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An equivalent formulation of 0-closed sesquilinear forms

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Abstract. In 1970, McIntosh introduced the so-called 0-closed sesquilinear forms and proved a corresponding representation theorem. In this paper, we give a simple equivalent formulation of 0-closed sesquilinear forms. The main underlying idea is to consider minimal pairs of non-negative dominating forms.

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1. Introduction. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\mathcal{D}_1, \mathcal{D}_2$ be two subspaces of $\mathcal{H}$. A *sesquilinear form* (or, more simply, a *form*) $\mathfrak{t}$ on $\mathcal{D}_1 \times \mathcal{D}_2$ is a map $\mathfrak{t} : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathbb{C}$ which is linear in the first component and anti-linear in the second one. If $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}$, we write $\mathfrak{t}[f] := \mathfrak{t}(f, f)$ for $f \in \mathcal{D}$. A sesquilinear form $\mathfrak{s}$ on $\mathcal{D} \times \mathcal{D}$ is said to be *non-negative* if $\mathfrak{s}[f] \geq 0$ for every $f \in \mathcal{D}$; *strictly-positive* if there exists $c > 0$ such that $\mathfrak{s}[f] \geq c\|f\|^2$ for every $f \in \mathcal{D}$. If $\mathfrak{s}$ is non-negative, we denote by $\ker(\mathfrak{s})$ the subspace $\{f \in \mathcal{D} : \mathfrak{s}[f] = 0\}$.

Given a sesquilinear form $\mathfrak{t}$ on $\mathcal{D}_1 \times \mathcal{D}_2$, with $\mathcal{D}_2$ is dense in $\mathcal{H}$, it is possible to construct an operator T with domain

$$D(T) = \{f \in \mathcal{D}_1 : \exists h \in \mathcal{H}, \mathfrak{t}(f, g) = \langle h, g \rangle, \forall g \in \mathcal{D}_2\} \tag{1.1}$$

and defined as $Tf = h$, for all $f \in D(T)$, where h is the element in (1.1). The operator T is called *associated* to $\mathfrak{t}$ and then the following representation holds

$$\mathfrak{t}(f, g) = \langle Tf, g \rangle, \quad \forall f \in D(T), g \in \mathcal{D}_2. \tag{1.2}$$

In the last decades, several theorems about the representation (1.2) have been given [1–5, 9, 11, 13–17]. The topic of the representation is connected to the Lebesgue decomposition (see [6–8, 12, 19]) as motivated in [8].

One of the classical representation theorems has been given for the so-called *closed* sesquilinear forms [14, Ch. VI]. We recall that a non-negative[1] sesquilinear form $\mathfrak{s}$ on $\mathcal{D}\times\mathcal{D}$ is closed if, for any sequence of vectors $\{f_n\}$ of $\mathcal{H}$ such that $f_n \to f$ and $\mathfrak{s}[f_n - f_m] \to 0$, one has $f \in \mathcal{D}$ and $\mathfrak{s}[f_n - f] \to 0$. The representation theorem for closed sesquilinear forms is useful, for instance, to define the Friedrichs extension of densely defined positive operators [14, Ch. VI] and a special sum of two operators [18].

In this paper, we specifically focus on 0-closed forms introduced and treated in [16,17] by McIntosh in 1970. Hence, first of all, we recall the definition. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be dense subspaces of $\mathcal{H}$. A sesquilinear form $\mathfrak{t}$ on $\mathcal{D}_1 \times \mathcal{D}_2$ is called 0-*closed* [16,17] if

- $\mathcal{D}_1$ and $\mathcal{D}_2$ can be made into Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ continuously embedded in $\mathcal{H}$ with inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ and norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively;
- $\mathfrak{t}$ is bounded with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$, i.e., there exists $C > 0$ such that
$$|\mathfrak{t}(f,g)| \leq C\|f\|_1\|g\|_2, \quad \forall f \in \mathcal{H}_1, g \in \mathcal{H}_2; \tag{1.3}$$
- the bounded operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ satisfying
$$\mathfrak{t}(f,g) = \langle Af, g\rangle_2, \quad \forall f \in \mathcal{H}_1, g \in \mathcal{H}_2, \tag{1.4}$$
is a bijection,[2]

For 0-closed forms, the following representation theorem holds.

