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On the representability of actions of non-associative algebras

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Abstract

Department of Mathematics and Computer Sciences

Doctor of Philosophy

On the representability of actions of non-associative algebras

by Manuel MANCINI

We study the categorical-algebraic condition of *internal actions being weakly representable* in the context of non-associative algebras over a field. It is known that such varieties are action accessible if and only if they are Orzech categories of interest and it is also known that both these conditions are implied by weakly representable actions in this context.

Our first aim is to give a complete characterization of action accessible, operadic quadratic varieties of non-associative algebras which satisfy an identity of degree two (so commutative or anti-commutative algebras) and to study the representability of actions for them. Moreover, we prove that the varieties of two-step nilpotent commutative and anti-commutative algebras are weakly action representable.

Our second aim is to work towards the construction, still within the context of algebras over a field, of a weakly representing object $\mathcal{E}(X)$ for the actions on (or split extensions of) an object X of a variety of non-associative algebras $\mathcal{V}$. We actually obtain a *partial* algebra $\mathcal{E}(X)$, which we call *external weak actor* of X, together with a natural monomorphism of functors $\mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}((X))$, where **PAlg** is the category of partial algebras and $U \colon \mathcal{V} \to \mathbf{PAlg}$ denotes the forgetful functor, which we study in detail in the case of Leibniz algebras, where $\mathcal{E}(X) \cong \mathrm{Bider}(X)$ is the Leibniz algebra of *biderivations* of X. Furthermore, the relations between the construction of the *universal strict general actor* $\mathrm{USGA}(X)$ and that of $\mathcal{E}(X)$ are thoroughly described.

Eventually, we study the representability of actions of the category of (non-commutative) Poisson algebras, showing a possible direction for the construction of the external weak actor for any action accessible variety of algebras with two non-necessarily associative bilinear operations. We conclude with some open problems.

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Contents

Chapter 1

Introduction

Internal actions of the objects of a category were defined in [14] by F. Borceux, G. Janelidze and G. M. Kelly with the aim of extending the correspondence between actions and split extensions from the context of groups and Lie algebras to arbitrary semi-abelian categories [49]. In some of those categories, internal actions are exceptionally well behaved, in the sense that the actions on each object X are *representable*: this means that there exists an object $[X]$ such that the functor $\mathrm{Act}(-, X) \cong \mathrm{SplExt}(-, X)$ which sends an object B to the set of actions of B on X (isomorphisms classes of split extensions of B by X) is naturally isomorphic to the functor $\mathrm{Hom}(-, [X])$. The context of action representable semi-abelian categories is further studied in [15], where it is explained for instance that the category of commutative associative algebras over a field is not action representable. Later it was shown that the only action representable varieties of non-associative algebras over an infinite field $\mathbb{F}$ of characteristic different from 2 are the category **Lie** of Lie algebras and the category **AbAlg** of abelian algebras [44]. The relative strength of the notion naturally led to the definition of closely related weaker notions.

In [21] D. Bourn and G. Janelidze introduced the concept of *action accessible* category in order to include relevant examples that do not fit into the frame of action representable categories (such as rings, associative algebras and Leibniz algebras amongst others). A. Montoli proved in [63] that all *Orzech categories of interest* [66] are action accessible. On the other hand, in [26] the authors showed that a weaker notion of actor (namely, the *universal strict general* actor, USGA for short) is available for any Orzech category of interest $\mathcal{C}$.

The present work focuses on a notion which have been more recently introduced, by G. Janelidze in [47]: *weakly representable actions*. Instead of asking that for each object X in a semi-abelian category $\mathcal{C}$ we have an object $[X]$ and a natural isomorphism $\mathrm{Act}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X])$, we require the existence of an object T and a natural monomorphism of functors

$$\tau\colon \mathrm{Act}(-, X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, T).$$

Such an object T is then called a *weak actor* of X, and when each X admits a weak actor, $\mathcal{C}$ is said to be *weakly action representable*. For instance, if in an Orzech category of interest each $\mathrm{USGA}(X)$ is an object of the category, then this category is weakly action representable [29]. This is the case of the category **Assoc** of associative algebras over a field [47], where $\mathrm{USGA}(X) = \mathrm{Bim}(X)$ is the associative algebra of *bimultipliers* of X (see [59]), or the category **Leib** of Leibniz algebras [29] over a field, where $\mathrm{USGA}(X) = \mathrm{Bider}(X)$ is the Leibniz algebra of *biderivations* of X (see [58] and [62]). J. R. A. Gray observed in [45] that an Orzech category of interest needs not be weakly action representable.

The main aim of this thesis is to study the representability of actions in the context of varieties of non-associative algebras over a field. (We recall basic definitions and results in Chapter 2 and Chapter 3.) It is known that a variety of non-associative algebras is action accessible if and only if it is an Orzech category of interest [41, 42], and it is also known that action accessibility is implied by weak action representability [47].

In Chapter 4 we give a complete classification of action accessible, operadic quadratic varieties of non-associative algebras with an identity of degree 2 (so commutative or anti-commutative algebras) and we study the representability of actions of each of them. Moreover, we prove that the variety of two-step nilpotent commutative algebras and that of two-step nilpotent anti-commutative algebras are weakly action representable, with a weak actor being in both cases an abelian algebras.

In Chapter 5, the representability of actions of the variety **Leib** of Leibniz algebras is studied: we prove that **Leib** is weakly action representable with a weak actor of an object $\mathfrak{g}$ being the Leibniz algebra $\mathrm{Bider}(\mathfrak{g})$ of biderivations of $\mathfrak{g}$ [29]. Moreover, we give the complete classifications of the Leibniz algebras of biderivations of low-dimensional (right) Leibniz algebras over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$, and we introduce an algorithm for finding biderivations of a Leibniz algebra as pairs of matrices with respect to a fixed basis [62].

In Chapter 6 we generalize the study of Chapter 4 and Chapter 5 to a general variety of non-associative algebras over a field: more in detail, we work towards the construction, still within the context of algebras over a field, of a weakly representing object $\mathcal{E}(X)$ for the actions on (split extensions by) an object X. We actually obtain a *partial algebra* $\mathcal{E}(X)$, which we call *external weak actor* of X [43], together with a monomorphism of functors

$$\mathrm{Act}(-, X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}(X)),$$

where **PAlg** is the category of partial algebras over $\mathbb{F}$ (see Section 3.2.4) and $U\colon \mathcal{V} \to \mathbf{PAlg}$ denotes the forgetful functor. Furthermore, we describe in detail the relations between the construction of $\mathcal{E}(X)$ and the one of the universal strict general actor $\mathrm{USGA}(X)$ given in [26].

In Chapter 7 we study the representability of actions of the category **Pois** of (non-commutative) Poisson algebras. We describe explicitly an external weak actor $\mathcal{E}(V)$, which turns out to be also a universal strict general actor, for any Poisson algebra V, and the corresponding monomorphism of functors

$$\tau\colon\ \mathrm{Act}(-, V) \rightarrowtail \mathrm{Hom}_{\mathbf{Alg_2}}(U(-), \mathcal{E}(V)),$$

where $\mathbf{Alg_2}$ is the category of algebras over $\mathbb{F}$ with two non-necessarily associative bilinear operations and $U\colon \mathbf{Pois} \to \mathbf{Alg_2}$ denotes the forgetful functor. This shows a possible direction for the construction of an external weak actor for any variety of algebras with two bilinear operations.

In Chapter 8 we end our work with some open problems and possible future directions.

Chapter 2

The semi-abelian context

Most of this work takes place in the context of *semi-abelian categories* which were introduced in [49] by G. Janelidze, L. Márki and W. Tholen in order to capture categorical-algebraic properties of non-abelian algebraic structures, such as groups, rings and algebras. A category is *semi-abelian* if it is *pointed*, admits *finite coproducts*, is *Barr exact* and *Bourn protomodular*. Pointed means that it has a zero object: an initial object that is also a terminal object. A category is Barr exact if it is *regular* (finitely complete with coequalizers of kernel pairs and pullback-stable regular epimorphisms) and such that any internal equivalence relation is a kernel pair [9]. A pointed and regular category is Bourn protomodular when the *Short Five Lemma* holds: for any commutative diagram

$$\begin{array}{ccccc} K' & \overset{k'}{\rightarrowtail} & E' & \overset{p'}{\twoheadrightarrow} & B' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ K & \underset{k}{\rightarrowtail} & E & \underset{p}{\twoheadrightarrow} & B \end{array}$$

where p and p' are regular epimorphisms, (K, k) and (K', k') are their kernels, if u and w being isomorphisms then so is v [19].

Well-known examples of semi-abelian categories are the category **Grp** of groups, the category **Rng** of not necessarily unitary rings, any variety $\mathcal{V}$ of non-associative algebras over a field $\mathbb{F}$, or any abelian category. A counterexample is given by the category **Ring** of unitary rings, which is not pointed (the initial object is the ring of integers $\mathbb{Z}$ and the terminal one is the zero ring 0).

2.1 Semi-abelian categories

In this section we give an overview of the categorical structures relevant to our work. We consider their most important properties and examples, and the connections between them. We refer to [9], [17], [19] and [49] for the definitions and the main results in this section.

2.1.1 Pointed categories

In an abelian category one frequently considers kernels and cokernels of morphisms, but in an arbitrary category, no such concepts exist. Pointed categories form the context where a categorical definition of kernels and cokernels is possible.

Definition 2.1. [17] A category $\mathcal{C}$ is *pointed* if it has an initial object 0 that is also terminal (i.e., the unique arrow $0 \to 1$ is an isomorphism).

Remark 2.2. There are two dual ways of turning a category $\mathcal{C}$ into a pointed category. If it has a terminal object 1, the coslice category $1/\mathcal{C}$ is pointed; it is usually called the *category of pointed objects in* $\mathcal{C}$. For instance, the category of pointed sets $\mathbf{Set}_*$ arises this way: if $1 = \{*\}$ is the singleton, then a function $1 \to X$ chooses a basepoint in X, and a morphism in $1/\mathbf{Set}$ is a function which preserves the basepoint. Analogously, if $\mathcal{C}$ has an initial object 0, we shall call *category of copointed objects in* $\mathcal{C}$ the slice category $\mathcal{C}/0$.

Definition 2.3. [17] Let $\mathcal{C}$ be a pointed category. Given a pullback square

$$\begin{array}{ccc} K[f] & \xrightarrow{\ker f} & A \\ \downarrow & \lrcorner & \downarrow f \\ 0 & \longrightarrow & B \end{array}$$

we call $\operatorname{Ker} f := (K[f], \ker f)$ a *kernel* of f. Given a pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \operatorname{coker} f \\ 0 & \longrightarrow & Q[f] \end{array}$$

we call $\operatorname{Coker} f := (Q[f], \operatorname{coker} f)$ a *cokernel* of f.

Definition 2.4. [17] A morphism $A \to B$ that factors trough 0 is denoted $0\colon A \to B$ and is called a *zero morphism*. At most one such morphism exists.

Most of the time, we shall refer to a kernel $\operatorname{Ker} f$ or a cokernel $\operatorname{Coker} f$ by just naming the object part $K[f]$, $Q[f]$ or the morphism part $\ker f$, $\operatorname{coker} f$.

Remark 2.5. In a diagram, an arrow $A \rightarrowtail B$ denotes a monomorphism and an arrow $A \twoheadrightarrow B$ denotes an epimorphism.

We recall from [17] that a *regular epimorphism* is a coequalizer of some parallel pair of morphisms.

Remark 2.6. Note that $\ker f$ is a monomorphism, since it is an equalizer of $f\colon A \to B$ and $0\colon A \to B$. Indeed, $f \circ \ker f$ is the zero morphism and for any other $k\colon K \to A$ such that $f \circ k = 0\colon K \to B$, there exists a unique arrow $u\colon K \to K[f]$ such that $k = \ker f \circ u$:

$$\begin{array}{ccccc} K[f] & \xrightarrow{\ker f} & A & \overset{f}{\underset{0}{\rightrightarrows}} & B \\ {\scriptstyle \exists!\, u}\uparrow & \nearrow_{k} & & & \\ K & & & & \end{array}$$

Analogously, $\operatorname{coker} f$ is a regular epimorphism since it is a coequalizer of $f\colon A \to B$ and $0\colon A \to B$. In fact, $\operatorname{coker} f \circ f$ is the zero morphism and for any other $q\colon B \to Q$ such that $q \circ f = 0\colon A \to Q$, there exists a unique morphism $v\colon Q[f] \to Q$ such that $q = v \circ \operatorname{coker} f$:

$$\begin{array}{ccccc} A & \overset{f}{\underset{0}{\rightrightarrows}} & B & \xrightarrow{\operatorname{coker} f} & Q[f] \\ & & & \searrow^{q} & \downarrow {\scriptstyle \exists!\, v} \\ & & & & Q \end{array}$$

We further recall the following definitions from [60].

Definition 2.7. [60] A category $\mathcal{C}$ is *finitely (co)complete* if it admits all finite (co)limits.

Definition 2.8. [60] A morphism $p\colon E \to B$ in a finitely complete category $\mathcal{C}$ is a *strong epimorphism* when for every commutative square

$$\begin{array}{ccc} E & \longrightarrow & A \\ {\scriptstyle p}\downarrow & \nearrow & \downarrow{\scriptstyle i} \\ B & \longrightarrow & X \end{array}$$

with i being a monomorphism, there exists a (unique) diagonal making the whole diagram commute.

2.1.2 Regular and Barr exact categories

Another useful feature of abelian categories is the existence of image factorizations: any morphism may be factored as a cokernel followed by a kernel. But in the category **Grp** this is no longer possible, as shows the example of the injection $H \rightarrowtail G$ of a subgroup H that is not normal in the group G. However, every group homomorphism has, up to isomorphism, a unique factorization as a regular epimorphism followed by a monomorphism. A category where such a factorization always exists is called *regular*.

Definition 2.9. [17] Given a pullback diagram

$$\begin{array}{ccc} \mathrm{Eq}(f) & \xrightarrow{k_1} & A \\ {\scriptstyle k_0}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

the triple $(\mathrm{Eq}(f), k_0, k_1)$ is called a *kernel pair* of f.

A finitely complete category $\mathcal{C}$ with coequalizers of kernel pairs is said to be *regular* when the pullback of a regular epimorphism along an arbitrary arrow is a regular epimorphism.

In a regular category, the *image factorization* $\mathrm{Im}\, f \circ p$ of a map $f\colon A \to B$ is obtained as follows: p is a coequalizer of a kernel pair $k_0, k_1\colon \mathrm{Eq}(f) \to A$ and the *image* $\mathrm{Im}\, f\colon I[f] \to B$ is the universally induced arrow [9, 12, 13]. Image factorizations are unique up to isomorphism and may be chosen in a functorial way. Using the image factorization, one can see that in a regular category, regular and strong epimorphisms coincide.

A related notion is that of direct images.

Definition 2.10. [17] In a regular category $\mathcal{C}$, consider a monomorphism $m\colon D \to E$ and a regular epimorphism $p\colon E \to B$. Taking the image factorization

$$\begin{array}{ccc} D & \dashrightarrow\!\!\!\!\rightarrow & p(D) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle p(m)=\mathrm{Im}(p\circ m)} \\ E & \xrightarrow[p]{} \!\!\!\!\rightarrow & B \end{array}$$

of $p \circ m$ yields a monomorphism $p(m)\colon p(D) \rightarrowtail B$ called the *direct image of m along p*.

We mentioned before that, even in a pointed and regular category like **Grp**, not every morphism factors as a regular epimorphism followed by a kernel. Nevertheless, this can happen for some of them.

Definition 2.11. [16] A morphism f in a pointed and regular category is called *proper* when its image $\mathrm{Im}\, f$ is a kernel. We call a subobject *proper* when any representing monomorphism is proper, i.e. it is a kernel.

Remark 2.12. In a pointed and finitely cocomplete regular category, the image $\mathrm{Im}\, f$ of a proper morphism f has a cokernel. It is easily seen that this cokernel is also a cokernel of f. Moreover, $\mathrm{Im}\, f$ is a kernel of $\mathrm{coker}\, f$.

A regular epimorphism is proper if and only if its codomain is copointed. For this to be the case, it is sufficient that such is its domain.

Another important aspect of regular categories is the behaviour of internal relations: they compose associatively. Recall from [17] that, in a category with finite limits, a *relation*

$$R = (R, r_0, r_1)\colon A \relbar\!\!\ominus\!\!\rightarrow B$$

from A to B is a subobject $(r_0, r_1)\colon R \to A \times B$. For instance, a kernel pair of some morphism $f\colon A \to B$ may be construed as a relation $(\mathrm{Eq}(f), k_0, k_1)$ from A to A (*on* A) called the *kernel relation* of f. If a map $(x_0, x_1)\colon X \to A \times B$ factors through (r_0, r_1), then the map $h\colon X \to R$ with $(x_0, x_1) = (r_0, r_1) \circ h$ is necessarily unique; we denote the situation by $x_0(R)x_1$. In a regular category, $SR\colon A \relbar\!\!\ominus\!\!\rightarrow C$ denotes the composition of $R\colon A \relbar\!\!\ominus\!\!\rightarrow B$ with $S\colon B \relbar\!\!\ominus\!\!\rightarrow C$.

Proposition 2.13. *[25] . Let $\mathcal{C}$ be a regular category.*

(1) A morphism $b\colon X \to B$ factorizes through the image of a morphism $f\colon A \to B$ if and only if there is a regular epimorphism $p\colon Y \twoheadrightarrow X$ and an arrow $a\colon Y \to A$ with $b \circ p = f \circ a$;

(2) given relations $R\colon A \relbar\!\!\ominus\!\!\rightarrow B$ and $S\colon B \relbar\!\!\ominus\!\!\rightarrow C$ and morphisms $a\colon X \to A$ and $c\colon X \to C$, $c(SR)a$ if and only if there is a regular epimorphism $p\colon Y \twoheadrightarrow X$ and an arrow $b\colon Y \to B$ with $b(R)a \circ p$ and $c \circ p(S)b$. □

It follows from (2) that in a regular category, the composition of relations is associative.

Recall from [60] that an object P in a category $\mathcal{C}$ is called *(regular) projective* when for every regular epimorphism $p\colon E \twoheadrightarrow B$ in $\mathcal{C}$, the function

$$p \circ - = \mathrm{Hom}_{\mathcal{C}}(P, p)\colon\ \mathrm{Hom}_{\mathcal{C}}(P, E) \to \mathrm{Hom}_{\mathcal{C}}(P, B)$$

is a surjection. A category $\mathcal{C}$ is said to have *enough (regular) projectives* when for every object A in $\mathcal{C}$ there exists a projective object P and a regular epimorphism $p\colon P \twoheadrightarrow A$.

Remark 2.14. In a regular category some arguments, which would otherwise involve projectives, may be avoided, and thus the requirement that sufficiently many projective objects exist.

For instance, given the assumptions of (2) in Proposition 2.13, in case X is a projective object, one easily sees that a morphism $b\colon X \to B$ exists such that $b(R)a$ and $c(S)b$. If now X is arbitrary, but $\mathcal{C}$ has enough projectives, one may take a projective object Y and a regular epimorphism $p\colon Y \twoheadrightarrow X$ to get the conclusion of (2). In case $\mathcal{C}$ lacks projectives, this property needs an alternative proof; the regularity of $\mathcal{C}$ allows us to prove it using the first statement of Proposition 2.13.

As for ordinary relations, the notion of equivalence relation has an internal categorical counterpart. Note that for every object X in a finitely complete category $\mathcal{C}$, the Hom functor $\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathbf{Set}$ maps internal relations in $\mathcal{C}$ to relations in $\mathbf{Set}$.

Definition 2.15. [17] A relation R on an object A of a finitely complete category $\mathcal{C}$ is called an *equivalence relation* if, for every object X of $\mathcal{C}$, its image through $\mathrm{Hom}_{\mathcal{C}}(X, -)$ is an equivalence relation on the set $\mathrm{Hom}_{\mathcal{C}}(X, A)$. We denote by $\mathbf{Eq}(\mathcal{C})$ the category of internal equivalence relations in $\mathcal{C}$

In a similar way, one can define the notions of reflexive, symmetric and transitive relations. It is easily seen that a kernel relation is always an equivalence relation. Such an equivalence relation is called *effective*. But in general, the converse is not necessarily true, whence the following definition.

Definition 2.16. [9] A regular category $\mathcal{C}$ is called *Barr exact* when in $\mathcal{C}$, every equivalence relation is a kernel pair.

Examples 2.17. Any category of algebras and any slice or coslice category in a Barr exact category is again Barr exact (in particular, $\mathbf{Set}_*^{\mathrm{op}} = \mathbf{Set}^{\mathrm{op}}/\{*\}$).

The category **Cat** of *small categories*, i.e. that categories where both the object classes and the hom classes are sets, and functors between them is not regular, since regular epimorphisms are not pullback-stable. This is also the case of the category **Top** of topological spaces and continuous maps [12]. However, both the category **CompHaus** of compact Hausdorff spaces and its dual (the category of commutative C^* algebras) are Barr exact [9].

Two examples of categories that are regular but not Barr exact are the category of torsion-free abelian groups [20] and that of topological groups [25].

2.1.3 Bourn protomodular categories

The last important ingredient for the semi-abelian context is *Bourn protomodularity*. This notion due to D. Bourn [19] is perhaps the most difficult and elusive ingredient. For a pointed category, it implies that every regular epimorphism is a cokernel; for a pointed and regular category, it is equivalent to the validity of the *Short Five Lemma*.

Definition 2.18. [60] A *split epimorphism* or *retraction* is a morphism $p\colon A \to B$ such that there is a morphism $s\colon B \to A$ (called a *splitting of* p) satisfying $p \circ s = 1_B$. Dually, then s is a *split monomorphism*.

Definition 2.19. [19, 20] A finitely complete category is called *(Bourn) protomodular* when, given a commutative diagram

$$\begin{array}{ccccc} A' & \xrightarrow{f} & B' & \xrightarrow{h} & C' \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow\uparrow{\scriptstyle s} & & \downarrow{\scriptstyle\gamma} \\ A & \xrightarrow[g]{} & B & \xrightarrow[l]{} & C \end{array}$$

where β is a split epimorphism and s is a splitting of p, if the outer rectangle and the left hand side square are pullbacks, then so is the right hand side square.

Proposition 2.20. *[13, 20] A pointed and regular category is protomodular if and only if the* Short Five Lemma *holds. This means that for any commutative diagram*

$$\begin{array}{ccccc} K[p'] & \xrightarrow{\ker p'} & E' & \xrightarrow{p'} & B' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ K[p] & \xrightarrow[\ker p]{} & E & \xrightarrow[p]{} & B \end{array}$$

where p and p' are regular epimorphisms, if u and w being isomorphisms then so is v. □

The categories of (abelian) groups, non-unitary rings, Lie algebras, crossed modules are all examples of protomodular categories.

Remark 2.21. In a finitely complete and pointed protomodular category, every regular epimorphism is a cokernel (of its kernel).

In a protomodular category $\mathcal{C}$, an intrinsic notion of normal monomorphism exists.

Definition 2.22. [18] An arrow $k\colon K \to A$ in a finitely complete category $\mathcal{C}$ is *normal to an equivalence relation R on A* when $k^{-1}(R)$ is the largest equivalence relation $\nabla_K = (K \times K, \pi_1, \pi_2)$ on K and the induced map $\nabla_K \to R$ in the category $\mathbf{Eq}(\mathcal{C})$ of internal equivalence relations in $\mathcal{C}$ is a *discrete fibration*.

This means that

(1) there is a morphism $\tilde{k}\colon K \times K \to R$ in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} K \times K & \xrightarrow{\tilde{k}} & R \\ {\scriptstyle \pi_1}\downarrow\downarrow{\scriptstyle \pi_2} & & {\scriptstyle d_0}\downarrow\downarrow{\scriptstyle d_1} \\ K & \xrightarrow[k]{} & A \end{array}$$

commutes;

(2) all the commutative squares in the diagram above are pullbacks.

One may prove that the arrow k is then necessarily a monomorphism; furthermore, when the category $\mathcal{C}$ is protomodular, a monomorphism can be normal to at most one equivalence relation, so that the fact of being normal becomes a property [18]. The notion of normal monomorphism gives an intrinsic way to express the fact that K is an equivalence class of R.

In a pointed finitely complete category every kernel is normal. In particular, $\ker f$ is normal with respect to $R[f]$. The converse is not true: if B is not copointed then the morphism 1_B is not a kernel, although it is normal with respect to ∇_B. In the pointed exact protomodular case, normal monomorphisms and kernels coincide.

There is a natural way to associate, with any equivalence relation (R, d_0, d_1) in a pointed and finitely complete category, a normal monomorphism k_R, called the *normalization* of R, or *the normal subobject associated with R*. It is defined as the composition $k_R = d_1 \circ \ker d_0$

$$K[d_0] \xrightarrow{\ker d_0} R \xrightarrow{d_1} A.$$

In the exact protomodular case, this construction determines a bijection between the (effective) equivalence relations on A and the proper subobjects of A (a proper monomorphism corresponds to the kernel pair of its cokernel). In the pointed protomodular case, it determines a bijection between the equivalence relations on A and the normal subobjects of A [18].

Proposition 2.23. *[17] Consider, in a regular and protomodular category, a commutative square with horizontal regular epimorphisms*

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ A & \xrightarrow[f]{} & B. \end{array}$$

If w is a monomorphism and v a normal monomorphism, then w is normal. □

In the pointed and exact case, it implies that the direct image of a kernel is a kernel.

2.1.4 Semi-abelian and abelian categories

As mentioned above, the ideal context for the results of this work is that of *semi-abelian* categories.

Definition 2.24. [49] A category $\mathcal{C}$ is *semi-abelian* when it is pointed, Barr exact, Bourn protomodular with finite coproducts.

There is an historical reason for the importance of this notion. Indeed, introducing semi-abelian categories, G. Janelidze, L. Márki and W. Tholen solved Mac Lane's long standing problem [61] of finding a framework that reflects the categorical properties of non-abelian groups as nicely as abelian categories do for abelian groups. But over the years, many different people came up with partial solutions to this problem, proving theorems starting from various sets of axioms, which all require "good behaviour" of normal monomorphisms and epimorphisms. In [49], the relationship between these "old-style" axioms and the semi-abelian context is explained, and thus the old results are incorporated into the new theory.

As we see in the next sections, in semi-abelian categories there is an intrinsic notion of internal action [14] and semi-direct product [22].

Finally, we give a quick overview of those techniques (one uses in the abelian context) that, when valid in a semi-abelian category, make it abelian.

Definition 2.25. [40] A category $\mathcal{C}$ is *abelian* when it is pointed, has finite products and coproducts, has kernels and cokernels, and is such that every monomorphism is a kernel and every epimorphism is a cokernel.

Examples of abelian categories include all categories of modules over a ring, such as the category **Ab** of abelian groups.

One has that a category is abelian if and only if it is Barr exact and *additive* [72], i.e. it has finite coproducts and it is *pre-additive* (which means that each hom sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is endowed with the structure of an abelian group such that the composition

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

is bilinear).

Hence, since a pointed category is additive if and only if it is protomodular and any monomorphism is normal (see [17, Proposition 6.2.3]), if in a semi-abelian category $\mathcal{C}$ every monomorphism is a kernel, then $\mathcal{C}$ is abelian.

The category **AbTop** of abelian topological groups is an example of a category that is additive and regular, though not Barr exact.

2.2 Internal actions and their representability

From now on, when we consider a category $\mathcal{C}$, we assume it to be semi-abelian. When the objects of $\mathcal{C}$ are non-associative algebras over a field $\mathbb{F}$, we assume that $\mathbb{F}$ is fixed, so that we may drop it from our notation.

A central notion which appears in the semi-abelian context is that of *internal actions*. We refer to [14], [15], [21] and [47] for the main results in this section.

For an object B in a semi-abelian category $\mathcal{C}$, let $(B\flat X, \kappa_{B,X})$ denote the kernel of the arrow $[1_B, 0]\colon B + X \to B$ induced by the identity morphism of B and the zero morphism $0\colon X \to B$, where $B + X$ is the *coproduct* of B and X. We can define the functor

$$B\flat(-)\colon \mathcal{C} \to \mathcal{C}\colon X \mapsto B\flat X,$$

which maps any object X of $\mathcal{C}$ to the object $B\flat X$, and the natural transformations $\eta^B\colon 1_{\mathcal{C}} \to B\flat(-)$ and $\mu^B\colon B\flat(B\flat(-)) \to B\flat(-)$ in the following way: for any object X of $\mathcal{C}$, the component μ^B_X is a restriction of the codiagonal $(B+B)+X \to B+X$, i.e. it maps $B\flat(B\flat X)$ to $B\flat X$, and η^B_X sends an element of X to $B\flat X$, i.e. it is the morphism induced by the coproduct injection $X \to B + X$.

Definition 2.26. [14] Let $\mathcal{C}$ be a semi-abelian category and let B be a object of $\mathcal{C}$. An *(internal) B-action* is a $B\flat(-)$-algebra, which is a pair (X, ξ) consisting of an object X of $\mathcal{C}$ and a morphism $\xi\colon B\flat X \to X$, called an *action* of B on X, such that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\eta^B_X} & B\flat X \\ & \searrow^{1_X} & \downarrow{\scriptstyle \xi} \\ & & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} B\flat(B\flat X) & \xrightarrow{\mu^B_X} & B\flat X \\ {\scriptstyle 1_B\flat\xi}\downarrow & & \downarrow{\scriptstyle \xi} \\ B\flat X & \xrightarrow[\xi]{} & X \end{array}$$

commute.

We write $\mathrm{Act}_{\mathcal{C}}(B, X)$ for the set of (internal) actions of B on X. When there is no ambiguity on the category $\mathcal{C}$, we denote it by $\mathrm{Act}(B, X)$. If we fix an object X of $\mathcal{C}$, actions on X give rise to a functor

$$\mathrm{Act}(-, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

which assigns to any object B of $\mathcal{C}$, the set $\mathrm{Act}(B, X)$ and, given any morphism $f\colon B' \to B$, $\mathrm{Act}(f, X)$ sends an action $\xi\colon B\flat X \to X$ to

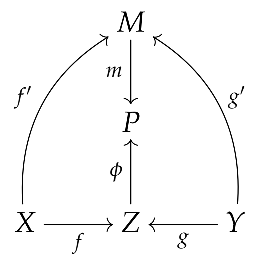
$$\mathrm{Act}(f, X)\xi := \xi \circ (f\flat X)\colon B'\flat X \to X.$$

where $f\flat X\colon B'\flat X \to B\flat X$ is the unique morphism in $\mathcal{C}$ such that $\kappa_{B,X} \circ (f\flat X) = (f + X) \circ \kappa_{B',X}$, where $f + X\colon B' + X \to B + X$ is induced by f on B and by 1_X on X.

One equivalent way of viewing actions is to use *split extensions*. We recall from [14] that a *split extension* of B by X, or with kernel X, is a diagram in $\mathcal{C}$

$$0 \longrightarrow X \xrightarrow{\ i\ } A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{2.2.1}$$

such that $\pi \circ s = 1_B$ and (X, i) is the kernel of π. Notice that protomodularity implies that the pair (i, s) is *jointly strongly epic* and π is indeed the cokernel of i. We recall from [28] that a *cospan* (f, g) over an object Z, i.e. a pair of morphisms having the same codomain Z, in an arbitrary category $\mathcal{C}$ is said to be *jointly strongly epic*, or *jointly strongly epimorphic*, when for each commutative diagram



if m is a monomorphism, then there exists a unique morphism $\varphi\colon Z \to M$ such that $m \circ \varphi = \phi$.

A morphism of split extensions from

$$0 \longrightarrow X \xrightarrow{\ i\ } A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

to

$$0 \longrightarrow X' \xrightarrow{\ i'\ } A' \underset{s'}{\overset{\pi'}{\rightleftarrows}} B' \longrightarrow 0$$

is a triple of morphisms in $\mathcal{C}$

$$(f\colon X \to X', \theta\colon A \to A', g\colon B \to B')$$

such that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\ i\ } & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & X' & \xrightarrow{\ i'\ } & A' & \underset{s'}{\overset{\pi'}{\rightleftarrows}} & B' & \longrightarrow & 0
\end{array}$$

Let us observe that $\theta = s' \circ g \circ \pi$ is uniquely determined by g and, again by protomodularity, a morphism of split extensions fixing X and B is necessarily an isomorphism.

For an object X of $\mathcal{C}$, we define the functor

$$\mathrm{SplExt}_{\mathcal{C}}(-, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

which assigns to any object B of $\mathcal{C}$, the set $\mathrm{SplExt}_{\mathcal{C}}(B, X)$ of isomorphism classes of split extensions of B by X in $\mathcal{C}$ and to any arrow $f\colon B' \to B$ the *change of base* $f^* := \mathrm{SplExt}_{\mathcal{C}}(f, X)\colon \mathrm{SplExt}_{\mathcal{C}}(B, X) \to \mathrm{SplExt}_{\mathcal{C}}(B', X)$ given by pulling back along

f. Again, if it is not confusing, we use the notation $\mathrm{SplExt}(-,X)$.

The connection between internal actions and split extensions in a semi-abelian category is explained by the following.

Lemma 2.27. *[14, 22] Given two objects B and X in $\mathcal{C}$, there is a bijection*

$$\tau_B \colon \mathrm{SplExt}(B,X) \cong \mathrm{Act}(B,X).$$

Proof. The bijection τ_B sends any split extension

$$0 \longrightarrow X \xrightarrow{\ i\ } A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{2.2.2}$$

in $\mathcal{C}$ to the action $\xi\colon B\flat X \to X$, where ξ is the unique morphism for which $i\circ\xi\colon B\flat X \to A$ is equal to the composition

$$B\flat X \xrightarrow{\kappa_{B,X}} B+X \xrightarrow{[s,i]} A\ .$$

Its inverse sends an action $\xi\colon B\flat X \to X$ to the split extension

$$0 \longrightarrow X \xrightarrow{i'_{(X,\xi)}} B\ltimes_\xi X \underset{s'_{(X,\xi)}}{\overset{\pi'_{(X,\xi)}}{\rightleftarrows}} B \longrightarrow 0$$

where $B\ltimes_\xi X$ is defined, together with a morphism $\sigma_{(X,\xi)}\colon B+X \to B\ltimes_\xi X$, via the coequalizer diagram

$$B+(B\flat X) \underset{1_B+\xi}{\overset{[i_{B,X},\kappa_{B,X}]}{\rightrightarrows}} B+X \xrightarrow{\sigma_{(X,\xi)}} B\ltimes_\xi X\ , \tag{2.2.3}$$

where $1_B+\xi$ is induced by 1_B on B and by ξ on $B\flat X$. Moreover, $i_{B,X}\colon B\to B+X$ is the coproduct injection, $s'_{(X,\xi)} = \sigma_{(X,\xi)}\circ i_{B,X}$, $\pi'_{(X,\xi)}$ is uniquely determined by $\pi'_{(X,\xi)}\circ\sigma_{(X,\xi)} = \pi_{B,X}$, where $\pi_{B,X} = [1_B,0]\colon B+X\to B$, and $i'_{(X,\xi)}$ is the kernel of $\pi'_{(X,\xi)}$.

Finally the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{i'_{(X,\xi)}} & B\ltimes_\xi X & \underset{s'_{(X,\xi)}}{\overset{\pi'_{(X,\xi)}}{\rightleftarrows}} & B & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_B} & & \\
0 & \longrightarrow & X & \xrightarrow{\ i\ } & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0
\end{array}$$

where $\theta = s\circ\pi'_{(X,\xi)}$, defines an isomorphism of split extensions between (2.2.3) and (2.2.2). □

Definition 2.28. [14] The object $B\ltimes_\xi X$ is called the *semi-direct product* of B with (X,ξ), or the semi-direct product of B and X with respect to ξ.

In [15] the authors proved that the bijection τ_B of Lemma 2.27 is natural in B.

Proposition 2.29. *[15] The bijection τ_B extends to a natural isomorphism*

$$\tau\colon \mathrm{SplExt}(-,X) \cong \mathrm{Act}(-,X).$$

Proof. Let $f\colon B' \to B$ be a morphism in $\mathcal{C}$ and suppose that $f^* = \mathrm{SplExt}(f, X)$ takes the isomorphism class of

$$0 \longrightarrow X \xrightarrow{\ i\ } A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

to that of

$$0 \longrightarrow X \xrightarrow{\ i'\ } A' \underset{s'}{\overset{\pi'}{\rightleftarrows}} B' \longrightarrow 0\,. \tag{2.2.4}$$

As above, the image of (2.2.4) under $\tau_{B'}$ is the action $\xi'\colon B' \flat X \to X$ where $i' \circ \xi' = [s', i'] \circ \kappa_{B',X}$. It follows that ξ' is equal to the composition

$$B' \flat X \xrightarrow{f \flat X} B \flat X \xrightarrow{\ \xi\ } X$$

since

$$\begin{aligned} i \circ \xi \circ (f \flat X) &= [s, i] \circ \kappa_{B,X} \circ (f \flat X) = \\ &= [s, i] \circ (f + X) \circ \kappa_{B',X} = \\ &= \theta \circ [s', i'] \circ \kappa_{B',X} = \theta \circ i' \circ \xi' = i \circ \xi', \end{aligned}$$

where $\theta\colon A' \to A$ is given by the functoriality of $\mathrm{SplExt}(-, X)$. Thus $\xi' = \mathrm{Act}(f, X)\xi$ and we have a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X).$$

□

2.2.1 Action representable and action accessible categories

In [14] F. Borceux, J. Janelidze and G. M. Kelly introduced the notions of *representable action* and *action representable* category.

Definition 2.30. [14] A semi-abelian category $\mathcal{C}$ is said to be *action representable* if for every object X in it the functor $\mathrm{Act}(-, X)$ is representable. In other words, there exists an object $[X]$ in $\mathcal{C}$, called the *actor* of X, and a natural isomorphism of functors

$$\mathrm{Act}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X]).$$

Basic examples of action representable categories are the category **Grp** of groups with the actor of X being the group of automorphisms $\mathrm{Aut}(X)$, the category **Lie** of Lie algebras with the actor of X being the Lie algebra of derivations $\mathrm{Der}(X)$, and any abelian category with the actor of X being the zero object.

Example 2.31. Recall that an (external) action of a group B on a group X is a map

$$\xi\colon B \times X \to X\colon (b, x) \mapsto b \cdot x$$

such that $(bb') \cdot x = b \cdot (b' \cdot x)$, $1_B \cdot x = x$ and $b \cdot (xx') = (b \cdot x)(b \cdot x')$, for any $b, b' \in B$ and for any $x, x' \in X$ (we use the multiplicative notation both for B and X).

Given an action ξ of B on X in **Grp**, one can construct the *semi-direct product* $B \ltimes X$ of B and X with respect to ξ, that is the group whose underline set is the cartesian product $B \times X$ and the multiplication is determined by

$$(b, x)(b', x') = (bb', x(b \cdot x')),$$

with $1_{B\ltimes X} = (1_B, 1_X)$ and $(b,x)^{-1} = (b^{-1}, b^{-1}\cdot x^{-1})$. Thus, we can associate with ξ the split extension in **Grp**

$$1 \longrightarrow X \xrightarrow{i_2} B \ltimes X \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 1 \tag{2.2.5}$$

where 1 denotes the trival group and i_1, i_2 and π_1 are the canonical injections and projection.

