption does not avoid the weight degeneracy of the main operator $\Delta^2_{k,p}$, but permits us to prove the result for certain special sequences of type $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^{\infty} V_n$. This is coherent with the setting of approximation theory and numerical analysis. Part of this strategy is dictated by the (S_+)-property of operators (see Definition 1) and its involvement in the proof of the theorem.

4. Proofs of results

We first establish the existence of an approximation sequence of solutions to the type equation (6), for all $n \in \mathbb{N}$. The proof develops a bound from below for an appropriate operator (see equation (8) of the proof), and then uses Proposition 3 to conclude. The arguments of proof are carried out in the finite dimensional space V_n (clearly that holds for each $n \in \mathbb{N}$).

Proof of Proposition 4. Given $n \in \mathbb{N}$, we introduce the operator $A_n : V_n \to V_n^*$ defined by

$$\langle A_n(u), w\rangle = \langle -\Delta^2_{k,p} u, w\rangle - \int_\Omega f(x, u, \nabla u, \Delta u) w dx \tag{8}$$

for all $u, w \in V_n$. Referring to the bound condition (4), if $\|w\| \geq \|\Delta w\|_{L^{p(x)}(\Omega)} > 1$ we have

$$\begin{aligned}
&\langle -A_n(w), w\rangle \\
&= \left(b \int_\Omega \frac{1}{p(x)} |\Delta w|^{p(x)} dx - a \right) \int_\Omega |\Delta w|^{p(x)} dx + \int_\Omega f(x, w, \nabla w, \Delta w) w dx \\
&\geq \left(b \int_\Omega \frac{1}{p(x)} |\Delta w|^{p(x)} dx - a \right) \int_\Omega |\Delta w|^{p(x)} dx - \int_\Omega |f(x, w, \nabla w, \Delta w) w| dx \\
&\geq \frac{b}{p^+} \|\Delta w\|^{2p^-}_{L^{p(x)}(\Omega)} - a \|\Delta w\|^{p^+}_{L^{p(x)}(\Omega)} - \lambda^* \|\Delta w\|^{p^+}_{L^{p(x)}(\Omega)} - \|\sigma_0\|_{L^1(\Omega)} \\
&\quad \text{(here we use (4))} \\
&\geq \frac{b}{p^+} \|\Delta w\|^{2p^-}_{L^{p(x)}(\Omega)} - (a + \lambda^*) \|\Delta w\|^{p^+}_{L^{p(x)}(\Omega)} - \|\sigma_0\|_{L^1(\Omega)} \\
&\geq \frac{b}{p^+} \|\Delta w\|^{2p^-}_{L^{p(x)}(\Omega)} - (a + \lambda^* + \|\sigma_0\|_{L^1(\Omega)}) \|\Delta w\|^{p^+}_{L^{p(x)}(\Omega)}
\end{aligned}$$

(here we use $\|\Delta w\|_{L^{p(x)}(\Omega)} > 1$)

$$\geq \frac{b}{p^+ c_2}\|w\|^{2p^-} - (a + \lambda^* + \|\sigma_0\|_{L^1(\Omega)})\|w\|^{p^+}$$

(here we use (2)).

Summing up, we obtain the inequality

$$\langle -A_n(w), w\rangle \geq \|w\|^{p^+}\left[\frac{b}{p^+ c_2}\|w\|^{2p^- - p^+} - a - \lambda^* - \|\sigma_0\|_{L^1(\Omega)}\right]$$

for all $w \in V_n$. Thus, we deduce the non-negativity condition

$$\langle -A_n(w), w\rangle \geq 0 \quad \text{if } \|w\| \geq \left[\frac{p^+ c_2}{b}(a + \lambda^* + \|\sigma_0\|_{L^1(\Omega)})\right]^{1/(2p^- - p^+)}.$$

Next, we prepare the application of Proposition 3, and hence we choose and fix a value $R > \max\left\{\left[\frac{p^+ c_2}{b}(a + \lambda^* + \|\sigma_0\|_{L^1(\Omega)})\right]^{1/(2p^- - p^+)}, 1\right\}$. Consequently, for each element w of the generic subspace V_n, in the Galerkin sequence, satisfying $\|w\| = R$ we get

$$\langle -A_n(w), w\rangle \geq 0.$$

These are the hypotheses of Proposition 3, and consequently the operator equation $-A_n(w) = 0$ is solved by a suitable $u_n \in V_n$. Clearly, the same conclusion will hold for its opposite counterpart $A_n(w) = 0$, and hence equation (6) is established. $\square$

We remark that Proposition 4 gives us a sequence of solutions of problems in the form (P), but restricted to finite dimensional spaces (namely, V_n for all $n \in \mathbb{N}$). But these finite dimensional spaces are linked each others since they are elements of the Galerkin basis $\{V_n\}_{n\in\mathbb{N}}$ of $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Next proof is aimed to show the boundedness of the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^{\infty} V_n$.