Theorem 1.1 ([16,17]). *Let $\mathfrak{t}$ be a 0-closed form on $\mathcal{D}_1 \times \mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are dense subspaces of $\mathcal{H}$. Then the operator T associated to $\mathfrak{t}$ is densely defined, closed, and 0 belongs to the resolvent set of T. Moreover, also the sesquilinear form $\mathfrak{t}^*$ given by*
$$\mathfrak{t}^*(f,g) = \overline{\mathfrak{t}(g,f)}, \quad \forall f \in \mathcal{D}_2, g \in \mathcal{D}_1,$$
is 0-closed and its associated operator is T^*.

The main scope of this paper is to prove an equivalent formulation of 0-closed forms (Theorem 2.4). In particular, we will employ the concept of minimal pairs of non-negative sesquilinear forms which dominate a given sesquilinear form (Definition 2.1). The auxiliary results Lemma 2.3 and Proposition 3.1 give some characterizations of minimal pairs assuming that the non-negative sesquilinear forms are closed.

2. The equivalent formulation. Throughout the paper, we denote by $\ker(S)$ and $R(S)$ the kernel and the range of an operator $S : D(S) \to \mathcal{H}_2$, respectively. We use the symbol $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ for the set of bounded operators $S : \mathcal{H}_1 \to \mathcal{H}_2$. Firstly, we introduce the set of pairs of dominating forms. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be subspaces of $\mathcal{H}$ and $\mathfrak{t}$ a sesquilinear form on $\mathcal{D}_1 \times \mathcal{D}_2$. We denote by $\mathcal{M}(\mathfrak{t})$ the set of pairs $(\mathfrak{s}_1, \mathfrak{s}_2)$ of non-negative sesquilinear forms such that

[1] Actually, the definition of closed forms can be given for a more general classes of forms, namely for semi-bounded or sectorial forms [14].

[2] The existence of the bounded operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ satisfying (1.4) is ensured by (1.3).

- $\mathfrak{s}_1 : \mathcal{D}_1 \times \mathcal{D}_1 \to \mathbb{C}$ and $\mathfrak{s}_2 : \mathcal{D}_2 \times \mathcal{D}_2 \to \mathbb{C}$;
- for every $f \in \mathcal{D}_1$ and $g \in \mathcal{D}_2$, one has

$$|\mathfrak{t}(f,g)| \leq \mathfrak{s}_1[f]^{\frac{1}{2}}\mathfrak{s}_2[g]^{\frac{1}{2}}. \tag{2.1}$$

Definition 2.1. We say that a pair $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathcal{M}(\mathfrak{t})$ is *minimal* in $\mathcal{M}(\mathfrak{t})$ if, for every $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{M}(\mathfrak{t})$ such that $\mathfrak{p} \leq \mathfrak{s}_1$ and $\mathfrak{q} \leq \mathfrak{s}_2$, we have $\mathfrak{p} = \mathfrak{s}_1$ and $\mathfrak{q} = \mathfrak{s}_2$.

Remarks 2.2. 1. If $\mathfrak{s}$ is a non-negative sesquilinear form on $\mathcal{D} \times \mathcal{D}$, then trivially $(\mathfrak{s}, \mathfrak{s})$ belongs to $\mathcal{M}(\mathfrak{s})$ by the Cauchy-Schwarz inequality

$$|\mathfrak{s}(f,g)| \leq \mathfrak{s}[f]^{\frac{1}{2}}\mathfrak{s}[g]^{\frac{1}{2}},$$

and it is also minimal in $\mathcal{M}(\mathfrak{s})$.