Conversely, if we start with a split extension in **Grp**

$$1 \longrightarrow X \xrightarrow{i} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 1 \tag{2.2.6}$$

then we can define an action of B on X by

$$B \times X \to X\colon (b,x) \mapsto s(b)i(x)s(b)^{-1} \tag{2.2.7}$$

and there is an isomorphism of split extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & X & \xrightarrow{i_2} & B \ltimes X & \underset{s_1}{\overset{\pi_1}{\rightleftarrows}} & B & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_B} & & \\ 1 & \longrightarrow & X & \xrightarrow{i} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 1 \end{array}$$

where $B \ltimes X$ is the semi-direct product associated with (2.2.7) and $\theta(b,x) = s(b) + i(x)$.

Moreover, split extensions of groups are in bijection with the group homomorphisms $B \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ denotes the group of *automorphisms* of X: given a split extension of groups as in (2.2.6), one can define the group homomorphism

$$\varphi\colon B \to \mathrm{Aut}(X)\colon b \mapsto \varphi_b,$$

where $\varphi_b(x) = s(b)i(x)s(b)^{-1}$. Conversely, any homomorphism $\varphi\colon B \to \mathrm{Aut}(X)$ defines a semi-direct product $B \ltimes X$ with

$$(b,x)(b',x') = (bb', x\varphi_b(x'))$$

and thus a split extension of B by X as in (2.2.5).

This bijection is natural in B, since the diagram in **Grp**

$$\begin{array}{ccc} \mathrm{SplExt}(B,X) & \longrightarrow & \mathrm{Hom}(B,\mathrm{Aut}(X)) \\ {\scriptstyle \mathrm{SplExt}(f,X)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(f,\mathrm{Aut}(X))} \\ \mathrm{SplExt}(B',X) & \longrightarrow & \mathrm{Hom}(B',\mathrm{Aut}(X)) \end{array}$$

is commutative, for any group homomorphism $f\colon B' \to B$. Thus, $\mathrm{Aut}(X)$ is the actor of X and the category **Grp** is action representable.

Example 2.32. The same argument can be used for the category of Lie algebras, replacing $\mathrm{Aut}(X)$ with the Lie algebra $\mathrm{Der}(X)$.

Recall from [38] that a Lie $\mathbb{F}$-algebra $(X, [-,-])$ is a vector space X over a field $\mathbb{F}$ (we assume $\mathrm{char}(\mathbb{F}) \neq 2$) together with a bilinear map

$$[-,-]\colon B \times X \to X,$$

called *commutator* or *Lie bracket*, which is *alternating*, i.e. $[x,x] = 0$ for any $x \in X$, or equivalently anti-commutative $[x,y] = -[y,x]$, and which satisfies the *Jacobi identity*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \quad \forall x,y,z \in X.$$

Denote by **Lie** the category of Lie algebras over $\mathbb{F}$, whose morphisms are the $\mathbb{F}$-linear maps which preserve the Lie bracket.

An action of a Lie algebra B on another Lie algebra X is given by a pair of bilinear maps

$$l\colon B \times X \to X, \quad r\colon X \times B \to X,$$

denoted by $[\![b,x]\!] = l(b,x)$ and $[\![x,b]\!] = r(x,b)$, such that

- $[\![x,b]\!] = -[\![b,x]\!]$;
- $[\![b,[x,x']]\!] = [[\![b,x]\!],y] + [x,[\![b,x']\!]]$;
- $[\![[b,b'],x]\!] = [\![b,[\![b',x]\!]]\!] - [\![b',[\![b,x]\!]]\!]$,

for any $b,b' \in B$ and for any $x,x' \in X$; the corresponding semi-direct product $B \ltimes X$ is the Lie algebra defined on the direct sum of vector spaces $B \oplus X$ with commutator

$$[(b,x),(b',x')] = ([b,b'],[x,x'] + [\![b,x']\!] - [\![b',x]\!]).$$

Thus, we can associate with the action (l,r) the split extension in **Lie**

$$0 \longrightarrow X \xrightarrow{i_2} B \ltimes X \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0 \tag{2.2.8}$$

where i_1, i_2 and π_1 are the canonical injections and projection.

Conversely, if we start with a split extension in **Lie**

$$0 \longrightarrow X \xrightarrow{i} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{2.2.9}$$

then we can define an action of B on X by

$$[\![b,x]\!] = [s(b),i(x)]_A, \quad [\![x,b]\!] = -[s(b),i(x)]_A. \tag{2.2.10}$$

and there is an isomorphism of split extension

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{i_2} & B \ltimes X & \underset{s_1}{\overset{\pi_1}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_B} & & \\
0 & \longrightarrow & X & \xrightarrow{i} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0
\end{array}$$

where $B \ltimes X$ is the semi-direct product associated with (2.2.10) and $\theta(b,x) = s(b) + i(x)$.

We recall from [38] that a derivation of a Lie algebra X is a linear endomorphism $d\colon X \to X$ such that $d([x,y]) = [d(x),y] + [x,d(y)]$, for any $x,y \in X$, and the vector space $\mathrm{Der}(X)$ of derivations of X is a Lie algebra with respect to the Lie bracket $[d, d'] = d \circ d' - d' \circ d$.

We have that the split extensions of Lie algebras are in bijection with the Lie algebra homomorphisms $B \to \mathrm{Der}(X)$: given a split extension of Lie algebra as in (2.2.9), one can define the Lie algebra homomorphism

$$\varphi\colon B \to \mathrm{Der}(X)\colon b \mapsto \varphi_b,$$

where $\varphi_b(x) = [s(b), i(x)]_A$. Conversely, any homomorphism $\varphi\colon B \to \mathrm{Der}(X)$ defines a semi-direct product $B \ltimes X$ with

$$[(b, x), (b', x')] = ([b, b'], [x, x'] + \varphi_b(x') - \varphi_{b'}(x))$$

and thus a split extension of B by X as in (2.2.8).

This bijection is natural in B, since the diagram in **Lie**

$$\begin{array}{ccc} \mathrm{SplExt}(B, X) & \longrightarrow & \mathrm{Hom}(B, \mathrm{Der}(X)) \\ {\scriptstyle \mathrm{SplExt}(f,X)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(f,\mathrm{Der}(X))} \\ \mathrm{SplExt}(B', X) & \longrightarrow & \mathrm{Hom}(B', \mathrm{Der}(X)) \end{array}$$

is commutative, for any Lie algebra homomorphism $f\colon B' \to B$. Thus, $\mathrm{Der}(X)$ is the actor of X and the category **Lie** is action representable.

A first example of a category which is not action representable is the category **CAssoc** of commutative associative algebras over a field $\mathbb{F}$: in [15] the authors proved that, for a commutative associative algebra X, there exists a natural isomorphism

$$\mathrm{SplExt}_{\mathbf{CAssoc}}(-, X) \cong \mathrm{Hom}_{\mathbf{Assoc}}(U(-), M(X))$$

where $U\colon \mathbf{CAssoc} \to \mathbf{Assoc}$ denotes the forgetful functor and

$$\mathrm{M}(X) := \{f \in \mathrm{End}(X) \mid f(xy) = f(x)y,\ \forall x \in X\}$$

is the associative algebra of *multipliers* of X [59], endowed with the usual composition of functions (see Lemma 4.4 for a detailed proof). Moreover, they proved that $\mathrm{M}(X)$ is a commutative algebra if and only if the functor $\mathrm{SplExt}_{\mathbf{CAssoc}}(-, X)$ is representable (see [15, Theorem 2.6]). Since there are examples of commutative associative algebras X such that $\mathrm{M}(X)$ is not commutative, such as the abelian two-dimensional algebra where $\mathrm{M}(X) = \mathrm{End}(X)$, it follows that the category **CAssoc** is not action representable. We give a detailed proof of this result in Chapter 4.

Remark 2.33. We recall from [60] that the *category of elements* $\mathbf{el}(F)$ of a functor $F\colon \mathcal{C} \to$ **Set** is the category whose objects are the pairs (X, x), where X is an object of $\mathcal{C}$ and $x \in F(X)$, and the morphisms $(X, x) \to (Y, y)$ are morphisms $f\colon X \to Y$ in $\mathcal{C}$ such that $F(f)(x) = y$.

Moreover, a functor $F\colon \mathcal{C} \to$ **Set** is representable, i.e. there exists a natural isomorphism $\tau\colon \mathrm{Hom}_{\mathcal{C}}(I, -) \cong F$ for some object I of $\mathcal{C}$, if and only its category of elements $\mathbf{el}(F)$ has an initial object, which is the pair (I, i) with $i = \tau_I(1_I)$.

Dually, a functor $\mathcal{C}^{\mathrm{op}} \to$ **Set** is representable if and only its category of elements has a terminal object (see [14]) and thus action representability is equivalent to the condition that for any object X in $\mathcal{C}$, the category $\mathbf{SplExt}(X)$ of split extensions in $\mathcal{C}$

with kernel X has a terminal object of the form

$$0 \longrightarrow X \longrightarrow [X] \ltimes X \rightleftarrows [X] \longrightarrow 0.$$

More precisely we have the following.

Theorem 2.34. *[14] A semi-abelian category $\mathcal{C}$ is action representable if and only if for every object X of $\mathcal{C}$, there exists an object $[X]$ of $\mathcal{C}$ and an internal action of $[X]$ on X with the following universal property: for any object B of X and for every split extension*

$$0 \longrightarrow X \xrightarrow{i} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

there is a unique morphism $B \to [X]$ such that there exists a morphism $A \to [X] \ltimes X$ making the following diagram commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & B \ltimes X & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_X} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & [X] \ltimes X & \rightleftarrows & [X] & \longrightarrow & 0. \end{array}$$

Proof. Let $\mathcal{C}$ be a semi-abelian action representable category. Let X be an object of $\mathcal{C}$ and let $[X]$ be its actor. Then the identity morphism $1_{[X]}$ defines an internal action ξ of $[X]$ on X. Denote by

$$0 \longrightarrow X \xrightarrow{i_X} [X] \ltimes X \underset{s_X}{\overset{\pi_X}{\rightleftarrows}} [X] \longrightarrow 0$$

the split extension of $[X]$ by X associated with ξ, which is given by $\eta^{-1}_{[X]}(\xi)$, where $\eta_{[X]}$ is the bijection of of Lemma 2.27.

The existence of a terminal object in $\mathbf{el}(\mathrm{SplExt}(-, X))$ implies that, for any object B of $\mathcal{C}$ and for any split extension in $\mathcal{C}$

$$0 \longrightarrow X \xrightarrow{i} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

there exists a unique morphism $b\colon B \to [X]$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & B \ltimes X & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & X & \xrightarrow{i_X} & [X] \ltimes X & \underset{s_X}{\overset{\pi_X}{\rightleftarrows}} & [X] & \longrightarrow & 0 \end{array}$$

where $\theta = s_X \circ b \circ \pi$. We observe that the right hand side square is a pullback.

Conversely, if for any object X of $\mathcal{C}$ there exists an object $[X]$ of $\mathcal{C}$ acting on X and which satisfies the universal property described above, then $[X]$ is the actor of X and a representation of the functor $\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X)$ is given by

$$\tau\colon \mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X]),$$

where, for any other object B of X, τ_B sends an isomorphism class of split extensions of B by X to the unique morphism $B \to [X]$ given by the univeral property. □

Example 2.35. Let $\mathcal{C} = \mathbf{Grp}$ and let X be a group. Then $\mathrm{Aut}(X)$ acts on X with

$$\mathrm{Aut}(X) \times X \to X \colon (\varphi, x) \mapsto \varphi(x).$$

The corresponding semidirect product $\mathrm{Aut}(X) \ltimes X$, which is called the *holomorph* of G and it is dentoted by $\mathrm{Hol}(G)$, has the multiplication

$$(\varphi, x)(\varphi', x') = (\varphi \circ \varphi', x\varphi(x')).$$

Example 2.36. Let $\mathcal{C} = \mathbf{Lie}$ and let X be a Lie algebra. The action of the Lie algebra $\mathrm{Der}(X)$ on X is defined by the pair of bilinear maps

$$\mathrm{Der}(X) \times X \to X \colon (d, x) \mapsto d(x)$$

and

$$X \times \mathrm{Der}(X) \to X \colon (x, d) \mapsto -d(x).$$

The Lie bracket on the associated semi-direct product $\mathrm{Der}(X) \ltimes X$ is

$$[(d, x), (d', x')] = (d \circ d' - d' \circ d, [x, x'] + d(x') - d'(x)).$$

We can weaken the condition of representable action, assuming instead that, for any object X of a semi-abelian category $\mathcal{C}$, every object in $\mathbf{SplExt}(X)$ is *accessible*, i.e. it has a morphism into a so-called *subterminal* or *faithful* object [21], that is an object whjch admits at most one morphism into it.

Definition 2.37. [21] A semi-abelian category $\mathcal{C}$ is *action accessible* if for any object X of it, every object of $\mathbf{SplExt}(X)$ is accessible.

The notion of *action accessibility*, which was introduced by D. Bourn and G. Janelidze in [21] in order to calculate centralizers of normal subobjects or of equivalence relations, allows us to encompass a wider class of categories which are not action representable, such as the category **CAssoc** of commutative associative algebras, the category **Pois** of (non-commutative) Poisson algebras (see Chapter 7) and, more in general, all *Orzech catgories of interest* (see Theorem 3.28).

Remark 2.38. By definition, the existence of a terminal object in $\mathbf{SplExt}(X)$ is stronger than every object being accessible. Thus, it immediately follows that

action representability $\Rightarrow$ *action accessibility.*

2.2.2 Weakly action representable categories

Recently in [47], G. Janelidze introduced the notion of *weakly representable actions*.

Definition 2.39. [47] A semi-abelian category $\mathcal{C}$ is said to be *weakly action representable* if for every object X in it, there exists an object T of $\mathcal{C}$ and a natural monomorphism of functors

$$\tau \colon \mathrm{Act}(-, X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, T).$$

The pair (T, τ) is called a *weak representation* of $\mathrm{Act}(-, X)$, the object T is called a *weak actor* of X and morphism $\varphi \colon B \to T$ in the image of τ_B is called *acting morphism*.

Remark 2.40. It is clear from Definition 2.39 that a weak actor needs not be unique. Indeed, if (T, τ) is a weak representation of $\mathrm{Act}(-, X)$ and there is a monomorphism in $\mathcal{C}$

$$i \colon T \rightarrowtail T',$$

then i induces a monomorphism of functors

$$i^*\colon \operatorname{Hom}_{\mathcal{C}}(-,T) \rightarrowtail \operatorname{Hom}_{\mathcal{C}}(-,T')$$

and $(T', i^* \circ \tau)$ is another weak representation of $\operatorname{Act}(-,X)$.

Example 2.41. Every action representable category is weakly action representable. In this case $T = [X]$ is the actor of X, τ is a natural isomorphism and every arrow $\varphi\colon B \to [X]$ is an acting morphism.

In [47] it is proven that the category **Assoc** of associative algebras over a field $\mathbb{F}$ is weakly action representable.

Example 2.42. Given a split extension in **Assoc**

$$0 \longrightarrow X \xrightarrow{\ i\ } A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{2.2.11}$$

we may define the pair of bilinear maps

$$l\colon B \times X \to X, \qquad r\colon X \times B \to X$$

by

$$b * x = s(b)i(x), \quad x * b = i(x)s(b), \quad \forall b \in B,\ \forall x \in X$$

where $b * - = l(b,-)$ and $- * b = r(-,b)$.

Moreover, we can endow the direct sum of vector spaces $B \oplus X$ with the bilinear multiplication

$$(b,x)\cdot(b',x') = (bb', xx' + b * x' + x * b'). \tag{2.2.12}$$

Then $(B \oplus X, \cdot)$ is an associative algebra, which is called the *semi-direct product* of B and X with respect to (l,r) and it is denoted by $B \ltimes X$, and the diagram

$$0 \longrightarrow X \xrightarrow{\ i_2\ } B \ltimes X \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0$$

with $i_2(x) = (0,x)$, $i_1(b) = (b,0)$ and $\pi_1(b,x) = b$, defines a split extension in **Assoc** which is isomorphic to (2.2.11) via

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\ i_2\ } & B \ltimes X & \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_B} & & \\
0 & \longrightarrow & X & \xrightarrow{\ i\ } & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0
\end{array}$$

where $\theta(b,x) = s(b) + i(x)$, for any $(b,x) \in B \oplus X$.

The representability of actions of the category **Assoc** of associative algebras over $\mathbb{F}$ was studied in [44, Proposition 1.11], where the authors proved the following.

Proposition 2.43. *[44] The category* **Assoc** *is not action representable.*

Proof. Suppose that the category **Assoc** is action representable, with the actor of X being the associative algebra $[X]$.

The natural isomorphism $\operatorname{Act}(-,X) \cong \operatorname{Hom}_{\mathbf{Assoc}}(-,[X])$ implies that

$$\operatorname{Act}(B_1 + B_2, X) \cong \operatorname{Act}(B_1, X) \times \operatorname{Act}(B_2, X),$$

for any objects B_1, B_2 of **Assoc**, which can be understood as the condition that having an action of the coproduct $B_1 + B_2$ on X is the same as having two actions on X, one of B_1 and the other of B_2. It follows that $(B_1 + B_2) \ltimes X$ is an object of **Assoc**.

Now, let B_1 and B_2 be the abelian one-dimensional algebras spanned by b_1 and b_2 respectively. Let X be the abelian algebra spanned by the elements $x, y_1, y_2, z_{12}, z_{21}$. We consider the pair of bilinear maps

$$l_i \colon B_i \times X \to X, \quad r_i \colon X \times B_i \to X, \quad i = 1, 2$$

defined by

$$b_i * x = x * b_i = y_i, \quad b_i * y_j = y_j * b_i = \begin{cases} z_{ij}, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases} \text{ and } b_i * z_{jk} = z_{jk} * b_i = 0$$

for any $i, j, k \in \{1, 2\}$ and $j \neq k$, where $b_i * - = l_i(b_i, -)$ and $- * b_i = r_i(-, b_i)$. These choices determine two split extension in **Assoc**

$$0 \longrightarrow X \xrightarrow{j_{i,2}} B_i \ltimes X \underset{j_{i,1}}{\overset{\pi_{i,1}}{\rightleftarrows}} B_i \longrightarrow 0,$$

where $j_{i,1}, j_{i,2}$ and $\pi_{i,1}$ are the canonical injections and projection and the bilinear multiplication on $B_i \ltimes X$ is defined by

$$(\alpha b_i, x) \cdot (\alpha' b_i, x') = (0, \alpha b_i * x' + \alpha' x * b_i),$$

for any $\alpha, \alpha' \in \mathbb{F}$, $x, x' \in X$ and $i = 1, 2$. Thus, by Lemma 2.27, we have well-defined actions of B_i on X.

Since we supposed that **Assoc** is action representable, also the algebra $B_1 + B_2$ acts on X. But the corresponding semi-direct product $(B_1 + B_2) \ltimes X$ is not an associative algebra, since

$$b_1 * (x * b_2) = z_{12} \neq z_{21} = (b_1 * x) * b_2$$

and then

$$(b_1, 0) \cdot [(0, x) \cdot (b_2, 0)] = (0, z_{12}) \neq (0, z_{21}) = [(b_1, 0) \cdot (0, x)] \cdot (b_2, 0).$$

Thus, we have a contradiction and **Assoc** is not action representable. □

We want now to prove that **Assoc** is a weakly action representable category.

We observe that the associativity of the operation (2.2.12) implies that the following equalities hold:

(i) $b * (xx') = (b * x)x'$;

(ii) $(xx') * b = x(x' * b)$;

(iii) $x(b * x') = (x * b) * x'$;

(iv) $(bb') * x = b * (b' * x)$;

(v) $x * (bb') = (x * b) * b'$;

(vi) $b * (x * b') = (b * x) * b'$,

for any $b, b' \in B$ and for any $x, x' \in X$. In fact, equation (i)-(vi) are equivalent to the following:

(i') $(b,0)\cdot[(0,x)\cdot(0,x')] = [(b,0)\cdot(0,x))]\cdot(0,x')$;

(ii') $[(0,x)\cdot(0,x')]\cdot(b,0) = (0,x)\cdot[(0,x')\cdot(b,0)]$;

(iii') $(0,x)\cdot[(b,0)\cdot(0,x')] = [(0,x)\cdot(b,0)]\cdot(0,x')$;

(iv') $[(b,0)\cdot(b',0)]\cdot(0,x) = (b,0)\cdot[(b',0)\cdot(0,x)]$;

(v') $(0,x)\cdot[(b,0)\cdot(b',0)] = [(0,x)\cdot(b,0)]\cdot(b',0)$;

(vi') $(b,0)\cdot[(0,x)\cdot(b',0)] = [(b,0)\cdot(0,x)]\cdot(b',0)$,

for any $b, b' \in B$ and for any $x, x' \in X$.

The first three equalities state that the pair $(b * -, - * b)$ belongs to the associative algebra

$$\mathrm{Bim}(X) = \{(f * -, - * f) \in \mathrm{End}(X) \times \mathrm{End}(X)^{\mathrm{op}} \mid \cdots$$
$$\cdots \mid f*(xy) = (f*x)y, (xy)*f = x(y*f), x(f*y) = (x*f)y,\ \forall x,y \in X\}$$

of *bimultipliers*, or *bimultiplications* of X [59], which is endowed with the bilinear multiplication

$$(f * -, - * f)\cdot(g * -, - * g) = (f*(g*-), (-*f)*g).$$

From equations (iii)-(iv), we have that the linear map

$$B \to \mathrm{Bim}(X) \colon b \mapsto (b * -, - * b)$$

is a homomorphism of associative algebras. Indeed

$$((bb') * -, - * (bb')) = (b * (b' * -), (- * b) * b') = (b * -, - * b)\cdot(b' * -, - * b'),$$

for any $b, b' \in B$. Moreover, thi homomorphism satisfies (vi), which states that the left multiplier $b * -$ and the right multiplier $- * b'$ are permutable.

On the other hand, given an associative algebra homomorphism

$$\varphi \colon B \to \mathrm{Bim}(X) \colon b \mapsto (b *_{\varphi} -, - *_{\varphi} b)$$

satisfying $b *_{\varphi} (x *_{\varphi} b') = (b *_{\varphi} x) *_{\varphi} b'$, we can associate the split extension in **Assoc**

$$0 \longrightarrow X \xrightarrow{i_2} (B \oplus X, \cdot_{\varphi}) \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0$$

where the associative algebra structure of $B \oplus X$ is given by

$$(b,x)\cdot_{\varphi}(b',x') = (bb', xx' + b *_{\varphi} x' + x *_{\varphi} b').$$

Remark 2.44. A generic homomorphism from B to $\mathrm{Bim}(X)$ needs not give rise to a split extension. For instance, if $B = \mathbb{F}\{b\}$ and $X = \mathbb{F}\{e_1, e_2\}$ are abelian algebras (i.e. the multiplications of B and X are trivial), then $\mathrm{Bim}(X) = \mathrm{End}(X) \times \mathrm{End}(X)^{\mathrm{op}}$ and the homomorphism $\varphi \colon B \to \mathrm{Bim}(X)$ defined by

$$b *_{\varphi} e_1 = e_2, \quad b *_{\varphi} e_2 = 0, \qquad e_1 *_{\varphi} b = e_1, \quad e_2 *_{\varphi} b = e_1$$

does not satisfy (vi) since

$$(b *_\varphi e_1) *_\varphi b = e_2 *_\varphi b = e_1 \neq e_2 = b *_\varphi e_1 = b *_\varphi (e_1 *_\varphi b).$$

We can now prove the following.

Theorem 2.45. *[47] Let B and X be Leibniz algebras over $\mathbb{F}$.*

(1) *The set of isomorphism classes of split extensions of B by X is in bijection with the set of associative algebra homomorphisms*

$$B \to \mathrm{Bim}(X) \colon b \mapsto (b * -, - * b)$$

which satisfy the condition

$$b * (x * b') = (b * x) * b', \qquad \forall b \in B, x \in X. \tag{2.2.13}$$

(2) *The category* **Assoc** *of associative algebras over $\mathbb{F}$ is weakly action representable and weak actor of an object$X$ of* **Assoc** *is the associative algebra* $\mathrm{Bim}(X)$*.*

Proof.

(1) The first statement follows from the analysis above.

(2) Given any associative algebra X, we take $T = \mathrm{Bim}(X)$ and we define τ in the following way: for every associative algebra B, the component

$$\tau_B \colon \ \mathrm{SplExt}(B, X) \to \mathrm{Hom}_{\mathbf{Assoc}}(B, \mathrm{Bim}(X)),$$

where $\mathrm{SplExt}(-, X) = \mathrm{SplExt}_{\mathbf{Assoc}}(-, X)$, is the morphism in **Set** which associates with any split extension

$$0 \longrightarrow X \xrightarrow{\ i\ } A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

the homomorphism $B \to \mathrm{Bim}(X) \colon b \mapsto (b * -, - * b)$ as above. The transformation τ is natural. Indeed, for every associative algebra homomorphism $f \colon B' \to B$, the following diagram in **Set**

$$\begin{array}{ccc} \mathrm{SplExt}(B, X) & \xrightarrow{\ \tau_B\ } & \mathrm{Hom}(B, \mathrm{Bim}(X)) \\ {\scriptstyle \mathrm{SplExt}(f,X)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(f,\mathrm{Bim}(X))} \\ \mathrm{SplExt}(B', X) & \xrightarrow{\ \tau_{B'}\ } & \mathrm{Hom}(B', \mathrm{Bider}(X)) \end{array}$$

is commutative. Moreover, for every associative algebra B, the morphism τ_B is an injection since every element of $\mathrm{SplExt}(B, X)$ is uniquely determined by the corresponding pair of bilinear maps

$$l \colon B \times X \to X, \qquad r \colon X \times B \to X\,.$$

Thus τ is a natural monomorphism of functors and the category **Assoc** is weakly action representable, with a weak actor of X being the associative algebra $\mathrm{Bim}(X)$.

$\square$

Equation (2.2.13) can be used to characterize the class of acting morphisms in the category **Assoc**. In [59] S. Mac Lane described, for a ring Λ, the $\Lambda-$bimodule structures over an abelian group K in terms of ring homomorphisms from Λ to the ring of bimultipliers of K. The following is a straightforward generalization to actions on an object which is not necessarily abelian.

Proposition 2.46. *[29] Let B and X be associative algebras over $\mathbb{F}$ and let*

$$\varphi\colon B \to \mathrm{Bim}(X)\colon b \mapsto (b *_\varphi -, - *_\varphi b)$$

be an morphism in **Assoc**. *Then φ is an* acting morphism *if and only if*

$$b *_\varphi (x *_\varphi b') = (b *_\varphi x) *_\varphi b' \tag{2.2.14}$$

for every $b, b' \in B$ and for every $x \in X$.

Proof. It follows from Theorem 2.45 that a morphism $\varphi \in \mathrm{Hom}_{\mathbf{Assoc}}(B, \mathrm{Bim}(X))$ is an acting morphism if and only if it defines a split extension of B by X in **Assoc**, i.e. if and only if it satisfies Equation (2.2.14). □

Remark 2.47. As observed in Remark 2.44, there are examples of associative algebra homomorphisms which do not satisfy Equation (2.2.14), i.e. which are not acting morphisms. Nevertheless, it is possible to find examples of associative algebras such that Equation (2.2.14) is always satisfied [26].

For instance, let X be an associative algebra such that the *annihilator*

$$\mathrm{Ann}(X) = \{x \in X \mid xy = yx = 0,\ \forall y \in X\}$$

is trivial, or such that X is a *perfect* algebra, i.e. $X^2 = X$, where X^2 is the subalgebra of X spanned by $\{xy \mid x, y \in X\}$. In this case we have

$$f * (x * f') = (f * x) * f' \tag{2.2.15}$$

for every $(f * -, - * f), (f' * -, - * f') \in \mathrm{Bim}(X)$ and $x \in X$.

In fact, if $\mathrm{Ann}(X) = 0$, then given $(f * -, - * f), (f' * -, - * f') \in \mathrm{Bim}(X)$, one has

$$f * (x * f')y = f((x * f')y) = f * (x(f' * y)) = (f * x)(f' * y) = ((f * x) * f')y$$

for any $x, y \in X$. Thus $f * (x * f') - (f * x) * f' \in \mathrm{Ann}(X) = 0$ for any $x \in X$, i.e. Equation (2.2.14) is satisfied.

In a similar way, if $X^2 = X$ we have

$$f * ((xy) * f') = f * (x(y * f')) = (f * x)(y * f') = ((f * x)y) * f' = (f * (xy)) * f'$$

for any $x, y \in X$ and $f * (x * f') = (f * x) * f'$.

In both cases, for any other associative algebra B, every arrow

$$\varphi\colon B \to \mathrm{Bim}(X)$$

is an acting morphism and we have a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Assoc}}(-, \mathrm{Bim}(X)),$$

i.e. $\mathrm{Bim}(X)$ is the actor of X.

Remark 2.48. [47] Let $\mathcal{C}$ be a weakly action representable category. From Definition 2.39 it follows that, given two objects in $\mathbf{SplExt}(X)$ and a morphism between them

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{i} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & X & \xrightarrow{i'} & A' & \underset{s'}{\overset{\pi'}{\rightleftarrows}} & B' & \longrightarrow & 0,
\end{array}$$

if $\varphi \colon B \to T$ and $\varphi' \colon B' \to T$ are the corresponding acting morphisms, with T being a weak actor of X, then $\varphi = \varphi' \circ f$. This is an immediate consequence of the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{SplExt}(B', X) & \xrightarrow{\tau_{B'}} & \mathrm{Hom}_{\mathcal{C}}(B', T) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(f,T)} \\
\mathrm{SplExt}(B, X) & \xrightarrow[\tau_B]{} & \mathrm{Hom}_{\mathcal{C}}(B, T)
\end{array}$$

where $f^* = \mathrm{SplExt}(f, X)$.

Another important observation made by G. Janelidze in [47] is the following.

Theorem 2.49. *[47] Every weakly action representable category $\mathcal{C}$ is action accessible.*

To prove this, we need two intermediate results.

Lemma 2.50. *[47] Let $\mathcal{C}$ be a weakly action representable category and let*

$$0 \longrightarrow X \xrightarrow{i} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{2.2.16}$$

be a split extension in $\mathcal{C}$ with associated acting morphism $\varphi \colon B \to T$, where T is a weak actor of X. If φ is a monomorphism, then the split extension (2.2.16) *is faithful.*

Proof. Let

$$0 \longrightarrow X \xrightarrow{i'} A' \underset{s'}{\overset{\pi'}{\rightleftarrows}} B' \longrightarrow 0 \tag{2.2.17}$$

be another split extension in $\mathcal{C}$ and suppose there exist two morphisms of split extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{i'} & A' & \underset{s}{\overset{\pi'}{\rightleftarrows}} & B' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_X} & & {\scriptstyle \theta}\downarrow\,\downarrow{\scriptstyle \theta'} & & {\scriptstyle f}\downarrow\,\downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & X & \xrightarrow{i} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0.
\end{array} \tag{2.2.18}$$

If $\varphi' \colon B' \to T$ is the acting morphism associated with (2.2.17), then $\varphi' = \varphi \circ f = \varphi \circ g$ with φ being a monomorphism. Thus $f = g$, $\theta = \theta' = s \circ f \circ \pi'$ and (2.2.17) is a faithful object in the category $\mathbf{SplExt}(X)$. □

Lemma 2.51. *[47] Let $\mathcal{C}$ be a weakly action representable category. Let B, B', X be objects of $\mathcal{C}$ and let T be a weak actor of X. If $f \colon B \twoheadrightarrow B'$ is a regular epimorphism, then a morphism $\varphi' \colon B' \to T$ is an acting morphism if so is the composition $\varphi' \circ f$.*

Proof. Let $\varphi := \varphi' \circ f$ be an acting morphism and let

$$0 \longrightarrow X \xrightarrow{i} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{2.2.19}$$

be the associated split extension. Since f is a regular epimorphism, the diagram

$$\begin{array}{ccc} \mathrm{SplExt}(B',X) & \xrightarrow{\tau_{B'}} & \mathrm{Hom}_{\mathcal{C}}(B',T) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(f,T)} \\ \mathrm{SplExt}(B,X) & \xrightarrow[\tau_B]{} & \mathrm{Hom}_{\mathcal{C}}(B,T) \end{array}$$

is a pullback. This means there exists a split extension

$$0 \longrightarrow X \xrightarrow{i'} A' \underset{s'}{\overset{\pi'}{\rightleftarrows}} B' \longrightarrow 0 \tag{2.2.20}$$

such that f^* takes the isomorphism class of (2.2.20) to that of (2.2.19) and φ' is the acting morphism associated with (2.2.20). □

We are now ready to prove Theorem 2.49.

Proof. Let X be an object of $\mathcal{C}$ and let

$$\tau\colon \mathrm{SplExt}(-,X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-,T)$$

be a weak representation of $\mathrm{SplExt}(-,X)$. Let

$$0 \longrightarrow X \xrightarrow{i} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

be a split extension in $\mathcal{C}$ with associated acting morphism $\varphi\colon B \to T$. We want to construct a morphism of split extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & X & \xrightarrow{i'} & A' & \underset{s'}{\overset{\pi'}{\rightleftarrows}} & B' & \longrightarrow & 0 \end{array} \tag{2.2.21}$$

with a faithful codomain.. We take $B' = \varphi(B)$ and $f\colon B \twoheadrightarrow B'$ to be the corestriction of φ to its image, so that $\varphi = \varphi' \circ f$ where $\varphi'\colon B' \rightarrowtail T$ is given by the *epi-mono factorization* of φ.

By Lemma 2.51, φ' is an acting morphism; thus it defines a split extension

$$0 \longrightarrow X \xrightarrow{i'} A' \underset{s'}{\overset{\pi'}{\rightleftarrows}} B' \longrightarrow 0 \tag{2.2.22}$$

Finally the split extension (2.2.22) is faithful since the acting morphism φ' is a monomorphism (see Lemma 2.50) and the category $\mathcal{C}$ is action accessible. □

We thus have that

action representability ⇒ *weak action representability* ⇒ *action accessibility.*

This allows us to present an example of a category which is not weakly action representable in the context of varieties of algebras: the category **Jord** of Jordan algebras.

Definition 2.52. A *Jordan algebra* over a field $\mathbb{F}$ is a non-associative commutative $\mathbb{F}$-algebra $(X, \cdot)$ over $\mathbb{F}$ which satisfies the *Jordan identity*

$$(xy)x^2 = x(yx^2), \quad \forall x, y \in X,$$

where the powers have precedence on the multiplications.

In [30] it was observed that the category **Jord** is not action accessible, hence it is not weakly action representable.

A first class of examples of categories which are action accessible but not weakly action representable was given by J. R. A. Gray in [45].

Chapter 3

Actions in categories of algebras

In this chapter, we explain how to describe internal actions and their representability in *Orzech categories of interest* [66] and *varieties of non-associative algebras* over a field [56].

3.1 Orzech categories of interest

We refer to [26], [29, Section 3], [34], [63] and [66] for the definitions and the results in this section.

Let $\mathcal{C}$ be a *category of groups with a set of operations* Ω and with a set of identities $\mathbb{E}$, such that $\mathbb{E}$ includes the group laws and the following conditions hold: if Ω_i is the set of i-ary operations in Ω, then

(i) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(ii) the group operations (written additively 0, $-$ and $+$) are elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \setminus \{+\}$, $\Omega_1' = \Omega_1 \setminus \{-\}$ and assume that if $* \in \Omega_2'$, then Ω_2' contain $*^{\mathrm{op}}$ defined by $x *^{\mathrm{op}} y := y * x$. Assume further that $\Omega_0 = \{0\}$;

(iii) for any $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $x * (y + z) = x * y + x * z$;

(iv) for any $\omega \in \Omega_1'$ and $* \in \Omega_2'$, $\mathbb{E}$ contains the identities $\omega(x + y) = \omega(x) + \omega(y)$ and $\omega(x * y) = \omega(x) * y$.

For an object X of $\mathcal{C}$ and $x_1, x_2, x_3 \in X$, we define

(v) Axiom 1: $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$, for each $* \in \Omega_2'$;

(vi) Axiom 2: for each ordered pairs $(*, \bar{*}) \in \Omega_2 \times \Omega_2$, there exists a word W in the variables

$$x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2,$$

where each juxtaposition represents the operation $*$ or the operation $\bar{*}$, and which we denote by $W(x_1; x_2, x_3; *, \bar{*})$, such that

$$(x_1 * x_2)\bar{*}x_3 = W(x_1; x_2, x_3; *, \bar{*}).$$

Definition 3.1. [66] A category of groups with operations satisfying conditions (i)-(vi) is called an *Orzech category of interest*, or simply a *category of interest*.

Let $\mathbb{E}'$ be the subset of identities of $\mathbb{E}$ which includes the group laws and the identities (iii) and (iv) and let $\mathcal{C}'$ be the corresponding category of groups with operations. Thus we have a full inclusion functor $\mathcal{C} \hookrightarrow \mathcal{C}'$. $\mathcal{C}'$ is called the *general category of groups with operations* of the Orzech category of interest $\mathcal{C}$.

Remark 3.2. If $\mathcal{C}$ is a category of abelian groups with operations, i.e. the group operation $+$ is commutative, then Axiom 1 is automatically satisfied.

We describe now some examples (and counterexamples) of Orzech categories of interest.

Example 3.3. The categories **Grp** of groups and **Ab** of abelian groups are Orzech categories of interest, with $\Omega_1' = \Omega_2' = \emptyset$.

Example 3.4. The categories **Rng** of (not necessarily unitary) rings and **CRng** of commutative (not necessarily unitary) rings are Orzech categories of interst, with $\Omega_1' = \emptyset, \Omega_2' = \{*, *^{\mathrm{op}}\}$, where $*$ denotes the multiplication of a ring, and Axiom 2 is the associativity of $*$.

Example 3.5. Let $\mathcal{C} = \mathbf{Alg}$ be the category of non-associative algebras over a field $\mathbb{F}$: its objects are $\mathbb{F}$-vector spaces X endowed with a bilinear operation

$$*\colon X \times X \to X\colon (x, y) \mapsto x * y.$$

and the morphism are $\mathbb{F}$-linear maps which preserve the bilinear operation (see Examples 3.46). Then **Alg** is a category of abelian groups with operations, with $\Omega_1' = \emptyset$, $\Omega_2' = \{*, *^{\mathrm{op}}\}$, but it is not an Orzech category of interest since Axiom 2 fails.

