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Fractal Calculus: Nonhomogeneous Linear Systems

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Abstract: In this paper, we present a concise overview of fractal calculus and explore the solution of non-homogeneous fractal differential equations. We analyze fractal homogeneous linear systems with initial conditions, introducing the fundamental matrix and special fundamental matrix, and demonstrate their applications in solving systems and analyzing the Jordan form of matrices. We propose the method of undetermined coefficients for solving non-homogeneous fractal linear differential equations and introduce the method of variation of parameters as a supplementary technique. To illustrate these methods, we apply them to the differential equations of RLC circuits, successfully solving the corresponding fractal differential equations. Additionally, we provide examples, solve systems with initial conditions, and present the results through plotted graphs.

Keywords: Fractal Differential Equations,Fractal Non-Homogeneous Systems, Method of Undetermined Coefficients, Variation of Parameters.

1 Introduction

Fractals, which are repeating patterns found in natural phenomena like blood vessels, mountains, clouds, and coastlines, have been thoroughly studied using fractal geometry. This field investigates the distinct properties and measurements of these intricate structures. Key characteristics of fractals include fractional dimensions, self-similarity, and fractal dimensions that exceed their topological dimensions [8, 9, 38, 39].

The analysis of fractals employs various techniques, such as fractional space [46], fractional calculus, stochastic processes [3], measure theory, and harmonic analysis [10, 35, 37, 47, 50].

Riemann-type integrals, including s-Riemann, s-HK, and s-first-return integrals, have been developed by adapting the Fundamental Theorem of Calculus. The relationships between these integrals and the Lebesgue integral concerning the Hausdorff measure have been explored in several studies [4–6, 34], which also describe the primitives of the s-HK integral.

A novel framework for fractal calculus has been established by extending classical calculus, providing an algorithmic, geometric, and physically meaningful approach. This generalization allows the analysis of functions with fractal characteristics, such as those observed in Cantor sets and Koch curves [17, 41, 42, 44]. Due to their complex geometric structures and scale-invariant self-similarity [36], fractals often require analytical methods that go beyond standard mathematics.

Fractal local derivatives have been used to study sub- and super-diffusion, preserving locality and the central limit theorem [12, 23]. Non-local fractal calculus has been introduced for modeling incompressible

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viscous fluids in fractal media and processes with memory [1, 2, 14, 31], analogous to the extensions of traditional local calculus seen in Riemann-Liouville and Caputo generalizations.

In physics, fractal calculus has led to new models of fractal space and time [7, 28, 40, 43, 45, 48, 49], often producing solutions that exhibit power laws and self-similarity.

The dynamics of systems featuring fractal times have been explored using fractal Laplace, Sumudu, and Fourier transforms applied to Cantor sets and fractal curves [11, 15, 20, 21, 24, 31, 32]. Additionally, fractal calculus has been applied to calculate the derivatives and integrals of complex functions, including the Weierstrass function [33].

Incorporating hyperreal and hyperinteger numbers, nonstandard analysis has been employed to define derivative structures and bounds on fractal curves, as well as fractal integral and differential forms [29]. The use of gauge functions has further generalized fractal calculus to cover unbounded functions [13]. Studies on the stability of fractal differential equations have outlined conditions that guarantee unique and stable solutions [27].

Methods such as the steps approach and Laplace transform have been utilized to solve fractal retarded, neutral, and renewal delay differential equations with constant coefficients [26]. Furthermore, a recent extension of mean square calculus has been proposed [30], introducing concepts like the mean square derivative, fractal mean square integral, fractal mean square continuity, and the order of random variables on fractal curves.

Fractional Brownian motion on thin Cantor-like sets has been formulated using non-local fractal derivatives. This work introduced the fractal Hurst exponent and established its connection to the order of non-local fractal derivatives [25].

New concepts such as Shannon entropy on fractal thin Cantor-like sets have been defined, and stable distributions on fractal sets have been proposed alongside related physical models [22]. Random walks on fractal middle-ξ Cantor sets have been analyzed, revealing that the corresponding variances follow power laws. Fractal sets have also seen a generalization of mean square stochastic calculus, with the standard case recoverable by adjusting the dimension [16].

This paper aims to present and solve fractal nonhomogeneous linear systems while exploring their practical applications.

The structure of the paper is organized as follows. Section 2 provides a brief overview of fractal calculus. In Section 3, we explore fractal homogeneous linear systems with initial conditions and provide detailed solutions. Section 4 focuses on fractal nonhomogeneous linear systems, exploring various methods for solving them. Section 5 demonstrates the application of these concepts by solving the fractal nonhomogeneous linear system associated with the RLC circuit. Finally, Section 6 offers concluding remarks for the paper.