2. If $\mathfrak{t}$ is a sesquilinear form and $(\mathfrak{s}_1, \mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{t})$, then also $(\alpha\mathfrak{s}_1, \alpha^{-1}\mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{t})$ for any $\alpha > 0$. Anyway, even in the non-negative case, there might exist other less trivial minimal pairs. To make an example, let $\mathcal{H} = \mathbb{C}^2$ and $\mathfrak{s}$ the non-negative form defined as follows

$$\mathfrak{s}(f,g) = 2f(1)\overline{g(1)} + 2f(2)\overline{g(2)}, \quad \forall f, g \in \mathbb{C}^2,$$

where $f = (f(1), f(2))$ and $g = (g(1), g(2))$. The pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ made with

$$\mathfrak{s}_1(f,g) = 4f(1)\overline{g(1)} + f(2)\overline{g(2)}, \quad \forall f, g \in \mathbb{C}^2,$$

and

$$\mathfrak{s}_2(f,g) = f(1)\overline{g(1)} + 4f(2)\overline{g(2)}, \quad \forall f, g \in \mathbb{C}^2,$$

is in $\mathcal{M}(\mathfrak{s})$. Indeed, by the Cauchy-Schwarz inequality, for every $f, g \in \mathbb{C}^2$,

$$\begin{aligned} |s(f,g)| &= |2f(1) \cdot \overline{g(1)} + f(2) \cdot 2\overline{g(2)}| \\ &\leq (4|f(1)|^2 + |f(2)|^2)^{\frac{1}{2}}(|g(1)|^2 + 4|g(2)|^2)^{\frac{1}{2}} \\ &= \mathfrak{s}_1[f]^{\frac{1}{2}}\mathfrak{s}_2[g]^{\frac{1}{2}}. \end{aligned}$$

Moreover, $(\mathfrak{s}_1, \mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{s})$. Indeed, let $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{M}(\mathfrak{s})$ with $\mathfrak{p} \leq \mathfrak{s}_1$ and $\mathfrak{q} \leq \mathfrak{s}_2$. We have

$$2 = \mathfrak{s}((1,0),(1,0)) \leq \mathfrak{p}[(1,0)]^{\frac{1}{2}}\mathfrak{q}[(1,0)]^{\frac{1}{2}} \leq \mathfrak{s}_1[(1,0)]^{\frac{1}{2}}\mathfrak{s}_2[(1,0)]^{\frac{1}{2}} = 2,$$

so $\mathfrak{p}[(1,0)]\mathfrak{q}[(1,0)] = 4$. Since $\mathfrak{p} \leq \mathfrak{s}_1$ and $\mathfrak{q} \leq \mathfrak{s}_2$, we have $\mathfrak{p}[(1,0)] = \mathfrak{s}_1[(1,0)] = 2$ and $\mathfrak{q}[(1,0)] = \mathfrak{s}_2[(1,0)] = 1$. In the same way, we can prove that $\mathfrak{p}[(0,1)] = \mathfrak{s}_1[(0,1)] = 1$ and $\mathfrak{q}[(0,1)] = \mathfrak{s}_2[(0,1)] = 2$. Thus the non-negative forms $\mathfrak{p}$ and $\mathfrak{q}$ are completely determined knowing $\mathfrak{p}((1,0),(0,1))$ and $\mathfrak{q}((1,0),(0,1))$. In particular, denoting by $\mathfrak{s}_1 - \mathfrak{p}$ the sequilinear form given by the difference between $\mathfrak{s}_1$ and $\mathfrak{p}$, we have

$$\begin{aligned} |\mathfrak{p}((1,0),(0,1))| &= |(\mathfrak{s}_1 - \mathfrak{p})((1,0),(0,1))| \\ &\leq (\mathfrak{s}_1 - \mathfrak{p})[(1,0)]^{\frac{1}{2}}(\mathfrak{s}_1 - \mathfrak{p})[(0,1)]^{\frac{1}{2}} = 0, \end{aligned}$$

where the first line holds because $\mathfrak{s}_1((1,0),(0,1)) = 0$ and the second line is valid by the Cauchy-Schwarz inequality since $\mathfrak{s}_1 - \mathfrak{p}$ is non-negative by the hypothesis on $\mathfrak{p}$. In conclusion, $\mathfrak{p} = \mathfrak{s}_1$. With similar arguments $\mathfrak{q} = \mathfrak{s}_2$, i.e., $(\mathfrak{s}_1, \mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{s})$.