Example 3.6. The category **Assoc** of associative $\mathbb{F}$-algebras over a field $\mathbb{F}$ is an Orzech category of interest: the group operation $+$ is commutative, $\Omega_1' = \emptyset, \Omega_2' = \{*, *^{\mathrm{op}}\}$, where $*$ denotes the associative multiplication, and Axiom 2 is the associativity

$$(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3).$$

Other examples are the category **Lie** of Lie $\mathbb{F}$-algebras, with $\Omega_1' = \emptyset$, $\Omega_2' = \{[-,-], [-,-]^{\mathrm{op}}\}$, where $[-,-]$ denotes the Lie bracket, and Axiom 2 is the Jacobi identity

$$[x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]] = 0,$$

or the category **Leib** of Leibniz $\mathbb{F}$-algebras (see Chapter 5). Moreover, we have that $\mathbf{Assoc}' = \mathbf{Lie}' = \mathbf{Leib}' = \mathbf{Alg}$. More precisely, $\mathcal{V}' = \mathbf{Alg}$ for any variety $\mathcal{V}$ of non-associative algebras over a field which is an Orzech category of interest (see Section 3.2).

Example 3.7. The category **Jord** of Jordan $\mathbb{F}$-algebras is not an Orzech category of interest since Axiom 2 fails. In Section 3.2 we see that for any variety of non-associative algebras over a field, action accessibility is equivalent to being an Orzech category of interest (see Theorem 3.51).

3.1.1 Derived actions in Orzech categories of interest

All the categories of groups with operations are semi-abelian and thus, split extensions are in natural bijection with internal actions. However, in this context, it is more convenient to describe internal actions in terms of the so-called *derived actions*.

Definition 3.8. Let $\mathcal{C}$ be a category of groups with operations and let

$$0 \longrightarrow X \xrightarrow{i} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \qquad (3.1.1)$$

be a split extension in $\mathcal{C}$. The collection of maps

$$f_* \colon B \times X \to X, \qquad \forall * \in \Omega_2.$$

defined by

$$b \cdot x = s(b) + i(x) - s(b), \qquad b * x = s(b) * i(x),$$

where

$$b \cdot x := f_+(b, x), \quad b * x := f_*(b, x), \quad \forall * \in \Omega_2',$$

is called the *derived action* of B on X associated with (3.1.1).

Given two objects B, X of $\mathcal{C}$ and a collection of maps $\{f_*\}_{*\in\Omega_2}$ denoted by

$$b \cdot x := f_+(b, x), \qquad b * x := f_*(b, x), \quad \forall * \in \Omega_2',$$

we may define the binary operations on the cartesian product $B \times X$

$$(b, x) + (b', x') = (b + b', x + b \cdot x')$$

and

$$(b, x) * (b', x') = (b * b', x * x' + b * x' + x * b'),$$

for any $* \in \Omega_2'$, where $x * b' := b' *^{\mathrm{op}} x$.

Definition 3.9. $(B \times X, +, \{*\}_{*\in\Omega_2'})$ is called the *semidirect product* of B and X with respect to $\{f_*\}_{*\in\Omega_2}$, and it is denoted by $B \ltimes X$.

Then, the collection $\{f_*\}_{*\in\Omega_2}$ defines a derived action of B on X if and only if $B \ltimes X$ is an object of $\mathcal{C}$. In this case, one can construct the split extension in $\mathcal{C}$

$$0 \longrightarrow X \xrightarrow{i_2} B \ltimes X \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0 \qquad (3.1.2)$$

with $i_2(x) = (0, x)$, $i_1(b) = (b, 0)$, $\pi_1(b, x) = b$, and $\{f_*\}_{*\in\Omega_2}$ is precisely the derived action associated with (3.1.2).

Remark 3.10. When the group operation $+$ is commutative, we have

$$(b, x) + (b', x') = (b', x') + (b, x).$$

Thus

$$b \cdot x = x, \qquad \forall b \in B,\ \forall x \in X$$

and a derived action of B on X is given by a collection of maps $\{f_*\}_{*\in\Omega_2'}$.

Example 3.11. Let $\mathcal{C} = \mathbf{Alg}$. Then, for any objects B, X of $\mathbf{Alg}$, a pair of maps

$$f_* \colon B \times X \to X, \qquad f_{*^{\mathrm{op}}} \colon B \times X \to X.$$

defines a derived action of B on X if and only if the vector space $B \oplus X \cong B \times X$, endowed with the binary operation

$$(b, x) * (b', x') = (b * b', x * x' + b * x' + x * b'),$$

is a non-associative algebra over $\mathbb{F}$. This happens if and only if the above operation is bilinear, i.e. if and only if f_* and $f_{*^{op}}$ are bilinear maps.

When $\mathcal{C}$ is an Orzech category of interest and B, X are object of $\mathcal{C}$, then $\{f_*\}_{*\in\Omega_2}$ defines a derived action of B on X in $\mathcal{C}$ if and only if the maps f_*, $* \in \Omega_2$, satisfy a set of suitable conditions, as explained in the following.

Proposition 3.12. *[34] Let $\mathcal{C}$ be an Orzech category of interest and let B, X be objects of $\mathcal{C}$. The collection of maps $\{f_*\}_{*\in\Omega_2}$, denoted by*

$$b \cdot x := f_+(b, x), \quad b * x := f_*(b, x), \quad \forall * \in \Omega_2',$$

defines a derived action of B on X in $\mathcal{C}$ if and only if the following conditions are satisfied:

(1) $0 \cdot x = x$;

(2) $b \cdot (x + x') = b \cdot x + b \cdot x'$;

(3) $(b + b') \cdot x = b \cdot (b' \cdot x)$;

(4) $b * (x + x') = b * x + b * x'$;

(5) $(b + b') * x = b * x + b' * x$;

(6) $b \cdot (x * x') = x * x'$;

(7) $b \cdot (x * b') = x * b'$;

(8) $(b * b') \cdot x = x$;

(9) $x * (b \cdot x') = x * x'$;

(10) $b * (b' \cdot x) = b * x$;

(11) $\omega(b \cdot x) = \omega(b) \cdot \omega(x)$;

(12) $\omega(x * b) = \omega(x) * b = x * \omega(b)$;

(13) $\alpha + \beta * \gamma = \beta * \gamma + \alpha$;

(14) for each ordered pairs $(*, \bar{*}) \in \Omega_2 \times \Omega_2$, there exists a word W such that

$$(x_1 * x_2) \bar{*} x_3 = W(x_1; x_2, x_3; *, \bar{*}),$$

for any $\omega \in \Omega_1'$, $* \in \Omega_2'$, $b, b' \in B$, $x, x' \in A$, $\alpha, \beta, \gamma \in B \cup X$ such that each side of (14) is well defined and $x_1, x_2, x_3 \in B \cup X$.

Proof. Let $\{f_*\}_{*\in\Omega_2}$ be a derived action of B on X in $\mathcal{C}$ and let $B \ltimes X$ be the corresponding semi-direct product, which is an object of $\mathcal{C}$. We thus have:

(1) $(0, 0) + (0, x) = (0, x) + (0, 0)$ implies $0 \cdot x = x$;

(2) from $[(b, 0) + (0, x)] + (0, x') = (b, 0) + (0, x + x')$, we obtain $b \cdot (x + x') = b \cdot x + b \cdot x'$;

(3) from $(b + b', 0) + (0, x) = (b, 0) + [(b', 0) + (0, x)]$, we have $(b + b') \cdot x = b \cdot (b' \cdot x)$;

(4) from (iii) of Definition 3.1, we have $(b, 0) * [(0, x) + (0, x')] = (b, 0) * (0, x) + (b, 0) * (0, x')$ and thus $b * (x + x') = b * x + b * x'$;

(5) again from (iii) of Definition 3.1, $[(b,0)+(b',0)]*(0,x) = (b,0)*(0,x) + (b',0)*(0,x)$ and $(b+b')*x = b*x + b'*x$;

(6) from Axiom 1, we have $(b,0)+[(0,x)*(0,x')] = [(0,x)*(0,x')]+(b,0)$ and thus $b\cdot(x*x') = x*x'$;

(7) again from Axiom 1, $(b,0)+[(0,x)*(b',0)] = [(0,x)*(b',0)]+(b,0)$ and thus $b\cdot(x*b') = x*b'$;

(8) in the same way, from $(0,x)+[(b,0)*(b',0)] = [(b,0)*(b',0)]+(0,x)$ we obtain $(b*b')\cdot x = x$;

(9) from (iii) of Definition 3.1, we have $(0,x)*[(b,0)+(0,x')] = (0,x)*(b,0) + (0,x)*(0,x')$ and thus $x*(b\cdot x') = x*x'$;

(10) in the same way, $(b,0)*[(b',0)+(0,x)] = (b,0)*(b'0)+(b,0)*(0,x)$ implies $b*(b'\cdot x) = b*x$;

(11) from (iv) of Definition 3.1, we have $\omega[(b,x)+(b',x')] = \omega(b,x)+\omega(b',x')$ and thus $\omega(b\cdot x) = \omega(b)\cdot\omega(x)$;

(12) again from (iv) of Definition 3.1, $\omega[(b,x)*(b',x')] = \omega(b,x)*(b',x') = (b,x)*\omega(b',x')$ and $\omega(x*b) = \omega(x)*b = x*\omega(b)$;

(13) from Axiom 1, we obtain $\alpha+\beta*\gamma = \beta*\gamma+\alpha$, for any $\alpha,\beta,\gamma \in B\cup X$ such that each side of the equation is well defined;

(14) from Axiom 2, we have that for each ordered pairs $(*,\bar{*}) \in \Omega_2\times\Omega_2$, there exists a word W such that

$$(x_1 * x_2)\bar{*}x_3 = W(x_1;x_2,x_3;*,\bar{*}),$$

for any $x_1,x_2,x_3 \in B\cup X$.

Conversely, if the collection of maps $\{f_*\}_{*\in\Omega_2}$ satisfies Equations (1)-(14), then the corresponding semi-direct product $B\ltimes X$ is an object of $\mathcal{C}$ and $\{f_*\}_{*\in\Omega_2}$ is the derived action of B on X in $\mathcal{C}$ associated with the split extension (3.1.2). □

Remark 3.13. We observe that an analogous result can be obtained for a category of groups with operations $\mathcal{C}$, as shown in [34, Proposition 1.1]. In this case, we have to ask that $\{f_*\}_{*\in\Omega_2}$ satisfies conditions (1)-(5) and (9)-(12) of Proposition 3.12; moreover conditions (6)-(8) are replaced by

$$(b*b')\cdot(x*x') = x*x'$$

and

$$(b*b')\cdot(x*b'') = x*b''$$

for any $b,b',b'' \in B$ and $x,x' \in X$, and (13) is replaced by

$$\alpha*\beta+\gamma*\delta = \gamma*\delta+\alpha*\beta$$

for any $\alpha,\beta,\gamma,\delta \in B\cup X$ such that both sides of the equation make sense. Note that (6)-(7)-(8) and (13) are consequences of Axiom 1, while (14) is implied by Axiom 2.

Corollary 3.14. *Let $\mathcal{C}$ be an Orzech category of interest whose objects are abelian groups with operations and let B, X be objects of $\mathcal{C}$. The collection $\{f_*\}_{*\in\Omega_2'}$ defines a derived action of B on X in $\mathcal{C}$ if and only if it satisfies conditions (4)-(5)-(11)-(12)-(14) of Proposition 3.12.*

Proof. The proof is an immediate consequence of Remark 3.10 and Proposition 3.12. □

Example 3.15. Let $\mathcal{C} = \mathbf{Lie}$. The group operation $+$ is commutative and a derived action of B on X is a pair of bilinear maps

$$f_{[-,-]} \colon B \times X \to X, \qquad f_{[-,-]^{\mathrm{op}}} \colon B \times X \to X$$

such that the bilinear operation

$$[(b,x),(b',x')] = ([b,b'],[x,x'] + [b,x'] + [x,b'])$$

where $[b,x] := f_{[-,-]}(b,x)$ and $[x,b] := f_{[-,-]^{\mathrm{op}}}(b,x)$, defines a Lie algebra structure on the vector space $B \oplus X \cong B \times X$. By Corollary 3.14, this happens if and only if

(1) $[x,b] = -[b,x]$;

(2) $[b,[x,x']] = [[b,x],x'] + [x,[b,x']]$;

(3) $[[b,b'],x] = [b,[b',x]] - [b',[b,x]]$;

for any $b, b' \in B$ and $x, x' \in X$.

For instance, if $B = \mathrm{Der}(X)$ is the Lie algebra of derivations of X, the canonical action of $\mathrm{Der}(X)$ on X defined by

$$[d,x] := d(x), \quad [x,d] := -d(x), \quad \forall d \in \mathrm{Der}(X),\ \forall x \in X$$

is a derived action in **Lie**, since the semi-direct product $\mathrm{Der}(X) \ltimes X$ is a Lie algebra.

This is not true for the canonical action of $\mathrm{Bim}(X)$ on X in the category **Assoc**, where the action is defined by

$$(f * -, - * f) * x := f * x$$

and

$$x * (f * -, - * f) := x * f,$$

for any $(f * -, - * f) \in \mathrm{Bim}(X)$ and for any $x \in X$. In this case a derived action of B on X in **Assoc** is given by a pair of bilinear maps

$$f_* \colon B \times X \to X, \qquad f_{*^{\mathrm{op}}} \colon B \times X \to X,$$

with $b * x := f_*(b,x)$ and $x * b := f_{*^{\mathrm{op}}}(b,x)$, such that

(1) $(b * b') * x = b * (b' * x)$;

(2) $x * (b * b') = (x * b) * b'$;

(3) $(b * x) * b' = b * (x * b')$;

(4) $b * (x * x') = (b * x) * x'$;

(5) $(x * x') * b = x * (x' * b)$;

(6) $x * (b * x') = (x * b) * x'$,

for any $b, b' \in B$ and $x, x' \in X$.

We observe that, given two bimultipliers $(f * -, - * f), (f' * -, - * f') \in \mathrm{Bim}(X)$, Equation (3) needs not in general be satisfied. For instance, let $X = \mathbb{F}\{e_1, e_2\}$ be an abelian algebra (i.e. $x * x' = 0$, for any $x, x' \in X$) and let $(f * -, - * f) \in \mathrm{Bim}(X) = \mathrm{End}(X) \times \mathrm{End}(X)^{\mathrm{op}}$ defined by

$$f * e_1 = e_2, \quad f * e_2 = 0, \qquad e_1 * f = e_1, \quad e_2 * f = e_1.$$

Then

$$(f * e_1) * f = e_2 * f = e_1 \neq e_2 = f * e_1 = f * (e_1 * f),$$

(3) is not satisfied and the semi-direct product $\mathrm{Bim}(X) \ltimes X$ is non an associative algebra. We thus have an example of a derived action in $\mathbf{Assoc}' = \mathbf{Alg}$ which is not a derived action in **Assoc**.

Remark 3.16. Summarizing, we defined derived actions in any category of groups with operations $\mathcal{C}$ and we saw that we can use them to describe split extensions / internal actions.

Starting with a split extension in $\mathcal{C}$

$$0 \longrightarrow X \xrightarrow{\;i\;} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0\,, \tag{3.1.3}$$

we can associate the derived action $\{f_*\}_{*\in\Omega_2}$ of B on X given by

$$b \cdot x = s(b) + i(x) - s(b), \qquad b * x = s(b) * i(x),$$

for any $b \in B$, $x \in X$ and $* \in \Omega_2'$. Conversely, given any collection of maps $\{f_*\}_{*\in\Omega_2}$, denoted by

$$b \cdot x := f_+(b, x), \qquad b * x := f_*(b, x), \quad \forall * \in \Omega_2',$$

one can construct the semi-direct product $B \ltimes X$ as above and $\{f_*\}_{*\in\Omega_2}$ is a derived action of B on X if and only if $B \ltimes X$ is an object of $\mathcal{C}$. In this case, $\{f_*\}_{*\in\Omega_2}$ is the derived action associated with the split extension in $\mathcal{C}$

$$0 \longrightarrow X \xrightarrow{\;i_2\;} B \ltimes X \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0 \tag{3.1.4}$$

where i_1, i_2 and π_1 are the canonical injections and projection, and the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\;i_2\;} & B \ltimes X & \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_B} & & \\
0 & \longrightarrow & X & \xrightarrow{\;i\;} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0
\end{array}$$

where $\theta(b, x) = s(b) + i(x)$, defines an isomorphism of split extensions between (3.1.3) and (3.1.4).

We thus have a bijection

$$\eta_B \colon \mathrm{SplExt}(B, X) \cong \mathrm{DAct}(B, X)$$

between the set $\mathrm{SplExt}(B, X)$ of isomorphism classes of split extensions of B by X and the set $\mathrm{DAct}(B, X) = \mathrm{DAct}_{\mathcal{C}}(B, X)$ of derived actions of B on X in any category $\mathcal{C}$ of groups with operations.

Now denote by $\mathrm{DAct}(-, X)$ the functor from $\mathcal{C}^{\mathrm{op}}$ to **Set** which assigns to any object B of $\mathcal{C}$, the set of $\mathrm{DAct}(B, X)$ and, for any morphism $g \colon B' \to B$, $\mathrm{DAct}(g, X)$ sends a derived action $\{f_*\}_{*\in\Omega_2}$ of B on X to the derived action $\{f'_*\}_{*\in\Omega_2}$ of B' on X, where $f'_* := f \circ (g \times 1_X)$, i.e.

$$f'_+(b',x) = f_+(g(b'),x), \qquad f'_*(b,x) = f_*(g(b'),x), \tag{3.1.5}$$

for any $* \in \Omega'_2$ and for any $b' \in B'$, $x \in X$. We want to show that, if $\{f_*\}_{*\in\Omega_2}$ is the derived action associated with a split extension

$$0 \longrightarrow X \xrightarrow{i} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{3.1.6}$$

then $\{f'_*\}_{*\in\Omega_2}$ is the derived action associated with the split extension

$$0 \longrightarrow X \xrightarrow{i'} A' \underset{s'}{\overset{\pi'}{\rightleftarrows}} B' \longrightarrow 0 \tag{3.1.7}$$

given by $g^* = \mathrm{SplExt}(g,X)$. In other words, we have the following.

Proposition 3.17. *The bijection η_B extends to a natural isomorphism of functors*

$$\eta\colon \mathrm{SplExt}(-,X) \cong \mathrm{DAct}(-,X).$$

Proof. Let $g\colon B' \to B$ be a morphism in $\mathcal{C}$ and suppose that

(1) $g^* = \mathrm{SplExt}(g,X)$ takes the isomorphism class of (3.1.6) to that of (3.1.7);

(2) $\mathrm{DAct}(g,X)$ sends the derived action $\{f_*\}_{*\in\Omega_2}$ associated with (3.1.6) to $\{f'_*\}_{*\in\Omega_2}$ defined as in (3.1.5).

We observe that, if $\theta\colon A' \to A$ is given by pulling back along g, then $\theta \circ s' = s \circ g$ and $\theta \circ i' = i$.

We have to show that the bijection μ_B is natural in B, i.e. the following diagram in **Set**

$$\begin{array}{ccc} \mathrm{SplExt}(B,X) & \xrightarrow{\mu_B} & \mathrm{DAct}(B,X) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle \mathrm{DAct}(g,X)} \\ \mathrm{SplExt}(B',X) & \xrightarrow[\mu_{B'}]{} & \mathrm{DAct}(B',X) \end{array}$$

is commutative.

The composition $\mu'_B \circ g^*$ sends (3.1.6) to the derived action of B' on X associated with (3.1.7), which is given by

$$F'_+(b',x) = s'(b') + i'(x) - s'(b'), \quad F'_*(b',x) = s'(b') * i'(x), \quad \forall * \in \Omega'_2.$$

We thus have

$$\begin{aligned} i(F'_+(b',x)) &= \theta(i'(s'(b') + i'(x) - s'(b'))) = \\ &= i(s(g(b')) + i(x) - s(g(b'))) = i(f_+(g(b'),x)) = i(f'_+(b',x)) \end{aligned}$$

and

$$\begin{aligned} i(F'_*(b',x)) &= \theta(i'(s'(b') * i'(x))) = \\ &= i(s(g(b')) * i(x)) = i(f_*(g(b'),x)) = i(f'_*(b',x)) \end{aligned}$$

for any $* \in \Omega'_2$, i.e.

$$F'_+(b',x) = f_+(g(b'),x) = f'_+(b',x), \quad F'_*(b',x) = f_*(g(b'),x) = f'_*(b',x)$$

for any $* \in \Omega_2'$, and $\mu_B' \circ f^* = \mathrm{DAct}(g, X) \circ \mu_B$, i.e. we have a natural isomorphism

$$\mu\colon \mathrm{SplExt}(-, X) \cong \mathrm{DAct}(-, X).$$

□

Finally, since any category of groups with operations $\mathcal{C}$ is semi-abelian, for any objects X of $\mathcal{C}$ we also have a natural isomorphism

$$\tau\colon \mathrm{Act}(-, X) \cong \mathrm{SplExt}(-, X),$$

as shown in Proposition 2.29. Since the composition of natural isomorphisms is still a natural isomorphism, we have that

$$\eta \circ \tau^{-1}\colon \mathrm{Act}(-, X) \cong \mathrm{DAct}(-, X),$$

i.e. internal actions are in natural bijection with derived actions.

We end this section by recalling two definitions and a result which are useful later.

Definition 3.18. [26] Let B, X be objects of a category of groups with operations $\mathcal{C}$. A derived action of B on X is said to be *strict* if for any $b, b' \in B$ the conditions $b \cdot x = b' \cdot x$, $\omega(b) \cdot x = \omega(b') \cdot x$ and $b * x = b' * x$, for any $x \in X$, $\omega \in \Omega_1'$ and $* \in \Omega_2'$, we have that $b = b'$.

Definition 3.19. [63] Let B, X be objects of a category of groups with operations $\mathcal{C}$. A derived action of B on X is said to be *super-strict* if for any $b, b' \in B$ the conditions $b \cdot x = b' \cdot x$ and $b * x = b' * x$, for any $x \in X$ and $* \in \Omega_2'$, we have that $b = b'$.

Remark 3.20. It is immediate to observe that a derived action of B on X is strict if and only if, for any $b \in B$ such that $b \cdot x = x$, $\omega(b) \cdot x = x$ and $b * x = 0$ for any $x \in X$, $\omega \in \Omega_1'$ and $* \in \Omega_2'$, we have $b = 0$.

Analogously, a derived action of B on X is super-strict if and only if, for any $b \in B$ such that $b \cdot x = x$ and $b * x = 0$, for any $x \in X$ and $* \in \Omega_2'$, we have $b = 0$.

For the next proposition, we need to recall the notion of *free object* on a set, which is available for any *concrete category*.

Definition 3.21. [60] Let $\mathcal{C}$ and $\mathcal{D}$ be *locally small* categories (i.e. the hom-classes of $\mathcal{C}$ and $\mathcal{D}$ are sets). A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *faithful* (resp. *fully*, resp. *fully faithful*) if the induced functions

$$F_{X,Y}\colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))\colon f \mapsto F(f)$$

are injections (resp. surjections, resp. bijections), for any pairs of object X, Y of $\mathcal{C}$.

Definition 3.22. [1] A category $\mathcal{C}$ is said to be *concrete* if it has a faithful functor $U\colon \mathcal{C} \to \mathbf{Set}$.

For instance, any category of groups with operations is concrete with $\mathcal{U}$ being the usual forgetful functor.

Definition 3.23. [2] Let $\mathcal{C}$ be a concrete category. A *free object* on a set X is a an object $A = F(X)$ of $\mathcal{C}$ together with a function $i\colon X \to U(A)$, called the *canonical injection*, which satisfy the following universal property: for any object B of $\mathcal{C}$ and any map

$g\colon X \to U(B)$, there exists a unique morphism $f\colon A \to B$ in $\mathcal{C}$ such that $g = U(f) \circ i$, i.e. the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & U(A) \\ & \searrow^{g} & \big\downarrow {\scriptstyle U(f)} \\ & & U(B) \end{array}$$

If free objects exist in $\mathcal{C}$, the universal property implies that every function between two sets induces a unique morphism between the associated free objects, and this defines a functor $F\colon \mathbf{Set} \to \mathcal{C}$, called the *free functor*.

This relation between the functors F and U is more clear when we recall the following general definition, of one of the key notions in Category Theory:

Definition 3.24. [60] Consider a pair of functors $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$. Then L is said to be *left adjoint* to R, and R is said to be *right adjoint* to L, when for every object C of $\mathcal{C}$ there exists a morphism $\eta_C\colon C \to R(L(C))$ in $\mathcal{C}$, such that for every object D of $\mathcal{D}$ and every morphism $f\colon C \to R(D)$ there exists a unique morphism $\overline{f}\colon L(C) \to D$ in $\mathcal{D}$ such that the triangle

$$\begin{array}{ccc} C & \xrightarrow{\;\eta_C\;} & R(L(C)) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle R(\overline{f})} & \\ & R(D) & \end{array}$$

commutes in $\mathcal{C}$. In other words, for every objects C of $\mathcal{C}$ and D of $\mathcal{D}$, we have a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(C, R(D)) \cong \mathrm{Hom}_{\mathcal{D}}(L(C), D).$$

The adjunction is denoted by $L \dashv R$. When they exist, adjoints are unique (up to isomorphism), so we may say that R *has a left adjoint* or L *has a right adjoint*. The collection of morphisms $\{\eta_C\}_C$ is called the *unit* of the adjunction, and always forms a natural transformation from $1_{\mathcal{C}}$ to $R \circ L$, which means that for every morphism $c\colon C \to C'$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} C & \xrightarrow{\;\eta_C\;} & R(L(C)) \\ {\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle R(L(c))} \\ C' & \xrightarrow[\;\eta_{C'}\;]{} & R(L(C')) \end{array}$$

in $\mathcal{C}$ is commutative.

We thus have that $F \dashv U$, i.e. there exists a bijection

$$\mathrm{Hom}_{\mathbf{Set}}(X, U(B)) \cong \mathrm{Hom}_{\mathcal{C}}(F(X), B)$$

for any set X and any object B of $\mathcal{C}$.

In Section 3.2.1 we describe in detail the construction of the free object on a set for the categories **Mag** of *magmas* and **Alg** of non-associative algebras over a field.

We are now ready to prove the following.

Proposition 3.25. *[63] Let*

$$0 \longrightarrow X \xrightarrow{\;i\;} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{3.1.8}$$

be a split extension in an Orzech category of interest $\mathcal{C}$. The following are equivalent:

(i) the derived actions associated with (3.1.8) *is super-strict;*

(ii) the derived actions associated with (3.1.8) *is strict;*

(iii) the split extension (3.1.8) *is a faithful object in the category* $\mathbf{SplExt}(X)$ *of split extensions in $\mathcal{C}$ with kernel X.*

Proof. (i) ⇒ (ii). Obvious.

(ii) ⇒ (iii). We recall from [21] that an object is said to be faithful if it admits at most one morphism onto it. Consider another split extension in $\mathcal{C}$

$$0 \longrightarrow X \xrightarrow{\;i'\;} A' \underset{s'}{\overset{\pi'}{\rightleftarrows}} B' \longrightarrow 0$$

and a pair of morphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\;i'\;} & A' & \underset{s'}{\overset{\pi'}{\rightleftarrows}} & B' & \longrightarrow & 0 \\
 & & {\scriptstyle 1_X}\big\downarrow & & {\scriptstyle \theta}\big\downarrow\big\downarrow{\scriptstyle \theta'} & & {\scriptstyle g}\big\downarrow\big\downarrow{\scriptstyle g'} & & \\
0 & \longrightarrow & X & \xrightarrow{\;i\;} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0.
\end{array} \tag{3.1.9}$$

We have to prove that $g = g'$ and $\theta = \theta'$.

Since the derived action of B on X is strict, one has the following equalities, for any $b' \in B'$ and $x \in X$:

(a) $s(g(b')) + i(x) - s(g(b')) = s(g'(b')) + i(x) - s(g'(b'))$;

(b) $s(\omega(g(b'))) + i(x) - s(\omega(g(b'))) = s(\omega(g'(b'))) + l(x) - s(\omega(g'(b')))$, for any $\omega \in \Omega_1'$;

(c) $s(g(b')) * x = s(g'(b')) * x, \quad \forall * \in \Omega_2'$.

The equality (i) is equivalent to

$$-s(g(b')) + s(g'(b')) + i(x) - s(g(b')) + s(g'(b')) = i(x);$$

since $s \circ g = \theta \circ s'$, $s \circ g' = \theta' \circ s'$ and $i = \theta \circ i' = \theta' \circ i'$, we obtain:

$$-\theta'(s'(b')) + \theta(s'(b')) + \theta(i'(x)) - \theta(s'(b')) + \theta'(s'(b')) = \theta'(i'(x)),$$

that is:

$$-\theta'(s'(b')) + \theta(s'(b') + i'(x) - s'(b')) + \theta'(s'(b')) = \theta(i'(x)).$$

We have that $s'(b') + i'(x) - s'(b') \in i'(X)$, so there exists $y \in X$ such that $s'(b') + i'(x) - s'(b') = i'(y)$ and

$$-\theta'(s'(b')) + \theta(i'(y)) + \theta'(s'(b')) = \theta'(i'(x))$$

if and only if

$$-\theta'(s'(b')) + \theta'(i'(y)) + \theta'(s'(b')) = \theta'(i'(x))$$

if and only if
$$\theta'(-s'(b')+i'(y)+s'(b'))=\theta'(i'(x)).$$
But
$$-s'(b')+i'(y)+s'(b')=-s'(b')+-s'(b')+i'(x)+s'(b')+s'(b)=i'(x)$$
and then the equality (i) holds for any $x \in X$.

Condition (b) is very similar to (a). In fact, we have
$$s(\omega(g(b')))+i(x)-s(\omega(g(b')))=s(\omega(g'(b')))+i(x)-s(\omega(g'(b')))$$
if and only if
$$-s(\omega(g'(b')))+s(\omega(g(b')))+i(x)-s(\omega(g(b')))+s(\omega(g'(b')))=i(x);$$
but $g\circ\omega=\omega\circ g$ and $g'\circ\omega=\omega\circ g'$. Thus the equality becomes
$$-s(g'(\omega(b')))+s(g(\omega(b')))+i(x)-s(g(\omega(b')))+s(g'(\omega(b')))=i(x)$$
if and only if
$$-\theta'(s'(\omega(b')))+\theta(s'(\omega(b')))+\theta(i'(x))-\theta'(s'(\omega(b')))+\theta'(s'(\omega(b')))=\theta'(i'(x))$$
if and only if
$$-\theta'(s'(\omega(b')))+\theta(s'(\omega(b'))+i'(x)-s'(\omega(b')))+\theta'(s'(\omega(b')))=\theta'(i'(x)).$$
Again, $s'(\omega(b'))+i'(x)-s'(\omega(b')) \in i'(X)$, so there exists $z \in i'(X)$ such that $s'(\omega(b'))+i'(x)-s'(\omega(b'))=i'(z)$ and
$$-\theta'(s'(\omega(b')))+\theta(i'(z))+\theta'(s'(\omega(b')))=\theta'(i'(x))$$
if and only if
$$-\theta'(s'(\omega(b')))+\theta'(i'(z))+\theta'(s'(\omega(b'))))=\theta'(i'(x))$$
if and only if
$$\theta'(-s'(\omega(b'))+i'(z)+s'(\omega(b')))=\theta'(i'(x)).$$
Since
$$\begin{aligned}&-s'(\omega(b'))+i'(z)+s'(\omega(b'))=\\=&-s'(\omega(b'))+s'(\omega(b'))+i'x()-s'(\omega(b'))+s'(\omega(b'))=i'(x),\end{aligned}$$
the equality (b) holds for any $x \in X$.

Concerning (c), we have
$$\pi'(s'(b)*i'(x))=\pi'(s'(b'))*\pi'(i'(x))=\pi'(s'(b'))*0=0,$$
thus there exists $y \in X$ such that $i'(y)=s'(b')*i'(x)$ and then
$$\begin{aligned}&s(g(b'))*i(x)=\theta(s'(b'))*\theta(i'(x))=\theta(s'(b')*i'(x))=\theta(i'(y))=\\=&\theta'(i'(y))=\theta'(s'(b')*i'(x))=\theta'(s'(b'))*\theta(i'(x))=s(g'(b'))*i(x).\end{aligned}$$

Since the derived action of B on X is strict, we also have that $g(b) = g'(b)$, for any $b \in B$, and this completes the proof, since f is uniquely determined by g.

(iii) $\Rightarrow$ (i). Fixed $b \in B$, it is possible to construct a split extension in $\mathcal{C}$

$$0 \longrightarrow X \xrightarrow{i'} A' \underset{s'}{\overset{\pi'}{\rightleftarrows}} B' \longrightarrow 0$$

and a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i'} & A' & \underset{s}{\overset{\pi'}{\rightleftarrows}} & B' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & X & \xrightarrow{i} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0. \end{array}$$

such that

- B' is the free object in $\mathcal{C}$ on the singleton $\{z\}$ (as observed [66], free objects exist in any Orzech category of interest).
- g is the unique morphism such that $g(z) = b$ (whose existence and unicity is given by the universal property of B');
- $A' = B' \ltimes X$, where the derived action of B' on X is induced, via g, by the action of B on X, that is

  $$b' \cdot x := g(b)' \cdot x, \quad b' * x := g(b') * x, \quad \forall * \in \Omega_2'.$$

  Hence, the operation in A' are defined as follows:

  $$(b_1', x_1) + (b_2', x_2) = (b_1' + b_2', x_1 + b_1' \cdot x_2)$$

  and
  $$(b_1', x_1) * (b_2', x_2) = (b_1' * b_2', x_1 * x_2 + b_1' * x_2 + x_1 * b_2'),$$
  where $x_1 * b_2' = b_2' *^{\mathrm{op}} x_1$;
- $\pi'(b', x) = b$, $s'(b') = (b', 0)$ $i'(x) = (0, x)$ and $\theta(b', x) = s(g(b')) + i(x)$, for any $b' \in B'$ and $x \in X$.

Now, let us construct another split extension

$$0 \longrightarrow X \xrightarrow{i''} A'' \underset{s''}{\overset{\pi''}{\rightleftarrows}} B'' \longrightarrow 0$$

and a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i''} & A' & \underset{s'}{\overset{\pi''}{\rightleftarrows}} & B'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle g'} & & \\ 0 & \longrightarrow & X & \xrightarrow{i} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0. \end{array}$$

as above, but choosing another element $\overline{b} \in B$ instead of b. We prove that, if b and $\overline{b}$ act in the same way on X, then the two split extensions defined above coincide. Of course $B' = B$ as objects in $\mathcal{C}$, $\pi' = \pi''$, $s' = s''$, $k' = k''$ and $A' = A''$ as sets; it remains to prove that $A' = A''$ as objects of $\mathcal{C}$, i.e. that the operations defined on A' and A'' are the same. This is a consequence of the identities

(1) $s(g(b')) + i(x) - s(g(b')) = s(g'(b')) + i(x) - s(g'(b'))$;

(2) $s(g(b')) * i(x) = s(g'(b')) * i(x)$,

which hold for any $b' \in B'$, $x \in X$ and $* \in \Omega_2'$. Indeed, by the hypothesis on b and $\overline{b}$, (a) and (b) are satisfied for $b' = z$. Moreover, the set Z of those elements $b' \in B'$ for which (i) and (ii) hold for every $x \in X$, is a subobject of B', i.e. it is a subgroup closed under the operations in B'.

Indeed, let $b_1, b_2 \in Z$, then for any $x \in X$

$$\begin{aligned} &s(g(b_1 - b_2)) + i(x) - s(g(b_1 - b_2)) = \\ =&s(g(b_1)) - s(g(b_2)) + i(x) + s(g(b_2)) - s(g(b_1)) = \\ =&s(g(b_1)) - s(g'(b_2)) + i(x) + s(g'(b_2)) - s(g(b_1)) = \\ =&s(g'(b_1)) - s(g'(b_2)) + i(x) + s(g'(b_2)) - s(g'(b_1)) = \\ =&s(g'(b_1 - b_2)) + i(x) - s(g'(b_1 - b_2)), \end{aligned}$$

where the third equality holds because $-s(g'(b_2)) + i(x) + s(g'(b_2)) \in l(X)$ and

$$\begin{aligned} &s(g(b_1 - b_2)) + i(x) = [s(g(b_1)) - s(g(b_2))] * i(x) = \\ =&s(g(b_1)) * i(x) - s(g(b_2)) * i(x) = s(g'(b_1)) * i(x) - s(g'(b_2)) * i(x) = \\ =&[s(g'(b_1)) - s(g'(b_1))] * i(x) = s(g'(b_1 - b_2)) * i(x). \end{aligned}$$

Then $b_1 - b_2 \in Z$ and Z is a subgroup of B. Moreover, if $b_1, b_2 \in Z$, $x \in X$ and $* \in \Omega_2'$, then

$$\begin{aligned} &s(g(b_1 * b_2)) + i(x) - s(g(b_1 * b_2)) = [s(g(b_1)) * s(g(b_2))] + i(x) - [s(g(b_1)) * s(g(b_2))] = \\ =&i(x) + [s(g(b_1)) * s(g(b_2))] - [s(g(b_1)) * s(g(b_2))] = i(x) = \\ =&i(x) + [s(g'(b_1)) * s(g'(b_2))] - [s(g'(b_1)) * s(g'(b_2))] = \\ =&[s(g'(b_1)) * s(g'(b_2))] + i(x) - [s(g'(b_1)) * s(g'(b_2))] = \\ =&s(g'(b_1 * b_2)) + i(x) - s(g(b_1 * b_2)), \end{aligned}$$

where the second and the fifth equalities follow from Axiom 1, and

$$\begin{aligned} &s(g(b_1 * b_2)) * i(x) = [s(g(b_1)) * s(g(b_2))] * i(x) = \\ =&W(s(g(b_1))(s(g(b_2))i(x)), (s(g(b_1))(i(x)s(g(b_2))), \\ &(s(g(b_2))i(x))(s(g(b_1)), (i(x)s(g(b_2)))(s(g(b_1)), \\ &tg(b_2)((s(g(b_1))i(x)), s(g'(b_2))(i(x)(s(g(b_1))), \\ &((s(g(b_1))i(x))s(g(b_2)), (i(x)(s(g(b_1)))s(g(b_2))) = \\ =&W((s(g'(b_1))(tg'(b2)i(x)), s(g'(b_1))(i(x)s(g'(b_2))), \\ &(s(g'(b_2))i(x))s(g'(b_1)), (i(x)tg'(b_2))s(g'(b_1)), \\ &s(g'(b_2))(s(g'(b_1))i(x)), s(g'(b_2))(i(x)s(g'(b_1))), \\ &(s(g'(b_1))i(x))s(g'(b_2)), (i(x)s(g'(b_1)))s(g'(b_2))) = \\ =&[s(g'(b_1)) * s(g'(b_2))] * i(x) = s(g(b_1 * b_2)) * i(x), \end{aligned}$$

where we are using Axiom 2 and the fact that $i(X)$ is an ideal of A. Hence Z is a subobject of B' and then $Z = B'$.