2 Fundamental definitions in fractal calculus

In this paper, we lay out the foundational definitions necessary for developing the key concepts discussed. We begin by providing an overview of fractal calculus as it applies to the Cantor set $F \subset [a, b] \subset \mathbb{R}$, following the work in [17, 42].

Definition 1. The indicator function $\mathbb{I}_F(J)$ for the set F is defined as follows:

$$\mathbb{I}_F(J) = \begin{cases} 1, & \text{if } F \cap J \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

where $J = [a, b] \subset \mathbb{R}$.

Definition 2. The coarse-grained measure $\mu_\delta^\alpha(F, a, b)$ of $F \cap [a, b]$ is given by:

$$\mu_\delta^\alpha(F, a, b) = \inf_{|\mathcal{Q}| \leq \delta} \sum_{j=0}^{n-1} \Gamma(\alpha + 1)(z_{j+1} - z_j)^\alpha \mathbb{I}_F([z_j, z_{j+1}]),$$

where $|\mathcal{Q}| = \max_{0\leq j\leq n-1}(z_{j+1} - z_j)$, $\mathcal{Q}_{[a,b]} = \{z_0 = a, z_1, ..., z_n = b\}$, $0 < \alpha \leq 1$, and $\Gamma(\cdot)$ is the Gamma function.

Definition 3. The measure function $\mu^\alpha(F, a, b)$ for F is defined by:

$$\mu^\alpha(F, a, b) = \lim_{\delta\to 0} \mu_\delta^\alpha(F, a, b).$$

Definition 4. The fractal η-dimension of $F \cap [a, b]$ is defined by:

$$\begin{aligned}\dim_\eta(F \cap [a, b]) &= \inf\{\alpha : \mu^\alpha(F, a, b) = 0\} \\ &= \sup\{\alpha : \mu^\alpha(F, a, b) = \infty\}.\end{aligned} \tag{1}$$

Definition 5. The integral staircase function $S_F^\alpha(z)$ is defined as:

$$S_F^\alpha(z) = \begin{cases} \mu^\alpha(F, a_0, z), & \text{if } z \geq a_0, \\ -\mu^\alpha(F, z, a_0), & \text{if } z < a_0, \end{cases}$$

where $a_0 \in \mathbb{R}$ is a fixed constant.

Definition 6. The characteristic function $\chi_F(z)$ on the fractal set F is defined by:

$$\chi_F(z) = \begin{cases} \Gamma(\alpha + 1), & \text{if } z \in F, \\ 0, & \text{otherwise}, \end{cases}$$

where α is the fractal dimension as defined in (1).

Definition 7. For a function $p : F \to \mathbb{R}$, the $F\_\lim$ of $p(z)$ at a point $z \in F$ is the number ℓ such that for every $\epsilon > 0$, there exists a $\delta > 0$ satisfying:

$$y \in F \text{ and } |y - z| < \delta \implies |p(y) - \ell| < \epsilon.$$

If such an ℓ exists, it is denoted by:

$$\ell = F\_\lim_{y\to z} p(y).$$

This definition does not involve the function values at y if $y \notin F$.

Definition 8. A function $p : F \to \mathbb{R}$ is said to be F-continuous at $z \in F$ if:

$$p(z) = F\_\lim_{y\to z} p(y), \tag{2}$$

whenever the $F\_\lim$ exists.

Definition 9. For a function p on a α-perfect fractal set F, the F^α-derivative of p at z is defined as:

$$D_F^\alpha p(z) = \begin{cases} F\_\lim_{y\to z} \frac{p(y)-p(z)}{S_F^\alpha(y)-S_F^\alpha(z)}, & \text{if } z \in F, \\ 0, & \text{otherwise}, \end{cases}$$

provided that the $F\_\lim$ exists.

Definition 10. The F^α-integral of a bounded function $p(z)$, where $p \in B(F)$ (i.e., p is bounded on F), is defined as:

$$\int_a^b p(z)\, d_F^\alpha z = \sup_{\mathcal{Q}_{[a,b]}} \sum_{j=0}^{n-1} \inf_{z\in F\cap J} p(z)\, (S_F^\alpha(z_{j+1}) - S_F^\alpha(z_j)) \tag{3}$$

$$= \inf_{\mathcal{Q}_{[a,b]}} \sum_{j=0}^{n-1} \sup_{z\in F\cap J} p(z)\, (S_F^\alpha(z_{j+1}) - S_F^\alpha(z_j)), \tag{4}$$

where $z \in F$, and the infimum or supremum is taken over all partitions $\mathcal{Q}_{[a,b]}$ [17, 42].