In the end of Section 3, we will give another example, but in infinite dimension, showing several minimal pairs. Now we are going to prove a preliminary result (Lemma 2.3) which will be useful in the proof of our main claim, i.e., Theorem 2.4. We start with a particular representation of a sesquilinear form $\mathfrak{t}$ on $\mathcal{D}_1 \times \mathcal{D}_2$ determined by a given pair of closed non-negative sesquilinear forms $(\mathfrak{s}_1, \mathfrak{s}_2)$ in $\mathcal{M}(\mathfrak{t})$. Let

$$\widetilde{\mathcal{D}_1} = \ker(\mathfrak{s}_1)^\perp \cap \mathcal{D}_1 \quad \text{and} \quad \widetilde{\mathcal{D}_2} = \ker(\mathfrak{s}_2)^\perp \cap \mathcal{D}_2, \tag{2.2}$$

where $\ker(\mathfrak{s}_1)^\perp$ and $\ker(\mathfrak{s}_2)^\perp$ are the orthogonal complements of $\ker(\mathfrak{s}_1)$ and $\ker(\mathfrak{s}_2)$ in $\mathcal{H}$, respectively. The restrictions $\widetilde{\mathfrak{s}_1}$, $\widetilde{\mathfrak{s}_2}$ of $\mathfrak{s}_1$, $\mathfrak{s}_2$ on $\widetilde{\mathcal{D}_1}$, $\widetilde{\mathcal{D}_2}$, respectively, are closed non-negative sesquilinear forms with $\ker(\widetilde{\mathfrak{s}_1}) = \{0\}$ and $\ker(\widetilde{\mathfrak{s}_2}) = \{0\}$. Now let

$$\langle f, f' \rangle_1 = \widetilde{\mathfrak{s}_1}(f, f') \ \text{ and } \ \langle g, g' \rangle_2 = \widetilde{\mathfrak{s}_2}(g, g'), \quad f, f' \in \widetilde{\mathcal{D}_1}, g, g' \in \widetilde{\mathcal{D}_2}, \tag{2.3}$$

$$\|f\|_1 = \widetilde{\mathfrak{s}_1}[f]^{\frac{1}{2}} \ \text{ and } \ \|g\|_2 = \widetilde{\mathfrak{s}_2}[g]^{\frac{1}{2}}, \quad f \in \widetilde{\mathcal{D}_1}, g \in \widetilde{\mathcal{D}_2}. \tag{2.4}$$

Since $\widetilde{\mathfrak{s}_1}$ and $\widetilde{\mathfrak{s}_2}$ are closed forms, $\widetilde{\mathcal{D}_1}$ and $\widetilde{\mathcal{D}_2}$ are complete with respect to the inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ in (2.3), respectively. We denote by $\mathcal{H}_1$ and $\mathcal{H}_2$ the Hilbert spaces made in this way. By (2.1), the restriction of $\mathfrak{t}$ on $\widetilde{\mathcal{D}_1} \times \widetilde{\mathcal{D}_2}$ can be considered as a bounded sesquilinear form with respect to the norms $\| \cdot \|_1$ and $\| \cdot \|_2$ in (2.4) induced by $\widetilde{\mathfrak{s}_1}$ and $\widetilde{\mathfrak{s}_2}$, respectively. Then, by Riesz's theorem, there exists an operator $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that

$$\mathfrak{t}(f, g) = \langle Bf, g \rangle_2, \quad \forall f \in \widetilde{\mathcal{D}_1}, g \in \widetilde{\mathcal{D}_2}. \tag{2.5}$$

In particular, again by (2.1), B has norm[3] $\|B\|$ less or equal 1. We have the following characterization.