Since the split extension (3.1.8) is faithful and the two split extensions defined above are the same, it follows that $g = g'$. Thus $b = b'$ and the derived action of B on X is super-strict. □

3.1.2 The universal strict general actor

In [26] the problem of the representability of actions for an Orzech category of interest was studied with a different approach than [47]

The authors of [26] proved that, for every object X in an Orzech category of interest $\mathcal{C}$, there exists an object $\mathrm{USGA}(X)$ of $\mathcal{C}'$, called *universal strict general actor* of X, with the following properties:

(1) $\mathrm{USGA}(X)$ is a *general actor* of X, i.e. $\mathrm{USGA}(X)$ has a derived action on X in $\mathcal{C}'$ and for any other object B in $\mathcal{C}$ and for every derived action $\{f_*\}_{*\in\Omega_2}$ of B on X in $\mathcal{C}$, there exists a unique morphism $\varphi\colon B \to \mathrm{USGA}(X)$ in $\mathcal{C}'$ such that $\{f_*\}_{*\in\Omega_2}$ is uniquely determined by the action of $\varphi(B)$ on X, i.e.

$$b \cdot x = \varphi(b) \cdot x, \quad b * x = \varphi(b) * x, \quad \forall * \in \Omega_2', \forall b \in B, \forall x \in X.$$

(2) The derived action of $\mathrm{USGA}(X)$ on X is strict.

(3) Let $\{B_j\}_{j\in J}$ be the set of all objects of $\mathcal{C}$ which have a derived action on X in $\mathcal{C}$ and let $\varphi_j\colon B_j \to \mathrm{USGA}(X)$ be the corresponding unique morphisms coming from the definition of general actor. The elements of $\mathrm{USGA}(X)$ satisfy the following equality:

$$(\varphi_i(b) * \varphi_j(b'))\bar{*}x = W(\varphi_i(b); \varphi_j(b'), x; *, \bar{*})$$

for any $b \in B_i$, $b' \in B_j$, $* \in \Omega_2$ and $i, j \in J$.

(4) $\mathrm{USGA}(X)$ satisfies the following universal property: for any strict general actor $\mathrm{SGA}(X)$ satisfying (3), there exists a unique morphism $\eta\colon \mathrm{USGA}(X) \to \mathrm{SGA}(X)$ in $\mathcal{C}'$, with $\eta \circ \varphi_j = \psi_j$, for any $j \in J$, where $\varphi_j\colon B_j \to \mathrm{USGA}(X)$ and $\psi_j\colon B_j \to \mathrm{SGA}(X)$ denote the corresponding unique morphisms from the definition of a general actor.

Remark 3.26. The condition (3) is formulated in terms of identities which are consequences of Axiom 2. For any Orzech category of interest $\mathcal{C}$ we can have different equivalent presentations, since the identities in Axiom 2 can be chosen in different equivalent ways; but the equalities in the corresponding condition (3) are not equivalent in general. Thus, universal strict general actors corresponding to different equivalent presentations of the same category $\mathcal{C}$ can be not isomorphic. This is the case of the category **Leib** of Leibniz algebras (see Example 6.6).

It was clear from the investigation of [26] that Orzech categories of interest are not action representable in general. In fact the authors proved that an object X in an Orzech category of interest $\mathcal{C}$ admits an actor $[X]$ if and only if $[X] \cong \mathrm{USGA}(X)$. More precisely, we have the following.

Theorem 3.27. *[26] Let $\mathcal{C}$ be an Orzech category of interest and let X be an object of $\mathcal{C}$. X admits an actor $[X]$ if and only if the semidirect product $\mathrm{USGA}(X) \ltimes X$ is an object of $\mathcal{C}$. If it is the case, then $[X] = \mathrm{USGA}(X)$.*

Proof. If X has an actor $[X]$, then given any universal strict general actor $\mathrm{USGA}(X)$, the unique morphism $\eta\colon \mathrm{USGA}(X) \to [X]$ is an isomorphism. Thus, $\mathrm{USGA}(X)$ is an actor as well and therefore $\mathrm{USGA}(X) \ltimes X$ is an object of $\mathcal{C}$.

Conversely, suppose that $\mathrm{USGA}(X) \ltimes X$ is an object of $\mathcal{C}$. Then, if $i_2\colon X \to \mathrm{USGA}(X) \ltimes X$ denotes the canonical injection on the second component, we have that $\mathrm{USGA}(X) = \mathrm{Coker}\, i$ and thus it is an object of $\mathcal{C}$. Moreover, $\mathrm{USGA}(X)$ has a derived action on X in $\mathcal{C}$ and, from the universal property as a general actor, we have that for any object B of $\mathcal{C}$ and for any derived action of B on X in $\mathcal{C}$, there exists a unique morphism in $\mathcal{C}'$ (which is obviously a morphism in $\mathcal{C}$) $\varphi\colon B \to \mathrm{USGA}(X)$ such that $b * x = \varphi(b) * x$, for any $b \in B$, $x \in X$ and $* \in \Omega_2'$. Thus, $\mathrm{USGA}(X)$ is the actor of X. □

It was then proven by A. Montoli in [63, Theorem 4.7] that any Orzech category of interest is action accessible.

Theorem 3.28. *[63] Every Orzech category of interest $\mathcal{C}$ is action accessible.*

Proof. Let

$$0 \longrightarrow X \xrightarrow{k} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{3.1.10}$$

be a split extension in $\mathcal{C}$. We have to construct a morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{k} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & X & \xrightarrow{k''} & A' & \underset{s'}{\overset{\pi'}{\rightleftarrows}} & B' & \longrightarrow & 0
\end{array}$$

with faithful codomain. Consider the subset of B

$$I := \{i \in B \mid s(i) + k(x) - s(i) = k(x),\ s(i) * k(x) = 0,\ \forall x \in X,\ \forall * \in \Omega_2'\}.$$

We show that I is an ideal of B, i.e. I is a normal subgroup of B and $i * b \in I$, for any $i \in I$, $b \in B$ and for any $* \in \Omega_2'$.

Let $i_1, i_2 \in I$. Then

$$s(i_1 - i_2) + k(x) - s(i_1 - i_2) = s(i_1) - s(i_2) + k(x) + s(i_2) - s(i_1) = k(x),$$

since $s(i_j) + k(x) - s(i_j) = k(x)$, for any $j = 1, 2$,

$$\begin{aligned} s(i_1 - i_2) * k(x) &= (s(i_1) - s(i_2)) * k(x) = \\ &= s(i_1) * k(x) - s(i_2) * k(x) = 0 - 0 = 0 \end{aligned}$$

and I is a subgroup of B. If $i \in I$ and $b \in B$, then

$$s(b + i - b) + k(x) - s(b + i - b) = s(b) + s(i) - s(b) + k(x) + s(b) - s(i) - s(b);$$

moreover $-s(b) + k(x) + s(b) \in k(X)$ (since $k(X)$ is a normal subgroup of A), so $-s(b) + k(x) + s(b) = k(y)$ for some $y \in X$; then

$$s(b) + s(i) - s(b) + k(x) + s(b) - s(i) - s(b) = s(b) + k(y) - s(b) = k(x),$$

where the first equality holds because $i \in I$; furthermore

$$\begin{aligned} s(b+i-b)*k(x) &= (s(b)+s(i)-s(b))*k(x) = \\ &= s(b)*k(x)+s(i)*k(x)-s(b)*k(x) = 0 \end{aligned}$$

since $i \in I$. Then I is a normal subgroup of B. Finally, let $i \in I$, $b \in B$ and $* \in \Omega_2'$; then

$$\begin{aligned} s(i*b)+k(x)-s(i*b) &= (s(i)*s(b))+k(x)-(s(i)*s(b)) = \\ &= k(x)+(s(i)*s(b))-(s(i)*s(b)) = k(x), \end{aligned}$$

where the second equality follows from Axiom 1 of the definition of Orzech category of interest. Furthermore, for any $*' \in \Omega_2'$, $s(i*b)*'k(x) = 0$ since, by Axiom 2, we have

$$\begin{aligned} s(i*b)*'k(x) &= (s(i)*s(b))*'k(x) = \\ &= W(s(i)(s(b)k(x)), s(i)(k(x)s(b)), (s(b)k(x))s(i), (k(x)s(b))s(i), \\ &\quad s(b)(s(i)k(x)), s(b)(k(x)s(i)), (s(i)k(x))s(b), (k(x)s(i))s(b)), \end{aligned}$$

and, in any component of the word W, $s(i)$ operates, with respect to the operations in Ω_2', on elements of $k(X)$, then any component is 0, so that $W = 0$. Thus, I is an ideal of B. We consider the quotien $B/I = \{b+I \mid b \in B\}$: it is an object of $\mathcal{C}$ with operations defined as

$$(b+I)+(b'+I) := (b+b')+I, \quad (b+I)*(b'+I) := b*b'+I$$

for any $b+I, b'+I \in B/I$ and $* \in \Omega_2'$.

Let us define now a derived action of B/I on X in $\mathcal{C}$ in the following way:

$$(b+I)\cdot x = b\cdot x = s(b)+k(x)-s(b)$$

and

$$(b+I)*x = b*x = s(b)*k(x),$$

for any $b+I \in B/I$, $x \in X$ and $* \in \Omega_2'$. The maps are well-defined; indeed, if $b_1+I = b_2+I$, i.e. $b_1-b_2 \in I$, then

$$\begin{aligned} &(s(b_1)+k(x)-s(b_1))-(s(b_2)+k(x)-s(b_2)) = \\ &= s(b_1)+k(x)-s(b_1)+s(b_2)-k(x)-s(b_2) = \\ &= s(b_2)-s(b_2)+s(b_1)+k(x)-s(b_1)+s(b_2)-k(x)-s(b_2) = \\ &= s(b_2)+s(-b_2+b_1)+k(x)-s(-b_2+b_1)-k(x)-s(b_2) = \\ &= s(b_2)+k(x)-k(x)-s(b_2) = 0 \end{aligned}$$

and then

$$s(b_1)+k(x)-s(b_1) = s(b_2)+k(x)-s(b_2).$$

In a similar way

$$s(b_1)*k(x)-s(b_2)*k(x) = (s(b_1)-s(b_2))*k(x) = s(b_1-b_2)*k(x) = 0$$

and thus

$$s(b_1)*k(x) = s(b_2)*k(x), \qquad \forall * \in \Omega_2'.$$

The collection of maps above is in fact a derived action of B/I on X, since the conditions needed to be satisfied are the same already satisfied by the action of B on X associated with (3.1.10):

$$b \cdot x = s(b) + k(x) - s(b), \quad b * x = s(b) * k(x), \quad \forall * \in \Omega_2'.$$

More precisely, it is the derived action associated with the split extension in $\mathcal{C}$

$$0 \longrightarrow X \xrightarrow{i_2} B/I \ltimes X \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B/I \longrightarrow 0$$

where $i_1(b+I) = (b+I, 0)$, $i_2(x) = (0, x)$ and $\pi_1(b+I, x) = b+I$. We define then $B' = B/I$, $A' = B/I \ltimes X$, with respect to the derived action defined above, $g = \pi_B \colon B \to B/I$ to be the canonical projection and $\varphi = (\pi_B \times 1_X) \circ \theta$, where θ is the canonical isomorphism $A \cong B \ltimes X$ of Remark 3.16. We thus have a morphism of split extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \pi_B} & & \\ 0 & \longrightarrow & X & \xrightarrow{i_2} & B/I \ltimes X & \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} & B/I & \longrightarrow & 0 \end{array}$$

and it only remains to show that the codomain is a faithful object in the category **SplExt**(X). Thanks to Proposition 3.25, this is equivalent to showing that the derived action of B/I on X is super-strict, i.e. for any $b+I \in B/I$ such that $(b+I) \cdot x = x$ and $(b+I) * x = 0$, for any $x \in X$ and $* \in \Omega_2'$, we have $b+I = I$. But, if $s(b) + k(x) - s(b) = k(x)$ and $s(b) * k(x) = 0$ for any $x \in X$ and $* \in \Omega_2'$, then $b \in I$, i.e. $b+I = I$. Thus, the split extension

$$0 \longrightarrow X \xrightarrow{i_2} B/I \ltimes X \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B/I \longrightarrow 0$$

is faithful and this concludes the proof. □

More recently J. R. A. Gray showed in [45] that an Orzech category of interest may not even be weakly action representable. In fact, he proved that the categories of n-solvable groups, $n \geq 3$, are not weakly action representable. However, by the results in [26], we can deduce the following.

Proposition 3.29. *[29] Let $\mathcal{C}$ be an Orzech category of interest and let X be an object of $\mathcal{C}$. Then there exists a monomorphism of functors*

$$\mu \colon \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}'}(U(-), \mathrm{USGA}(X)),$$

where $\mathrm{SplExt}(-, X) = \mathrm{SplExt}_{\mathcal{C}}(-, X)$ *and $U \colon \mathcal{C} \to \mathcal{C}'$ denotes the forgetful functor. If moreover* $\mathrm{USGA}(X)$ *is an object of $\mathcal{C}$, then the pair* $(\mathrm{USGA}(X), \mu)$ *is a weak representation of* $\mathrm{SplExt}(-, X)$.

Proof. By the above discussion, for every object B in $\mathcal{C}$, there exists an injection

$$\mu_B \colon \mathrm{SplExt}(B, X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}'}(B, \mathrm{USGA}(X)),$$

which sends a derived action of B on X to the corresponding (unique) morphism $\varphi \colon B \to \mathrm{USGA}(X)$ in $\mathcal{C}'$ that comes out from the definition of general actor. We want to prove that the collection $\{\tau_B\}_{B\in\mathcal{C}}$ gives rise to a natural transformation τ.

Consider in $\mathcal{C}$ a morphism $g \colon B' \to B$ and a split extension of B by X with associated derived action $\xi = \{f_*\}_{*\in\Omega_2}$. The naturality of μ is equivalent to saying that

$$\mu_{B'}(g^*(\xi)) = (\mu_B(\xi)) \circ g,$$

for every such g and ξ, where $g^* = \mathrm{SplExt}(g, X)$ and we identify the derived action ξ with its corresponding split extension. This follows immediately from the definition of general actor.

Since $\mathcal{C}$ is a full subcategory of $\mathcal{C}'$, when $\mathrm{USGA}(X)$ belongs to $\mathcal{C}$, the pair $(\mathrm{USGA}(X), \mu)$ is a weak representation of the functor $\mathrm{Act}(-, X)$. □

Corollary 3.30. *[29] Let $\mathcal{C}$ be an Orzech category of interest. If* $\mathrm{USGA}(X)$ *is an object of $\mathcal{C}$ for every X in $\mathcal{C}$, then $\mathcal{C}$ is a weakly action representable category.* □

Remark 3.31. We observe that in the proof of Proposition 3.29 we only used the condition (1) of the definition of universal strict general actor. Thus, Proposition 3.29 and Corollary 3.30 can be extended to any general actor $\mathrm{GA}(X)$ of any object X of $\mathcal{C}$.

In view of the last results, an explicit description of a universal strict general actor in concrete cases is very useful for studying the representability of actions. Two examples were given in [26]:

- the category **Assoc** of associative algebras, where $\mathrm{USGA}(X) = \mathrm{Bim}(X)$ is the associative algebra of bimultipliers of X (see Example 2.42);
- the category **Leib** of Leibniz algebras, where $\mathrm{USGA}(\mathfrak{g}) = \mathrm{Bider}(\mathfrak{g})$ is the Leibniz algebra of biderivations of $\mathfrak{g}$ (see Chapter 5).

In Chapter 7 we present the construction of a universal strict general actor for the category **Pois** of Poisson algebras.

3.2 Varieties of non-associative algebras

We now describe the algebraic setting we are working in: *varieties of non-associative algebras* over a field $\mathbb{F}$. We think of those as collections of algebras satisfying a chosen set of polynomial equations. We refer to [41], [42], [43, Section 1], [44] and [56] for the definitions and the main results in this section.

Definition 3.32. [56] A *(non-associative) algebra* $(X, \cdot)$ is vector space X over a field $\mathbb{F}$ equipped with a bilinear operation $\cdot \colon X \times X \to X \colon (x, y) \mapsto x \cdot y$ which we call the *multiplication* or the *product*.

Sometimes, as in the case of Lie and Leibniz algebras (see Chapter 5), we refer at the multiplication as the *commutator* or the *bracket* and we denote it by $[x, y]$. Otherwise, unless when this would be confusing, we denote the multiplication $x \cdot y$ by xy and we write X for an algebra $(X, \cdot)$.

The existence of a unit element is not assumed, nor are any other conditions on the multiplication besides its bilinearity. Let **Alg** denote the category of non-associative algebras where morphisms are the $\mathbb{F}$-linear maps $f \colon X \to Y$ which preserve the multiplication:

$$f(x+y) = f(x) + f(y), \quad f(\lambda x) = \lambda f(x), \quad f(xy) = f(x)f(y),$$

for any $x, y \in X$ and $\lambda \in \mathbb{F}$.

3.2.1 Free non-associative algebras, polynomials and identities

Definition 3.33. A *magma* is a set X endowed with a binary operation

$$\cdot\colon X \times X \to X\colon x \mapsto xy.$$

A *morphism of magmas* $f\colon (X, \cdot) \to (Y, *)$ is a function $f\colon X \to Y$ which preserves the binary operation: $f(xy) = f(x) * f(y)$, for any $x, y \in X$. Magmas and their morphisms form a category denoted **Mag**.

We observe that any non-associative algebra X has two underlying magma structures (associated with $+$ and $\cdot$).

We want now to describe in detail the construction of the free object on a set (see Definition 3.23) in the categories **Mag** and **Alg**.

Definition 3.34. [56] Let S be a set. A *non-associative word* in the alphabet S is a finite sequence of elements of S (called the *letters of the word*) and brackets "(" and ")" of the following kinds (and no others):

(1) for every element $\in S$, (s) is a non-associative word;

(2) if w_1 and w_2 are non-associative words, then the string (w_1w_2) is a non-associative word.

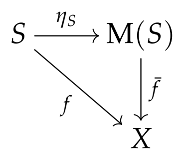
To avoid a rapidly increasing number of brackets, we don't write the outer brackets in a non-associative word.

We write $\mathrm{M}(S)$ for the set of non-associative words in the alphabet S. The *text* of a word is the number of letters (in S, brackets don't count) that it consists of, and the *degree* of a letter is the number of times it occurs in the given word. We sometimes write $\varphi(x_1, \ldots, x_n)$ for a word in which the elements $x_1, \ldots, x_n$ of S (and no others) appear.

The rule (2) allows us to define a binary operation $(\cdot)$ on $\mathrm{M}(S)$ making it into a magma which we call the *free magma* on S. The canonical inclusion of S into $\mathrm{M}(S)$ obtained from (1) is written $\eta_S\colon S \to \mathrm{M}(S)$.

Example 3.35. If $S = \{x, y, z\}$, the strings $x, xy, (xy)z, x(yz), x(xy), (x(yz))x$ and $(xy)(zx)$ are elements of $\mathrm{M}(S)$. The strings $x(yz)x$ and xyz are not in $\mathrm{M}(S)$, and neither is the string $()$. (This "empty string" appears when considering free unitary magmas.)

Proposition 3.36. *[56] For every magma $(X, \cdot)$ and every function $f\colon S \to X$ there exists a unique morphism of magmas $f\colon (\mathrm{M}(S), \cdot) \to (X, \cdot)$ such that the triangle*



commutes.

Proof. The function f must send a word (s), where $s \in S$, to $f(s)$ in order to make the triangle commute. It must preserve products, so a string w_1w_2 is sent to $f(w_1) \cdot f(w_2)$. □

The free magma determines a functor $\mathrm{M}\colon \mathbf{Set} \to \mathbf{Mag}$ which sends a set S to $\mathrm{M}(S)$, and a function $f\colon S \to T$ to the morphism of magmas $\mathrm{M}(f)\colon \mathrm{M}(S) \to \mathrm{M}(T)$ induced by $\eta_T \circ f\colon S \to \mathrm{M}(T)$. On the other hand, there is the forgetful functor $\mathbf{Mag} \to \mathbf{Set}$ which forgets about multiplications, taking a magma and sending it to its underlying set.

In other words, the free magma functor is left adjoint to the forgetful functor to **Set**, and in fact this is the reason why it carries that name: it plays the same role as the free group functor, for instance.

There are many equivalent ways to phrase adjointness, and going into the general theory of adjunctions here would lead us too far. However, we meet some examples, starting with the following one, which makes one of the relationships between non-associative algebras and magmas explicit.

Example 3.37. For any field $\mathbb{F}$, the forgetful functor $\mathbf{Alg} \to \mathbf{Mag}$, which takes an algebra $(X, \cdot)$ and sends it to the underlying set of the vector space X, equipped with the multiplication $(\cdot)$, has a left adjoint denoted $\mathbb{F}[-]\colon \mathbf{Mag} \to \mathbf{Alg}$ and called the *magma algebra functor*.

The functor $\mathbb{F}[-]$ takes a magma $(X, \cdot)$ and sends it to the $\mathbb{F}$-vector space $\mathbb{F}[X]$ with basis X, whose elements are finite linear combinations of the elements of X, equipped with the multiplication $\cdot\colon \mathbb{F}[X] \times \mathbb{F}[X] \to \mathbb{F}[X]$ defined by

$$(\sum_{i=1}^{n} \lambda_i x_i, \sum_{j=1}^{k} \mu_j y_j) \mapsto \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda_i \mu_j (x_i \cdot y_j)$$

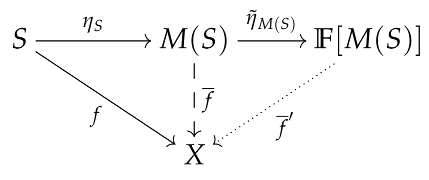
for any $x_i, y_j \in X$ and $\lambda_i, \mu_j \in \mathbb{F}$, $i = 1, \ldots, n$, $j = 1, \ldots, k$. Note that its bilinearity is obvious.

$\mathbb{F}[-]$ satisfies the universal property of a left adjoint, because for the natural inclusion $\tilde{\eta}_{(X,\cdot)}\colon (X, \cdot) \to (\mathbb{F}[X], \cdot)$, we have that any given morphism of magmas $f\colon X \to A$, where $(A, \cdot)$ is a non-associative algebra, extends to a unique morphism of algebras $f'\colon \mathbb{F}[X] \to A$ such that $f' \circ \tilde{\eta}_X = f$ in **Mag**. Indeed, this just follows from the fact that X is a basis of $\mathbb{F}[X]$ and the definition of the multiplication of that algebra. As in the case of magmas, this determines how the functor $\mathbb{F}[-]$ should act on morphisms.

Example 3.38. The composition of adjunctions is again an adjiunction, and thus we find the construction of the *free non-associative* $\mathbb{F}$*-algebra* on a set:

$$\mathbf{Set} \underset{\text{Forget}}{\overset{M}{\underset{\perp}{\rightleftarrows}}} \mathbf{Mag} \underset{\text{Forget}}{\overset{\mathbb{F}[-]}{\underset{\perp}{\rightleftarrows}}} \mathbf{Alg}$$

The functors to the left first forget the vector space structure of an algebra X, then the multiplication of the underlying magma $(X, \cdot)$, so that we obtain the underlying set of X. Looking at the diagram in **Set**



it is easy to see that the composite functor to the right does indeed satisfy the universal property of a left adjoint.

Summarizing, we constructed the *free algebra* functor $\mathbb{F}[M(-)]\colon \mathbf{Set} \to \mathbf{Alg}$ which sends a set S to the *free algebra* generated by elements of S. This functor has the forgetful functor as a right adjoint. Moreover, it factorises through the *free magma functor* $\mathrm{M}\colon \mathbf{Set} \to \mathbf{Mag}$, which sends a set S to the magma $\mathrm{M}(S)$ of non-associative words in S, and the *magma algebra functor* $\mathbb{F}[-]\colon \mathbf{Mag} \to \mathbf{Alg}$.

For a given set S, an element of $\mathbb{F}[\mathrm{M}(S)]$ is an $\mathbb{F}$-linear combination of non-associative words in the alphabet S. In other words we have the following.

Definition 3.39. [56] Let S be a set. An element φ of $\mathbb{F}[\mathrm{M}(S)]$ is called a *non-associative polynomial* on S. We say that such a polynomial is a *monomial* if it is a scalar multiple of an element in $\mathrm{M}(S)$.

For example, if $S = \{x, y, z, t\}$, then $(xy)t + (zy)x$, $xx + yz$ and $(xt)(yz)$ are polynomials in S and only the last one is a monomial.

For a monomial φ on a set $\{x_1, \ldots, x_n\}$, we define its *type* as the n-tuple $(k_1, \ldots, k_n) \in \mathbb{N}^n$, where k_i is the number of times x_i appears in φ, and its *degree* as the natural number $k_1 + \cdots + k_n$. A polynomial is said to be

(1) *homogeneous* if all its monomials are of the same type;

(2) *multilinear* if all its monomials are of the type $(1, \ldots, 1)$.

Among the examples above, only the last one is multilinear.

Remark 3.40. Our algebras need not have units, and so our polynomials have no constant terms.

Definition 3.41. [56] A *(polynomial) identity* of a non-associative algebra X is a non-associative polynomial $\varphi = \varphi(x_1, \ldots, x_n)$ such that $\varphi(x_1, \ldots, x_n) = 0$, for every $x_1, \ldots, x_n \in X$. We say that the algebra A *satisfies* the identity φ.

Definition 3.42. [56] Let I be a subset of $\mathbb{F}[\mathrm{M}(S)]$ with S being a set of variables. The *variety of non-associative algebras* $\mathcal{V}$ determined by I is the class of all algebras which satisfy all the identities in I. We say that a variety of non-associative algebra *satisfies the identities in* I if all algebras in this variety satisfy the given identities.

Definition 3.43. Let $\mathcal{V}$ be a variety non-associative of algebras. We say that

(1) $\mathcal{V}$ is *operadic* [70] if it is determined by a set of multilinear polynomials;

(2) $\mathcal{V}$ is *quadratic* [37] if there exists a set of identities of degree 2 and 3 that generate all the identities of $\mathcal{V}$.

Of course, any variety of non-associative $\mathcal{V}$ algebras can be seen as a category where the morphisms are the same ones as in $\mathbf{Alg}$ and we have a full inclusion functor $\mathcal{V} \hookrightarrow \mathbf{Alg}$. In particular, any such variety is a semi-abelian category.

Remark 3.44. Whenever the characteristic of the field $\mathbb{F}$ is zero, any variety of non-associative algebras over $\mathbb{F}$ is operadic. This is due to the well-known multilinearisation process, see [67, Corollary 3.7].

Remark 3.45. We recall that, when we consider a variety of non-associative algebras $\mathcal{V}$, we assume the field $\mathbb{F}$ is fixed, so that we may drop it from our notation.

Examples 3.46.

(1) We write **AbAlg** for the variety of *abelian* algebras determined by the identity $xy = 0$ [56]. Seen as a category, this variety is isomorphic to the category **Vec** of vector spaces over $\mathbb{F}$ and thus it is an abelian category; this explains the terminology.

In fact, any vector space V may be considered as a non-associative algebra, by imposing a trivial multiplication $xy = 0$. If the functor that equips a vector space with the trivial multiplication is denoted by $T\colon \mathbf{Vec} \to \mathbf{AbAlg}$, and $U\colon \mathbf{AbAlg} \to \mathbf{Vec}$ is the functor which forgets the multiplication of a trivial algebra, then clearly $T \circ U = 1_{\mathbf{AbAlg}}$ and $U \circ T = 1_{\mathbf{Vec}}$.

We observe that **AbAlg** is the only variety of non-associative algebras which is an abelian category. In fact, let $\mathcal{V}$ be a variety of non-associative algebras and suppose that $\mathcal{V} \neq \mathbf{AbAlg}$. Then, there exists a non-abelian algebra X which is an object of $\mathcal{V}$.

If $\mathcal{V}$ is an abelian category, then $\mathcal{V}$ is action representable and

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathcal{V}}(-, 0)$$

where 0 denotes the zero algebra. Thus, for any other object B of $\mathcal{V}$, any split extension of B by X in $\mathcal{V}$ is represented by the zero-morphism $B \to 0$. In other words, since $\mathcal{V}$ is a category of group with operations and split extensions are in bijection with semi-direct products (see Remark 3.16), this is equivalent to saying that the only semi-direct product $B \ltimes X$ which is an object of $\mathcal{V}$ is the direct product $B \times X$, i.e. the direct sum of vector spaces $B \oplus X$ with the multiplication defined by $(b, x) \cdot (b', x') = (bb', xx')$.

We reach a contraddiction, since the semi-direct product $X \ltimes X = (X \oplus X, \cdot)$ defined by

$$(x, y) \cdot (x', y') = (xx', yy' + xy' + yx')$$

is always an object of $\mathcal{V}$ (because it satisfies all the identities which define the variety $\mathcal{V}$ since X is an object of $\mathcal{V}$) and it is not the direct product $X \times X$ since X is not an abelian algebra (i.e. there exists $x, y \in X$ such that $xy \neq 0$ and thus $(x, 0)(0, y) = (0, xy) \neq (0, 0)$).

(2) We write **Assoc** for the variety of *associative* algebras determined by the identity of *associativity* which is $x(yz) - (xy)z = 0$, or equivalently $x(yz) = (xy)z$. Relevant examples of associative algebras are given by the vector space $\mathrm{M}_n(\mathbb{F})$ of $n \times n$ matrices over $\mathbb{F}$ endowed with the usual multiplication between matrices and, for any vector space V, by the space $\mathrm{End}(V)$ of linear endomorphisms of V, endowed with the usual composition of functions. We recall that, if $\dim_{\mathbb{F}} V = n$, fixed a basis $\{e_1, \ldots, e_n\}$ of V, there exists a canonical isomorphism between $\mathrm{End}(V)$ and $\mathrm{M}_n(\mathbb{F})$.

(3) We write **AAssoc** for the variety of *anti-associative* algebras, determined by the *anti-associative* identity $x(yz) + (xy)z = 0$, or equivalently $x(yz) = -(xy)z$.

(4) We write **Com** for the variety of *commutative* algebras determined by the identity of *commutativity* which is $xy - yx = 0$, or equivalently $xy = yx$.

(5) We write **ACom** for the variety of *anti-commutative* algebras determined by *anti-commutativity* which is $xy + yx = 0$, or equivalently $xy = -yx$.

(6) We write **CAssoc** for the variety of commutative associative algebras.

(7) We write **Lie** for the variety of *Lie algebras* determined by $x^2 = 0$ (i.e. the algebra is *alternating* or *skew-symmetric*) and the *Jacobi identity* $x(yz) + y(zx) + z(xy) = 0$. If $\mathrm{char}(\mathbb{F}) \neq 2$, skew-symmetry is equivalent to *anti-commutativity* $xy + yx = 0$: if $x^2 = 0$ for every x in X, then

$$0 = (x+y)(x+y) = x^2 + xy + yx + y^2 = xy + yx$$

for all $x, y \in X$. Conversely, since we can take $x = y$, the equation $xy = -yx$ implies $x^2 = -x^2$, hence $0 = x^2 + x^2 = 2x^2$. So unless $0 = 2$ in the field $\mathbb{F}$, this implies that $x^2 = 0$.

However, the two identities are not equivalent in general: the simplest example of a field of characteristic 2 is the field $\mathbb{F}_2 = \{0, 1\}$ of integers modulo 2. Over $\mathbb{F}_2$, the two-dimensional vector space with basis $\{x, y\}$ becomes an anti-commutative algebra which is not alternating if we define its multiplication as $x^2 = y$ and $xy = yx = y^2 = 0$. Note that $x^2 = y = -y = -x^2$.

Lie algebras are notorious because of their connection with *Lie groups*, which are smooth manifolds endowed with a (compatible) group structure. Actually, each Lie group induces a Lie algebra over $\mathbb{R}$, and this process gives rise to a non-trivial equivalence between the category **Lie** and the category of simply-connected real Lie groups.

Another source of Lie algebras (over any field) are those coming from associative algebras. There is a functor $G\colon \mathbf{Assoc} \to \mathbf{Lie}$ which takes an associative algebra $(X, \cdot)$ and sends it to the couple $(X, [-,-])$ where

$$[-,-]\colon X \times X \to X\colon (x,y) \mapsto [x,y] = xy - yx.$$

It is easy to check that this bracket does indeed define a Lie algebra structure on X. The functor G sends a homomorphism of associative algebras to the same linear map, now a homomorphism of Lie algebras, since it automatically preserves the bracket.

For instance, $\mathfrak{gl}(n, \mathbb{F})$ and $\mathfrak{gl}(V)$ are the Lie algebras coming respectively from the associative algebras $\mathrm{M}_n(\mathbb{F})$ and $\mathrm{End}(V)$, for some vector space V.

Note that two elements x, y of $(X, \cdot)$ commute ($xy = yx$) if and only if their bracket vanishes ($[x,y] = 0$); so the associative algebra $(X, \cdot)$ is commutative if and only if the Lie algebra $(X, [-,-])$ is abelian.

The functor G is not an equivalence of categories, but it has a left adjoint **Lie** $\to$ **Assoc** which is called the *universal enveloping algebra* functor (see [38, Chapter 15]).

As mentioned in Example 2.32, another example of a Lie algebra is the *Lie algebra of derivations* $\mathrm{Der}(X)$ of a non-associative algebra X. A *derivation* of X is an $\mathbb{F}$-linear map $d\colon X \to X$ such that

$$d(xy) = d(x)y + xd(y), \qquad \forall x, y \in X.$$

Given $d, d' \in \mathrm{Der}(X)$, their commutator $[d, d'] = d \circ d' - d' \circ d$ is still a derivation. Thus, it is possible define the Lie algebra $\mathrm{Der}(X)$ as the Lie subalgebra of $\mathfrak{gl}(X)$ consisting of all derivations of X. We already saw that, for any Lie algebra X, $\mathrm{Der}(X)$ is the actor of X.

(8) Instead of being alternating, we may ask that the multiplication of an algebra satisfying the Jacobi identity is anti-commutative ($xy = -yx$). Then this

algebra is called a *quasi-Lie algebra*. The variety **qLie** of quasi-Lie algebras coincides with **Lie** as long as the characteristic of the field $\mathbb{F}$ is different from 2. However, when $\text{char}(\mathbb{F}) = 2$, the variety **Lie** is strictly contained in **qLie**: the algebra over $\mathbb{F}_2$ described above is a quasi-Lie algebra which is not a Lie algebra.

(9) We write **ACAAssoc** for the variety of anti-commutative anti-associative algebras. We observe that in this case, if $\text{char}(\mathbb{F}) \neq 2$, anti-associativity $x(yz) + (xy)z = 0$ is equivalent to the identity $x(xy) = 0$. In fact, starting with $x(yz) + (xy)z = 0$ and $x^2 = 0$, we obtain

$$x(xy) = -x^2y = 0,$$

where the powers have precedence on the multiplications. We observe this implies also $y(xy) = 0$. Conversely, the identity $x(xy) = 0$ implies that

$$\begin{aligned} 0 =&(x+y)[(x+y)z] = x[(x+y)z] + y[(x+y)z] = \\ =&x(xz) + x(yz) + y(xz) + y(yz) = x(yz) + y(xz) \end{aligned}$$

and thus, using anti-commutativity and $x(yz) = -y(xz)$, we obtain

$$x(yz) = -y(xz) = y(zx) = -z(yx) = z(xy) = -(xy)z.$$

(10) One can see that all the previous examples are operadic varieties. Let us provide a non-operadic example: the variety **Bool** of *Boolean rings*, which may be seen as associative $\mathbb{F}_2$-algebras satisfying $x^2 = x$. This variety is action representable with the actor of a Boolean ring X being the Boolean ring $\text{End}(X)$ (see [15, Proposition 3.1]).

(11) We write **Leib** for the variety of *(right) Leibniz algebras* determined by the *(right) Leibniz identity* which is $(xy)z - (xz)y - x(yz) = 0$. It is easy to see that an anti-commutative algebra is a Leibniz algebra if and only if it is a quasi-Lie algebra. We study in detail the representability of actions of **Leib** in Chapter 5.

(12) We write **Jord** for the variety of *Jordan algebras* determined by *commutativity* and the *Jordan identity* $(xy)x^2 = x(yx^2)$. This variety is not action accessible [30] and, if we have an associative algebra X, then the bilinear map

$$(x, y) \mapsto x \circ y = xy + yx$$

define Jordan algebra structure on X.

(13) We write **JJord** for the variety of *Jacobi-Jordan algebras* which is determined by commutativity and the Jacobi identity. Jacobi-Jordan algebras are the commutative counterpart of Lie algebras and, over a field of characteristic 2, they coincide with quasi-Lie algebras (since commutativity coincides with anti-commutativity). In particular then, they are Jordan algebras: indeed, the Jacobi identity implies that $3x(x^2) = 0$, so $x(x^2) = 0$ and then

$$(xy)x^2 = x(yx^2) + (xx^2)y = x(yx^2).$$

This justifies the name of Jordan in the definition. Sometimes this variety is also known as *mock-Lie* algebras.

(14) We write **Alt** for the variety of *alternative algebras* [4], which is determined by the identities $(yx)x - yx^2 = 0$ and $x(xy) - x^2y = 0$. Every associative algebra is obviously alternative and an example of an alternative algebra which is not associative is given by the *octonions* $\mathbb{O}$ [7], that is the eight-dimensional algebra with basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ and multiplication table

$$e_ie_j = \begin{cases} e_j, & \text{if } i = 1 \\ e_i, & \text{if } j = 1 \\ -\delta_{ij}e_1 + \varepsilon_{ijk}e_k, & \text{otherwise,} \end{cases}$$

where δ_{ij} is the *Kronecker delta* and ε_{ijk} a *completely antisymmetric tensor* with value 1 when $ijk = 123, 145, 176, 246, 257, 347, 365$. Notice that e_1 is the unit of the algebra $\mathbb{O}$.

Alternative algebras are strictly related with *loop theory* and *quasigroup theory*, since the set of units of a commutative alternative algebras forms an *algebraic commutative Moufang loop* (see [65, Section 5]).

When $\text{char}(\mathbb{F}) \neq 2$, the multilinearisation process shows that **Alt** is equivalent to the variety defined by

$$(xy)z + (xz)y - x(yz) - x(zy) = 0$$

and

$$(xy)z + (yx)z - x(yz) - y(xz) = 0.$$

Indeed, from $(yx)x - yx^2 = 0$, we have

$$\begin{aligned} 0 =& [x(y+z)](y+z) - x(y+z)^2 = \\ =& [x(y+z)]y + [x(y+z)]z - xy^2 - x(yz) - x(zy) - xz^2 = \\ =& (xy)y + (xz)y + (xy)z + (xz)z - xy^2 - x(yz) - x(zy) - xz^2 = \\ =& (xz)y + (xy)z - x(yz) - x(zy). \end{aligned}$$

In the same way, from $x(xy) - x^2y = 0$, we obtain

$$\begin{aligned} 0 =& (x+y)[(x+y)z] - (x+y)^2z = \\ =& x[(x+y)z] + y[(x+y)z] - x^2z - (xy)z - (yx)z - y^2z = \\ =& x(xz) + x(yz) + y(xz) + y(yz) - x^2z - (xy)z - (yx)z - y^2z = \\ =& x(yz) + y(xz) - (xy)z - (yx)z. \end{aligned}$$

Conversely, starting from $(xy)z + (xz)y - x(yz) - x(zy) = 0$, when $z = y$ one obtains

$$2[(xy)y - xy^2] = 0.$$

Analogously, when $y = x$, the identity $(xy)z + (yx)z - x(yz) - y(xz) = 0$ becomes

$$2[x(xz) - x^2z] = 0.$$

(15) The largest variety of non-associative $\mathbb{F}$-algebras is **Alg** itself (no conditions) and the smallest one is the *trivial variety* 0 (consisting of the zero algebra only, satisfying all equations possible, including $x = 0$).