Proof of Proposition 5. The proof requires estimates of the quantity $\|\Delta u_n\|_{L^{p(x)}(\Omega)}$, in the form of bounds from above. Precisely, we will show that

$$\|\Delta u_n\|_{L^{p(x)}(\Omega)} \leq \max\left\{\left[\frac{p^+}{b}(a + \lambda^* + \|\sigma_0\|_{L^1(\Omega)})\right]^{1/(2p^- - p^+)}, 1\right\} \quad \text{for all } n \in \mathbb{N}. \tag{9}$$

If $\|\Delta u_n\|_{L^{p(x)}(\Omega)} \leq 1$ for all $n \in \mathbb{N}$, then the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Whenever $\|\Delta u_n\|_{L^{p(x)}(\Omega)} > 1$ (for some $n \in \mathbb{N}$), we note that

$$\|\Delta u_n\|^{2p^- - p^+}_{L^{p(x)}(\Omega)} \leq \frac{p^+}{b}\left(a + \lambda^* + \|\sigma_0\|_{L^1(\Omega)}\right).$$

In fact, referring to equation (6) (that is starting from the result of Proposition 4), for the choice $w = u_n$ we deduce that

$$\begin{aligned}
\frac{b}{p^+}\|\Delta u_n\|^{2p^-}_{L^{p(x)}(\Omega)} &\leq a\|\Delta u_n\|^{p^+}_{L^{p(x)}(\Omega)} - \int_\Omega f(x, u_n, \nabla u_n, \Delta u_n)u_n dx \\
&\leq a\|\Delta u_n\|^{p^+}_{L^{p(x)}(\Omega)} + \int_\Omega |f(x, u_n, \nabla u_n, \Delta u_n)u_n| dx \\
&\leq a\|\Delta u_n\|^{p^+}_{L^{p(x)}(\Omega)} + \lambda^*\|\Delta u_n\|^{p^+}_{L^{p(x)}(\Omega)} + \|\sigma_0\|_{L^1(\Omega)}
\end{aligned}$$

(here we use the estimate (4)).

We assumed before that $\|\Delta u_n\|_{L^{p(x)}(\Omega)} > 1$, and hence we have

$$\frac{b}{p^+}\|\Delta u_n\|^{2p^-}_{L^{p(x)}(\Omega)} \leq \left(a + \lambda^* + \|\sigma_0\|_{L^1(\Omega)}\right)\|\Delta u_n\|^{p^+}_{L^{p(x)}(\Omega)},$$

and dividing both sides of the inequality by $\frac{b}{p^+}\|\Delta u_n\|^{p^+}_{L^{p(x)}(\Omega)}$, we get

$$\|\Delta u_n\|^{2p^- - p^+}_{L^{p(x)}(\Omega)} \leq \frac{p^+}{b}\left(a + \lambda^* + \|\sigma_0\|_{L^1(\Omega)}\right).$$

But this implies that (9) holds true. Of course, it follows trivially that the Galerkin sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^{\infty} V_n$ is bounded in the Banach space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. □

We are ready for the proof of our convergence result.

Proof of Theorem 3. Starting from the boundedness of the approximation sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^{\infty} V_n$ in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ (as follows by Proposition 5), upon appealing to the reflexivity of this Banach space we note that for some $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, we can assume

$$u_n \xrightarrow{w} u \text{ in } W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \text{ and } u_n \to u \text{ in } L^{\alpha(x)}(\Omega). \tag{10}$$

Referring to the boundedness of the Nemitsky map in (7), it follows that the sequence

$$\{N_f(u_n)\}_{n\in\mathbb{N}} \quad \text{is bounded in } \mathbb{W}(\Omega).$$

Additionally, the operator $-\Delta^2_{k,p} : W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \to \mathbb{W}(\Omega)$ is bounded too, and hence also the sequence

$$\{-\Delta^2_{k,p} u_n - N_f(u_n)\}_{n\in\mathbb{N}} \quad \text{is bounded in } \mathbb{W}(\Omega). \tag{11}$$