Lemma 2.3. *Let $\mathfrak{t}$ be a sesquilinear form on $\mathcal{D}_1 \times \mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are subspaces of $\mathcal{H}$, and let $(\mathfrak{s}_1, \mathfrak{s}_2)$ be a pair of closed non-negative sesquilinear forms in $\mathcal{M}(\mathfrak{t})$. Then $(\mathfrak{s}_1, \mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{t})$ if and only if the operator B in (2.5) is a unitary operator.*

Proof. Let us assume that $(\mathfrak{s}_1, \mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{t})$. By (2.5),

$$|\mathfrak{t}(f, g)| \leq \|Bf\|_2 \|g\|_2 = \|Bf\|_2 \mathfrak{s}_2[g]^{\frac{1}{2}}, \quad \forall f \in \widetilde{\mathcal{D}_1}, g \in \widetilde{\mathcal{D}_2}. \tag{2.6}$$

Let $I : \mathcal{H} \to \mathcal{H}$ be the identity operator, $O_1 : \mathcal{H} \to \mathcal{H}$ and $O_2 : \mathcal{H} \to \mathcal{H}$ the orthogonal projections onto $\ker(\mathfrak{s}_1)^\perp$ and $\ker(\mathfrak{s}_2)^\perp$, respectively. We note that $\ker(\mathfrak{s}_1), \ker(\mathfrak{s}_2)$ are closed since $\mathfrak{s}_1, \mathfrak{s}_2$ are closed forms and that for $f \in \mathcal{D}_1, g \in \mathcal{D}_2$, we have $(I - O_1)f \in \ker(\mathfrak{s}_1) \in \mathcal{D}_1, (I - O_2)g \in \ker(\mathfrak{s}_2) \in \mathcal{D}_2$, so $O_1 f \in \widetilde{\mathcal{D}_1}, O_2 g \in \widetilde{\mathcal{D}_2}$. Let $\mathfrak{p}$ be the sesquilinear form defined as follows

$$\mathfrak{p}(f, g) = \langle BO_1 f, BO_1 g \rangle_2, \quad \forall f, g \in \mathcal{D}_1.$$

Hence, the inequality (2.6) can be rewritten as

$$|\mathfrak{t}(f, g)| \leq \mathfrak{p}[f]^{\frac{1}{2}} \mathfrak{s}_2[g]^{\frac{1}{2}}, \quad \forall f \in \widetilde{\mathcal{D}_1}, g \in \widetilde{\mathcal{D}_2}. \tag{2.7}$$

[3]For simplifying the notation, we do not add a symbol to $\|B\|$ to specify the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We are going to do this also for other operators.

Considering that $\mathfrak{t}(f,g)=0$ if $f\in\ker(\mathfrak{s}_1)$ or $g\in\ker(\mathfrak{s}_2)$ and that $\mathfrak{p}[O_1f]=\mathfrak{p}[f],\mathfrak{s}_2[O_2g]=\mathfrak{s}_2[g]$ for every $f\in\mathcal{D}_1,g\in\mathcal{D}_2$, the inequality (2.7) extends by linearity for every $f\in\mathcal{D}_1$ and $g\in\mathcal{D}_2$. In other words, $(\mathfrak{p},\mathfrak{s}_2)\in\mathcal{M}(\mathfrak{t})$.
As said before, $\|B\|\leq 1$ and then $\mathfrak{p}[f]^{\frac{1}{2}}=\|BO_1f\|_2\leq\|O_1f\|_1=\mathfrak{s}_1[O_1f]^{\frac{1}{2}}=\mathfrak{s}_1[f]^{\frac{1}{2}}$ for every $f\in\mathcal{D}_1$. Since $(\mathfrak{s}_1,\mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{t})$, we have $\|Bf\|_2=\mathfrak{p}[f]^{\frac{1}{2}}=\mathfrak{s}_1[f]^{\frac{1}{2}}=\|f\|_1$ for every $f\in\widetilde{\mathcal{D}_1}$, i.e., B is an isometry. Because

$$|\mathfrak{t}(f,g)|\leq\|f\|_1\|B^*g\|_1,\quad\forall f\in\widetilde{\mathcal{D}_1},g\in\widetilde{\mathcal{D}_2},$$

holds too, in the same way, we have that B^* is an isometry. In conclusion, B is unitary.