(15) *Unitary* (or *unital*) algebras, i.e. those algebras $(X, \cdot)$ which have an element

$1 = 1_X$ for which $x \cdot 1 = x = 1 \cdot x$, for any $x \in X$) do not form a variety in our sense, since the *existence* of 1 cannot be expressed as an equational condition. This does not mean that an algebra cannot have a unit. On the other hand, even between algebras with units, *a priori* there is no reason why a morphism of algebras should preserve this unit.

(16) Taking any variety $\mathcal{V}$, one can look at a subvariety of it by adding further identities to be satisfied. For example, let $\mathcal{V}$ be a variety determined by a set of identities I and let k be any positive integer, then we write $\mathbf{Nil}_k(\mathcal{V})$ for the variety of *k-step nilpotent algebras in* $\mathcal{V}$ determined by the identities in I and the identities of the form $x_1 \ldots x_{k+1} = 0$ with all possible choices of parentheses. We remark that, when $\mathcal{V} = \mathbf{Assoc}$, any proper subvariety of it is equal to $\mathbf{Nil}_k(\mathcal{V})$, for some positive integer k.

3.2.2 Actions in varieties of non-associative algebras

We now want to explain how to describe actions in a variety of non-associative algebras. As we already mentioned before, in a semi-abelian category, actions are in natural bijection with split extensions and, since any variety of non-associative algebras over a field is a category of groups with operations, we can use derived actions in place of internal actions, since we have a natural isomorphism $\mathrm{Act}(-,X) \cong \mathrm{DAct}(-,X)$.

In Section 3.1, we already described explicitly what is a derived action in some varieties of non-associative algebras, such as **Alg**, **Assoc** and **Lie**. We want now to make this description more clear for a a general variety of non-associative algebras over a field.

Definition 3.47. [34, 66] Let $\mathcal{V}$ be a variety of non-associative algebras and let

$$0 \longrightarrow X \xrightarrow{\ i\ } A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{3.2.1}$$

be a split extension in $\mathcal{V}$. The pair of bilinear maps

$$l\colon B \times X \to X, \qquad r\colon X \times B \to X$$

defined by

$$b * x = s(b)i(x), \quad x * b = i(x)s(b), \quad \forall b \in B,\ \forall x \in X$$

where $b * - = l(b,-)$ and $- * b = r(-,b)$, is called the *derived action* of B on X associated with (3.2.1).

Given a pair of bilinear maps

$$l\colon B \times X \to X, \qquad r\colon X \times B \to X$$

where B, X are objects of $\mathcal{V}$, we may define a bilinear multiplication on the direct sum / cartesian product of vector spaces $B \oplus X \cong B \times X$ by

$$(b,x) \cdot (b',x') = (bb', xx' + b * x' + x * b') \tag{3.2.2}$$

with $b * x' := l(b, x')$ and $x * b' := r(x, b')$. This construction allows us to build the split extension in **Alg**

$$0 \longrightarrow X \xrightarrow{i_2} (B \oplus X, \cdot) \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0 \tag{3.2.3}$$

with $i_2(x) = (0, x)$, $i_1(b) = (b, 0)$ and $\pi_1(b, x) = b$. This is a split extension in $\mathcal{V}$ if and only if $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$, i.e. it satisfies the identities which determine $\mathcal{V}$. In other words, we have the following result analogous to [34, Proposition 1.1], [44, Lemma 1.8] and [66, Theorem 2.4]:

Lemma 3.48. *[43] In a variety of non-associative algebras $\mathcal{V}$, given a pair of bilinear maps*

$$l \colon B \times X \to X, \qquad r \colon X \times B \to X,$$

we define the multiplication on $B \oplus X$ as above in (3.2.2)*. Then, the pair (l, r) is a derived action of B on X if and only if $(B \oplus X, \cdot)$ is in $\mathcal{V}$. In this case, we call $(B \oplus X, \cdot)$ the* semi-direct product *of B and X (with respect to the derived action (l, r)) and we denote it by $B \ltimes X$.*

Proof. ($\Rightarrow$) If the pair (l, r) is a derived action of B on X associated with a split extension

$$0 \longrightarrow X \xrightarrow{i} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

in $\mathcal{V}$, then the linear map

$$(B \oplus X, \cdot) \to A \colon (b, x) \to s(b) + i(x)$$

is an isomorphism. Thus $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$.

($\Leftarrow$) Conversely, if $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$, then the derived action associated with the split extension

$$0 \longrightarrow X \xrightarrow{i_2} B \ltimes X \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0$$

where $i_1(b) = (b, 0)$, $i_2(x) = (0, x)$, $\pi_1(b, x) = b$, is given by the pair (l, r). □

Remark 3.49. As mentioned in Lemma 3.48, for any split extension (3.2.1) and the corresponding derived action (l, r), there is an isomorphism of split extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i_2} & B \ltimes X & \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_B} & & \\ 0 & \longrightarrow & X & \xrightarrow{i} & A & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \end{array}$$

where $\theta(b, x) = s(b) + i(x)$, for any $(b, x) \in B \oplus X$. Thus, when we write $b * x$ (resp. $x * b$), one can think of it as the multiplication $(b, 0) \cdot (0, x)$ (resp. $(0, x) \cdot (b, 0)$) in $B \ltimes X$.

Remark 3.50. In order to prove that $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$, we need to check that it satisfies all the identities $\varphi = \varphi(\alpha_1, \ldots, \alpha_n)$ which determine the variety $\mathcal{V}$. Since B and X are objects of $\mathcal{V}$ and every element $(b, x) \in B \oplus X$ can be written as the sum $(b, 0) + (0, x)$, it is sufficient to prove this when at least one of the $\alpha_1, \ldots, \alpha_n$ is an element of the form $(0, x)$, with $x \in X$, and the others are of the form $(b, 0)$, with $b \in B$.

For instance, using the notation of the previous remark, if $\mathcal{V}$ is a subvariety of **Com**, then $(B \oplus X, \cdot)$ is commutative if and only if $b * x = x * b$, for any $b \in B$ and $x \in X$. Instead, if $\mathcal{V}$ is a subvariety of **Assoc**, then $(B \oplus X, \cdot)$ is an associative algebra if and only if $b * (xx') = (b * x)x'$, $(xx') * b = x(x' * b)$, $x(b * x) = (x * b)x$ and $b * (x * b') = (b * x) * b'$, for any $b, b' \in B$ and $x, x' \in X$.

3.2.3 Categorical consequences

We recall from [28] that a semi-abelian category is *algebraically coherent* if for any morphism $f \colon X \to Y$ in $\mathcal{C}$, the change of base functor

$$f^* \colon \mathbf{Pt}_Y(\mathcal{C}) \to \mathbf{Pt}_X(\mathcal{C})$$

is *coherent*, which means that it preserves finite limits and jointly strongly epimorphic pairs of arrows. In the case of a variety of algebras $\mathcal{V}$, this is equivalent to saying that the comparison morphism

$$B\flat X + B\flat Y \to B\flat(X + Y)$$

is a surjective algebra homomorphism, fon any algebras B, X, Y of $\mathcal{V}$ (see [42]).

We explain two results which are useful for understanding the rest of the paper.

Theorem 3.51. *[41, 42] Let $\mathcal{V}$ be an operadic variety of non-associative algebras. The following conditions are equivalent:*

(1) $\mathcal{V}$ is algebraically coherent;

(2) $\mathcal{V}$ is an Orzech category of interest;

(3) $\mathcal{V}$ is action accessible;

(4) there exist $\lambda_1, \dots, \lambda_8, \mu_1, \dots, \mu_8 \in \mathbb{F}$ such that

$$\begin{aligned} x(yz) = {} & \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3 z(xy) + \lambda_4 z(yx) \\ & + \lambda_5(xz)y + \lambda_6(zx)y + \lambda_7 y(xz) + \lambda_8 y(zx) \end{aligned}$$

and

$$\begin{aligned} (yz)x = {} & \mu_1(xy)z + \mu_2(yx)z + \mu_3 z(xy) + \mu_4 z(yx) \\ & + \mu_5(xz)y + \mu_6(zx)y + \mu_7 y(xz) + \mu_8 y(zx) \end{aligned}$$

are identities in $\mathcal{V}$.

Proof. From the results of [28] we have that (2) implies (1).

We address the reader to [41] for a detailed proof that (1) implies (4).

It follows immediately from the definition of an Orzech category of interest that (4) implies (2).

We already proved in Theorem 3.28 that any Orzech category of interest is action accessible, so that (2) implies (3).

Finally, as explained in Remark 2.3 of [44], a non-trivial use of [31, Lemma 2.9] shows that (3) implies (1). □

Following [41] and [42], we call the two previous identities together the λ/μ-*rules*. Since the weak representability of actions implies action accessibility in general, the existence of the λ/μ-rules is a necessary condition for the variety $\mathcal{V}$ to be weakly action representable.

Remark 3.52. When a variety of non-associative algebras $\mathcal{V}$ is an Orzech category of interest, the corresponding category of groups with operations $\mathcal{V}'$ is the variety **Alg** (see Example 3.6) and the notion of derived actions introduced in Definition 3.47 coincides with the one of Definition 3.8.

Theorem 3.53. *[44] Let $\mathcal{V}$ be a variety of non-associative algebras over an infinite field $\mathbb{F}$, with* $\operatorname{char}(\mathbb{F}) \neq 2$. *The following conditions are equivalent:*

(1) $\mathcal{V}$ is action representable;

(2) $\mathcal{V}$ is either the variety **Lie** *or the variety* **AbAlg**.

Proof. We already now that the both the varieties **Lie** and **AbAlg** are action representable.

Now let $\mathcal{V}$ be an action representable variety of non-associative algebras and suppose that $\mathcal{V}$ is not the variety **AbAlg** of abelian algebras. As shown in Proposition 3.1 and Proposition 3.4 of [44], $\mathcal{V}$ satisfies a non-trivial identity of degree two, i.e. $\mathcal{V}$ is a variety of commutative or anti-commutative algebras.

If $xy - yx = 0$ is an identity of $\mathcal{V}$, then considering the actions of the abelian one-dimensional algebras B_i, $i = 1, 2, 3$, on the 79-dimensional algebra X defined as in [44, Proposition 3.1], and the fact that $(B_1 + B_2 + B_3) \ltimes X$ is an object of $\mathcal{V}$, one obtains that $x(yz) = -(xy)z - y(xz)$ is also an identity of $\mathcal{V}$ and we have a contradiction, since $\mathcal{V} \neq$ **AbAlg** (see [44, Proposition 4.1]).

If $xy + yx = 0$ is an identity of $\mathcal{V}$, then using the same actions corrected for anti-commutativity, one obtains that the Jacobi identity holds and $\mathcal{V}$ is a subvariety of **Lie** (see [44, Proposition 4.3]). Finally, as shown in [44, Proposition 5.1], a proper subvariety of **Lie** is action representable if and only if it is an abelian variety. Thus, we can conclude that $\mathcal{V} =$ **Lie**. □

Theorem 3.53 helps motivating our interest in the study of weakly representable actions. In fact, in our context, there is only one non-trivial example of a variety which is action representable. Therefore, in order to study the representability of actions, it makes sense to weaken our assumptions.

The next results explains one way of understanding (weak) action representability for any variety of non-associative algebras.

Proposition 3.54. *[43] A variety of non-associative algebras $\mathcal{V}$ is weakly action representable if and only if for any object X in it, there exists an object $T = T_X$ of $\mathcal{V}$ such that for every derived action of an object B of $\mathcal{V}$ on X*

$$l \colon B \times X \to X, \qquad r \colon X \times B \to X,$$

there exists a unique morphism $\varphi \in \operatorname{Hom}_{\mathcal{V}}(B, T)$ and a derived action (l', r') of $\varphi(B)$ on X such that

$$l'(\varphi(b), x) = l(b, x), \quad r'(x, \varphi(b)) = r(x, b),$$

for every $b \in B$ and for every $x \in X$.

Proof. ($\Rightarrow$) If $\mathcal{V}$ is weakly action representable, then for any object X in it there exists a weak representation (T, τ). Let B be an object of $\mathcal{V}$ which acts on X and let $\varphi \colon B \to T$

be the corresponding acting morphism. Consider the split extension diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & B \ltimes X & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_X} & & \vdots{\scriptstyle \exists!\theta} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & X & \xrightarrow{i'} & \varphi(B) \ltimes X & \underset{s'}{\overset{\pi'}{\rightleftarrows}} & \varphi(B) & \longrightarrow & 0 \end{array} \tag{3.2.4}$$

where f is the corestriction of φ to its image, $i'(x) = (0, x)$, $s'(\varphi(b)) = (\varphi(c), 0)$, where $(c, 0) = s(b)$, and $\theta(b, x) = (\varphi(b), x)$. Then the action of $\varphi(B)$ on X is defined by the pair of bilinear maps

$$l' \colon \varphi(B) \times X \to X, \qquad r' \colon X \times \varphi(B) \to X$$

where

$$l'(\varphi(b), x) = s'(\varphi(b))i'(x) = s(b)i(x) = l(b, x)$$

and

$$r'(\varphi(b), x) = i(x)s'(\varphi(b)) = i(x)s(b) = r(b, x),$$

for every $b \in B$ and for every $x \in X$.

($\Leftarrow$) Conversely, given an object X of $\mathcal{V}$, a weak representation of $\mathrm{SplExt}(-, X)$ is given by (T, τ), where the component

$$\tau_B \colon \mathrm{SplExt}(B, X) \rightarrowtail \mathrm{Hom}_{\mathcal{V}}(B, T)$$

sends every action of B on X to the corresponding morphism φ. Moreover, τ_B is an injection since the morphism φ is uniquely determined by the action of B on X. Thus τ is a monomorphism of functors. □

When $\mathcal{V}$ is action representable, one can use the terminal object

$$0 \longrightarrow X \longrightarrow [X] \ltimes X \rightleftarrows [X] \longrightarrow 0.$$

of the category $\mathbf{SplExt}(X)$ to obtain the following.

Corollary 3.55. *[44] A variety of non-associative algebras $\mathcal{V}$ is action representable if and only if for any object X in it, there exists an object $[X]$ of $\mathcal{V}$ acting on X with the following property: for every derived action of an object B of $\mathcal{V}$ on X*

$$l \colon B \times X \to X, \qquad r \colon X \times B \to X,$$

there exists a unique morphism $\varphi \in \mathrm{Hom}_{\mathcal{V}}(B, [X])$ such that

$$l(b, x) = \varphi(b) * x, \quad r(x, b) = x * \varphi(b),$$

where $\varphi(b) * x$ and $x * \varphi(b)$ are given by the action of $[X]$ on X.

Proof. ($\Rightarrow$) Let X be an object of $\mathcal{C}$ and let $[X]$ be the actor of X. Let B be an object of $\mathcal{V}$ which acts on X and let

$$0 \longrightarrow X \xrightarrow{i} B \ltimes X \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{3.2.5}$$

be the corresponding split extension. Then there exits a unique morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & B \ltimes X & \underset{s}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & X & \longrightarrow & [X] \ltimes X & \rightleftarrows & [X] & \longrightarrow & 0 \end{array}$$

and the action of B on X is uniquely determined by

$$l(b, x) = \varphi(b) * x, \quad r(x, b) = x * \varphi(b).$$

($\Leftarrow$) Conversely, given an object X of $\mathcal{V}$, one can define a natural transformation

$$\tau\colon \text{SplExt}(-, X) \to \text{Hom}_{\mathcal{V}}(-, [X])$$

in the following way: for any object B of $\mathcal{V}$, the component τ_B sends any action of B on X to the unique morphism $\varphi\colon B \to [X]$. The uniqueness of φ implies that τ_B is an injection. Moreover, τ_B is surjective since any morphism $\varphi \in \text{Hom}_{\mathcal{V}}(B, [X])$ defines a derived action (l, r) of B on X by

$$l(b, x) = \varphi(b) * x, \quad r(x, b) = x * \varphi(b).$$

Thus τ is a natural isomorphism and $[X]$ is the actor of X. □

3.2.4 Partial algebras

We end this chapter with a notion we shall use throughout the text.

Definition 3.56. [43] Let X be an $\mathbb{F}$-vector space. A *bilinear partial operation* on X is a map

$$\cdot\colon \Omega \to X,$$

where Ω is a vector subspace of $X \times X$, which is bilinear on Ω, i.e.

$$(\alpha_1 x_1 + \alpha_2 x_2) \cdot y = \alpha_1 x_1 \cdot y + \alpha_2 x_2 \cdot y$$

for any $\alpha_1, \alpha_2 \in \mathbb{F}$ and $x_1, x_2, y \in X$ such that $(x_1, y), (x_2, y) \in \Omega$ and

$$x \cdot (\beta_1 y_1 + \beta_2 y_2) = \beta_1 x \cdot y_1 + \beta_2 x \cdot y_2$$

for any $\beta_1, \beta_2 \in \mathbb{F}$ and $x, y_1, y_2 \in X$ such that $(x, y_1), (x, y_2) \in \Omega$.

Definition 3.57. [43] A *partial algebra* over $\mathbb{F}$ is an $\mathbb{F}$-vector space X endowed with a bilinear partial operation

$$\cdot\colon \Omega \to X.$$

We denote it by $(X, \cdot, \Omega)$. When $\Omega = X \times X$ we say that the algebra is *total*.

Let $(X, \cdot, \Omega)$ and $(X', *, \Omega')$ be partial algebras over $\mathbb{F}$. A homomorphism of partial algebras is an $\mathbb{F}$-linear map $f\colon X \to X'$ such that $f(x \cdot y) = f(x) * f(y)$ whenever $(x, y) \in \Omega$, which tacitly implies that $(f(x), f(y)) \in \Omega'$ (i.e. both $x \cdot y$ and $f(x) * f(y)$ are defined). We denote by **PAlg** the category whose objects are partial algebras and whose morphisms are partial algebra homomorphisms.

Definition 3.58. [43] We say that a partial algebra $(X, \cdot, \Omega)$ *satisfies an identity* when that identity holds wherever the bilinear partial operation is well defined.

For instance, a partial algebra $(X, \cdot, \Omega)$ is associative if

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

for every $x, y, z \in X$ such that (x, y), (y, z), (x, yz), $(xy, z) \in \Omega$.

Remark 3.59. We observe that clearly any variety of non-associative algebras $\mathcal{V}$ has an obvious forgetful functor $U \colon \mathcal{V} \to \mathbf{PAlg}$.

Chapter 4

Commutative and anti-commutative algebras

In this chapter we aim to study the representability of actions of some varieties of non-associative algebras, over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$, which satisfy the commutative law or the anti-commutative law.

As explained in Chapter 3, we may assume our variety to satisfy the λ/μ-rules, or equivalently to be action accessible. When $\mathcal{V}$ is either a variety of commutative or anti-commutative algebras, i.e. $xy = \varepsilon yx$ is an identity of $\mathcal{V}$ with $\varepsilon = \pm 1$, the λ/μ-rules reduce to

$$x(yz) = \alpha(xy)z + \beta(xz)y,$$

for some α, $\beta \in \mathbb{F}$ and we have the following.

Proposition 4.1. *[43] Let $\mathcal{V}$ be an action accessible, operadic variety of non-associative algebras and suppose that $\mathcal{V}$ is not the variety* **AbAlg** *of abelian algebras.*

(1) If $\mathcal{V}$ is a variety of commutative algebras, then $\mathcal{V}$ is a either a subvariety of **CAssoc** *or a subvariety of* **JJord**.

(2) If $\mathcal{V}$ is a variety of anti-commutative algebras, then $\mathcal{V}$ is either a subvariety of **Lie** *or a subvariety of* **ACAAssoc**.

Proof. For (1), let $\mathcal{V}$ be a variety of commutative algebras. If $\alpha = \beta = 0$, then $x(yz) = 0$ and $\mathcal{V}$ is a subvariety of both **CAssoc** and **JJord**. If $(\alpha, \beta) \neq (0, 0)$, then

$$x(yz) = \alpha(xy)z + \beta(xz)y,$$
$$y(xz) = \alpha(yx)z + \beta(yz)x$$

and

$$(1+\beta)x(yz) = (1+\beta)(xz)y.$$

If $\beta \neq -1$, then $x(zy) = x(yz) = (xz)y$ and $\mathcal{V}$ is a subvariety of **CAssoc**. If $\beta = -1$, then

$$(1+\alpha)x(yz) = (1+\alpha)(xy)z.$$

If $\alpha \neq -1$, then $x(yz) = (xy)z$ as before. If $\alpha = -1$, then the Jacobi identity holds and $\mathcal{V}$ is a subvariety of **JJord**.

For the proof of (2), let $\mathcal{V}$ be a variety of anti-commutative algebras. Since $\mathrm{char}(\mathbb{F}) \neq 2$, this is equivalent to saying that $\mathcal{V}$ is *skew-symmetric*, i.e. $x^2 = 0$ is an identity of $\mathcal{V}$. If $\alpha = \beta = 0$, then $x(yz) = 0$ and $\mathcal{V}$ is subvariety of both **Lie** and **ACAAssoc**. If $(\alpha, \beta) \neq (0, 0)$, then

$$0 = xy^2 = \alpha(xy)y + \beta(xy)y,$$

thus $(\alpha+\beta)(xy)y=0$. If $\beta=-\alpha$, then

$$x(xz)=\alpha\left[x^2z-(xz)x\right]=-\alpha(xz)x=\alpha x(xz),$$

hence $(\alpha-1)x(xz)=0$. If $\alpha=1$, then the Jacobi identity holds and $\mathcal{V}$ is a subvariety of **Lie**. If $x(xz)=0$ is an identity, then using the multilinearisation process we have

$$\begin{aligned}0=&(x+y)[(x+y)z]=x[(x+y)z]+y[(x+y)z]=\\=&x(xz)+x(yz)+y(xz)+y(yz)=x(yz)+y(xz)\end{aligned}$$

and hence $x(yz)+y(xz)=-x(zy)-(xz)y=0$. Thus $x(zy)=-(xz)y$ and $\mathcal{V}$ is a subvariety of **ACAAssoc**.

Finally, if $(xy)y=0$, again using the multilinearisation process, we obtain

$$\begin{aligned}0=&[x(y+z)](y+z)=[x(y+z)]y+[x(y+z)]z=\\=&(xy)y+(xz)y+(xy)z+(xz)z=(xz)y+(xy)z.\end{aligned}$$

Thus $(xz)y+(xy)z=-(zx)y-z(xy)=0$ and again $\mathcal{V}$ is a subvariety of **ACAAssoc**. □

Remark 4.2. We observe that $\mathbf{Nil}_2(\mathbf{Com})$ is a subvariety of both **CAssoc** and **JJord**: in fact, from $x(yz)=(xy)z=0$ we have that associativity holds and the Jacobi identity is satisfied

$$x(yz)+y(zx)+z(xy)=0+0+0=0.$$

If $\mathrm{char}(\mathbb{F})\neq 3$, $\mathbf{Nil}_2(\mathbf{Com})$ is precisely the intersection of the varieties **CAssoc** and **JJord**. Indeed, let $\mathcal{V}$ be a subvariety of both **CAssoc** and **JJord**. Since commutativity, associativity and the Jacobi identity hold in $\mathcal{V}$, we have

$$(xy)z=x(yz)=-y(zx)-z(xy)=-x(yz)-(xy)z=-2(xy)z$$

and thus $3(xy)z=3x(yz)=0$.

An example of an algebra which lies in the intersection of **CAssoc** and **JJord** but which is not two-step nilpotent is the two-dimensional $\mathbb{F}_3$-algebra with basis $\{e_1,e_2\}$ and multiplication determined by

$$e_1^2=e_1e_2=e_2e_1=e_2^2=e_2.$$

In a similar way, $\mathbf{Nil}_2(\mathbf{ACom})$ is a subvariety of both **Lie** and **ACAAssoc**: in fact, from $x(yz)=(xy)z=0$ we have that anti-associativity holds and the Jacobi identity is satisfied

$$x(yz)+y(zx)+z(xy)=0+0+0=0.$$

If $\mathrm{char}(\mathbb{F})\neq 3$, $\mathbf{Nil}_2(\mathbf{ACom})$ coincides with the intersection of the varieties **Lie** and **ACAAssoc**. Indeed, let $\mathcal{V}$ be a subvariety of both **Lie** and **ACAAssoc**. Since anti-commutativity, anti-associativity and the Jacobi identity hold in $\mathcal{V}$, we have

$$(xy)z=-x(yz)=-(xy)z-y(xz)=-(xy)z+(yx)z=-2(xy)z$$

and thus $3(xy)z=-3x(yz)=0$.

When $\mathrm{char}(\mathbb{F})=3$, it is possible to construct an algebra that lies in the intersection of **Lie** and **ACAAssoc** but which is not two-step nilpotent. Indeed, let $\mathfrak{g}$ be the algebra of dimension 7 over $\mathbb{F}_3$ with basis

$$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

and bilinear multiplication $[-,-]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ determined by

$$[e_1, e_2] = -[e_2, e_1] = e_4, \ [e_1, e_3] = -[e_3, e_1] = -e_6, \ [e_2, e_3] = -[e_3, e_2] = e_5$$

and

$$[e_1, e_5] = -[e_5, e_1] = [e_2, e_6] = -[e_6, e_2] = [e_3, e_4] = -[e_4, e_3] = e_7.$$

Then $(\mathfrak{g}, [-,-])$ is a Lie algebra such that

$$[x, [x, y]] = 0$$

for any $x, y \in \mathfrak{g}$ (we recall this identity is equivalent to anti-associativity if the characteristic of the field is different from 2). Indeed, let

$$x = \sum_{i=1}^{7} \alpha_i e_i, \quad y = \sum_{i=1}^{7} \beta_i e_i$$

with $\alpha_i, \beta_i \in \mathbb{F}$, for every $i = 1, \dots, 7$. Then

$$[x, y] = (\alpha_1\beta_2 - \alpha_2\beta_1)e_4 + (\alpha_2\beta_3 - \alpha_3\beta_2)e_5 + (\alpha_3\beta_1 - \alpha_1\beta_3)e_6 + \\ + (\alpha_1\beta_5 + \alpha_2\beta_6 + \alpha_3\beta_4 - \alpha_4\beta_3 - \alpha_5\beta_1 - \alpha_6\beta_2)e_7$$

and

$$[x, [x, y]] = [\alpha_3(\alpha_1\beta_2 - \alpha_2\beta_1) + \alpha_1(\alpha_2\beta_3 - \alpha_3\beta_2) + \alpha_2(-\alpha_1\beta_3 + \alpha_3\beta_1)]e_7 = 0.$$

Furthermore, $\mathfrak{g}$ is not two-step nilpotent since

$$[e_1, [e_2, e_3]] = [e_1, e_5] = e_7.$$

We thank Gabor P. Nagy for suggesting this example.

Corollary 4.3. *[43] Let $\mathcal{V}$ be an action accessible, operadic, quadratic variety of non-associative algebras and suppose that $\mathcal{V}$ is not the variety* **AbAlg** *of abelian algebras.*

(1) If $\mathcal{V}$ is commutative, then it has to be one of the following varieties: **CAssoc**, **JJord**, *their intersection, or* $\mathbf{Nil}_2(\mathbf{Com})$.

(2) If $\mathcal{V}$ is anti-commutative, then it has to be one of the following varieties: **Lie**, **ACAAssoc**, *their intersection, or* $\mathbf{Nil}_2(\mathbf{ACom})$.

Proof.

(1) By Proposition 4.1, any identity of degree 3 in $\mathcal{V}$ is either associativity, or the Jacobi identity or $x(yz) = 0$.

(2) Again by Proposition 4.1, any identity of degree 3 in $\mathcal{V}$ is either the Jacobi identity, or the anti-associativity or $x(yz) = 0$.

□

We already know that **Lie** is action representable and that the actor of a Lie algebra X is the Lie algebra $\mathrm{Der}(X)$ of derivations of X. Therefore, we shall study the representability of actions of the varieties **CAssoc**, **JJord**, $\mathbf{Nil}_2(\mathbf{Com})$, **ACAAssoc** and $\mathbf{Nil}_2(\mathbf{ACom})$.

4.1 Commutative associative algebras

As already mentioned in Chapter 2, the representability of actions of the variety of commutative associative algebras over a field was studied in [15], where F. Borceux, G. Janelidze and G. M. Kelly proved that **CAssoc** is not action representable.

Lemma 4.4. *[15] Let X be a commutative associative algebra. There exists a natural isomorphism*

$$\mathrm{SplExt}(-,X) \cong \mathrm{Hom}_{\mathbf{Assoc}}(U(-),\mathrm{M}(X)),$$

where $\mathrm{SplExt}(-,X) = \mathrm{SplExt}_{\mathbf{CAssoc}}(-,X)$, $U\colon \mathbf{CAssoc} \to \mathbf{Assoc}$ *denotes the forgetful functor and*

$$\mathrm{M}(X) = \{f \in \mathrm{End}(X) \mid f(xy) = f(x)y, \quad \forall x,y \in X\}$$

is the associative algebra of multipliers *of X, endowed with a product induced by the usual composition of functions (see [26, 59]).*

Proof. Given a split extension

$$0 \longrightarrow X \xrightarrow{\;i\;} A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0 \tag{4.1.1}$$

in **CAssoc**, there exists an isomorphism of commutative associative algebra split extensions as in Remark 3.49 and $A \cong B \ltimes X = (B \oplus X, \cdot)$, where

$$(b,x)\cdot(b',x') = (bb', xx' + b * x' + b' * x).$$

Thus we have a homomorphism of associative algebras

$$B \to \mathrm{M}(X)$$

which maps an element $b \in B$ onto the linear map $b * -$ given by the action of B on X . On the other hand, starting with a homomorphism

$$B \to \mathrm{M}(X)\colon b \mapsto b * -$$

we can build the split extension in **CAssoc**

$$0 \longrightarrow X \xrightarrow{\;i_2\;} B \ltimes X \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0$$

where $B \ltimes X = (B \oplus X, \cdot)$ as before. Finally, one can check that the bijection $\mathrm{Act}(B,X) \cong \mathrm{Hom}_{\mathbf{Assoc}}(U(B),\mathrm{M}(X))$ is natural in B since the diagram in **Set**

$$\begin{array}{ccc} \mathrm{SplExt}(B,X) & \longrightarrow & \mathrm{Hom}(U(B),\mathrm{M}(X)) \\ {\scriptstyle \mathrm{SplExt}(f,B)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(U(f),\mathrm{M}(X))} \\ \mathrm{SplExt}(B',X) & \xrightarrow{\;\tau_{B'}\;} & \mathrm{Hom}(U(B'),\mathrm{M}(X)) \end{array}$$

is commutative for any morphism $f\colon B' \to B$ in **CAssoc**. □

We observe that $\mathrm{M}(X)$ in general needs not be a commutative algebra. For instance, let $X = \mathbb{F}^2$ be the abelian two-dimensional algebra, then $\mathrm{M}(X) = \mathrm{End}(X)$ which is not commutative. However there are special cases where $\mathrm{M}(X)$ is an object of **CAssoc** (see [15] and [26]), such as when the *annihilator* of X

$$\mathrm{Ann}(X) = \{x \in X \mid xy = 0, \ \forall y \in X\}$$

is trivial or when $X^2 = X$, where X^2 denotes the subalgebra of X generated by the products xy, with $x, y \in X$.

In fact, if $\mathrm{Ann}(X) = 0$, then given two elements $f, g \in \mathrm{M}(X)$, we have

$$f(g(x))y = f(g(x)y) = f(xg(y)) = f(x)g(y) = g(f(x))y$$

for any $x, y \in X$. Thus $f(g(x)) - g(f(x)) \in \mathrm{Ann}(X) = 0$, i.e. $f \circ g = g \circ f$.

Instead, if $X^2 = X$ we have

$$f(g(xy)) = f(xg(y)) = f(x)g(y) = g(f(x)y) = g(f(xy))$$

for any $x, y \in X$. Again $f \circ g = g \circ f$ and $\mathrm{M}(X)$ is a commutative associative algebra.

In [15, Theorem 2.6] the authors also proved the following characterization.

Theorem 4.5. *[15] Let X be a commutative associative algebra. Then* $\mathrm{M}(X)$ *is a commutative associative algebra if and only if it the functor* $\mathrm{SplExt}(-, X)$ *is representable.*

Proof. If $\mathrm{M}(X)$ is an object of **CAssoc**, then we deduce that

$$\begin{aligned}\mathrm{SplExt}(-, X) &= \mathrm{Hom}_{\mathbf{Assoc}}(U(-), U(\mathrm{M}(X))) \\ &\cong \mathrm{Hom}_{\mathbf{CAssoc}}(-, \mathrm{M}(X))\end{aligned}$$

since the forgetful functor $U\colon \mathbf{CAssoc} \to \mathbf{Assoc}$ is fully faithful. Thus, $M(X)$ is the actor of X and $\mathrm{SplExt}(-, X)$ is representable.

Conversely, suppose that the functor $\mathrm{SplExt}(-, X)$ is representable by a commutative associative algebra $[X]$. Then we have that

$$\mathrm{Hom}_{\mathbf{CAssoc}}(-, [X]) \cong \mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Assoc}}(U(-), \mathrm{M}(X)).$$

In particular the identity morphism $1_{[X]}$ by these bijections to an associative algebra homomorphism $u\colon [X] \to \mathrm{M}(X)$. For any other commutative associative algebra B, the composition with u induces thus a bijection

$$\mathrm{Hom}_{\mathbf{Assoc}}(U(B), U([X])) = \mathrm{Hom}_{\mathbf{CAssoc}}(-, [X]) \cong \mathrm{Hom}_{\mathbf{Assoc}}(U(-), \mathrm{M}(X)).$$

The free non necessarily commutative associative algebra on one generator is the algebra $\mathbb{F}^*[t]$ of polynomials with coefficients in $\mathbb{F}$ and a zero constant term. But this algebra is also commutative, thus it can be chosen as the object B in the bijection above. Since it is a strong generator in the category **Assoc**, it follows that u is an isomorphism. Thus, $M(X)$ is an object of **CAssoc**. $\square$

Since there are examples of commutative associative algebras X such that $\mathrm{M}(X)$ is not commutative, it follows that the category **CAssoc** is not action representable.

4.2 Jacobi-Jordan algebras

We know that every split extension of B by X in the category **Lie** is represented by a Lie algebra homomorphism $B \to \mathrm{Der}(X)$. For Jacobi-Jordan algebras, the role the derivations have in **Lie** is played by the so-called *anti-derivations*.

Definition 4.6. [24] Let X be a Jacobi-Jordan algebra. An *anti-derivation* is a linear map $d\colon X \to X$ such that

$$d(xy) = -d(x)y - d(y)x, \quad \forall x, y \in X.$$

The (left) multiplications L_x, with $x \in X$, are particular anti-derivations, called *inner anti-derivations*. We denote by $\mathrm{ADer}(X)$ the space of anti-derivations of X and by $\mathrm{Inn}(X)$ the subspace of the inner anti-derivations. Anti-derivations play a significant role in the study of cohomology of Jacobi-Jordan algebras: see [8] for further details.

We now want to make explicit what are the derived actions in the category **JJord** and how they are related with the anti-derivations. The following is an immediate application of Lemma 3.48 and Remark 3.50.

Proposition 4.7. *[43] Let X and B be two Jacobi-Jordan algebras. Given a pair of bilinear maps*

$$l\colon B \times X \to X, \qquad r\colon X \times B \to X,$$

we construct $(B \oplus X, \cdot)$ as in Equation (3.2.2)*. Then $(B \oplus X, \cdot)$ is a Jacobi-Jordan algebra if and only if*

(1) $b * x = x * b$;

(2) $b * (xx') = -(b * x)x' - (b * x') * x$;

(3) $(bb') * x = -b * (b' * x) - b' * (b * x)$;

for every b, $b' \in B$ and x, $x' \in X$.

Proof. Using Remark 3.50, $(B \oplus X, \cdot)$ is a commutative algebra if and only if (1) is satisfied. Moreover, the Jacobi identity holds in $(B \oplus X, \cdot)$ if and only if the following equations are satisfied for any b, $b' \in B$ and x, $x' \in X$:

- $(b, 0) \cdot [(0, x) \cdot (0, x')] = -[(b, 0) \cdot (0, x)] \cdot (0, x') - [(b, 0) \cdot (0, x')] \cdot (0, x)$, which is equivalent to (2);
- $[(b, 0) \cdot (b', 0)] \cdot (0, x) = -(b, 0) \cdot [(b', 0) \cdot (0, x)] - (b', 0) \cdot [(b, 0) \cdot (0, x)]$, which is equivalent to (3).

□

In an equivalent way, a derived action of B on X in the variety **JJord** is given by a linear map

$$B \to \mathrm{ADer}(X)\colon b \mapsto b * -$$

which satisfies

$$(bb') * - = -\langle b * -, b' * - \rangle, \quad \forall b, b' \in B, \tag{4.2.1}$$

where

$$\langle -, - \rangle\colon\ \mathrm{ADer}(X) \times \mathrm{ADer}(X) \to \mathrm{End}(X), \quad \langle f, f' \rangle = -f \circ f' - f' \circ f$$

denotes the *anti-commutator* between two anti-derivations of X.

Remark 4.8. The vector space $\mathrm{ADer}(X)$ endowed with the anti-commutator is not in general a Jacobi-Jordan algebra. For instance, if $X = \mathbb{F}$ is the abelian one-dimensional algebra, then $\mathrm{ADer}(X) = \mathrm{End}(X) \cong \mathbb{F}$ (every linear endomorphism of X is of the form $\varphi_\alpha \colon x \mapsto \alpha x$, for some $\alpha \in \mathbb{F}$) does not satisfy the Jacobi identity. Nevertheless, there are some subspaces of $\mathrm{ADer}(X)$ that are Jacobi-Jordan algebras. For instance, the subspace $\mathrm{Inn}(X)$ of all inner anti-derivations of X. Indeed, the linear map

$$X \to \mathrm{Inn}(X) \colon x \mapsto L_x$$

is a Jacobi-Jordan algebra homomorphism. This is true in general for the image of any linear map $B \to \mathrm{ADer}(X)$ satisfying Equation (4.2.1).

Thus we need to use an algebraic structure which includes the space of anti-derivations endowed with the anti-commutator and which allows us to describe categorically the representability of actions of the variety **JJord**. The answer is given by *partial algebras*.

Indeed, the vector space $\mathrm{ADer}(X)$ endowed with the anti-commutator $\langle -, - \rangle$ is a commutative partial algebra. In this case Ω is the preimage

$$\langle -, - \rangle^{-1}(\mathrm{ADer}(X))$$

of the inclusion $\mathrm{ADer}(X) \hookrightarrow \mathrm{End}(X)$.