If necessary, we can consider a relabeled subsequence of (11) to conclude that

$$-\Delta^2_{k,p} u_n - N_f(u_n) \xrightarrow{w} g \text{ in } \mathbb{W}(\Omega), \text{ for some } g \in \mathbb{W}(\Omega), \tag{12}$$

this is an immediate consequence of the fact that the dual space $\mathbb{W}(\Omega)$ is reflexive. Moreover, we can select w in $\cup_{n=1}^{\infty} V_n$, so that there exists an index $n(w) \in \mathbb{N}$ satisfying

$$w \in V_{n(w)}.$$

Of course, Proposition 4 says us that equation (6) remains true for each $n \geq n(w)$. We pass n to infinity in the same (6) to get

$$\langle g, w\rangle = 0 \quad \text{for all } w \in \cup_{n=1}^{\infty} V_n.$$

Referring to the properties of the Galerkin basis $\{V_n\}_{n\in\mathbb{N}}$ (see Section 3) we know that $\cup_{n=1}^{\infty} V_n$ is dense in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Therefore, this leads to the conclusion $g = 0$, and using (12) we get

$$-\Delta^2_{k,p} u_n - N_f(u_n) \xrightarrow{w} 0 \text{ in } \mathbb{W}(\Omega). \tag{13}$$

Turning to equation (6), we consider $w = u_n$ and obtain

$$\langle -\Delta^2_{k,p} u_n - N_f(u_n), u_n\rangle = 0 \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

By (13) we have

$$\langle -\Delta^2_{k,p} u_n - N_f(u_n), u\rangle \to 0 \quad \text{as } n \to +\infty,$$

and using (14) we get

$$\lim_{n\to+\infty} \langle -\Delta^2_{k,p} u_n - N_f(u_n), u_n - u\rangle = 0. \tag{15}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded, then $\{N_f^*(u_n)\}_{n\in\mathbb{N}}$ is bounded too. Using this fact along with Hölder's inequality and the compact embedding $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ (see Proposition 1), we get

$$\left|\int_\Omega f(x, u_n, \nabla u_n, \Delta u_n)(u_n - u)\, dx\right|$$

$$\leq 2\|N_f^*(u_n)\|_{L^{\alpha'(x)}(\Omega)}\|u-u_n\|_{L^{\alpha(x)}(\Omega)}$$
$$\leq 2\left(\sup_{n\in\mathbb{N}}\|N_f^*(u_n)\|_{L^{\alpha'(x)}(\Omega)}\right)\|u-u_n\|_{L^{\alpha(x)}(\Omega)}\to 0 \quad \text{as } n\to+\infty.$$

It follows that

$$\lim_{n\to+\infty}\langle -\Delta^2_{k,p}u_n, u_n-u\rangle = 0 \quad \text{(recall (15))}. \tag{16}$$

Combining (10), (13) and (16) we conclude that $u\in W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$ is an approximated solution to (P), in the sense that u is the weak limit of the Galerkin (approximation) sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq\cup_{n=1}^{\infty}V_n$.

Without any loss of generality, from the assumption $\liminf_{n\to+\infty}|K(p,\Delta u_n)|>0$ we can consider the case where

$$\liminf_{n\to+\infty}K(p,\Delta u_n)>0,$$

that is we remove the absolute value above (however, the other case can be concluded in a similar fashion). Thus, we can find a relabeled subsequence of $\{u_n\}_{n\in\mathbb{N}}$ satisfying the limit condition

$$K(p,\Delta u_n)\to K_0>0 \quad \text{as } n\to+\infty. \tag{17}$$

Now, (16) jointly with (17) leads to the limit

$$\lim_{n\to+\infty}\langle -\Delta^2_{p(x)}u_n, u_n-u\rangle\leq 0,$$

which gives us the $(S)_+$-property of the $p(x)$-biharmonic operator (see [1], Proposition 4.2 (iii)), provided that $u_n\to u$ in $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$, as n goes to infinity. Using the convergence (13), we know that

$$-\Delta^2_{k.p}u_n - f(x,u_n,\nabla u,\Delta u_n)\xrightarrow{w}0 \text{ in } \mathbb{W}(\Omega).$$

This means that the following equality occurs

$$-\Delta^2_{k,p}u - f(x,u,\nabla u,\Delta u)=0.$$

Consequently, we get that $u\in W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$ solves problem (P), in the sense of weak solutions (namely, we retrieve (5)). This completes the proof. □