Now assume that B is a unitary operator. Let $(\mathfrak{p},\mathfrak{q})\in\mathcal{M}(\mathfrak{t})$ be such that $\mathfrak{p}\leq\mathfrak{s}_1$ and $\mathfrak{q}\leq\mathfrak{s}_2$. Then there exist $P\in\mathcal{B}(\mathcal{H}_1)$ and $Q\in\mathcal{B}(\mathcal{H}_2)$ satisfying

$$\|P\|\leq 1,\quad\|Q\|\leq 1,\tag{2.8}$$

$\mathfrak{p}[f]=\|Pf\|_1^2$ for every $f\in\widetilde{\mathcal{D}_1}$, and $\mathfrak{q}[g]=\|Qg\|_2^2$ for every $g\in\widetilde{\mathcal{D}_2}$. By [14, Ch. VI, Lemma 3.1], there exists $R\in\mathcal{B}(\mathcal{H}_\mathfrak{p},\mathcal{H}_\mathfrak{q})$[4] such that

$$\|R\|\leq 1,\tag{2.9}$$

$$\mathfrak{t}(f,g)=\langle RPf,Qg\rangle_2,\quad\forall f\in\widetilde{\mathcal{D}_1},g\in\widetilde{\mathcal{D}_2}.$$

Therefore, by uniqueness of the associated operator, $B=Q^*RP$. Moreover,

$$\|f\|_1=\|Bf\|_2=\|Q^*RPf\|_2\leq\|Pf\|_1\leq\|f\|_1,\quad\forall f\in\widetilde{\mathcal{D}_1},$$

by (2.8) and (2.9), i.e., $\mathfrak{p}[f]=\|Pf\|_1^2=\|f\|_1^2=\mathfrak{s}_1[f]$ for every $f\in\widetilde{\mathcal{D}_1}$. Furthermore, on $\ker(\mathfrak{s}_1)$ (which, as said before, is closed), $\mathfrak{s}_1$ and $\mathfrak{p}$ are null. Hence, by linearity, $\mathfrak{p}=\mathfrak{s}_1$. Working in a similar way with B^*, we also find that $\mathfrak{q}=\mathfrak{s}_2$. Thus $(\mathfrak{s}_1,\mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{t})$. □

Now we are ready to prove the main result of the paper.

Theorem 2.4. *Let $\mathfrak{t}$ be a sesquilinear form on $\mathcal{D}_1\times\mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are dense subspaces of $\mathcal{H}$. The following statements are equivalent.*
- (i) *$\mathfrak{t}$ is 0-closed;*
- (ii) *there exists a pair $(\mathfrak{s}_1,\mathfrak{s}_2)$ of closed strictly-positive sesquilinear forms minimal in $\mathcal{M}(\mathfrak{t})$.*

Proof. (i) $\Longrightarrow$ (ii) Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ be the inner products which make $\mathcal{D}_1$ and $\mathcal{D}_2$ two Hilbert spaces based on the definition of 0-closed forms. Let $A=U|A|=|A^*|U$ be the polar decomposition of the bounded operator A in (1.4). In particular, since A is bijective, U is a unitary operator and $|A|^{\frac{1}{2}}:\mathcal{H}_1\to\mathcal{H}_1$, $|A^*|^{\frac{1}{2}}:\mathcal{H}_2\to\mathcal{H}_2$ are bijective self-adjoint positive operators. By [10, Theorem 2.7], we also have $A=|A^*|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. Let us define

$$\mathfrak{s}_1(f,f')=\langle|A|^{\frac{1}{2}}f,|A|^{\frac{1}{2}}f'\rangle_1,\quad f,f'\in\mathcal{D}_1,$$
$$\mathfrak{s}_2(g,g')=\langle|A^*|^{\frac{1}{2}}g,|A^*|^{\frac{1}{2}}g'\rangle_2,\quad g,g'\in\mathcal{D}_2.$$

[4] $\mathcal{H}_\mathfrak{p}$ and $\mathcal{H}_\mathfrak{q}$ are the completions of $\widetilde{\mathcal{D}_1}$ and $\widetilde{\mathcal{D}_2}$ with respect to the norms induced by $\mathfrak{p}$ and $\mathfrak{q}$, respectively.