Theorem 4.9. *[43] Let X be a Jacobi-Jordan algebra.*

(1) There exists a natural isomorphism

$$\rho \colon \mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{PAlg}}(U(-), \mathrm{ADer}(X)),$$

where $\mathrm{SplExt}(-, X) = \mathrm{SplExt}_{\mathbf{JJord}}(-, X)$ *and* $U \colon \mathbf{JJord} \to \mathbf{PAlg}$ *denotes the forgetful functor;*

(2) if $\mathrm{ADer}(X)$ *is a Jacobi-Jordan algebra, then the functor* $\mathrm{SplExt}(-, X)$ *is representable and* $\mathrm{ADer}(X)$ *is the actor of* X*;*

Proof. (1) For a Jacobi-Jordan algebra B, we define the component

$$\rho_B \colon \mathrm{SplExt}(B, X) \to \mathrm{Hom}_{\mathbf{PAlg}}(U(B), \mathrm{ADer}(X))$$

as the functor which sends any split extension

$$0 \longrightarrow X \xrightarrow{\ i\ } A \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

to the homomorphism $B \to \mathrm{ADer}(X) \colon b \mapsto b * -$. The transformation ρ is natural. Indeed, for any Jacobi-Jordan algebra homomorphism $f \colon B' \to B$, it is easy to check that the diagram in **Set**

$$\begin{array}{ccc} \mathrm{SplExt}(B, X) & \xrightarrow{\rho_B} & \mathrm{Hom}(U(B), \mathrm{ADer}(X)) \\ {\scriptstyle \mathrm{SplExt}(f, X)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(U(f), \mathrm{ADer}(X))} \\ \mathrm{SplExt}(B', X) & \xrightarrow{\rho_{B'}} & \mathrm{Hom}(U(B'), \mathrm{ADer}(X)) \end{array}$$

where $\mathrm{Hom}(U(-), -) = \mathrm{Hom}_{\mathbf{PAlg}}(U(-), -)$, is commutative. Moreover, for any Jacobi-Jordan algebra B, the morphism ρ_B is an injection, as each element of

$\mathrm{SplExt}(B, X)$ is uniquely determined by the corresponding action of B on X. Thus ρ is a monomorphism of functors. Finally ρ is a natural isomorphism since, given any Jacobi-Jordan algebra B and any homomorphism of partial algebras $\varphi\colon B \to \mathrm{ADer}(X)$, the bilinear maps $l_\varphi\colon B \times X \to X\colon (b, x) \mapsto \varphi(b)(x)$, $r_\varphi = l_\varphi$ define a (unique) derived action of B on X such that $\rho_B(l_\varphi, r_\varphi) = \varphi$.

(2) If $\mathrm{ADer}(X)$ is a Jacobi-Jordan algebra, then by (1) we have a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{JJord}}(-, \mathrm{ADer}(X)),$$

hence $\mathrm{ADer}(X)$ is the actor of X. □

4.3 Two-step nilpotent commutative algebras

We now analyse the case where $\mathcal{V}$ is the variety $\mathbf{Nil}_2(\mathbf{Com})$ of two-step nilpotent commutative algebras, i.e. $\mathcal{V}$ is determined by the identities $xy = yx$ and $x(yz) = 0$. An example of such an algebra is the *Kronecker algebra* $\mathfrak{k}_1$ [53], which is the three-dimensional algebra with basis $\{e_1, e_2, e_3\}$ and multiplication determined by $e_1e_2 = e_2e_1 = e_3$.

We shall show that $\mathbf{Nil}_2(\mathbf{Com})$ is a weakly action representable, operadic, quadratic variety of commutative algebras.

Proposition 4.10. *[43] Let X and B be two algebras in $\mathbf{Nil}_2(\mathbf{Com})$. Given a pair of bilinear maps*

$$l\colon B \times X \to X, \qquad r\colon X \times B \to X,$$

we construct $(B \oplus X, \cdot)$ as in Equation (3.2.2). Then $(B \oplus X, \cdot)$ is in $\mathbf{Nil}_2(\mathbf{Com})$ if and only if

(1) $b * x = x * b$;

(2) $b * (xx') = (b * x)x' = 0$;

(3) $(bb') * x = b * (b' * x) = 0$;

for every $b, b' \in B$ and $x, x' \in X$.

Proof. The proof is a straightforward application of Lemma 3.48 and Remark 3.50. In fact, the bilinear map $(\cdot)$ is commutative if and onyl if (1) is satisfied. Moreover, $(B \oplus X, \cdot)$ is a two-step nilpotent algebra if and only the following equations hold for any $b, b' \in B$ and $x, x' \in X$:

- $(b, 0) \cdot [(0, x) \cdot (0, x')] = [(b, 0) \cdot (0, x)] \cdot (0, x') = (0, 0)$, which is equivalent to (2);
- $[(b, 0) \cdot (b', 0)] \cdot (0, x) = (b, 0) \cdot [(b', 0) \cdot (0, x)] = (0, 0)$, which is equivalent to (3).

□

The second equation of Proposition 4.10 states that, for every $b \in B$, the linear map $b * -$ belongs to the vector space

$$[X]_2 = \{f \in \mathrm{End}(X) \mid f(xy) = f(x)y = 0,\ \forall x \in X\}.$$

Moreover, seeing $[X]_2$ as an abelian algebra (i.e., we endow $[X]_2$ with the bilinear multiplication $\langle f, g\rangle = 0_{\mathrm{End}(X)}$, for any $f, g \in [X]_2$), from the third equation we deduce that the linear map

$$B \to [X]_2 \colon b \mapsto b * -$$

is an algebra homomorphism.

On the other hand, given a homomorphism of algebras

$$\varphi\colon B \to [X]_2, \quad \varphi(b) = b * -$$

satisfying

$$b * (b' * x) = 0, \quad \forall b, b' \in B,\ \forall x \in X,$$

we can consider the split extension

$$0 \longrightarrow X \xrightarrow{\ i\ } (B \oplus X, *_\varphi) \underset{s}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

where the two-step nilpotent commutative algebra structure of $B \oplus X$ is given by

$$(b, x) *_\varphi (b', x') = (bb', xx' + b * x' + b' * x), \quad \forall (b, x), (b', x') \in B \oplus X.$$

We can now claim the following result.

Theorem 4.11. *[43]*

(1) *Let B and X be two-step nilpotent commutative algebras. The set of isomorphism classes of split extensions of B by X is in bijection with set of the algebra homomorphisms*
$$B \to [X]_2 \colon b \mapsto b * -$$
satisfying
$$b * (b' * x) = 0, \quad \forall b, b' \in B,\ \forall x \in X. \tag{4.3.1}$$

(2) *The variety $\mathbf{Nil}_2(\mathbf{Com})$ is weakly action representable. For any object X of $\mathbf{Nil}_2(\mathbf{Com})$, a weak representation of $\mathrm{SplExt}(-, X) = \mathrm{SplExt}_{\mathbf{Nil}_2(\mathbf{Com})}(-, X)$ is given by*
$$\tau\colon \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathbf{Nil}_2(\mathbf{Com})}(-, [X]_2),$$
where τ_B is the injection which sends any split extension of B by X to the corresponding homomorphism $B \to [X]_2$, defined by $b \mapsto b * -$ as above.

(3) *A homomorphism $B \to [X]_2$ is an acting morphism if and only if it satisfies Equation* (4.3.1).

Proof.

(1) It follows from the analysis above.

(2) We observe that τ is a natural transformation. Indeed, for every morphism $f\colon B' \to B$ in $\mathbf{Nil}_2(\mathbf{Com})$, one can check that the diagram in $\mathbf{Set}$

$$\begin{array}{ccc} \mathrm{SplExt}(B, X) & \xrightarrow{\tau_B} & \mathrm{Hom}(B, [X]_2) \\ {\scriptstyle \mathrm{SplExt}(f, X)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(f, [X]_2)} \\ \mathrm{SplExt}(B', X) & \xrightarrow{\tau_{B'}} & \mathrm{Hom}(B', [X]_2) \end{array}$$

is commutative. Moreover τ_B is an injection since every isomorphism class of split extensions of B by X is uniquely determined by the corresponding derived action. Thus τ is a monomorphism of functors and $\mathbf{Nil}_2(\mathbf{Com})$ is a weakly action representable category.

(3) Finally, $\varphi\colon B \to [X]_2$ is an acting morphism if and only if it defines a split extension of B by X in $\mathbf{Nil}_2(\mathbf{Com})$, i.e. if and only if it satisfies Equation (4.3.1).

□

Let us observe that not every homomorphism $B \to [X]_2$ defines a split extension of B by X. For instance, if $B = \mathbb{F}\{b, b'\}$ and $X = \mathbb{F}\{x\} \cong \mathbb{F}$ are abelian algebras, then $[X]_2 = \mathrm{End}(X)$ and the homomorphism $\varphi\colon B \to [X]_2$, defined by

$$\varphi(b) = \varphi(b') = 1_X$$

is not an acting morphism. Indeed,

$$\varphi(b)(\varphi(b')(x)) = 1_X(1_X(x)) = x \neq 0.$$

4.4 Anti-commutative anti-associative algebras

For the variety **ACAAssoc** of anti-commutative anti-associative algebras, a similar description of split extensions and derived actions can be made as for the variety **JJord**. The role of the anti-derivations is played here by the endomorphisms in the associative partial algebra

$$\mathrm{AM}(X) := \{f \in \mathrm{End}(X) \mid f(xy) = -f(x)y,\ \forall x \in X\},$$

of *anti-multipliers* of X, whose bilinear partial operation is given by

$$\langle f, g \rangle = -f \circ g.$$

It is easy to see that $\langle -, - \rangle$ does not define, in general, a total algebra structure on $\mathrm{AM}(X)$, nor need it be anti-commutative or anti-associative. An example is given by the abelian two-dimensional algebra $X = \mathbb{F}^2$, where $\mathrm{AM}(X) = \mathrm{End}(X)$.

We may check that a derived action of B by X in the variety **ACAAssoc** is the same thing as a partial algebra homomorphism

$$B \to \mathrm{AM}(X)\colon b \mapsto b * -$$

which satisfies

$$(bb') * - = -b * (b' * -), \quad \forall b, b' \in B.$$

Moreover, we obtain the following result whose proof is similar to the one of Theorem 4.9.

Theorem 4.12. *[43] Let X be a an object of* **ACAAssoc**.

(1) There exists a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{PAlg}}(U(-), \mathrm{AM}(X)),$$

where $\mathrm{SplExt}(-, X) = \mathrm{SplExt}_{\mathbf{ACAAssoc}}(-, X)$ *and* $U\colon \mathbf{ACAAssoc} \to \mathbf{PAlg}$ *denotes the forgetful functor;*

(2) *if* $[X]$ *is an anti-commutative anti-associative algebra, then the functor* $\mathrm{SplExt}(-, X)$ *is representable and* $\mathrm{AM}(X)$ *is the actor of* X*;* □

4.5 Two-step nilpotent anti-commutative algebras

We conclude this section by studying the representability of actions of the variety $\mathbf{Nil}_2(\mathbf{ACom})$, which is determined by the identities $xy = -yx$ and $x(yz) = 0$. An important example of a two-step nilpotent anti-commutative algebra is the $(2n+1)$-dimensional *Heisenberg algebra* $\mathfrak{h}_{2n+1}$, that is the algebra with basis

$$\{e_1, \ldots, e_n, f_1, \ldots, f_n, h\}$$

and non-trivial products $e_i f_j = -f_j e_i = \delta_{ij} h$, for every $i, j = 1, \ldots, n$, where δ_{ij} is the *Kronecker delta*.

A similar analysis can be done as in the case of two-step nilpotent commutative algebras, so we may simply state the following theorem:

Theorem 4.13. *[43]*

(1) *Let* B *and* X *be two-step nilpotent anti-commutative algebras. The set of isomorphism classes of split extensions of* B *by* X *is in bijection with the of algebra homomorphisms*

$$B \to [X]_2 \colon b \mapsto b * -$$

where $[X]_2$ *is defined as in the commutative case, which satisfy the condition*

$$b * (b' * x) = 0, \quad \forall b, b' \in B, \ \forall x \in X. \tag{4.5.1}$$

(2) *The variety* $\mathbf{Nil}_2(\mathbf{ACom})$ *is weakly action representable. For any object* X *of* $\mathbf{Nil}_2(\mathbf{ACom})$*, a weak representation of* $\mathrm{SplExt}(-, X) = \mathrm{SplExt}_{\mathbf{Nil}_2(\mathbf{ACom})}(-, X)$ *is given by*

$$\tau \colon \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathbf{Nil}_2(\mathbf{ACom})}(-, [X]_2),$$

where τ_B *is the injection which associates with any split extension of* B *by* X*, the corresponding homomorphism* $B \to [X]_2 \colon b \mapsto b * -$ *as in (1).*

(3) *A homomorphism* $B \to [X]_2$ *is an acting morphism if and only if it satisfies Equation* (4.5.1). □

Again, if $B = \mathbb{F}\{b, b'\}$ is the abelian two-dimensional algebra and $X = \mathbb{F}$ is the abelian one-dimensional algebra, the linear map $\varphi \colon B \to [X]_2 = \mathrm{End}(X)$, defined by $\varphi(b) = \varphi(b') = 1_X$ is an example of a morphism in $\mathbf{Nil}_2(\mathbf{ACom})$ which is not an acting morphism.

Chapter 5

Leibniz algebras

The aim of this chapter is to study the representability of actions of the variety **Leib** of Leibniz algebras and to give another example of a weakly action representable variety of non-associative algebras (see [29, Section 2]).

Leibniz algebras were introduced by J.-L. Loday in [58] as a not skew-symmetric version of Lie algebras. Such algebraic structures were considered earlier by A. Blokh, who called them *D-algebras* [11] for their strict connection with derivations. Leibniz algebras play a significant role in several areas of mathematics and physics (e.g. [10, 39, 50, 51, 57, 64]).

5.1 Definitions and main properties

Again, we assume that $\mathbb{F}$ is a field with $\mathrm{char}(\mathbb{F}) \neq 2$. We refer to [6] and [58] for the definitions and the results in this section.

Definition 5.1. [58] A *(right) Leibniz algebra* over $\mathbb{F}$ is a vector space $\mathfrak{g}$ over $\mathbb{F}$ endowed with a bilinear map (called *commutator* or *bracket*) $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the *(right) Leibniz identity*

$$[[x,y],z] = [[x,z],y] + [x,[y,z]], \quad \forall x,y,z \in \mathfrak{g}.$$

In the same way we can define a left Leibniz algebra, using the left Leibniz identity

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]], \quad \forall x,y,z \in \mathfrak{g}.$$

A Leibniz algebra that is both left and right is called *symmetric Leibniz algebra*.

Every Lie algebra is a Leibniz algebra and every Leibniz algebra with skew-symmetric commutator is a Lie algebra. The full inclusion $i \colon$ **Lie** $\to$ **Leib** has a left adjoint $\pi \colon$ **Leib** $\to$ **Lie** that associates, with every Leibniz algebra $\mathfrak{g}$, its quotient $\mathfrak{g}/\mathfrak{g}^{\mathrm{ann}}$, where $\mathfrak{g}^{\mathrm{ann}} = \langle [x,x] \mid x \in \mathfrak{g} \rangle$ is called the *Leibniz kernel* of $\mathfrak{g}$.

We define the left and the right center of a Leibniz algebra

$$\mathrm{Z}_l(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,\mathfrak{g}] = 0\}, \quad \mathrm{Z}_r(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [\mathfrak{g},x] = 0\}$$

and we observe that they coincide when $\mathfrak{g}$ is a Lie algebra. The *center* of $\mathfrak{g}$ is $\mathrm{Z}(\mathfrak{g}) = \mathrm{Z}_l(\mathfrak{g}) \cap \mathrm{Z}_r(\mathfrak{g})$. In general $\mathrm{Z}_r(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, while the left center may not even be a subalgebra. Note that $\mathfrak{g}^{\mathrm{ann}} \subseteq \mathrm{Z}_r(\mathfrak{g})$ since

$$[x,[y,y]] = [[x,y],y] - [[x,y],y] = 0$$

and thus $\mathfrak{g}^{\mathrm{ann}}$ is an abelian algebra.

Finally we recall the definitions of solvable and nilpotent Leibniz algebras.

Definition 5.2. [6] Let $\mathfrak{g}$ be a right Leibniz algebra over $\mathbb{F}$ and let

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k], \quad \forall k \geq 0,$$

be the *derived series of* $\mathfrak{g}$. $\mathfrak{g}$ is $n-$*step solvable* if $\mathfrak{g}^{n-1} \neq 0$ and $\mathfrak{g}^n = 0$.

Definition 5.3. [6] Let $\mathfrak{g}$ be a right Leibniz algebra over $\mathbb{F}$ and let

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad L^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}], \quad \forall k \geq 0,$$

be the *lower central series of* $\mathfrak{g}$. $\mathfrak{g}$ is $n-$*step nilpotent* if $\mathfrak{g}^{(n-1)} \neq 0$ and $\mathfrak{g}^{(n)} = 0$.

Many results of Lie algebras are still valid for Leibniz algebras. One of them is the *Levi decomposition*, which states that any Leibniz algebra $\mathfrak{g}$ over a field $\mathbb{F}$ of characteristic zero is the semidirect product of its radical $\mathrm{rad}(\mathfrak{g})$, which is the largest solvable ideal of $\mathfrak{g}$, and a semisimple Lie algebra. This clarifies the importance of the problem of classification of solvable and nilpotent Leibniz algebras. For instance, in [46], [52], [53] and [54] *two-step nilpotent algebras* and their derivations were studied and classified, while in [35] and [36] the authors classified non-nilpotent Leibniz algebras $\mathfrak{g}$ with one-dimensional derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$.

5.2 Derivations, anti-derivations and biderivations

The definition of *derivation* for a Leibniz algebra is the same as in the case of Lie algebras.

Definition 5.4. [58] Let $\mathfrak{g}$ be a Leibniz algebra over $\mathbb{F}$. A *derivation* of $\mathfrak{g}$ is a linear map $d\colon \mathfrak{g} \to \mathfrak{g}$ such that

$$d([x,y]) = [d(x), y] + [x, d(y)], \quad \forall x, y \in \mathfrak{g}.$$

The right multiplications of $\mathfrak{g}$ are particular derivations called *inner derivations* and an equivalent way to define a Leibniz algebra is to say that the (right) adjoint map $\mathrm{ad}_x = [-, x]$ is a derivation, for every $x \in \mathfrak{g}$. On the other hand the left adjoint maps are not derivations in general.

With the usual bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$, the set $\mathrm{Der}(\mathfrak{g})$ is a Lie algebra and the set $\mathrm{Inn}(\mathfrak{g})$ of all inner derivations of $\mathfrak{g}$ is an ideal of $\mathrm{Der}(\mathfrak{g})$. Furthermore, $\mathrm{Aut}(\mathfrak{g})$ is a Lie group and the associated Lie algebra is $\mathrm{Der}(\mathfrak{g})$.

The definitions of *anti-derivation* and *biderivation* for a Leibniz algebra were introduced by J.-L. Loday in [58].

Definition 5.5. [58] Let $\mathfrak{g}$ be a (right) Leibniz algebra. An *anti-derivation* of $\mathfrak{g}$ is a linear map $D\colon \mathfrak{g} \to \mathfrak{g}$ such that

$$D([x,y]) = [D(x), y] - [D(y), x], \quad \forall x, y \in \mathfrak{g}.$$

For a left Leibniz algebra $\mathfrak{g}$, we have to ask that

$$D([x,y]) = [x, D(y)] - [y, D(x)], \quad \forall x, y \in \mathfrak{g}.$$

One can check that, for every $x \in \mathfrak{g}$, the left multiplication $\mathrm{Ad}_x = [x, -]$ defines and anti-derivation. We observe that in the case of Lie algebras, there is no difference between a derivation and an anti-derivation.

Remark 5.6. The set of anti-derivations of a Leibniz algebra $\mathfrak{g}$ has a $\mathrm{Der}(\mathfrak{g})$-module structure with the multiplication

$$d \cdot D := [D, d] = D \circ d - d \circ D,$$

for every $d \in \mathrm{Der}(\mathfrak{g})$ and for every anti-derivation D.

Remark 5.7. Let $D\colon \mathfrak{g} \to \mathfrak{g}$ be an anti-derivation. Then, for every $x \in \mathfrak{g}$, we have

$$D([x,x]) = [D(x), x] - [D(x), x] = 0,$$

i.e. $D(\mathfrak{g}^{\mathrm{ann}}) = 0$.

Definition 5.8. [58] Let $\mathfrak{g}$ be a (right) Leibniz algebra. A *biderivation* of $\mathfrak{g}$ is a pair (d, D) where d is a derivation and D is an anti-derivation, such that

$$[x, d(y)] = [x, D(y)], \quad \forall x, y \in \mathfrak{g} \tag{5.2.1}$$

i.e. $d(y) - D(y) \in \mathrm{Z}_r(\mathfrak{g})$, for any $y \in \mathfrak{g}$.

Remark 5.9. For a left Leibniz algebra $\mathfrak{g}$, Equation (5.2.1) becomes

$$[d(x), y] = [D(x), y], \quad \forall x, y \in \mathfrak{g}.$$

The set of all biderivations of $\mathfrak{g}$, denoted by $\mathrm{Bider}(\mathfrak{g})$, has a Leibniz algebra structure with the bracket

$$[(d, D), (d', D')] = (d \circ d' - d' \circ d, D \circ d' - d' \circ D)$$

and it is possibile to define a Leibniz algebra homomorphism

$$\mathfrak{g} \to \mathrm{Bider}(\mathfrak{g})\colon x \mapsto (-\operatorname{ad}_x, \operatorname{Ad}_x).$$

The pair $(-\operatorname{ad}_x, \operatorname{Ad}_x)$ is called the *inner biderivation* associated with $x \in \mathfrak{g}$ and the set of all inner biderivations forms a Leibniz subalgebra of $\mathrm{Bider}(\mathfrak{g})$.

Independently of its intrinsic interest, derivations and biderivations find concrete applications in representation theory (cf. [57]), (sub-)Riemannian geometry and control theory (see [10] and the bibliography therein).

Now we give an example of computation of the biderivations of a Leibniz algebra.

Example 5.10. Let $\mathfrak{d}_1$ be the four-dimensional *Dieudonné algebra* (see [53] for more details), i.e. $\mathfrak{d}_1$ is the four-dimensional (symmetric) Leibniz algebra with basis $\{e_1, e_2, e_3, z\}$ and non-zero commutators

$$[e_1, e_3] = [e_2, e_3] = -[e_3, e_1] = [e_3, e_2] = z.$$

We want to find the Leibniz algebra $\mathrm{Bider}(\mathfrak{d}_1)$ of biderivations of $\mathfrak{d}_1$. Let $d \in \mathrm{gl}(\mathfrak{d}_1)$ be a derivation, then d is represented, with respect to the basis $\{e_1, e_2, e_3, z\}$, by a matrix of the type

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_1 & a_2 & a_3 & \gamma \end{pmatrix},$$

where $D(z) = \gamma z$ because $[\mathfrak{d}_1, \mathfrak{d}_1] = \mathbb{F}\{z\}$ and $d([\mathfrak{d}_1, \mathfrak{d}_1]) \subseteq [\mathfrak{d}_1, \mathfrak{d}_1]$. The entries a_{ij}, $i, j = 1, 2, 3$, satisfy the following set of equations

$$\begin{aligned}
a_{31} + a_{32} &= 0\\
a_{11} + a_{21} + a_{33} &= \gamma\\
a_{31} - a_{32} &= 0\\
a_{32} &= 0\\
a_{12} + a_{22} + a_{33} &= \gamma\\
-a_{11} + a_{21} - a_{33} &= \gamma\\
-a_{12} + a_{22} + a_{33} &= \gamma\\
a_{23} &= 0.
\end{aligned}$$

Thus we have

$$\mathrm{Der}(\mathfrak{d}_1) = \left\{ \begin{pmatrix} x & 0 & \alpha & 0\\ 0 & x & 0 & 0\\ 0 & 0 & y & 0\\ a_1 & a_2 & a_3 & x+y \end{pmatrix} \middle| x, y, \alpha, a_1, a_2, a_3 \in \mathbb{F} \right\}.$$

Now let $D \in \mathrm{gl}(\mathfrak{d}_1)$ be an anti-derivation, then D is represented by a matrix of the type

$$\begin{pmatrix} a'_{11} & a'_{12} & a'_{13} & 0\\ a'_{21} & a'_{22} & a'_{23} & 0\\ a'_{31} & a'_{32} & a'_{33} & 0\\ A_1 & A_2 & A_3 & 0 \end{pmatrix},$$

where $D(z) = 0$ because $\mathrm{Leib}(\mathfrak{d}_1) = [\mathfrak{d}_1, \mathfrak{d}_1] = \mathbb{F}z$. The entries a'_{ij}, $i, j = 1, 2, 3$, must satisfy the following equations

$$\begin{aligned}
a'_{31} + a'_{32} &= 0\\
a'_{11} + a'_{21} + a'_{33} &= 0\\
a'_{12} + a'_{22} - a'_{33} &= 0\\
-a'_{12} - a'_{22} - a'_{33} &= 0.
\end{aligned}$$

Thus, a general anti-derivation of $\mathfrak{d}_1$ is represented by

$$\begin{pmatrix} a'_{11} & a'_{12} & a'_{13} & 0\\ a'_{21} & a'_{11} + a'_{21} - a'_{12} & a'_{23} & 0\\ a'_{31} & -a'_{31} & a'_{11} + a'_{21} & 0\\ A_1 & A_2 & A_3 & 0 \end{pmatrix}$$

and, by applying *Equation* (5.2.1), we obtain

$$a'_{31} = 0,\ \alpha = a'_{13} - a'_{23},\ y = a'_{11} + a'_{21}$$

and

$$x = a'_{11} - a'_{21} = a'_{11} + a'_{21} - 2a'_{12}.$$

We conclude that

$$\mathrm{Bider}(\mathfrak{d}_1)=\left\{\left(\begin{pmatrix} x & 0 & \alpha & 0\\ 0 & x & 0 & 0\\ 0 & 0 & y & 0\\ a_1 & a_2 & a_3 & x+y\end{pmatrix},\begin{pmatrix} \frac{y+x}{2} & \frac{y-x}{2} & \alpha+\beta & 0\\ \frac{y-x}{2} & \frac{y+x}{2} & \beta & 0\\ 0 & 0 & y & 0\\ A_1 & A_2 & A_3 & 0\end{pmatrix}\right)\middle| x,y,\alpha,\beta,a_i,A_j\in\mathbb{F}\right\}$$

and the inner biderivations are represented by the pairs of matrices

$$\left(\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ a_1 & a_1 & a_3 & 0\end{pmatrix},\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ a_1 & -a_1 & A_3 & 0\end{pmatrix}\right).$$

5.3 Biderivations of low-dimensional Leibniz algebras

We present now the complete classification of the Leibniz algebras of biderivations of low-dimensional Leibniz algebras over a general field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F})\neq 2$ [62].

There is no non-trivial Leibniz algebra in dimension 1, thus we start with two-dimensional Leibniz algebras.

5.3.1 Biderivations of two-dimensional Leibniz algebras

Let $\dim_{\mathbb{F}}\mathfrak{g}=2$, i.e. $\mathfrak{g}=\mathbb{F}\{e_1,e_2\}$. Then, as shown in [33] by C. Cuvier, up to isomorphism we have only two non-Lie Leibniz algebra structures on $\mathfrak{g}$.

(1) $\mathfrak{g}_1$: nilpotent Leibniz algebra with non-trivial bracket $[e_2,e_2]=e_1$;

(2) $\mathfrak{g}_2$: solvable Leibniz algebra with non-trivial brackets $[e_1,e_2]=[e_2,e_2]=e_1$.

Notice that $\mathfrak{g}_1$ is a symmetric Leibniz algebra, while $\mathfrak{g}_2$ is only a right Leibniz algebra. It turns out that

$$\mathrm{Der}(\mathfrak{g}_1)=\left\{\begin{pmatrix} 2a & b\\ 0 & a\end{pmatrix}\middle| a,b\in\mathbb{F}\right\}$$

and

$$\mathrm{Der}(\mathfrak{g}_2)=\left\{\begin{pmatrix} a & a\\ 0 & 0\end{pmatrix}\middle| a\in\mathbb{F}\right\}.$$

Moreover it is easy to check that the set of anti-derivations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are both represented by the matrices of the form

$$\begin{pmatrix} 0 & x\\ 0 & y\end{pmatrix}.$$

Equation (5.2.1) implies that $y=a$ for $\mathfrak{g}_1$ and $y=0$ for $\mathfrak{g}_2$, thus we have

$$\mathrm{Bider}(\mathfrak{g}_1)=\left\{\left(\begin{pmatrix} 2a & b\\ 0 & a\end{pmatrix},\begin{pmatrix} 0 & x\\ 0 & a\end{pmatrix}\right)\middle| a,b,x\in\mathbb{F}\right\}$$

and

$$\mathrm{Bider}(\mathfrak{g}_2)=\left\{\left(\begin{pmatrix} a & a\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & x\\ 0 & 0\end{pmatrix}\right)\middle| a,x\in\mathbb{F}\right\}.$$

Finally the inner biderivations of $\mathfrak{g}_1$ are represented by the pairs of matrices

$$\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}\right),$$

meanwhile the biderivations of $\mathfrak{g}_2$ are all inner.

5.3.2 Biderivations of three-dimensional Leibniz algebras

Three-dimensional complex Leibniz algebras and their derivations were classified in [27] and [68], while the more general classification of three-dimensional right Leibniz algebras over a field $\mathbb{F}$, with $\operatorname{char}(\mathbb{F}) \neq 2$, can be found in [5] and [69].

Let $\dim_{\mathbb{F}} \mathfrak{g} = 3$ and let $\{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$ over $\mathbb{F}$. The list of non-isomorphic three-dimensional right Leibniz algebras over $\mathbb{F}$ is the following.

Leibniz algebra	Non-zero brackets
$\mathfrak{g}_1$	$[e_1, e_3] = -2e_1,\ [e_2, e_2] = e_1,\ [e_3, e_2] = -[e_2, e_3] = e_2$
$\mathfrak{g}_2(\alpha),\ \alpha \neq 0$	$[e_1, e_3] = \alpha e_1,\ [e_3, e_2] = -[e_2, e_3] = e_2$
$\mathfrak{g}_3$	$[e_3, e_2] = -[e_2, e_3] = e_2,\ [e_3, e_3] = -e_1$
$\mathfrak{g}_4$	$[e_2, e_2] = e_1,\ [e_3, e_3] = e_1$
$\mathfrak{g}_5$	$[e_2, e_2] = e_1,\ [e_3, e_3] = -e_1$
$\mathfrak{g}_7(\alpha),\ \alpha \neq 0$	$[e_2, e_2] = [e_2, e_3] = e_1,\ [e_3, e_3] = \alpha e_1$
$\mathfrak{g}_8$	$[e_2, e_3] = e_1$
$\mathfrak{g}_9$	$[e_1, e_3] = e_2,\ [e_2, e_3] = e_1$
$\mathfrak{g}_{10}$	$[e_1, e_3] = e_2,\ [e_2, e_3] = -e_1$
$\mathfrak{g}_{12}(\alpha),\ \alpha \neq 0$	$[e_1, e_3] = e_2,\ [e_2, e_3] = \alpha e_1 + e_2$
$\mathfrak{g}_{13}$	$[e_1, e_3] = e_1,\ [e_2, e_3] = e_2$
$\mathfrak{g}_{14}$	$[e_1, e_3] = e_2,\ [e_3, e_3] = e_1$
$\mathfrak{g}_{15}$	$[e_1, e_3] = e_1 + e_2,\ [e_3, e_3] = e_1$

Here we use the same numbering of [5], but we do not report the algebras $\mathfrak{g}_6(\alpha)$ and $\mathfrak{g}_{11}(\alpha)$, where $\alpha \neq 0$, which are isomorphic to $\mathfrak{g}_4$ and $\mathfrak{g}_9$ respectively. We want to extend the results of [68] by completing the classification of the Lie algebras of derivations of three-dimensional Leibniz algebras over a general field $\mathbb{F}$, with $\operatorname{char}(\mathbb{F}) \neq 2$, and by finding the biderivations of this class of Leibniz algebras.

Remark 5.11. [62] We present the following algorithm for finding derivations and anti-derivations. Let $\mathfrak{g}$ be a Leibniz algebra and let $(d, D) \in \operatorname{Bider}(\mathfrak{g})$. Then, for every $x, y \in \mathfrak{g}$, we have

$$d([x,y]) = [d(x), y] + [x, d(y)], \quad D([x,y]) = [D(x), y] - [D(y), x]$$

if and only if

$$(d \circ \operatorname{ad}_y)(x) = (\operatorname{ad}_y \circ d)(x) + \operatorname{ad}_{d(y)}(x), \quad (D \circ \operatorname{ad}_y)(x) = (\operatorname{ad}_y \circ D)(x) - \operatorname{Ad}_{D(y)}(x),$$

thus

$$[d, \operatorname{ad}_y] = \operatorname{ad}_{d(y)}, \quad [D, \operatorname{ad}_y] = -\operatorname{Ad}_{D(y)}.$$

Fixed a basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$, we have that the biderivation (d, D) is represented by a pair of $n \times n$ matrices $((d_{i,j})_{i,j}, (D_{i,j})_{i,j})$ and for every $i = 1, \ldots, n$

$$[d, \operatorname{ad}_{e_i}] = \operatorname{ad}_{d(e_i)}, \quad [D, \operatorname{ad}_{e_i}] = -\operatorname{Ad}_{D(e_i)}$$

which are equations in the entries of the matrices representing d and D. By solving this set of equations, and after imposing Equation (5.2.1), we find the matrices $(d_{i,j})_{i,j}, (D_{i,j})_{i,j}$.

A straightforward application of the above algorithm produces the following complete classification of biderivations of three-dimensional right Leibniz algebras over $\mathbb{F}$. In particular we find that the dimension of these biderivation algebras lies between three and six and there are only two parameterized families of Leibniz algebra of biderivations of three-dimensional Leibniz algebras over $\mathbb{F}$.

Theorem 5.12. *[62] Let $\mathbb{F}$ be a field with $\operatorname{char}(\mathbb{F}) \neq 2$. The Leibniz algebras of biderivations of three-dimensional right Leibniz algebras over $\mathbb{F}$ can be described as follows:*

- $$\operatorname{Bider}(\mathfrak{g}_1) = \left\{ \left(\begin{pmatrix} 2x & y & 0 \\ 0 & x & y \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -y & a \\ 0 & x & y \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a \in \mathbb{F} \right\};$$
- $$\operatorname{Bider}(\mathfrak{g}_2(\alpha)) = \left\{ \left(\begin{pmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & y & z \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, z, a, b \in \mathbb{F} \right\},$$
  if $\alpha \neq -1$ and
  $$\operatorname{Bider}(\mathfrak{g}_2(-1)) = \left\{ \left(\begin{pmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & b \\ 0 & y & z \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, z, b \in \mathbb{F} \right\};$$
- $$\operatorname{Bider}(\mathfrak{g}_3) = \left\{ \left(\begin{pmatrix} 0 & 0 & y \\ 0 & x & z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & x & z \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, z, a \in \mathbb{F} \right\};$$
- $$\operatorname{Bider}(\mathfrak{g}_4) = \left\{ \left(\begin{pmatrix} 2x & y & z \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \right) \middle| x, y, z, a, b \in \mathbb{F} \right\};$$
- $$\operatorname{Bider}(\mathfrak{g}_5) = \left\{ \left(\begin{pmatrix} 2x & y & t \\ 0 & x & -z \\ 0 & z & x \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & x & -z \\ 0 & z & x \end{pmatrix} \right) \middle| x, y, z, t, a, b \in \mathbb{F} \right\};$$
- $$\operatorname{Bider}(\mathfrak{g}_7(\alpha)) = \left\{ \left(\begin{pmatrix} \gamma x & y & z \\ 0 & x & \frac{x}{2} \\ 0 & -\frac{x}{2\alpha} & (\gamma-1)x \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & x & \frac{x}{2} \\ 0 & -\frac{x}{2\alpha} & (\gamma-1)x \end{pmatrix} \right) \middle| x, y, z, a, b \in \mathbb{F} \right\},$$
  where $\gamma = \dfrac{4\alpha - 1}{2\alpha}$;
- $$\operatorname{Bider}(\mathfrak{g}_8) = \left\{ \left(\begin{pmatrix} x+y & z & t \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}, \begin{pmatrix} 0 & z & t \\ 0 & 0 & a \\ 0 & 0 & y \end{pmatrix} \right) \middle| x, y, z, t, a \in \mathbb{F} \right\};$$
- $$\operatorname{Bider}(\mathfrak{g}_9) = \left\{ \left(\begin{pmatrix} x & y & 0 \\ y & x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a, b \in \mathbb{F} \right\};$$
- $$\operatorname{Bider}(\mathfrak{g}_{10}) = \left\{ \left(\begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a, b \in \mathbb{F} \right\};$$

- $\mathrm{Bider}(\mathfrak{g}_{12}(\alpha)) = \left\{ \left(\begin{pmatrix} x & \alpha y & 0 \\ y & x+y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a, b \in \mathbb{F} \right\};$

- $\mathrm{Bider}(\mathfrak{g}_{13}) = \left\{ \left(\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a, b \in \mathbb{F} \right\};$

- $\mathrm{Bider}(\mathfrak{g}_{14}) = \left\{ \left(\begin{pmatrix} 2x & 0 & y \\ y & 3x & z \\ 0 & 0 & x \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & x \end{pmatrix} \right) \middle| x, y, z, a, b \in \mathbb{F} \right\};$

- $\mathrm{Bider}(\mathfrak{g}_{15}) = \left\{ \left(\begin{pmatrix} x & 0 & x \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| x, y, a, b \in \mathbb{F} \right\}.$

□

5.4 Representability of actions of Leibniz algebras

By studying biderivations of a Leibniz algebra $\mathfrak{h}$, we can classify the split extensions with kernel $\mathfrak{h}$. This relies on the correspondence between actions and split extensions available in any semi-abelian category, as explained in Section 2.2. Again, since the variety of Leibniz algebra is an Orzech category of interest, we can use derived actions in place of internal actions.

Remark 5.13. We recall from Definition 3.47 that, given a split extension in the category **Leib**

$$0 \longrightarrow \mathfrak{h} \xrightarrow{i} \hat{\mathfrak{g}} \underset{s}{\overset{\pi}{\rightleftarrows}} \mathfrak{g} \longrightarrow 0 \tag{5.4.1}$$

the pair of bilinear maps

$$l\colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}, \qquad r\colon \mathfrak{h} \times \mathfrak{g} \to \mathfrak{h}$$

defined by

$$l_x(b) = [s(x), i(b)], \quad r_y(a) = [i(a), s(y)]$$

where $l_x = l(x, -)$ and $r_y = r(-, y)$, is called the *derived action* of $\mathfrak{g}$ on $\mathfrak{h}$ associated with (5.4.1).

Given a pair of bilinear maps

$$l\colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}, \qquad r\colon \mathfrak{h} \times \mathfrak{g} \to \mathfrak{h},$$

we define a bilinear operation on the direct sum of vector spaces $\mathfrak{g} \oplus \mathfrak{h}$

$$[(x, a), (y, b)]_{(l,r)} = ([x, y], [a, b] + l_x(b) + r_y(a)).$$

By Theorem 2.4 in [66], this defines a Leibniz algebra structure on $\mathfrak{g} \oplus \mathfrak{h}$ if and only if the pair (l, r) is a derived action of $\mathfrak{g}$ on $\mathfrak{h}$. This in turn is equivalent to the following proposition, which is an application of Lemma 3.48 and Remark 3.50.