Example 1. Consider $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ given as follows:

$$f(x,u,\nabla u,\Delta u)=f_1(x,u)+f_2(x,\nabla u)+f_3(x,\Delta u)$$

where $f_1,f_3:\Omega\times\mathbb{R}\to\mathbb{R}$ and $f_2:\Omega\times\mathbb{R}^N\to\mathbb{R}$ are continuous functions, with f_1 positive function that grows slower than a suitable power of the unknown variable u, f_2 bounded from above by a gradient term, and f_3 bounded from above by a Laplacian term. For these functions, we assume that we can find $\sigma_i\in L^{\alpha'(x)}(\Omega)$ $(i=1,2,3)$, $\alpha\in C(\overline{\Omega})$ with $1<\alpha(x)\leq p(x)$ for all $x\in\overline{\Omega}$, such that:

$(H1)$ there exists $b_1>0$ satisfying

$$0<f_1(x,z)\leq\sigma_1(x)+b_1|z|^{\alpha(x)-1} \quad \text{for a.a. } x\in\Omega, \text{ all } z\in\mathbb{R};$$

$(H2)$ there exists $b_2\geq 0$ satisfying

$$0\leq f_2(x,y)\leq\sigma_2(x)+b_2|y|^{\frac{p(x)}{\alpha'(x)}} \quad \text{for a.a. } x\in\Omega, \text{ all } y\in\mathbb{R}^N;$$

$(H3)$ there exists $b_3\geq 0$ satisfying

$$0\leq f_3(x,v)\leq\sigma_3(x)+b_3|v|^{\frac{p(x)}{\alpha'(x)}} \quad \text{for a.a. } x\in\Omega, \text{ all } v\in\mathbb{R}.$$

It is clear that assumptions like $(H1)-(H3)$ can be seen as an immediate way to decompose the effects of a global reaction f and identify its basic components (for example, this is of a certain interest in population models to better control the dynamics of reaction-diffusion processes). On the other hand, combining assumptions $(H1)-(H3)$, it follows easily that f satisfies $(A2)-(A3)$.

Remark 2. In the case of positive nonlinearity (for example, refer to the situation in Example 1, by assumption $(H1)$), Theorem 3 ensures the existence of a weak solution to problem (P), namely $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ with $u \neq 0$.

5. Case of non-degenerate Kirchhoff term

In this section, we consider the case of a non-degenerate (constant sign) Kirchhoff term of the form

$$K_+(p, \Delta u) = a + b \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx, \quad \text{for some } a, b > 0, \tag{18}$$

and hence we assume

$$K_+(p, \Delta u) \geq a > 0 \quad \text{for all } u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega).$$

This time, we consider the operator $-\Delta_{k,p}^{2,+} : W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \to \mathbb{W}(\Omega)$ defined by

$$\langle -\Delta_{k,p}^{2,+} u, w \rangle = K_+(p, \Delta u) \langle -\Delta_{p(x)}^2 u, w \rangle = K_+(p, \Delta u) \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta w dx$$

for all $u, w \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Thus, we discuss the existence of weak solutions to the Navier problem

$$-\Delta_{k,p}^{2,+} u(x) = f(x, u(x), \nabla u(x), \Delta u(x)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = 0. \tag{19}$$

We derive the definition of weak solution to (19) as follows

$u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ is a weak solution to (19) if

$$\langle -\Delta_{k,p}^{2,+} u, w \rangle = \int_\Omega f(x, u, \nabla u, \Delta u) w dx$$

for all $w \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, and $u = \Delta u = 0$ on $\partial\Omega$.

We note that $-\Delta_{p(x)}^2$ is continuous, bounded, strictly monotone and of type $(S)_+$. Thus, the new operator $-\Delta_{k,p}^{2,+}$ is also bounded, continuous and satisfies the property $(S)_+$ (recall, that based on the assumption $K_+(p, \Delta u) \geq a > 0$, the last operator $-\Delta_{k,p}^{2,+}$ can be considered as positive-weight version of the variable exponent $p(x)$-biharmonic operator.

Since gradient and Laplacian dependences are again a main feature of our nonlinearity, clearly we cannot adopt variational tools. Thus, we revisit the theory of pseudo-monotone operators to develop a topological approach. For the Nemitsky map $N_f : W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \to \mathbb{W}(\Omega)$, we consider the operator $A : W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \to \mathbb{W}(\Omega)$ given as

$$A(u) = -\Delta_{k,p}^{2,+} u - N_f(u) \quad \text{for all } u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega). \tag{20}$$

Clearly, this operator is bounded and continuous. Additionally, we show that (20) is coercive and pseudo-monotone.