The sesquilinear forms $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are strictly-positive and closed forms because $|A|^{\frac{1}{2}}$, $|A^*|^{\frac{1}{2}}$ are bijective and $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ are strictly-positive and closed forms. Moreover,

$$|\mathfrak{t}(f,g)| = |\langle |A^*|^{\frac{1}{2}} U |A|^{\frac{1}{2}} f, g \rangle| = |\langle U |A|^{\frac{1}{2}} f, |A^*|^{\frac{1}{2}} g \rangle| \leq \mathfrak{s}_1[f]^{\frac{1}{2}} \mathfrak{s}_2[g]^{\frac{1}{2}},$$

i.e., $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathcal{M}(\mathfrak{t})$. Finally, $(\mathfrak{s}_1, \mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{t})$ by Lemma 2.3 because the operator B constructed as in (2.5) coincides with U which is a unitary operator.

(ii) $\Longrightarrow$ (i) Let

$$\langle f, f' \rangle_1 = \mathfrak{s}_1(f, f') \;\text{ and }\; \langle g, g' \rangle_2 = \mathfrak{s}_2(g, g'), \quad f, f' \in \mathcal{D}_1, g, g' \in \mathcal{D}_2.$$

Since $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are closed strictly-positive sesquilinear forms, $\mathcal{D}_1$ and $\mathcal{D}_2$ turn into Hilbert spaces continuously embedded in $\mathcal{H}$ with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Moreover, $\mathfrak{t}$ is bounded with respect to the norms of these spaces. Finally, the last condition required in the definition of 0-closed forms is satisfied as a consequence of Lemma 2.3. □

3. A supplementary result. In this section, we prove another characterization of minimal forms concerning a different representation in comparison to (2.5).

As previously, let $\mathfrak{t}$ be a sesquilinear form on $\mathcal{D}_1 \times \mathcal{D}_2$ and $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathcal{M}(\mathfrak{t})$ with $\mathfrak{s}_1$ and $\mathfrak{s}_2$ non-negative and closed. By Kato's second representation theorem [14, Theorem VI.2.23], there exist positive self-adjoint operators H_1 and H_2 in $\mathcal{H}$ with $D(H_1) = \mathcal{D}_1$ and $D(H_2) = \mathcal{D}_2$ such that $\mathfrak{s}_1[f] = \|H_1 f\|^2$ and $\mathfrak{s}_2[g] = \|H_2 g\|^2$ for every $f \in \mathcal{D}_1$ and $g \in \mathcal{D}_2$. We write $\overline{R(H_1)}$ and $\overline{R(H_2)}$ for the closures of $R(H_1)$ and $R(H_2)$ in $\mathcal{H}$, respectively. By (2.1) and [14, Ch. VI, Lemma 3.1], there exists a bounded operator $Q \in \mathcal{B}(\overline{R(H_1)}, \overline{R(H_2)})$ with $\|Q\| \leq 1$ such that

$$\mathfrak{t}(f,g) = \langle Q H_1 f, H_2 g \rangle, \quad f \in \mathcal{D}_1, g \in \mathcal{D}_2.$$

Moreover, $\ker(H_1) = \ker(\mathfrak{s}_1)$, $\ker(H_2) = \ker(\mathfrak{s}_2)$, so the restrictions $\widetilde{H_1}$ and $\widetilde{H_2}$ of H_1 and H_2 on $\widetilde{\mathcal{D}_1}$ and $\widetilde{\mathcal{D}_2}$ (which are defined in (2.2)), respectively, are injective and we can also write

$$\mathfrak{t}(f,g) = \langle Q \widetilde{H_1} f, \widetilde{H_2} g \rangle, \quad f \in \widetilde{\mathcal{D}_1}, g \in \widetilde{\mathcal{D}_2}. \tag{3.1}$$

Proposition 3.1. *Let $\mathfrak{t}$ be a sesquilinear form on $\mathcal{D}_1 \times \mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are dense subspaces of $\mathcal{H}$ and let $(\mathfrak{s}_1, \mathfrak{s}_2)$ be a pair of closed non-negative sesquilinear forms in $\mathcal{M}(\mathfrak{t})$. Then $(\mathfrak{s}_1, \mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{t})$ if and only if Q in (3.1) is a unitary operator.*