Proposition 5.14. *[29]* $(\mathfrak{g} \oplus \mathfrak{h}, [-, -]_{(l,r)})$ *is a Leibniz algebra if and only if*

(L1) $r_x([a, b]) = [r_x(a), b] + [a, r_x(b)]$;

(L2) $l_x([a,b]) = [l_x(a), b] - [l_x(b), a]$;

(L3) $[a, r_x(b) + l_x(b)] = 0$;

(L4) $r_{[x,y]} = [r_y, r_x] = r_y \circ r_x - r_x \circ r_y$;

(L5) $l_{[x,y]} = [r_y, l_x] = r_y \circ l_x - l_x \circ r_y$;

(L6) $l_x \circ (l_y + r_y) = 0$;

for every $x, y \in \mathfrak{g}$ and for every $a, b \in \mathfrak{h}$. The resulting Leibniz algebra is the semi-direct product *of $\mathfrak{g}$ and $\mathfrak{h}$ (with respect to the derived action (l, r)) and it is denoted by $\mathfrak{g} \ltimes \mathfrak{h}$.*

Proof. By Remark 3.50, $(\mathfrak{g} \oplus \mathfrak{h}, [-,-]_{(l,r)})$ is a Leibniz algebra if and only if it satisfies the (right) Leibniz identity

$$[[\alpha_1, \alpha_2], \alpha_3] = [[\alpha_1, \alpha_3], \alpha_2] + [\alpha_1, [\alpha_2, \alpha_3]]$$

when at least one of the α_i is an element of the form $(x, 0)$, with $x \in \mathfrak{g}$, and the others are of the form $(0, a)$, with $a \in \mathfrak{h}$. Thus, we need to ask that the following equations hold for any $x, y \in \mathfrak{g}$ and $a, b \in \mathfrak{h}$:

- $[[(0,a),(0,b)],(x,0)] = [[(0,a),(x,0)],(0,b)] + [(0,a),[(0,b),(x,0)]]$, which is equivalent to (L1);
- $[[(x,0),(0,a)],(0,b)] = [[(x,0),(0,b)],(0,a)] + [(x,0),[(0,a),(0,b)]]$, which is equivalent to (L2);
- $[[(0,a),(x,0)],(0,b)] = [[(0,a),(0,b)],(x,0)] + [(0,a),[(x,0),(0,b)]]$, which in turn is equivalent to $r_x([a,b]) = [r_x(a), b] - [a, l_x(b)]$. Combining with (L1), we deduce that (L3) holds;
- $[[(0,a),(x,0)],(y,0)] = [[(0,a),(y,0)],(x,0)] + [(0,a),[(x,0),(y,0)]]$, which is equivalent to (L4), i.e. $r_{[x,y]}(a) = r_y(r_x(a)) - r_x(r_y(a))$;
- $[[(x,0),(y,0)],(0,a)] = [[(x,0),(0,a)],(y,0)] + [(x,0),[(y,0),(0,a)]]$, which is equivalent to (L5), i.e. $l_{[x,y]}(a) = r_y(l_x(a)) - l_x(r_y(a))$;
- $[[(x,0),(y,0)],(0,a)] = [[(x,0),(0,a)],(y,0)] + [(x,0),[(y,0),(0,a)]]$ which in turn is equivalent to $l_{[x,y]} = r_y \circ l_x + l_x \circ l_y$. Combining with (L5), we obtain that (L6) holds.

□

Remark 5.15. The first three equations of Proposition 5.14 state that, for every $x \in \mathfrak{g}$, the pair

$$(-r_x, l_x)$$

is a biderivation of the Leibniz algebra $\mathfrak{h}$. Moreover, from equalities (L4)-(L5), we have that the linear map

$$\varphi \colon \mathfrak{g} \to \mathrm{Bider}(\mathfrak{h}) \colon x \mapsto (-r_x, l_x)$$

is a Leibniz algebra homomorphism. Indeed

$$\varphi([x,y]) = (-r_{[x,y]}, l_{[x,y]}) = (-[r_y, r_x], [r_y, l_x])$$

and

$$[\varphi(x), \varphi(y)] = [(-r_x, l_x), (-r_y, l_y)] = ([-r_x, -r_y], [l_x, -r_y]) = \\ = ([r_x, r_y], -[l_x, r_y]) = (-[r_y, r_x], [r_y, l_x]).$$

On the other hand, given a Leibniz algebra homomorphism

$$\varphi\colon \mathfrak{g} \to \mathrm{Bider}(\mathfrak{h})\colon x \mapsto ([\![-, x]\!], [\![x, -]\!])$$

satisfying

$$[\![x, [\![y, a]\!] - [\![a, y]\!]]\!] = 0, \quad \forall x, y \in \mathfrak{g},\ \forall a \in \mathfrak{h},$$

we can associate the split extension

$$0 \longrightarrow \mathfrak{h} \xrightarrow{i_2} (\mathfrak{g} \oplus \mathfrak{h}, [-,-]_\varphi) \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} \mathfrak{g} \longrightarrow 0$$

where i_1, i_2 and π_1 are the canonical injections and projection, and the Leibniz algebra structure of $\mathfrak{g} \oplus \mathfrak{h}$ is given by

$$[(x, a), (y, b)]_\varphi = ([x, y], [a, b] + [\![x, b]\!] - [\![a, y]\!]).$$

However a generic homomorphism from $\mathfrak{g}$ to $\mathrm{Bider}(\mathfrak{h})$ needs not give rise to a split extension, as the following example shows.

Example 5.16. [32] Let $\mathfrak{g} = \mathbb{F}$ be the abelian one-dimensional algebra. Then the homomorphism $\varphi\colon \mathbb{F} \to \mathrm{Bider}(\mathbb{F}) = \mathrm{End}(\mathbb{F})^2\colon a \mapsto (d_a, D_a)$, where

$$d_a(x) = -ax, \quad D_a(x) = ax.$$

does not define a split extension of $\mathbb{F}$ by itself. Indeed in general

$$D_a(D_b(x) - d_b(x)) = a(bx - (-bx)) = 2abx \neq 0.$$

Example 5.17. Let $\mathfrak{g}$ be a Leibniz algebra and consider the canonical action of $\mathfrak{g}$ on itself given by the pair of linear maps

$$r_x = \mathrm{ad}_x = [-, x], \quad \forall x \in \mathfrak{g},$$

$$l_y = \mathrm{Ad}_y = [y, -], \quad \forall y \in \mathfrak{g}.$$

We have a split extension of $\mathfrak{g}$ by itself with associated morphism

$$\mathfrak{g} \to \mathrm{Bider}(\mathfrak{g})\colon x \mapsto (-\mathrm{ad}_x, \mathrm{Ad}_x)$$

which obviously satisfies the condition

$$\mathrm{Ad}_x \circ (\mathrm{Ad}_y + \mathrm{ad}_y) = 0, \quad \forall x, y \in \mathfrak{g}.$$

Indeed, for every $z \in \mathfrak{g}$

$$[x, [y, z] + [z, y]] = [x, [y, z]] + [x, [z, y]] = \\ = [[x, y], z] - [[x, z], y] + [[x, z], y] - [[x, y], z] = 0$$

Thus the Leibniz algebra homomorphism which defines the inner biderivations of $\mathfrak{g}$ is associated with the canonical extension of $\mathfrak{g}$ by itself.

Example 5.18. Let $\mathfrak{h}$ be a Leibniz algebra. It is well known that, if $\mathfrak{h}$ has trivial center or if $\mathfrak{h}$ is a *perfect* algebra, i.e. $[\mathfrak{h},\mathfrak{h}]=\mathfrak{h}$, then

$$D(D'(x)-d'(x))=0,$$

for every $x\in\mathfrak{h}$ and for every $(d,D),(d',D')\in \mathrm{Bider}(\mathfrak{h})$ [26].

In fact, let $\mathrm{Z}(\mathfrak{h})=0$ and let $(d,D),(d',D')\in \mathrm{Bider}(\mathfrak{h})$. Then

$$\begin{aligned}[y,D(D'(x))]&=[y,d(D'(x))]=d([y,D'(x)])-[d(y),D'(x)]=\\&=d([y,d'(x)])-[d(y),d'(x)]=[y,d(d'(x))]=[y,D(d'(x))]\end{aligned}$$

for any $x,y\in\mathfrak{h}$. Thus $D(D'(x)-d'(x))\in \mathrm{Z}_r(\mathfrak{h})$, for any $x\in\mathfrak{h}$. Similar computations show that $D(D'(x)-d'(x))\in \mathrm{Z}_l(\mathfrak{h})$, thus $D(D'(x)-d'(x))=0$ and one can obtain the same result starting with a perfect Leibniz algebra $\mathfrak{h}$.

Then, given any other Leibniz algebra $\mathfrak{g}$, we can associate a split extension of $\mathfrak{g}$ by $\mathfrak{h}$ with any homomorphism

$$\mathfrak{g}\to \mathrm{Bider}(\mathfrak{h})$$

since equation (L6) is always satisfied and $\mathrm{Bider}(\mathfrak{h})$ is the actor of $\mathfrak{h}$.

Remark 5.19. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and let

$$0\longrightarrow \mathfrak{h}\xrightarrow{\ i\ }\hat{\mathfrak{g}}\underset{s}{\overset{\pi}{\rightleftarrows}}\mathfrak{g}\longrightarrow 0$$

be a Lie algebra split extension. Then, as observed above, we have that

$$\hat{\mathfrak{g}}\cong(\mathfrak{g}\oplus\mathfrak{h},[-,-]_r),$$

where the Lie bracket is defined by

$$[(x,a),(y,b)]_r=([x,y],[a,b]-r_x(b)+r_y(a)).$$

In this case anti-commutativity implies that the left component of the action of $\mathfrak{g}$ on $\mathfrak{h}$ is defined by

$$l_x(b)=-r_x(b),\quad \forall x\in\mathfrak{g},\ \forall b\in\mathfrak{h},$$

thus equation (L6) is automatically satisfied and every homomorphism

$$\mathfrak{g}\to \mathrm{Bider}(\mathfrak{h})\colon x\mapsto([\![-,x]\!],[\![-,x]\!])$$

represents a split extension of $\mathfrak{g}$ by $\mathfrak{h}$ in the category **Lie**. Moreover the subalgebra of $\mathrm{Bider}(\mathfrak{h})$

$$\{(d,d)\mid d\in \mathrm{Der}(\mathfrak{h})\}$$

is a Lie algebra isomorphic to $\mathrm{Der}(\mathfrak{h})$.

We can now claim the following result.

Theorem 5.20. *[29] Let $\mathfrak{g}$ and $\mathfrak{h}$ be Leibniz algebras over $\mathbb{F}$.*

(1) The set of isomorphism classes of split extensions of $\mathfrak{g}$ by $\mathfrak{h}$ is in bijection with the set of Leibniz algebra homomorphisms

$$\varphi\colon \mathfrak{g}\to \mathrm{Bider}(\mathfrak{h})\colon x\mapsto([\![-,x]\!],[\![x,-]\!])$$

which satisfy the condition

$$[\![x,[\![y,a]\!]-[\![a,y]\!]]\!]=0,\quad \forall x,y\in\mathfrak{g},\ \forall a\in\mathfrak{h}. \tag{5.4.2}$$

(2) *The category* **Leib** *of Leibniz algebras over* $\mathbb{F}$ *is weakly action representable and a weak actor of an object* $\mathfrak{h}$ *in* **Leib** *is the Leibniz algebra* $\mathrm{Bider}(\mathfrak{h})$*.*

(3) $\varphi\colon \mathfrak{g}\to\mathrm{Bider}(\mathfrak{h})$ *is an acting morphism if and only if it satisfies Equation* (5.4.2).

Proof.

(1) The first statement follows from Remark 5.15.

(2) Given any Leibniz algebra $\mathfrak{h}$, we take $T=\mathrm{Bider}(\mathfrak{h})$ and we define τ in the following way: for every Leibniz algebra $\mathfrak{g}$, the component

$$\tau_{\mathfrak{g}}\colon\ \mathrm{SplExt}(\mathfrak{g},\mathfrak{h})\to\mathrm{Hom}_{\mathbf{Leib}}(\mathfrak{g},\mathrm{Bider}(\mathfrak{h}))$$

is the morphism in **Set** which associates with any split extension

$$0\longrightarrow\mathfrak{h}\xrightarrow{\ i\ }\hat{\mathfrak{g}}\underset{s}{\overset{\pi}{\rightleftarrows}}\mathfrak{g}\longrightarrow 0$$

the homomorphism $\varphi_{(l,r)}\colon\mathfrak{g}\to\mathrm{Bider}(\mathfrak{h})\colon x\mapsto(-r_x,l_x)$ (see Remark 5.13). The transformation τ is natural. Indeed, for every Leibniz algebra homomorphism $f\colon\mathfrak{g}'\to\mathfrak{g}$, the following diagram in **Set**

$$\begin{array}{ccc}\mathrm{SplExt}(\mathfrak{g},\mathfrak{h}) & \xrightarrow{\tau_{\mathfrak{g}}} & \mathrm{Hom}(\mathfrak{g},\mathrm{Bider}(\mathfrak{h}))\\ {\scriptstyle\mathrm{SplExt}(f,\mathfrak{h})}\downarrow & & \downarrow{\scriptstyle\mathrm{Hom}(f,\mathrm{Bider}(\mathfrak{h}))}\\ \mathrm{SplExt}(\mathfrak{g}',\mathfrak{h}) & \xrightarrow{\tau_{\mathfrak{g}'}} & \mathrm{Hom}(\mathfrak{g}',\mathrm{Bider}(\mathfrak{h}))\end{array}$$

is commutative. Moreover, for every Leibniz algebra $\mathfrak{g}$, the morphism $\tau_{\mathfrak{g}}$ is an injection since every element of $\mathrm{SplExt}(\mathfrak{g},\mathfrak{h})$ is uniquely determined by the corresponding action of $\mathfrak{g}$ on $\mathfrak{h}$, i.e. by the pair of bilinear maps

$$l\colon\mathfrak{g}\times\mathfrak{h}\to\mathfrak{h},\qquad r\colon\mathfrak{h}\times\mathfrak{g}\to\mathfrak{h}\,.$$

Thus τ is a monomorphism of functors, the category **Leib** is weakly action representable and a *weak actor* of an object $\mathfrak{h}$ is the Leibniz algebra of biderivations $\mathrm{Bider}(\mathfrak{h})$.

(3) Finally $\varphi\in\mathrm{Hom}_{\mathbf{Leib}}(\mathfrak{g},\mathrm{Bider}(\mathfrak{h}))$ is an acting morphism if and only if it defines a split extension of $\mathfrak{g}$ by $\mathfrak{h}$, i.e. if and only if it satisfies the condition

$$[\![x,[\![y,a]\!]-[\![a,y]\!]]\!]=0,\quad \forall x,y\in\mathfrak{g},\ \forall a\in\mathfrak{h}.$$

□

Chapter 6

Representability of actions of non-associative algebras

We want to extend the results obtained in the previous chapters by studying the representability of actions of a general variety of non-associative algebras over a field $\mathbb{F}$ (see [43, Section 3]). Again, we assume that $\mathcal{V}$ is an action accessible, operadic variety of non-associative algebras over $\mathbb{F}$. Thus $\mathcal{V}$ satisfies a set of multilinear identities

$$\Phi_{k,i}(x_1, \ldots, x_k) = 0, \quad i = 1, \ldots, n,$$

where k is the degree of the polynomial $\Phi_{k,i}$. We fix $\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8 \in \mathbb{F}$ which determine a choice of λ/μ rules, i.e.

$$\begin{aligned} x(yz) = {} & \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3 z(xy) + \lambda_4 z(yx) \\ & + \lambda_5(xz)y + \lambda_6(zx)y + \lambda_7 y(xz) + \lambda_8 y(zx) \end{aligned}$$

and

$$\begin{aligned} (yz)x = {} & \mu_1(xy)z + \mu_2(yx)z + \mu_3 z(xy) + \mu_4 z(yx) \\ & + \mu_5(xz)y + \mu_6(zx)y + \mu_7 y(xz) + \mu_8 y(zx) \end{aligned}$$

which are identities in $\mathcal{V}$. Note that these are not unique, but fixed for our purposes.

6.1 The external weak actor

Let $\mathcal{V}$ be a variety of non-associative algebras as above. For any object X of $\mathcal{V}$, we look for a vector space $\mathcal{E}(X)$ such that

$$\mathrm{Inn}(X) \leq \mathcal{E}(X) \leq \mathrm{End}(X)^2,$$

where $\mathrm{Inn}(X) = \{(L_x, R_x) \mid x \in X\}$ is the vector space of left and right multiplications of X, and we want to endow it with a bilinear partial operation

$$\langle -, - \rangle \colon \Omega \to X,$$

where Ω is a vector subspace of $X \times X$, such that we can associate in a natural way a homomorphism of partial algebras $B \to \mathcal{E}(X)$, with every split extension of B by X in $\mathcal{V}$. To do this, we need to describe derived actions in $\mathcal{V}$ in a similar fashion as in the previous sections.

Proposition 6.1. *[43] Let X and B be two algebras in $\mathcal{V}$. Given a pair of bilinear maps*

$$l \colon B \times X \to X, \qquad r \colon X \times B \to X,$$

we construct $(B \oplus X, \cdot)$ *as in* (3.2.2). *Then* $(B \oplus X, \cdot)$ *is an object of* $\mathcal{V}$ *if and only if*

$$\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0, \quad \forall i = 1, \ldots, n,$$

where at least one of the $\alpha_1, \ldots, \alpha_k$ *is an element of of the form* $(0, x)$*, with* $x \in X$*, and the others are of the form* $(b, 0)$*, with* $b \in B$*. The resulting algebra is the* semi-direct product *of* B *and* X *and it is denoted by* $B \ltimes X$*.*

Proof. The proof is a straightforward application of Lemma 3.48 and Remark 3.50. □

Using the same notation of Remark 3.49, we obtain the following:

Corollary 6.2. *[43] When every identity of* $\mathcal{V}$ *can be deduced from the* λ/μ *rules,* $(B \oplus X, \cdot)$ *is an object of* $\mathcal{V}$ *if and only if*

(1) $b * (xx') = \lambda_1 (b * x)x' + \cdots + \lambda_8 x(x' * b)$;

(2) $(xx') * b = \mu_1 (b * x)x' + \cdots + \mu_8 x(x' * b)$;

(3) $x(x' * b) = \lambda_1 (xx') * b + \cdots + \lambda_8 x'(b * x)$;

(4) $(x' * b)x = \mu_1 (xx') * b + \cdots + \mu_8 x'(b * x)$;

(5) $x(b * x') = \lambda_1 (x * b)x' + \cdots + \lambda_8 b * (x'x)$;

(6) $(b * x')x = \mu_1 (x * b)x' + \cdots + \mu_8 b * (x'x)$;

(7) $x * (bb') = \lambda_1 (x * b) * b' + \cdots + \lambda_8 b * (b' * x)$;

(8) $(bb') * x = \mu_1 (x * b) * b' + \cdots + \mu_8 b * (b' * x)$;

(9) $b * (b' * x) = \lambda_1 (bb') * x + \cdots + \lambda_8 b' * (x * b)$;

(10) $(b' * x) * b = \mu_1 (bb') * x + \cdots + \mu_8 b' * (x * b)$;

(11) $b * (x * b') = \lambda_1 (b * x) * b' + \cdots + \lambda_8 x * (b'b)$;

(12) $(x * b) * b' = \mu_1 (b * x) * b' + \cdots + \mu_8 x * (b'b)$,

for every $b, b' \in B$ *and* $x, x' \in X$. □

Definition 6.3. [43] For every object X of $\mathcal{V}$, we define $\mathcal{E}(X)$ as the subset of all pairs $(f * -, - * f) \in \mathrm{End}(X)^2$ satisfying

$$\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0, \quad \forall i = 1, \ldots, n,$$

for each choice of $\alpha_j = f$ and $\alpha_t \in X$, where $t \neq j \in \{1, \ldots, k\}$ and $fx := f * x$, $xf := x * f$. We observe that $\mathcal{E}(X)$ is a linear subspace of $\mathrm{End}(X)^2$ since it is non-empty and it is determined by a collection of multilinear identities. We endow $\mathcal{E}(X)$ with the bilinear map $\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2$

$$\langle (f * -, - * f), (g * -, - * g) \rangle = (h * -, - * h),$$

where

$$x * h = \lambda_1 (x * f) * g + \lambda_2 (f * x) * g + \lambda_3 g * (x * f) + \lambda_4 g * (f * x) \\ + \lambda_5 (x * g) * f + \lambda_6 (g * x) * f + \lambda_7 f * (x * g) + \lambda_8 f * (g * x)$$

and

$$h * x = \mu_1(x * f) * g + \mu_2(f * x) * g + \mu_3 g * (x * f) + \mu_4 g * (f * x) \\ + \mu_5(x * g) * f + \mu_6(g * x) * f + \mu_7 f * (x * g) + \mu_8 f * (g * x).$$

When every identity of $\mathcal{V}$ is a consequence of the λ/μ rules, $\mathcal{E}(X)$ becomes the subspace of all pairs $(f * -, - * f) \in \operatorname{End}(X)^2$ satisfying

(1) $f * (xx') = \lambda_1(f * x)x' + \cdots + \lambda_8 x(x' * f)$;

(2) $(xx') * f = \mu_1(f * x)x' + \cdots + \mu_8 x(x' * f)$;

(3) $x(x' * f) = \lambda_1(xx') * f + \cdots + \lambda_8 x'(f * x)$;

(4) $(x' * f)x = \mu_1(xx') * f + \cdots + \mu_8 x'(f * x)$;

(5) $x(f * x') = \lambda_1(x * f)x' + \cdots + \lambda_8 f * (x'x)$;

(6) $(f * x')x = \mu_1(x * f)x' + \cdots + \mu_8 f * (x'x)$,

for every $x, x' \in X$.

Remark 6.4. Note that the choice of λ/μ rules does not affect the definition of the underlying vector space of $\mathcal{E}(X)$, but it does play an important role in the definition of the bilinear map $\langle -, - \rangle$.

In general, the vector space $\mathcal{E}(X)$ endowed with the bilinear map $\langle -, - \rangle$ is not an object of $\mathcal{V}$. It may happen that $\langle -, - \rangle$ does not even define a bilinear operation on $\mathcal{E}(X)$, i.e. there exist $(f * -, - * f),(g * -, - * g) \in \mathcal{E}(X)$ such that

$$\langle (f * -, - * f), (g * -, - * g) \rangle \notin \mathcal{E}(X)$$

or that $(\mathcal{E}(X), \langle -, - \rangle)$ is a non-associative algebra which does not satisfy some identity of $\mathcal{V}$.

Example 6.5. If $\mathcal{V} = \mathbf{Assoc}$, then $\mathcal{E}(X) \cong \operatorname{Bim}(X)$ as vector spaces. Moreover, with the standard choice of λ/μ rules $\lambda_1 = \mu_8 = 1$ and $\lambda_i = \mu_j = 0$, for any $i \in \{2, \dots, 8\}$ and $j \in \{1, \dots, 7\}$, we get also an isomorphism of associative algebras.

Example 6.6. If $\mathcal{V} = \mathbf{Leib}$, then

$$\mathcal{E}(X) = \{(f * -, - * f) \in \operatorname{End}(X)^2 \mid f * [x, y] = -[f * x, y] + [f * y, x], \\ [x, y] * f = [x * f, y] + [x, y * f],\ [x, f * y] + [x, y * f] = 0\}$$

and $\operatorname{Bider}(X) \cong \mathcal{E}(X)$ as vector spaces, where an explicit isomorphism is given by

$$(d, D) \mapsto (-D, d), \quad \forall (d, D) \in \operatorname{Bider}(X).$$

Choosing the λ/μ rules as

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \\ [[y, z], x] = [[y, x], z] - [y, [x, z]],$$

we get the standard multiplication defined in $\operatorname{Bider}(X)$ as in [58], that defines a weak actor in **Leib** (see Theorem 5.20). On the other hand, choosing the λ/μ rules as

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \\ [[y, z], x] = [[y, x], z] + [y, [z, x]],$$

we get the non-associative algebra structure defined in [26, Definition 5.2]

$$\{(d,D),(d',D')\} = (d\circ d' - d'\circ d, D\circ D' + d'\circ D)$$

which, in general, does not define a Leibniz algebra structure on $\mathrm{Bider}(X)$.

Example 6.7. If $\mathcal{V} = \mathbf{Nil}_k(\mathbf{Assoc})$, with $k \geq 3$, then

$$\mathcal{E}(X) = \{(f*-,-*f) \in \mathrm{Bim}(X) \mid f*(x_1\cdots x_k) = (x_1\cdots x_k)*f = 0\}.$$

With the same choice of λ/μ rules as in Example 6.5, the bilinear map $\langle -,-\rangle$ becomes

$$\langle (f*-,-*f),(g*-,-*g)\rangle = (f*(g*-),(-*f)*g)$$

which makes $\mathcal{E}(X)$ an associative algebra, but not a k-step nilpotent algebra. For instance, let $X = \mathbb{F}$ be the abelian one-dimensional algebra, then

$$\mathcal{E}(X) = \mathrm{End}(X)\times \mathrm{End}(X)^{\mathrm{op}} \cong \mathbb{F}^2$$

which is not nilpotent. Indeed, every linear endomorphism of X is of the form $\varphi_\alpha\colon x \mapsto \alpha x$, for some $\alpha \in \mathbb{F}$ and

$$\langle (\varphi_\alpha,\varphi_\beta),(\varphi_{\alpha'},\varphi_{\beta'})\rangle = (\varphi_\alpha\circ\varphi_{\alpha'},\varphi_{\beta'}\circ\varphi_\beta) = (\varphi_{\alpha\alpha'},\varphi_{\beta'\beta}).$$

Example 6.8. If $\mathcal{V} = \mathbf{Nil}_2(\mathbf{Alg})$ is the variety of two-step nilpotent algebras (see [52], [53] and [54]), i.e. $\mathcal{V}$ is defined by the identities $x(yz) = 0 = (yz)x$, then

$$\mathcal{E}(X) = \{(f*-,-*f) \in \mathrm{End}(X)^2 \mid f*(xy) = (xy)*f = (f*x)y = x(y*f) = 0\}$$

and the bilinear map is given by

$$\langle (f*-,-*f),(g*-,-*g)\rangle = (0,0).$$

Thus $\mathcal{E}(X)$ is an abelian algebra.

Example 6.9. If $\mathcal{V} = \mathbf{Alt}$ is the variety of alternative algebras over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ (see Examples 3.46), then $\mathcal{E}(X)$ consists of the pairs $(f*-,-*f) \in \mathrm{End}(X)^2$ satisfying

$$f*(xy) = (x*f)y + (f*x)y - x(f*y),$$

$$(xy)*f = x(f*y) + x(y*f) - (x*f)y,$$

$$x(y*f) = (yx)*f + (xy)*f - y(x*f)$$

and

$$(f*x)y = f*(yx) + f*(xy) - (f*y)x$$

for any $x,y \in X$, and the bilinear map

$$\langle (f*-,-*f),(g*-,-*g)\rangle = (h*-,-*h)$$

is given by

$$h*x = -(f*x)*g + f*(g*x) + f*(x*g)$$

and

$$x*h = (x*f)*g + (f*x)*g - f*(x*g).$$

One can check that $\langle -, - \rangle$ does not define an algebra structure. Nevertheless, it is possible to find examples where $\mathcal{E}(X)$ is an alternative algebra.

For instance, if X is a *unitary* alternative algebra (i.e. there exists an element $e \in X$ such that $xe = ex = x$, for any $x \in X$), such as the algebra of octonions $\mathbb{O}$, then the elements of $\mathcal{E}(X)$ satisfy the following set of equations

$$\begin{aligned} f * x =&(x * f)e + (f * x)e - x(f * e), \\ x * f =&e(f * x) + e(x * f) - (e * f)x, \\ x * f =&x * f + x * f - x(e * f), \\ f * x =&f * x + f * x - (f * e)x, \end{aligned}$$

for any $x \in X$. Thus, if $\alpha := f * e$ and $\beta := e * f$, one has

$$f * x = \alpha x = \beta x, \qquad x * f = x\alpha = x\beta$$

and, for $x = e$, one obtains $\alpha = \beta$. In other words, an element of $\mathcal{E}(X)$ is uniquely determined by an element $\alpha = f * e = e * f$ of X, i.e.

$$\mathcal{E}(X) \cong \{(\alpha, \alpha) \mid \alpha \in X\} \cong X.$$

is an object of **Alt**.

Remark 6.10. The same result can be obtained for unitary algebras in the variety **Assoc**. In fact, let X be a unitary associative algebra and let $(f * -, - * f) \in \mathrm{Bim}(X)$. Thus

$$\begin{aligned} f * x &= f * (ex) = \alpha x, \\ x * f &= (xe) * f = x\beta \end{aligned}$$

and

$$x\alpha = (x * f)e = x\beta,$$

where $\alpha := f * e$ and $\beta := e * f$. For $x = e$, we obtain $\alpha = \beta$ and

$$\mathrm{Bim}(X) \cong \{(\alpha, \alpha) \mid \alpha \in X\} \cong X.$$

Since unitary algebras are perfect and with trivial center, from Remark 2.47 we have a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Assoc}}(-, X)$$

for any unitary associative algebra X.

The construction of $\mathcal{E}(X)$ gives rise to an alternative characterisation of split extensions in $\mathcal{V}$. In fact, a split extension of B by X in $\mathcal{V}$ is the same as a linear map

$$B \to \mathcal{E}(X) \colon b \mapsto (b * -, - * b),$$

such that $((bb') * -, - * (bb')) = \langle (b * -, - * b), (b' * -, - * b') \rangle$ and

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad i = 1, \dots, n,$$

where $\alpha_1, \dots, \alpha_k$ are as in Proposition 6.1.

We remark also that the bilinear map

$$\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2$$

defines a partial operation $\langle -,-\rangle \colon \Omega \to \mathcal{E}(X)$, where Ω is the preimage

$$\langle -,-\rangle^{-1}(\mathcal{E}(X))$$

of the inclusion $\mathcal{E}(X) \hookrightarrow \mathrm{End}(X)^2$.

Now we are ready to announce and prove our main result about the weak representability of actions of non-associative algebras.

Theorem 6.11. *[43] Let $\mathcal{V}$ be an action accessible, operadic variety of non-associative algebras over a field $\mathbb{F}$.*

(1) Let X be an object of $\mathcal{V}$. There exists a monomorphism of functors

$$\tau \colon \mathrm{SplExt}(-,X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-),\mathcal{E}(X)),$$

*where $\mathrm{SplExt}(-,X) = \mathrm{SplExt}_{\mathcal{V}}(-,X)$, $U \colon \mathcal{V} \to$ **PAlg** denotes the forgetful functor and, for every B of $\mathcal{V}$, τ_B is the injection which sends an element of $\mathrm{SplExt}(B,X)$ to the corresponding partial algebra homomorphism*

$$B \to \mathcal{E}(X) \colon b \mapsto (b * -, - * b).$$

(2) Let B, X be objects of $\mathcal{V}$. The homomorphism of partial algebras

$$B \to \mathcal{E}(X) \colon b \mapsto (b * -, - * b)$$

belongs to $\mathrm{Im}(\tau_B)$ if and only if $\Phi_{k,i}(\alpha_1,\ldots,\alpha_k) = 0$, as in Proposition 6.1.

(3) If $(\mathcal{E}(X), \langle -,-\rangle)$ is an object of $\mathcal{V}$, then $(\mathcal{E}(X),\tau)$ becomes a weak representation of $\mathrm{SplExt}(-,X)$.

(4) If $\mathcal{V}$ is a variety of commutative or anti-commutative algebras, then $\mathcal{E}(X)$ is isomorphic to the partial algebra

$$\{f \in \mathrm{End}(X) \mid \Phi_{k,i}(f,x_2,\ldots,x_k) = 0,\ \forall x_2,\ldots,x_k \in X\}$$

endowed with the bilinear partial operation $\langle f,g\rangle = \alpha(f\circ g) + \beta(g \circ f)$, where $\alpha,\beta \in \mathbb{F}$ are given by the λ/μ rules.

Proof.

(1) The collection $\{\tau_B\}_B$ gives rise to a natural transformation since, for every algebra homomorphism $f \colon B' \to B$, the diagram in **Set**

$$\begin{array}{ccc} \mathrm{SplExt}(B,X) & \xrightarrow{\tau_B} & \mathrm{Hom}(U(B),\mathcal{E}(X)) \\ {\scriptstyle \mathrm{SplExt}(U(f),X)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(f,\mathcal{E}(X))} \\ \mathrm{SplExt}(B',X) & \xrightarrow{\tau_{B'}} & \mathrm{Hom}(U(B'),\mathcal{E}(X)) \end{array}$$

where $\mathrm{Hom}(U(-),-) = \mathrm{Hom}_{\mathbf{PAlg}}(U(-),-)$, is commutative. Moreover, for every object B of $\mathcal{V}$, the map τ_B is an injection, since every element of $\mathrm{SplExt}(B,X)$ is uniquely determined by the corresponding derived action of B on X, i.e. by the pair of bilinear maps

$$l \colon B \times X \to X, \qquad r \colon X \times B \to X$$

defined as in Definition 3.47. Thus τ is a monomorphism of functors.

(2) Let B, X be objects of $\mathcal{V}$. A homomorphism of partial algebras $B \to \mathcal{E}(X)$ belongs to $\mathrm{Im}(\tau_B)$ if and only if it defines a split extension of B by X in $\mathcal{V}$. This is equivalent to saying that

$$\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0, \quad \forall i = 1, \ldots, n,$$

where $\alpha_1, \ldots, \alpha_k$ are as in Proposition 6.1.

(3) If $(\mathcal{E}(X), \langle -, - \rangle)$ is an object of $\mathcal{V}$, then we have a monomorphism of functors

$$\tau \colon \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathcal{V}}(-, \mathcal{E}(X)),$$

and $(\mathcal{E}(X), \tau)$ is a weak representation of $\mathrm{SplExt}(-, X)$.

(4) If $\mathcal{V}$ is a variety of commutative (resp. anti-commutative) algebras, then for every object X of $\mathcal{V}$, $\mathcal{E}(X)$ consists of pairs of the form $(f * -, - * f)$ with $x * f = f * x$ (resp. $x * f = -f * x$), for every $x \in X$. Thus, an explicit isomorphism

$$\{f \in \mathrm{End}(X) \mid \Phi_{k,i}(f, x_2, \ldots, x_k) = 0\} \to \mathcal{E}(X)$$

is given by $f \mapsto (f, f)$ (resp. $f \mapsto (f, -f)$).

□

Because of these results, we can give the following definitions.

Definition 6.12. [43] Let X be an object of an action accessible and operadic variety of non-associative algebras $\mathcal{V}$ with a choice of λ/μ rules. The partial algebra $\mathcal{E}(X)$ is called *external weak actor* of X. The pair $(\mathcal{E}(X), \tau)$ is called *external weak representation* of the functor $\mathrm{SplExt}(-, X)$. When τ is a natural isomorphism, we say that $\mathcal{E}(X)$ is an *external actor* of X.

Proposition 6.13. *Let $\mathcal{V}$ be an action accessible and operadic variety of non-associative algebras over $\mathbb{F}$ and let X be an object of $\mathcal{V}$. We have that, with the* obvious *choices of the λ/μ rules:*

(1) if $\mathcal{V} = \mathbf{AbAlg}$, then $\mathcal{E}(X) = 0$ is the actor of X;

(2) if $\mathcal{V} = \mathbf{CAssoc}$, then $\mathcal{E}(X) \cong \mathrm{M}(X)$ is an external actor of X. Moreover, if X is a perfect algebra or it has trivial center, such as when it is unitary, then $\mathcal{E}(X)$ is the actor of X;

(3) if $\mathcal{V} = \mathbf{JJord}$, then $\mathcal{E}(X)$ is isomorphic to the commutative partial algebra $\mathrm{ADer}(X)$ of anti-derivations of X and it is an external actor;

(4) if $\mathcal{V} = \mathbf{Lie}$, then $\mathcal{E}(X) \cong \mathrm{Der}(X)$ is the actor of X;

(5) if $\mathcal{V} = \mathbf{ACAAssoc}$, then $\mathcal{E}(X)$ is isomorphic to the associative partial algebra $\mathrm{AM}(X)$ of anti-multipliers of X and it is an external actor;

(6) if $\mathcal{V} = \mathbf{Nil}_2(\mathbf{Com})$ or $\mathcal{V} = \mathbf{Nil}_2(\mathbf{ACom})$, then $\mathcal{E}(X) \cong [X]_2$ is a weak actor of X;

(7) if $\mathcal{V} = \mathbf{Nil}_2(\mathbf{Alg})$, then $\mathcal{E}(X)$ is an abelian algebra and and it is a weak actor of X;

(8) if $\mathcal{V} = \mathbf{Alt}$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and X is unitary, then $\mathcal{E}(X) \cong X$ is an alternative algebra and we have a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Alt}}(-, X)$$

i.e. X is the actor of itself. In particular, the algebra of octonions $\mathbb{O}$ has representable actions.

Proof.

(1) Let X be an abelian algebra and let $(f * -, - * f) \in \mathcal{E}(X)$. Then

$$f * x = x * f = 0,$$

for every $x \in X$. Thus $\mathcal{E}(X) = 0$ and we have a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{AbAlg}}(-, 0)$$

(see Examples 3.46), i.e. $\mathcal{E}(X) = 0$ is the actor of X.

(2) It follows from Lemma 4.4. If X is perfect or it has trivial center, then we have a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{CAssoc}}(-, \mathrm{M}(X))$$

and $\mathcal{E}(X) \cong \mathrm{M}(X)$ is the actor of X. In the case that X is unitary, then using the same argument of Remark 6.10, one proves that $\mathcal{E}(X) \cong X$.

(3) It is a consequence of Theorem 4.9.

(4) It follows from Example 2.32.

(5) It is an immediate consequence of Theorem 4.12.

(6) It follows from Theorem 4.11 and Theorem 4.13.

(7) It is an immediate consequence of Example 6.8;

(8) It easily follows from Example 6.9. In this case, every algebra homomorphism $B \to X$ defines a split extension of B by X and we have a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Alt}}(-, X).$$

Finally, since $\mathbb{O}$ is a unitary alternative algebras, it has representable actions.

□

Remark 6.14. The construction of the vector space $\mathcal{E}(X)$ can be done also in a variety of non-associative algebras $\mathcal{V}$ which is not action accessible. However, it is not clear how to endow $\mathcal{E}(X)$ with a bilinear map $\langle -, - \rangle$ as in Definition 6.3. So we only have a monomorphism of functors

$$\tau\colon \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathbf{Vec}}(U(-), \mathcal{E}(X))$$

where **Vec** is the category of $\mathbb{F}$-vector spaces and $U\colon \mathcal{V} \to \mathbf{Vec}$ denotes the forgetful functor.