Starting from the coercivity proof, using assumption $(A3)$, for all $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ with $\|\Delta u\|_{L^{p(x)}(\Omega)} > 1$ we get

$$\langle A(u), u \rangle = \left(a + b \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx \right) \int_\Omega |\Delta u|^{p(x)} dx - \int_\Omega f(x, u, \nabla u, \Delta u) u dx$$

$$\geq \frac{b}{p^+}\|\Delta u\|_{L^{p(x)}(\Omega)}^{2p^-} - (a+\lambda^* + \|\sigma_0\|_{L^1(\Omega)})\|\Delta u\|_{L^{p(x)}(\Omega)}^{p^+}$$
$$\geq \frac{b}{p^+ c_2}\|u\|^{2p^-} - c_4\|u\|^{p^+} \quad \text{for some } c_4 > 0$$

(here we use the inequality (2)).

Therefore the coercivity of (20) follows immediately since $p^+ < 2p^-$.

Next, we conclude the pseudo-monotonicity of (20), using the following arguments.

Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ satisfy

$$u_n \xrightarrow{w} u \text{ in } W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0. \tag{21}$$

On the other hand, requirement (21) implies that

$$\limsup_{n\to+\infty} \big[\langle -\Delta_{k,p}^{2,+} u_n, u_n - u\rangle - \int_\Omega f(x, u_n, \nabla u_n, \Delta u_n)(u_n - u)dx\big] \leq 0. \tag{22}$$

Moreover, we note that $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ and it is bounded. Then, we deduce that the sequence $\{N_f^*(u_n)\}_{n\in\mathbb{N}}$ is bounded too. An application of Hölder's inequality, jointly with compactness of the embedding $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ (we refer to Proposition 1), are sufficient enough to conclude that

$$\int_\Omega f(x, u_n, \nabla u_n, \Delta u_n)(u_n - u)dx \to 0 \qquad \text{as } n \to +\infty. \tag{23}$$

By (22) we derive the (strong) convergence of $\{u_n\}_{n\in\mathbb{N}}$ as follows

$$\begin{aligned} &\limsup_{n\to+\infty}\langle -\Delta_{k,p}^{2,+} u_n, u_n - u\rangle \leq 0, \\ \Rightarrow\quad & u_n \to u \text{ in } W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \\ & (\text{since } -\Delta_{k,p}^{2,+} \text{ has the } (S)_+\text{-property}). \end{aligned} \tag{24}$$

The convergence in (24) and the fact that the operator (20) is continuous, give us

$$A(u_n) \to A(u), \quad \langle A(u_n), u_n\rangle \to \langle A(u), u\rangle,$$

and therefore (20) is pseudo-monotone.

Based on the above properties of the operator (20) we establish the following existence theorem.

Theorem 4. *If assumptions* $(A1) - (A3)$ *hold, then* (19) *admits at least a weak solution.*

The proof of Theorem 4 is a consequence of the application of Theorem 2 to the operator (20). Indeed, Theorem 2 ensures that the pseudo-monotone, bounded and coercive operator (20) defined on the real and reflexive Banach space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ is such that the equation

$$A(\widehat{u}) = 0 \quad (\text{here, for the special choice } b = 0 \in \mathbb{W}(\Omega))$$

admits a solution $\widehat{u} \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Of course, that $\widehat{u} \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ is a weak solution to (19).

Remark 3. It is obvious that Theorem 4 can be seen as a byproduct of Theorem 3, but this time the proof does not use the approximation arguments and can be developed just adapting the theory of operators of monotone type in Banach spaces.

As a special case of nonlinearity $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, one can consider the following function

$$f(x, u, \nabla u, \Delta u) = f_1(x, u, \nabla u, \Delta u) - f_2(x, u, \nabla u, \Delta u),$$

which means a gradient and Laplacian dependent logistic-type nonlinearity. To recover our framework, we suppose that $f_1, f_2 : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the following assumptions:

$(L1)$ $f_1(x, z, y, v) - f_2(x, z, y, v) \geq 0$ for a.a. $x \in \Omega$, all $z \in \mathbb{R}$, all $y \in \mathbb{R}^N$, all $v \in \mathbb{R}$;