Proof. Let H_1 and H_2 be the operators introduced above. The operators Q and B of (2.5) are connected by the following relation

$$Q = \widetilde{H_2} B \widetilde{H_1}^{-1} \quad \text{on } R(H_1).$$

Comparing the representations (2.5) and (3.1) for $\mathfrak{t}^*$ instead of $\mathfrak{t}$, we also have

$$Q^* = \widetilde{H_1} B^* \widetilde{H_2}^{-1} \quad \text{on } R(H_2),$$

so

$$QQ^* = \widetilde{H_2} BB^* \widetilde{H_2}^{-1} \text{ on } R(H_2), \quad Q^*Q = \widetilde{H_1} B^* B \widetilde{H_1}^{-1} \text{ on } R(H_1),$$

and

$$BB^* = \widetilde{H_2}^{-1} QQ^* \widetilde{H_2} \text{ on } \widetilde{\mathcal{D}_2}, \quad B^*B = \widetilde{H_1}^{-1} Q^* Q \widetilde{H_1} \text{ on } \widetilde{\mathcal{D}_1}.$$

Thus, by Lemma 2.3 and taking into account that $R(H_1)$, $R(H_2)$ are dense in their corresponding closures, $(\mathfrak{s}_1, \mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{t})$ if and only if Q is unitary. □

Example. Let us consider $\mathcal{H} = \mathcal{D} = L^2(\mathbb{R})$ and a bounded measurable function $r : \mathbb{R} \to \mathbb{C}$. We write $N = \{x \in \mathbb{R} : r(x) = 0\}$. Let

$$\mathfrak{t}(f, g) = \int_{\mathbb{R}} r(x) f(x) g(x) dx, \quad \forall f, g \in \mathcal{H}. \tag{3.2}$$

The sesquilinear form $\mathfrak{t}$ is bounded, and it is non-negative if and only if $r(x)$ is non-negative for a.e. $x \in \mathbb{R}$. For any measurable function $p : \mathbb{R} \to [0, +\infty)$, with $c \leq p(x) \leq d$ for some $c, d > 0$ and every $x \in \mathbb{R}$, we define

$$\mathfrak{s}_1(f, g) = \int_{\mathbb{R}} |r(x)| p(x) f(x) g(x) dx, \quad \forall f, g \in \mathcal{H}$$

and

$$\mathfrak{s}_2(f, g) = \int_{\mathbb{R}} |r(x)| p(x)^{-1} f(x) g(x) dx, \quad \forall f, g \in \mathcal{H}.$$

The sesquilinear forms $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-negative (trivially closed by boundedness) and, by the Cauchy-Schwarz inequality, $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathcal{M}(\mathfrak{t})$. Moreover, again independently by p, we have $\ker(\mathfrak{s}_1) = \ker(\mathfrak{s}_2) = \{f \in \mathcal{H} : f(x) = 0 \text{ for a.e. } x \in \mathbb{R} \backslash N\} \equiv L^2(N)$, and so $\widetilde{\mathcal{D}_1} = \widetilde{\mathcal{D}_2} = \{f \in \mathcal{H} : f(x) = 0 \text{ for a.e. } x \in N\} \equiv L^2(\mathbb{R} \backslash N)$. The pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ is minimal in $\mathcal{M}(\mathfrak{t})$ for any choice of p by Proposition 3.1. Indeed, one easily checks that the operators $\widetilde{H_1}$ and $\widetilde{H_2}$ are the multiplication operators by $|r(x)|^{\frac{1}{2}} p(x)^{\frac{1}{2}}$ and $|r(x)|^{\frac{1}{2}} p(x)^{-\frac{1}{2}}$ on the domain $L^2(\mathbb{R} \backslash N)$, respectively. Hence, by comparing (3.2) and (3.1), the operator Q is the multiplication operator by $\frac{r}{|r|}$ on the domain $L^2(\mathbb{R} \backslash N)$ and it is in particular unitary.

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