6.2 Relations between the universal strict general actor and the external weak actor

As described in Section 3.1.2, for every Orzech category of interest $\mathcal{C}$ and for every object X of $\mathcal{C}$, it is possible to define a monomorphism of functors

$$\mu\colon \mathrm{SplExt}(-,X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}'}(V(-),\mathrm{USGA}(X)),$$

where $\mathrm{SplExt}(-,X) = \mathrm{SplExt}_{\mathcal{C}}(-,X)$, $\mathcal{C}'$ is a category which contains $\mathcal{C}$ as a full subcategory, $V\colon \mathcal{C} \to \mathcal{C}'$ denotes the forgetful functor and $\mathrm{USGA}(X)$ is an object of $\mathcal{C}'$ called the *universal strict general actor* of X [26]. We further recall that $\mathrm{USGA}(X)$ is unique up to isomorphism, once the presentation of the Orzech category of interest $\mathcal{C}$ is fixed (see Remark 3.26).

We now claim the following.

Proposition 6.15. *Let $\mathcal{V}$ be an action accessible and operadic variety of non-associative algebras over a field $\mathbb{F}$ with a choice of constants $\lambda_1,\dots,\lambda_8$, $\mu_1,\dots,\mu_8 \in \mathbb{F}$ which determine the λ/μ rules. For any object X of $\mathcal{V}$, there exist two monomorphism of functors*

$$\mu\colon \mathrm{SplExt}(-,X) \rightarrowtail \mathrm{Hom}_{\mathbf{Alg}}(V(-),\mathrm{USGA}(X))$$

and

$$\tau\colon \mathrm{SplExt}(-,X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-),\mathcal{E}(X)),$$

*where $V\colon \mathcal{V} \to$ **Alg** and $U\colon \mathcal{V} \to$ **PAlg** are the forgetful functors and $\mathcal{E}(X)$ is the external weak actor of X. Moreover, if the bilinear partial operation $\langle -,- \rangle$ is well defined on $\mathcal{E}(X) \times \mathcal{E}(X)$, then $\mathcal{E}(X) = \mathrm{USGA}(X)$ and $\tau = \mu$.*

Proof. We recall from Theorem 3.51 that a variety of non-associative algebras is an Orzech category of interest if and only if it is action accessible. In this case, a presentation of $\mathcal{V}$ is given by a choice of constants $\lambda_1,\dots,\lambda_8$, $\mu_1,\dots,\mu_8 \in \mathbb{F}$ which determine the λ/μ rules and, as observed in Example 3.6, it turns out that $\mathcal{V}' =$ **Alg**. Thus, by Proposition 3.29, we have a monomorphism of functors

$$\mu\colon \mathrm{SplExt}(-,X) \rightarrowtail \mathrm{Hom}_{\mathbf{Alg}}(V(-),\mathrm{USGA}(X))$$

and, by Theorem 6.11, another monomorphism of functors

$$\tau\colon \mathrm{SplExt}(-,X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-),\mathcal{E}(X)).$$

When the bilinear partial operation $\langle -,- \rangle$ is well defined on $\mathcal{E}(X) \times \mathcal{E}(X)$, i.e. when $\mathcal{E}(X)$ is a total algebra, then $\mathrm{USGA}(X) = \mathcal{E}(X)$ and $\tau = \mu$. This happens, for instance, in the varieties **Assoc**, **Leib** and **Lie**. □

However, it is often more convenient to work with the external weak actor $\mathcal{E}(X)$, since it is easier to construct than the universal strict general actor $\mathrm{USGA}(X)$ (see [26, Section 4]).

Chapter 7

Poisson algebras

In this chapter we aim to study the representability of actions of the variety of Poisson algebras by determining explicitly an *external weak actor*, which in this case is also a *universal strict general actor* (see [29, Section 4]). This traces a possible direction for the construction of the external weak actor for any action accessible and operadic variety of algebras with two non-necessary associative bilinear operations.

Furthermore we prove that the full subcategory of commutative Poisson algebras is not weakly action representable and we conclude with an open problem.

7.1 Definitions and main properties

Let $\mathbb{F}$ be a field with $\operatorname{char}(\mathbb{F}) \neq 2$. We denote by $\mathbf{Alg_2}$ the variety of algebras with two non-necessary associative bilinear operations over $\mathbb{F}$, i.e. the category whose objects are $\mathbb{F}$-vector spaces endowed with two bilinear maps and the morphisms are the $\mathbb{F}$-linear maps which preserve both of them.

Definition 7.1. [23] A *Poisson algebra* over $\mathbb{F}$ is a vector space P over $\mathbb{F}$ endowed with two bilinear maps

$$\cdot\colon P \times P \to P$$

$$[-,-]\colon P \times P \to P$$

such that $(P, \cdot)$ is an associative algebra, $(P, [-,-])$ is a Lie algebra and the *Poisson identity* holds:

$$[p, qt] = [p, q]t + q[p, t], \quad \forall p, q, t \in P$$

i.e. the adjoint map $\mathrm{ad}_p := [p, -]\colon P \to P$ is a derivation of the associative algebra $(P, \cdot)$. A Poisson algebra P is said to be commutative if $(P, \cdot)$ is a *commutative* associative algebra.

We denote by **Pois** the category of Poisson algebras and by **CPois** the full subcategory of commutative Poisson algebras.

Remark 7.2. Both **Pois** and **CPois** are Orzech categories of interest. Ideed, using the notation of Section 3.1, they are both categories of abelian groups with operations, where $\Omega_1' = \emptyset$, $\Omega_2' = \{\cdot, \cdot^{\mathrm{op}}, [-,-], [-,-]^{\mathrm{op}}\}$ and Axiom 2

$$(x_1 * x_2) \bar{*} x_3 = W(x_1; x_2, x_3; *, \bar{*}).$$

is given by associativity, when $*, \bar{*} \in \{\cdot, \cdot^{\mathrm{op}}\}$, by the Jacobi identity, when $*, \bar{*} \in \{[-,-], [-,-]^{\mathrm{op}}\}$, and by the Poisson identity, if $* \in \{\cdot, \cdot^{\mathrm{op}}\}$ and $\bar{*} \in \{[-,-], [-,-]^{\mathrm{op}}\}$, or vice-versa.

We can thus describe internal actions / split extension in terms of derived actions.

Definition 7.3. [29, 34, 66] Let

$$0 \longrightarrow V \xrightarrow{\ i\ } \hat{P} \underset{s}{\overset{\pi}{\rightleftarrows}} P \longrightarrow 0 \tag{7.1.1}$$

be a split extension of Poisson algebras. The triple of bilinear maps

$$l\colon P \times V \to V, \qquad r\colon V \times P \to V, \qquad [\![-,-]\!]\colon P \times V \to V$$

defined by

$$p * y = s(p) \cdot i(y), \quad x * q = i(x) \cdot s(q), \quad [\![p,y]\!] = [s(p), i(x)]$$

where $p * - = l(p,-)$ and $- * q = r(-,q)$, is called the *derived action* of P on V associated with (7.1.1).

Given a triple of bilinear maps

$$l\colon P \times V \to V, \qquad r\colon V \times P \to V, \qquad [\![-,-]\!]\colon P \times V \to V,$$

one can define two bilinear operations on $P \oplus V \cong P \times V$

$$(p,x) \diamond (q,y) = (pq, xy + p * y + x * q)$$

and

$$\{(p,x),(q,y)\} = ([p,q],[x,y] + [\![p,y]\!] - [\![q,x]\!]),$$

for every $(p,x),(q,y) \in P \oplus V$, and this defines a Poisson algebra structure on the vector space $P \oplus V$ if and only if the triple $(l,r,[\![-,-]\!])$ is a derived action of P on V.

This is equivalent to a set of conditions on $(l,r,[\![-,-]\!])$, as explained in the following proposition (again, see [34, Proposition 1.1] and [66, Theorem 2.4]).

Proposition 7.4. *[29, 34, 66] $(P \oplus V, \diamond, \{-,-\})$ is a Poisson algebra if and only if*

(P1) $(P \oplus V, \diamond)$ is an associative algebra, i.e. the following equalities hold

- $p * (xy) = (p * x)y$;
- $(xy) * p = x(y * p)$;
- $x(p * y) = (x * p)y$;
- $(p * x) * q = p * (x * q)$;
- $(pq) * x = p * (q * x)$;
- $x * (pq) = (x * p) * q$;

(P2) $(P \oplus V, \{-,-\})$ is a Lie algebra, i.e.

- $[\![p,[x,y]]\!] = [[\![p,x]\!],y] + [x,[\![p,y]\!]]$;
- $[\![[p,q],x]\!] = [\![p,[\![q,x]\!]]\!] - [\![q,[\![p,x]\!]]\!]$;

(P3) $[\![pq,x]\!] = p * [\![q,x]\!] + [\![p,x]\!] * q$;

(P4) $[p,q] * x = p * [\![q,x]\!] - [\![q,p * x]\!]$;

(P5) $x * [p,q] = [\![q,x]\!] * p - [\![q,x * p]\!]$;

(P6) $p * [x,y] = [p * x,y] - [\![p,y]\!]x$;

(P7) $[x, y] * p = [x * p, y] - x[\![p, y]\!]$;

(P8) $[\![p, xy]\!] = [\![p, x]\!]y + x[\![p, y]\!]$;

for every $p, q \in P$ and for every $x, y \in V$. The resulting Poisson algebra is the semi-direct product *of P and V (with respect to the derived action $(l, r, [\![-,-]\!])$) and it is denoted by $P \ltimes V$.*

Proof. We already know that $(P \oplus V, \diamond)$ is an associative algebra (resp. $(P \oplus V, \{-,-\})$ is a Lie algebra) if and only if the equations in (P1) (resp. (P2)) are satisfied. We only need to check that the Poisson identity

$$\{\alpha_1, \alpha_2 \diamond \alpha_3\} = \{\alpha_1, \alpha_2\} \diamond \alpha_3 + \alpha_2 \diamond \{\alpha_1, \alpha_3\}$$

holds in $(P \oplus V, \diamond, \{-,-\})$. Following a similar strategy of Remark 3.50, it is sufficient to check the Poisson identity when at least one of the α_i is of the form $(p, 0)$, with $p \in P$, and the others are of the form $(0, x)$, with $x \in V$. Thus $(P \oplus V, \diamond, \{-,-\})$ satisfies the Poisson identity if and only if the following equations hold, for any $p, q \in P$ and $x, y \in V$:

- $\{(0, x), (p, 0) \diamond (q, 0)\} = \{(0, x), (p, 0)\} \diamond (q, 0) + (p, 0) \diamond \{(0, x), (q, 0)\}$, which is equivalent to (P3);
- $\{(q, 0), (p, 0) \diamond (0, x)\} = \{(q, 0), (p, 0)\} \diamond (0, x) + (p, 0) \diamond \{(q, 0), (0, x)\}$, which is equivalent to (P4);
- $\{(q, 0), (0, x) \diamond (p, 0)\} = \{(q, 0), (0, x)\} \diamond (p, 0) + (0, x) \diamond \{(q, 0), (p, 0)\}$, which in turn is equivalent to (P5);
- $\{(0, y), (p, 0) \diamond (0, x)\} = \{(0, y), (p, 0)\} \diamond (0, x) + (p, 0) \diamond \{(0, y), (0, x)\}$, which is equivalent to (P6);
- $\{(0, y), (0, x) \diamond (p, 0)\} = \{(0, y), (0, x)\} \diamond (p, 0) + (0, x) \diamond \{(0, y), (p, 0)\}$, which is equivalent to (P7);
- $\{(p, 0), (0, x) \diamond (0, y)\} = \{(p, 0), (0, x)\} \diamond (0, y) + (0, x) \diamond \{(p, 0)(0, y)\}$, which in turn is equivalent to (P8).

□

We observe that an analogous study was done in [3] in the case of non-split extensions of Poisson algebras.

Remark 7.5. For any split extension (7.1.1), we have an isomorphism of split extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \xrightarrow{i_2} & P \ltimes V & \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_V} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_P} & & \\
0 & \longrightarrow & V & \xrightarrow{i} & \hat{P} & \underset{s}{\overset{\pi}{\rightleftarrows}} & P & \longrightarrow & 0
\end{array}$$

where i_1, i_2, π_1 are the canonical injections and projection and $\theta\colon P \ltimes V \to \hat{P}$ is defined by $\theta(p, x) = s(p) + i(x)$, for every $(p, x) \in P \oplus V$.

7.2 Representability of actions of Poisson algebras

The category **Pois** has two obvious forgetful functors to the categories **Assoc** and **Lie**. Now, the category of Lie algebras is action representable: any split extension of a Lie algebra P by another Lie algebra V corresponds to a Lie algebra homomorphism $\varphi\colon P \to \mathrm{Der}(V)$. On the other hand, we know that **Assoc** is a weakly action representable category and a split extension of an associative algebra P by another associative algebra V corresponds to an associative algebra homomorphism $\varphi\colon P \to \mathrm{Bim}(V)$. Notice that $\mathrm{Der}(V)$ is an actor, while $\mathrm{Bim}(V)$ is only a weak actor (see [47]), in fact they are both universal strict general actors in the sense of [26]. It is not clear whether the category **Pois** is weakly action representable, therefore we start by describing an external weak actor, which in this case turns out to be also a universal strict general actor. As explained in Section 3.1, in general the universal strict general actor lies in a larger category $\mathcal{C}'$, which in this case is the category $\mathbf{Alg_2}$ of algebras over $\mathbb{F}$ with two not necessarily associative bilinear operations.

Given a Poisson algebra V, we look for a suitable subspace

$$\mathcal{E}(V) \leq \mathrm{Bim}(V) \times \mathrm{Der}(V)$$

and this must be endowed with two bilinear operations

$$\cdot_{\mathcal{E}(V)}, [-,-]_{\mathcal{E}(V)}\colon \mathcal{E}(V) \times \mathcal{E}(V) \to \mathcal{E}(V)$$

such that we can associate with every split extension of P by V in **Pois** a morphism in $\mathbf{Alg_2}$

$$\phi\colon P \to \mathcal{E}(V)\colon p \mapsto (p * -, - * p, [\![p,-]\!]).$$

Thus

$$\phi(pq) = \phi(p) \cdot_{\mathcal{E}(V)} \phi(q)$$

and

$$\phi([p,q]) = [\phi(p), \phi(q)]_{\mathcal{E}(V)}.$$

In other words, using Proposition 7.4, the operations in $\mathcal{E}(V)$ must satisfy the following conditions

- $(p * -, - * p, [\![p,-]\!]) \cdot_{\mathcal{E}(V)} (q * -, - * q, [\![q,-]\!]) =$
  $= ((pq) * -, - * (pq), p * [\![q,-]\!] + [\![p,-]\!] * q)$
- $[(p * -, - * p, [\![p,-]\!]), (q * -, - * q, [\![q,-]\!])]_{\mathcal{E}(V)} =$
  $= (p * [\![q,-]\!] - [\![q, p * -]\!], [\![q,-]\!] * p - [\![q, - * p]\!], [\![p,[\![q,-]\!]]\!] - [\![q,[\![p,-]\!]]\!])$

for every $p, q \in P$.

Definition 7.6. [29] Let V be a Poisson algebra. We define $\mathcal{E}(V)$ as the subspace of all triples (f, F, d) of $\mathrm{Bim}(V) \times \mathrm{Der}(V)$ satisfying the following set of equations:

(V1) $f([x,y]) = [f(x), y] - d(y)x$;

(V2) $F([x,y]) = [F(x), y] - xd(y)$;

(V3) $d(xy) = d(x)y + xd(y)$,

for every $x, y \in V$.

Remark 7.7. The subspace $\mathcal{E}(V)$ is not empty, since

$$(L_x, R_x, \mathrm{ad}_x) \in \mathcal{E}(V)$$

for every $x \in V$, where L_x and R_x denote respectively the left and the right mulitplication of the associative algebra $(V, \cdot)$ and $\mathrm{ad}_x = [x, -]$. These triples are called *inner multipliers of* V.

Now we are ready to enunciate and prove the following.

Theorem 7.8. *[29] Let V be a Poisson algebra.*

(1) The vector space $\mathcal{E}(V)$ with the bilinear operations

$$(f, F, d) \cdot_{\mathcal{E}(V)} (f', F', d') = (f \circ f', F' \circ F, f \circ d' + F' \circ d)$$

$$[(f, F, d), (f', F', d')]_{\mathcal{E}(V)} = (f \circ d' - d' \circ f, F \circ d' - d' \circ F, d \circ d' - d' \circ d)$$

is an object of $\mathbf{Alg_2}$*;*

(2) The set $\mathrm{Inn}(V)$ *of all inner multipliers of V is a subalgebra of* $\mathcal{E}(V)$ *and it is a Poisson algebra itself;*

(3) For every object $(P, \cdot, [-, -])$ *in* **Pois***, the set of isomorphism classes of split extension of P by V is in bijection with the set of homomorphisms*

$$\phi = (\phi_1, \phi_2, \phi_3) \colon P \to \mathcal{E}(V)$$

in $\mathbf{Alg_2}$*, such that* $(\phi_1, \phi_2) \colon P \to \mathrm{Bim}(V)$ *is an acting morphism in the category* **Assoc***.*

(4) There exists a monomorphism of functors

$$\tau \colon \mathrm{SplExt}(-, V) \rightarrowtail \mathrm{Hom}_{\mathbf{Alg_2}}(U(-), \mathcal{E}(V)),$$

where $\mathrm{SplExt}(-, V) = \mathrm{SplExt}_{\mathbf{Pois}}(-, V)$ *and* $U \colon \mathbf{Pois} \to \mathbf{Alg_2}$ *denotes the forgetful functor, such that an arrow* $(\phi \colon P \to \mathcal{E}(V)) \in \mathrm{Im}(\tau_P)$ *if and only if* (ϕ_1, ϕ_2) *is an acting morphism in* **Assoc***.*

(5) If $(\mathcal{E}(V), \cdot_{\mathcal{E}(V)}, [-, -]_{\mathcal{E}(V)})$ *is a Poisson algebra, then the pair* $(\mathcal{E}(V), \tau)$ *becomes a weak representation for the functor* $\mathrm{SplExt}(-, V)$*.*

Proof.

(1) In order to show that $\mathcal{E}(V)$ is an object of $\mathbf{Alg_2}$, we have to prove that the bilinear operations are well defined. We observe that

$$(f \circ d' - d' \circ f, F \circ d' - d' \circ F) \in \mathrm{Bim}(V)$$

and

$$f \circ d' + F' \circ d \in \mathrm{Der}(V),$$

for every $(f, F, d), (f', F', d') \in \mathcal{E}(V)$. This follows from equations (V1)-(V2)-(V3), since

$$(f \circ d' - d' \circ f)(xy) = (f \circ d' - d' \circ f)(x)y,$$

$$(F \circ d' - d' \circ F)(xy) = x(F \circ d' - d' \circ F)(y),$$

$$x(f \circ d' - d' \circ f)(y) = (F \circ d' - d' \circ F)(x)y$$

and

$$(f \circ d' + F' \circ d)([x, y]) =$$

$$= [(f \circ d' + F' \circ d)(x), y] + [x, (f \circ d' + F' \circ d)(y)],$$

for every $x, y \in V$. Moreover the resulting triples

$$(f \circ f', F' \circ F, f \circ d' + F' \circ d)$$

$$(f \circ d' - d' \circ f, F \circ d' - d' \circ F, d \circ d' - d' \circ d)$$

belong to $\mathcal{E}(V)$, i.e. they satisfy equations (V1)-(V2)-(V3). Here we show this statement only for the second triple, since for the first one the computations are similar. We have that

$$\begin{aligned}
&(f \circ d' - d' \circ f)([x, y]) = \\
=&f([d'(x), y] + [x, d'(y)]) - d'([f(x), y] - d(y)x) = \\
=&[f(d'(x)), y] - d(y)d'(x) + [f(x), d'(y)] - d(d'(y))x + \\
-&[d'(f(x)), y] - [f(x), d'(y)] + d'(d(y))x + d(y)d'(x) = \\
=&[(F \circ d' - d' \circ F)(x), y] - (d \circ d' - d' \circ d)(y)x.
\end{aligned}$$

In the same way, one obtains

$$\begin{aligned}
&(F \circ d' - d' \circ F)([x, y]) = \\
=&F([d'(x), y] + [x, d'(y)]) - d'([F(x), y] - xd(y)) = \\
=&[F(d'(x)), y] - d'(x)d(y) + [F(x), d'(y)] - xd(d'(y)) + \\
-&[d'(F(x)), y] - [F(x), d'(y)] + d'(x)d(y) + xd'(d(y)) = \\
=&[(F \circ d' - d' \circ F)(x), y] - x(d \circ d' - d' \circ d)(y).
\end{aligned}$$

Finally

$$\begin{aligned}
&(d \circ d' - d' \circ d)(xy) = \\
=&d(d'(x)y + xd'(y)) - d'(d(x)y + xd(y)) = \\
=&d(d'(x))y + xd(d'(y)) - d'(d(x))y - xd'(d(y)) = \\
=&(d \circ d' - d' \circ d)(x)y + x(d \circ d' - d' \circ d)(y).
\end{aligned}$$

Thus $\mathcal{E}(V)$ is an object of $\mathbf{Alg_2}$.

(2) The subspace $\mathrm{Inn}(V)$ is precisely the image of the homomorphism

$$\mathrm{Inn}\colon V \to \mathcal{E}(V)\colon x \mapsto (L_x, R_x, \mathrm{ad}_x).$$

(3) We associate with any split extension.

$$0 \longrightarrow V \xrightarrow{\ i\ } \hat{P} \underset{s}{\overset{\pi}{\rightleftarrows}} P \longrightarrow 0$$

in the category **Pois** the homomorphism

$$P \to \mathcal{E}(V)\colon p \to (p * -, - * p, [\![p, -]\!])$$

where the bimultiplier $(p * -, - * p)$ and the derivation $[\![p, -]\!]$ are as in Definition 7.3. From the charaterization of the acting morphisms in the category **Assoc** (see Proposition 2.46), we have that

$$p * (x * q) = (p * x) * q,$$

for every $p, q \in P$ and $x \in V$. Conversely, given a Poisson algebra P and a homomorphism $\phi = (\phi_1, \phi_2, \phi_3) \in \mathrm{Hom}_{\mathbf{NAlg}_2}(P, \mathcal{E}(V))$ defined by

$$\phi(p) = (p *_\phi -, - *_\phi p, [\![p, -]\!]_\phi), \quad \forall p \in P,$$

such that $(\phi_1, \phi_2) \colon P \to \mathrm{Bim}(V)$ is an acting morphism in **Assoc**, we can associate with ϕ the split extension of Poisson algebras

$$0 \longrightarrow V \xrightarrow{\ i\ } (P \oplus V, \diamond_{(\phi_1,\phi_2)}, \{-,-\}_{\phi_3}) \underset{s}{\overset{\pi}{\rightleftarrows}} P \longrightarrow 0$$

where

$$(p, x) \diamond_{(\phi_1,\phi_2)} (q, y) = (pq, xy + p *_\phi y + x *_\phi q)$$

and

$$\{(p, x), (q, y)\}_{\phi_3} = ([p, q], [x, y] + [\![p, y]\!]_\phi - [\![q, x]\!]_\phi),$$

for every $(p, x), (q, y) \in P \oplus V$. One can check that these bilinear operations define a Poisson algebra structure on $P \oplus V$.

(4) We define

$$\tau \colon \mathrm{SplExt}(-, V) \rightarrowtail \mathrm{Hom}_{\mathbf{Alg}_2}(U(-), \mathcal{E}(V))$$

in the following way: for every object P in **Pois**, τ_P associates with any split extensions of P by V the homomorphism

$$P \to \mathcal{E}(V)$$

defined as in (3). By the description of split extensions in Definition 7.3, each component τ_P is injective since every homomorphism which belongs to $\mathrm{Im}(\tau_P)$ determines a unique split extension of P by V. Moreover, the family of injections

$$\tau_P \colon \mathrm{SplExt}(P, V) \rightarrowtail \mathrm{Hom}_{\mathbf{Alg}_2}(U(P), \mathcal{E}(V))$$

is natural in P and by (iii), an arrow $\phi = (\phi_1, \phi_2, \phi_3) \in \mathrm{Hom}_{\mathbf{Alg}_2}(U(P), \mathcal{E}(V))$ belongs to $\mathrm{Im}(\tau_P)$ if and only if $(\phi_1, \phi_2) \in \mathrm{Hom}_{\mathbf{Assoc}}(\hat{U}(P), \mathrm{Bim}(V))$ is an acting morphism, where $\hat{U} \colon$ **Pois** $\to$ **Assoc** denotes the forgetful functor.

(5) The last statement follows from Proposition 3.29, since the bilinear operations of $\mathcal{E}(V)$ are well defined and $\mathcal{E}(V) = \mathrm{USGA}(V)$.

□

The following example shows that $(\mathcal{E}(V), \cdot_{\mathcal{E}(V)}, [-,-]_{\mathcal{E}(V)})$ is not in general a Poisson algebra.

Example 7.9. Let $V = \mathbb{F}^2$ be the the abelian two-dimensional algebra (i.e. $xy = [x, y] = 0$, for every $x, y \in V$). It turns out that

$$\mathcal{E}(V) = \mathrm{End}(V)^3 \cong \mathrm{M}_2(\mathbb{F})^3,$$

as vector spaces, since every linear endomorphism of V is represented by a 2×2 matrix with respect to a fixed basis. Then the bilinear operations of $\mathcal{E}(V)$ can be represented as

$$(A, B, C) \cdot_{\mathcal{E}(V)} (A', B', C') = (AA', B'B, AC' + B'C),$$

$$[(A,B,C),(A',B',C')]_{\mathcal{E}(V)} = (AC' - C'A, BC' - C'B, CC' - C'C),$$

for every $(A,B,C),(A',B',C') \in \mathrm{M}_2(\mathbb{F})^3$ and one can check that $\mathcal{E}(V)$ is not a Poisson algebra since, for instance, the bracket $[-,-]_{\mathcal{E}(V)}$ is not skew-symmetric.

By Theorem 3.9 of [26], we can deduce that the category **Pois** is not action representable. Indeed, since for a Poisson algebra V, $\mathcal{E}(V) = \mathrm{USGA}(V)$ is not in general a Poisson algebra, then V does not admit an actor.

The following remark shows that there are special cases where τ becomes a natural isomorphism.

Remark 7.10. Let V be a Poisson algebra such that the annihilator

$$\mathrm{Ann}(V) = \{x \in V \mid xy = yx = 0,\ \forall y \in V\}$$

of the associative algebra $(V,\cdot)$ is trivial or $(V^2,\cdot) = (V,\cdot)$. From Remark 2.47 we have

$$f \circ F' = F' \circ f \tag{7.2.1}$$

for any $(f,F),(f',F') \in \mathrm{Bim}(V)$. Thus, if P is another Poisson algebra, any homomoprhism

$$\phi \colon P \to \mathcal{E}(V)$$

belongs to $\mathrm{Im}(\tau_P)$ and we have a natural isomorphism

$$\mathrm{SplExt}(-,V) \cong \mathrm{Hom}_{\mathbf{Alg_2}}(U(-),\mathcal{E}(V)).$$

Notice that the conditions $\mathrm{Ann}(V) = 0$ and $V^2 = V$ are not necessary to obtain Equation (7.2.1). For instance, if $V = \mathbb{F}$ is the abelian one-dimensional algebra, then $\mathrm{Ann}(V) = V$, $V^2 = 0$, $\mathcal{E}(V) \cong \mathbb{F}^3$ as vector spaces (every linear endomorphism of V is of the form $\varphi_a \colon x \mapsto ax$, with $a \in \mathbb{F}$) and every left multiplier of V commutes with every right multiplier. Moreover it turns out that the multiplication

$$(\varphi_a,\varphi_b,\varphi_c)\cdot_{\mathcal{E}(V)}(\varphi_{a'},\varphi_{b'},\varphi_{c'}) = (\varphi_{aa'},\varphi_{b'b},\varphi_{ac'+b'c})$$

is associative and

$$[(\varphi_a,\varphi_b,\varphi_c),(\varphi_{a'},\varphi_{b'},\varphi_{c'})]_{\mathcal{E}(V)} = (0,0,0).$$

Thus $\mathcal{E}(V)$ is a Poisson algebra and

$$\mathrm{SplExt}(-,V) \cong \mathrm{Hom}_{\mathbf{Pois}}(-,\mathcal{E}(V)),$$

i.e. $\mathcal{E}(V)$ is the actor of V.

7.3 Commutative Poisson algebras

If V is a commutative Poisson algebra, an analogous description of derived actions and split extensions can be done as in Definition 7.3 by asking that $p * x = x * p$.

Then we can define $\mathcal{E}(V)$ as the algebra of all pairs $(f,d) \in \mathrm{M}(V) \times \mathrm{Der}(V)$, where $\mathrm{M}(V)$ is the associative algebra of multipliers of $(V,\cdot)$, such that

(V1) $f([x,y]) = [f(x),y] - xd(y)$;

(V2) $d(xy) = d(x)y + xd(y)$;

endowed with the two bilinear operations

$$(f,d)\cdot_{\mathcal{E}(V)}(f',d') = (f\circ f', f\circ d' + f'\circ d),$$

$$[(f,d),(f',d')]_{\mathcal{E}(V)} = (f\circ d' - d'\circ f, d\circ d' - d'\circ d),$$

for every $(f,d),(f',d')\in\mathcal{E}(V)$.

Using the notation of Theorem 7.8, one can associate, with any split extension of P by V in **CPois**, a morphism

$$\phi\colon P\to\mathcal{E}(V)\colon p\mapsto (p*-, [\![p,-]\!])$$

in $\mathbf{Alg}_2$. Conversely, if P and V are commutative Poisson algebras, every homomorphism $\phi\colon P\to\mathcal{E}(V)$ in $\mathbf{Alg}_2$ defines a commutative Poisson algebra split extension. Indeed, by (iii) of Theorem 7.8, such $\phi\in\operatorname{Im}(\tau_P)$ if and only if $p\mapsto p*-$ defines an action in the category **CAssoc** of commutative associative algebra over $\mathbb{F}$, and moreover $\mathrm{SplExt}_{\mathbf{CAssoc}}(-,V)\cong\mathrm{Hom}_{\mathbf{Assoc}}(\overline{U}(-),\mathrm{M}(V))$ (see Lemma 4.4), where $\overline{U}\colon\mathbf{CAssoc}\to\mathbf{Assoc}$ denotes the forgetful functor. Thus there exists a natural isomorphism

$$\mathrm{SplExt}(-,V)\cong\mathrm{Hom}_{\mathbf{Alg}_2}(\tilde{U}(-),\mathcal{E}(V)),$$

where $\mathrm{SplExt}(-,V)=\mathrm{SplExt}_{\mathbf{CPois}}(-,V)$ and $\tilde{U}\colon\mathbf{CPois}\to\mathbf{Alg}_2$ is the forgetful functor.

Finally, we observe that also in this case $\mathcal{E}(V)$ needs not be an object of **CPois**. For instance, if $V=\mathbb{F}^2$ is the abelian two-dimensional algebra, then

$$\mathcal{E}(V)=\mathrm{M}(V)\times\mathrm{Der}(V)=\mathrm{End}(V)^2$$

as a vector space, and it is easy to check that the bilinear operation

$$(f,d)\cdot_{\mathcal{E}(V)}(f',d')=(f\circ f', f\circ d' + f'\circ d)$$

is not commutative.

Chapter 8

Conclusions and future directions

In conclusion, we studied the representability of actions in the context of varieties of non-associative algebras over a field.

We gave a complete characterization of action accessible, operadic quadratic varieties of commutative and anti-commutative algebras and we studied the representability of actions for each of them. Moreover, we proved the variety $\mathbf{Nil}_2(\mathbf{Com})$ of two-step nilpotent commutative algebras and the variety $\mathbf{Nil}_2(\mathbf{ACom})$ of two-step nilpotent anti-commutative algebras are weakly action representable with a weak actor of an object X being in both cases the abelian algebra $[X]_2$.

Then, for an action accessible and operadic variety of non-associative algebras $\mathcal{V}$ and an object X of it, we proved that it is always possible to construct a partial algebra $\mathcal{E}(X)$, called *external weak actor* of X, together with a monomorphism of functors

$$\tau\colon \operatorname{SplExt}(-, X) \rightarrowtail \operatorname{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}(X)),$$

where $\mathbf{PAlg}$ is the category of partial algebras over $\mathbb{F}$ and $U\colon \mathcal{V} \to \mathbf{PAlg}$ denotes the forgetful functor. The pair $(\mathcal{E}(X), \tau)$ is called *external weak representation* of the functor $\operatorname{Act}(-, X)$. Moreover, for any other object B in $\mathcal{V}$, we provided a complete description of the morphisms

$$(B \to \mathcal{E}(X)) \in \operatorname{Im}(\tau_B),$$

i.e. of the homomorphisms of partial algebras which identify the actions of B on X in $\mathcal{V}$.

We described this construction in detail in the case of Leibniz algebras, where $\mathcal{E}(X) \cong \operatorname{Bider}(X)$ is the Leibniz algebra of *biderivations* of X and we proved that the category **Leib** is weakly action representable. Furthermore, we provided the complete classifications of the Leibniz algebras of biderivations of low-dimensional Leibniz algebras over a field $\mathbb{F}$, with $\operatorname{char}(\mathbb{F}) \neq 2$, introducing an algorithm for finding biderivations of a Leibniz algebra as pairs of matrices with respect to a fixed basis.

Finally, we studied the representability of actions of the categories **Pois** and **CPois** of (commutative) Poisson algebras by determining explicitly an *external weak actor*, which is also a *universal strict general actor*.

We want to conclude our work by presenting some potential future directions for investigating the representability of actions in the context of non-associative algebras.

Converse of the implication "*weakly action representable category* $\Rightarrow$ *action accessible category*"

We studied the representability of actions of a general operadic variety of non-associative algebras over a field but we were not able to find an example of an action accessible variety which is not weakly action representable. Does the converse of the implication

weakly action representable category $\Rightarrow$ *action accessible category*

hold in this context?

Categorical properties of PAlg

It is easy to show that the category **PAlg** of *partial algebras* over a field is pointed with the zero object being the zero algebra. One of the next goals is to investigate the algebraic and categorical properties and to describe internal actions in the category **PAlg**.

Representability of actions of $\mathbf{CAssoc} \cap \mathbf{JJord}$ and $\mathbf{Lie} \cap \mathbf{ACAAssoc}$ in characteristic 3

We know that the variety $\mathbf{Nil}_2(\mathbf{Com})$ (resp. $\mathbf{Nil}_2(\mathbf{ACom})$) coincides with the intersection of **CAssoc** and **JJord** (resp. **Lie** and **ACAAssoc**) if $\mathrm{char}(\mathbb{F}) \neq 3$. One could study the representability of actions of the varieties $\mathbf{CAssoc} \cap \mathbf{JJord}$ and $\mathbf{Lie} \cap \mathbf{ACAAssoc}$ when $\mathrm{char}(\mathbb{F}) = 3$ to obtain other examples of weakly action representable categories or new counterexamples of action accessible categories which are not weakly action representable.

Converse to point (3) of Theorem 6.11

Let $\mathcal{V}$ be a variety of non-associative algebras over a field and let X be an object of $\mathcal{V}$. We know that if the external weak actor $\mathcal{E}(X)$ is an object of the variety $\mathcal{V}$ for each X in $\mathcal{V}$, then $\mathcal{V}$ is a weakly action representable category. Is the converse true? In Example 6.7 and Example 6.9 we have two instances of this situation for the varieties $\mathbf{Nil}_k(\mathbf{Assoc})$ of k-step nilpotent algebras and **Alt** of alternative algebras and we are not able to show if these are weakly action representable categories or not.

Representability of actions of subvarieties

We do not know how the condition of weakly representable actions behaves under taking subvarieties (especially in the non-quadratic case when the degree of the identities may be higher than 3). For instance, we know that the variety **Assoc** is weakly action representable, but we do not know whether or not the subvariety $\mathbf{Nil}_k(\mathbf{Assoc})$ with $k \geq 3$ satisfies the same condition. We recall that in this case

$$\mathcal{E}(X) = \{(f * -, - * f) \in \mathrm{Bim}(X) \mid f * (x_1 \cdots x_k) = (x_1 \cdots x_k) * f = 0\}.$$

is an associative algebra, but it is not k-step nilpotent in general (see Example 6.7).

External weak actor for varieties of algebras with two bilinear operations

Starting with the results obtained in Chapter 7 for Poisson algebras, we could try to generalize the costruction of the external weak actor $\mathcal{E}(X)$ introduced in Chapter 6 to any operadic and action accessible variety of algebras with two not necessarily associative bilinear operations.

Representability of actions of *unitary* algebras

In a recent paper [48], G. Janelidze introduced the notion of *ideally exact* category, with the aim of generalizing semi-abelian categories and to include relevant examples of *non-pointed* categories, such as the categories **Ring** of rings with unit and **CRing** of commutative rings with unit.

A category $\mathcal{C}$ is *ideally exact* when it is Barr-exact, Bourn-protomodular with finite coproducts and the unique morphism $0 \to 1$ in $\mathcal{C}$ is a regular epimorphism. Thus, any semi-abelian category is a pointed ideally exact category.

G. Janelidze also extended the notions of action representabity and weak action representability to ideally exact categories, showing that the categories **Ring** and **CRing** are action representable, with the actor of a (commutative) unitary ring X being isomoprhic to X itself. We do not recall the construction here, since it is essentialy the same of what we saw for an associative algebra with unit in Remark 6.10.

We recall from [71] that a variety of non-associative algebras $\mathcal{V}$ is said to be *unitary closed*, if for any object X ot it, the algebra $\tilde{X}$ spanned by X and the element 1, together with the multiplication $x \cdot 1 = 1 \cdot x = x$ for any $x \in X$, is still an object of $\mathcal{V}$. For instance, **Assoc** and **Alt** are unitary closed (see Example 6.9 and Remark 6.10) and the category **Leib**, or any variety of anti-commutative algebras over a field of characteristic different from 2, such as **Lie**, are examples of non-unitary closed varieties. Indeed, if there would exists a non-abelian Leibniz algebra with unit, then from the Leibniz identity we would get

$$[[x,1],1] = [[x,1],1] + [x,[1,1]]$$

and thus $[x,1] = 0$, for any $x \in \mathfrak{g}$. In a similar way, for any subvariety of **ACom**, the condition $x \cdot 1 = 1 \cdot x = x$, together with anti-commutativity, would imply $x \cdot 1 = 0$. Thus, the condition of being unitary closed depends on the set of identities which determine the variety $\mathcal{V}$.

When a variety of algebras $\mathcal{V}$ is unitary closed, one can consider the subcategory $\mathcal{V}_1$ of unitary algebras of $\mathcal{V}$ with the arrows being the algebra morphisms of $\mathcal{V}$ that preserve the unit. Of course, $\mathcal{V}_1$ is an ideally-exact category and it is not pointed.

One of the next goal is to characterize the operadic varieties of non-associative algebras which are unitary closed and to develop the study of representability of actions for them. Moreover, we may study the representability of actions of the ideally exact category **MVAlg** of *MV-algebras* [55], providing a new example outside the context of varieties of non-associative algebras over a field.

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