$(L2)$ $f_i(x, z, y, v) = 0$ for a.a. $x \in \Omega$ $(i = 1, 2)$, all $z \leq 0$, all $y \in \mathbb{R}^N$, all $v \in \mathbb{R}$, and there exist $\sigma_i \in L^\infty$ $(i = 1, 2)$ and $\alpha \in C(\overline{\Omega})$ with $1 < \alpha(x) \leq p(x)$ for all $x \in \overline{\Omega}$ such that

$$|f_i(x, z, y, v)| \leq \sigma_i(x)(1 + |z|^{\alpha(x)-1} + |y|^{\frac{p(x)}{\alpha'(x)}} + |v|^{\frac{p(x)}{\alpha'(x)}})$$

for a.a. $x \in \Omega$, all $z \geq 0$, all $y \in \mathbb{R}^N$, all $v \in \mathbb{R}$.

In the following example, we remove the gradient and Laplacian dependences in the nonlinearity.

Example 2. Consider $f : \Omega \times \mathbb{R} \to \mathbb{R}$ given as follows:

$$f(x, z) = f_1(x, z) - f_2(x, z),$$

where $f_1, f_2 : \Omega \times \mathbb{R} \to [0, +\infty)$ are defined by:

$$f_1(x, z) = \begin{cases} z^{\beta(x)-1} & \text{for } z > 0, \\ 0 & \text{for } z \leq 0, \end{cases}$$

and

$$f_2(x, z) = \begin{cases} z^{\beta(x)-1} \ln z & \text{for } z > 1, \\ z^{\beta(x)-1} & \text{for } z \in (0, 1], \\ 0 & \text{for } z \leq 0, \end{cases}$$

with $\beta \in C(\overline{\Omega})$ bounded away from 1. If $1 < \beta(x) < \alpha(x) \leq p(x)$ for all $x \in \overline{\Omega}$, then we have

$$|f(x, z, y, v)| \leq b_0 z^{\alpha(x)-1} \text{ and } |f(x, z, y, v)|z \leq b_1 z^{p(x)},$$

for a.a. $x \in \Omega$, all $z \geq 0$, all $y \in \mathbb{R}^N$, all $v \in \mathbb{R}$, some $b_0, b_1 > 0$.

In the next example, we depict a situation where we deal with a Laplacian term competing against a gradient dependent term.

Example 3. Consider $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given as follows:

$$f(x, u, \nabla u, \Delta u) = \lambda |\Delta u|^{\frac{p(x)-\alpha'(x)}{\alpha'(x)}} \Delta u - h(x, u, \nabla u), \quad \lambda > 0,$$

where $\alpha \in C(\overline{\Omega})$ with $1 < \alpha(x) \leq p(x)$ for all $x \in \overline{\Omega}$, and $h : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is the following continuous function:

$$h(x, z, y) = b_1 |z|^{p(x)-2} z - b_2 |y|^{\frac{\alpha(x)}{\alpha'(x)}}, \quad b_1 \geq 0,\ b_2 > 0.$$

We remark that

$$|h(x, z, y)| \leq b_1 |z|^{\alpha(x)-1} + b_2 |y|^{\frac{\alpha(x)}{\alpha'(x)}} \quad \text{and} \quad |h(x, z, y)z| \leq c_5(|z|^{\alpha(x)} + |y|^{p(x)}),$$

for all $x \in \Omega$, all $z \in \mathbb{R}$, all $y \in \mathbb{R}^N$, some $c_5 = c_5(b_1, b_2, \alpha^-, (\alpha')^-) > 0$. Then, the assumptions $(A_2) - (A3)$ hold true easily.

In the above examples we linked the exponents $\alpha, \beta \in C(\overline{\Omega})$ directly to $p \in C(\overline{\Omega})$ instead than to the critical Sobolev exponent $p_2^*(\cdot)$ (recall definition (3)). Adopting a similar setting in our assumptions, we are able to perform an asymptotic analysis of our problem. Thus, we revise assumption $(A3)$ as follows:

$(A3)'$ there exist $\sigma_0 \in L^1(\Omega)$, $\beta \in C(\overline{\Omega})$ with $1 < \beta(x) \le \beta^+ < p^- \le p(x)$ for all $x \in \overline{\Omega}$ and $b_1, b_2, b_3 \ge 0$ such that

$$|f(x,z,y,v)z| \le \sigma_0(x) + b_1|z|^{\beta(x)} + b_2|y|^{\beta(x)} + b_3|v|^{p(x)}$$

for a.a. $x \in \Omega$, all $z, v \in \mathbb{R}$, all $y \in \mathbb{R}^N$.

Clearly, Theorem 4 remains true if we change assumption $(A3)$ by $(A3)'$, as assumption $(A3)'$ implies $(A3)$.

Referring to the presence of the nonlocal term $b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx$ $(b > 0)$, which changes the geometry of problem (19) respect to the case where $b = 0$ in (18), we note that it is interesting to regard b as a parameter and investigate the asymptotic behavior of weak solutions to (19) as $b \downarrow 0$. The similar idea and convergence study are proposed in Shuai [23] and in a series of subsequent papers. To prepare the setting, we introduce the sets

$$\mathbb{S}_b = \text{solution set to (19), as } b \ge 0 \text{ is fixed},$$
$$\mathbb{S} = \cup_{b\ge 0}\, \mathbb{S}_b = \text{solution set to (19)}.$$

If assumptions (A_1), (A_2) and $(A3)'$ hold, we note that $\mathbb{S}_b$ and $\mathbb{S}$ are bounded in $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$, provided that $a > b_3$ (recall (18)). Fixed $b \ge 0$, without loss of generality, we choose a solution $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ of (19) satisfying the norm inequality $\|u\| > 1$. Since u is a weak solution, for a test function $w = u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ we obtain the estimates

$$\begin{aligned}
a\int_\Omega |\Delta u|^{p(x)}dx &\le \langle -\Delta_{k,p}^{2,+}u, u\rangle \\
&= \int_\Omega |f(x,u,\nabla u,\Delta u)u|dx \\
&\le \int_\Omega \left(\sigma_0(x) + b_1|u|^{\beta(x)} + b_2|\nabla u|^{\beta(x)} + b_3|\Delta u|^{p(x)}\right)dx \\
&\quad \text{(here we use } (A3)'\text{)} \\
&\le \|\sigma_0\|_{L^1(\Omega)} + (\lambda^* - b_3)\|\Delta u\|_{L^{p(x)}(\Omega)}^{\beta^+} + b_3\int_\Omega |\Delta u|^{p(x)}dx, \\
&\quad \text{(recall } \lambda^* = (b_1+b_2)c_3 + b_3, \text{ for some } c_3 > 0), \\
\Rightarrow \quad \int_\Omega |\Delta u|^{p(x)}dx &\le \frac{\|\sigma_0\|_{L^1(\Omega)} + (\lambda^* - b_3)\|\Delta u\|_{L^{p(x)}(\Omega)}^{\beta^+}}{a - b_3}.
\end{aligned}$$

Summing up, we deduce that

$$\|\Delta u\|_{L^{p(x)}(\Omega)}^{p^-} \le \frac{\|\sigma_0\|_{L^1(\Omega)} + (\lambda^* - b_3)\|\Delta u\|_{L^{p(x)}(\Omega)}^{\beta^+}}{a - b_3},$$

and hence

$$\|u\|^{p^-} \le c_2\frac{\|\sigma_0\|_{L^1(\Omega)} + (\lambda^* - b_3)\|u\|^{\beta^+}}{a - b_3}. \tag{25}$$

Since $\beta^+ < p^-$ by $(A3)'$ we conclude that $\mathbb{S}_b$ is bounded in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Next, we note that (25) does not dependent on b, and hence this inequality can be established for every $u \in \mathbb{S}$. It follows that the set $\mathbb{S} = \cup_{b\ge 0}\mathbb{S}_b$ is bounded in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

Based on the above properties (that is, the boundedness of the sets $\mathbb{S}$ and $\mathbb{S}_b$), we note that $u_n \in \mathbb{S}_{b_n}$ for all $n \in \mathbb{N}$ implies that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$. Then, we establish the following convergence theorem depicting the behavior of problem (19) in the case $b \downarrow 0$.

Theorem 5. *Let assumptions (A_1), (A_2) and $(A3)'$ with $b_3 < a$ hold. Given a sequence of parameters $\{b_n\}_{n\in\mathbb{N}}$ converging to 0^+, and a sequence $\{u_n\}_{n\in\mathbb{N}}$ of solutions to* (19) *such that $u_n \in \mathbb{S}_{b_n}$ for all $n \in \mathbb{N}$, then there is a relabeled subsequence of $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \to u$ in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ with $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ solution to* (19)*, whenever $b = 0$ in* (18).

The proof of Theorem 5 uses the similar arguments in establishing that (20) is pseudo-monotone. Indeed, since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, then we can find a relabeled subsequence of $\{u_n\}_{n\in\mathbb{N}}$ such that

$$u_n \xrightarrow{w} u \text{ in } W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \text{ and } u_n \to u \text{ in } L^{\alpha(x)}(\Omega), \text{ for some } u \in W_0^{1,p(x)}(\Omega).$$

Thus we get easily (refer to (23)) the convergence

$$\int_\Omega f(x, u_n, \nabla u_n, \Delta u_n)(u_n - u)dx \to 0 \quad \text{as } n \to +\infty,$$

whenever $u_n \to u$ in $L^{\alpha(x)}(\Omega)$ (by assumption (A_2)). Next, $u_n \in \mathbb{S}_{b_n}$ for all $n \in \mathbb{N}$, gives us

$$\langle -\Delta^{2,+}_{k,p} u_n, w\rangle = \int_\Omega f(x, u_n, \nabla u_n, \Delta u_n) w dx \tag{26}$$

for all $w \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. We choose $w = u_n - u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ in (26), and hence we get

$$\langle -\Delta^{2,+}_{k,p} u_n, u_n - u\rangle = \int_\Omega f(x, u_n, \nabla u_n, \Delta u_n)(u_n - u)dx \quad \text{for all } n \in \mathbb{N}. \tag{27}$$

Letting $n \to +\infty$ in (27), since $b_n \downarrow 0$ we obtain

$$\begin{aligned} &\lim_{n\to+\infty} a\langle -\Delta^2_{p(x)} u_n, u_n - u\rangle = 0, \\ \Rightarrow \quad & u_n \to u \text{ in } W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \\ & (\text{since } -\Delta^2_{p(x)} \text{ has the } (S)_+\text{-property}). \end{aligned}$$

From (A_2) we know that $N_f : W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \to \mathbb{W}(\Omega)$ defined by $N_f = i^* \circ N_f^*$ is bounded and continuous (recall the discussion about equation (7)). Thus, we have

$$\langle N_f(u_n), w\rangle \to \langle N_f(u), w\rangle \quad \text{in } \mathbb{W}(\Omega).$$

Since $\langle -\Delta^2_{p(x)} u_n, w\rangle \to \langle -\Delta^2_{p(x)} u, w\rangle$ in $\mathbb{W}(\Omega)$ and

$$\int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} dx \int_\Omega |\Delta u_n|^{p(x)-2} \Delta u_n \Delta w dx \text{ is bounded},$$

then taking the limit in (26) for $n \to +\infty$, we deduce that $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ is a weak solution to (19), whenever $b = 0$ in (18). Such a $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ is a weak solution to the Navier $p(x)$-biharmonic problem

$$-\Delta^2_{p(x)} u(x) = \frac{1}{a} f(x, u(x), \nabla u(x, \Delta u(x))) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = 0.$$

6. Conclusions

This manuscript proposed a topological approach in solving certain classes of boundary value problems. The new leading operator in the elliptic equation is named Kirchhoff type $p(x)$-biharmonic operator. It merges the features of a fourth order operator (namely, the biharmonic operator), constructed over the anisotropic $p(x)$-Laplace operator (in the case $p \in C(\overline{\Omega})$ is bounded and bounded away from 1), and of a nonlocal term (namely, a Kirchhoff type term). The investigated toy problems involve a Navier boundary condition, which gives us that the unknown variable and its Laplacian are null on the boundary of the domain Ω ($\subseteq \mathbb{R}^N$ and bounded). The main results established the existence of at least a weak solution, following two different strategies. The first one is originated by a Galerkin method for numerical approximation of solutions to continuous problems by corresponding discrete finite-dimensional problems. The second one is originated by the classical theory of pseudo-monotone operators, and is applied to a more classical non-degenerate Kirchhoff term (that is, bounded away from a positive value). Summing up, we focused on the impact that a gradient and Laplacian dependent nonlinearity has in the well-posedness of the problem and in the control of its growth via global a priori estimates. The similar results in the paper apply to different boundary conditions, without the need to change the variable space framework $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$. For example, we mention the well-known no-flux condition

$$u\big|_{\partial\Omega} = \text{constant},\ \Delta u\big|_{\partial\Omega} = 0,$$
$$\int_{\partial\Omega} \frac{\partial}{\partial\nu}(|\Delta u|^{p(x)-2}\Delta u)dS = 0.$$

This type of condition is useful to model practical situations of electrorheological and thermorheological fluids, whenever the surfaces are impermeable to certain contaminants (see again [3]).

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