3 Fractal Homogeneous Linear Systems

In this section, we extend the methods for solving fractal homogeneous linear systems with given initial conditions. Consider the fractal homogeneous system of linear differential equations defined by [19]:

$$D_F^\alpha \mathbf{x}(t) = \mathbf{P}(t)\mathbf{x}(t), \tag{5}$$

that defined on the interval $a < t < b$. Let $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ be a fundamental set of solutions to this system. The matrix formed by these solutions is called the fundamental matrix of the system:

$$\mathbf{\Phi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}. \tag{6}$$

Using this fundamental matrix, the general solution of the differential equation in Eq.(5) can be expressed as:

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t),$$

which can be compactly written in matrix form as:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c}, \tag{7}$$

where $\mathbf{c}$ is a constant vector with components $c_1, \ldots, c_n$. This representation provides a concise way to describe the general solution of the initial value problem using the fundamental matrix $\mathbf{\Phi}(t)$.

For an initial value problem defined by Eq.(5) with the initial condition:

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{8}$$

where t_0 is within the interval $a < t < b$ and $\mathbf{x}_0$ is the specified initial vector, the vector $\mathbf{c}$ in Eq.(7) must be determined so that the initial condition in Eq.(8) is satisfied. This leads to the equation:

$$\mathbf{\Phi}(t_0)\mathbf{c} = \mathbf{x}_0.$$

Since $\mathbf{\Phi}(t_0)$ is nonsingular, we can solve for $\mathbf{c}$ as follows:

$$\mathbf{c} = \mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0.$$

Therefore, the solution to the initial value problem defined by Eqs.(5) and (8) can be written as:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0.$$

In this expression, each column of the fundamental matrix $\mathbf{\Phi}(t)$ represents a solution to Eq.(5). Thus, $\mathbf{\Phi}(t)$ satisfies the matrix differential equation:

$$D_F^\alpha \mathbf{\Phi}(t) = \mathbf{P}(t)\mathbf{\Phi}(t).$$

This formulation provides a clear and efficient way to express the solution to the initial value problem using the fundamental matrix approach. To find the special fundamental matrix $\mathbf{\Psi}(t)$ that satisfies the condition $\mathbf{\Psi}(t_0) = I$ (where I is the identity matrix), we can normalize $\mathbf{\Phi}(t)$ by multiplying it with the inverse of $\mathbf{\Phi}(t_0)$:

$$\mathbf{\Psi}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0). \tag{9}$$

This approach ensures that $\mathbf{\Psi}(t_0) = I$, thereby simplifying the matrix to a standard form. The columns of $\mathbf{\Psi}(t)$ correspond to the particular solutions $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ that satisfy the initial conditions:

$$\mathbf{x}^{(j)}(t_0) = \mathbf{e}^{(j)},$$

where $\mathbf{e}^{(j)}$ are the standard basis vectors. This method provides a straightforward way to represent the general solution of the initial value problem as:

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{x}_0, \tag{10}$$

which offers a clear and efficient expression for solving the initial value problem using the special fundamental matrix $\mathbf{\Psi}(t)$. This approach highlights the utility of $\mathbf{\Psi}(t)$ in simplifying the solution process by directly aligning it with the initial conditions.

Example 1. Consider the following system of fractal linear differential equations:

$$D_F^{\alpha}\mathbf{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{x},$$

with the initial condition:

$$\mathbf{x}(t_0) = \mathbf{x}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \tag{11}$$

where the coefficient matrix is given by $\mathbf{P}(t) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

To find the fundamental matrix, we first need the general solution of the system. We assume solutions of the form:

$$\mathbf{x} = \exp(\lambda S_F^{\alpha}(t))\mathbf{v},$$

where λ is an eigenvalue and $\mathbf{v}$ is the corresponding eigenvector. The characteristic equation of the matrix $\mathbf{P}(t)$ is determined by:

$$\det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda) - 1 \times 1 = \lambda^2 - 4\lambda + 3 = 0.$$

Solving this characteristic equation, we obtain the eigenvalues and corresponding eigenvectors as:

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad \lambda_2 = 3, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The solutions corresponding to the eigenvalues and eigenvectors are given by:

$$\mathbf{x}^{(1)}(t) = \exp(S_F^{\alpha}(t))\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{x}^{(2)}(t) = \exp(3S_F^{\alpha}(t))\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus, the fundamental matrix $\mathbf{\Phi}(t)$ is:

$$\mathbf{\Phi}(t) = \begin{bmatrix} \exp(S_F^{\alpha}(t)) & \exp(3S_F^{\alpha}(t)) \\ -\exp(S_F^{\alpha}(t)) & \exp(3S_F^{\alpha}(t)) \end{bmatrix}.$$

Using Eq. (11), we have:

$$\mathbf{\Phi}(0)\mathbf{c} = \mathbf{x}_0. \tag{12}$$

At $t = 0$, the fundamental matrix becomes:

$$\mathbf{\Phi}(0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Solving Eq. (12), we find:

$$\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, the solution to the initial value problem is:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} = \begin{bmatrix} \exp(S_F^\alpha(t)) & \exp(3S_F^\alpha(t)) \\ -\exp(S_F^\alpha(t)) & \exp(3S_F^\alpha(t)) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \exp(S_F^\alpha(t)) + \exp(3S_F^\alpha(t)) \\ -\exp(S_F^\alpha(t)) + \exp(3S_F^\alpha(t)) \end{bmatrix}.$$

To find the special fundamental matrix $\mathbf{\Psi}(t)$, we use Eq. (9):

$$\mathbf{\Psi}(t) = \begin{bmatrix} \frac{1}{2}(\exp(S_F^\alpha(t)) + \exp(3S_F^\alpha(t))) & \frac{1}{2}(\exp(3S_F^\alpha(t)) - \exp(S_F^\alpha(t))) \\ \frac{1}{2}(-\exp(S_F^\alpha(t)) + \exp(3S_F^\alpha(t))) & \frac{1}{2}(\exp(S_F^\alpha(t)) + \exp(3S_F^\alpha(t))) \end{bmatrix}.$$

Using this special fundamental matrix, the solution $\mathbf{x}(t)$ is:

$$\mathbf{x}(t) = \begin{bmatrix} \frac{1}{2}(\exp(S_F^\alpha(t)) + \exp(3S_F^\alpha(t))) & \frac{1}{2}(\exp(3S_F^\alpha(t)) - \exp(S_F^\alpha(t))) \\ \frac{1}{2}(-\exp(S_F^\alpha(t)) + \exp(3S_F^\alpha(t))) & \frac{1}{2}(\exp(S_F^\alpha(t)) + \exp(3S_F^\alpha(t))) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} \exp(S_F^\alpha(t)) + \exp(3S_F^\alpha(t)) \\ -\exp(S_F^\alpha(t)) + \exp(3S_F^\alpha(t)) \end{bmatrix}. \tag{14}$$

This expression satisfies the given initial condition.

Example 2. Consider the system of fractal linear differential equations given by:

$$D_F^\alpha \mathbf{x} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \mathbf{x},$$

with the initial condition:

$$\mathbf{x}(t_0) = \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{15}$$

Solving the characteristic equation, we obtain the eigenvalues and their corresponding eigenvectors as:

$$\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}; \quad \lambda_2 = 5, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{16}$$

Using Eq.(6), the general form of the fundamental matrix $\mathbf{\Phi}(t)$ is:

$$\mathbf{\Phi}(t) = \begin{bmatrix} \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) \end{bmatrix} = \begin{bmatrix} 2\exp(2S_F^\alpha(t)) & \exp(5S_F^\alpha(t)) \\ -\exp(2S_F^\alpha(t)) & \exp(5S_F^\alpha(t)) \end{bmatrix}.$$

Evaluating $\mathbf{\Phi}(0)$, we obtain:

$$\mathbf{\Phi}(0) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

Using Eq.(9), we find the special fundamental matrix $\mathbf{\Psi}(t)$ as:

$$\mathbf{\Psi}(t) = \frac{1}{3} \begin{bmatrix} 2\exp(2S_F^\alpha(t)) + \exp(5S_F^\alpha(t)) & -2\exp(2S_F^\alpha(t)) + 2\exp(5S_F^\alpha(t)) \\ -\exp(2S_F^\alpha(t)) + \exp(5S_F^\alpha(t)) & \exp(2S_F^\alpha(t)) + 2\exp(5S_F^\alpha(t)) \end{bmatrix}.$$

Applying Eq.(10), the solution to the initial value problem is:

$$\mathbf{x}(t) = \frac{1}{3} \begin{bmatrix} 2\exp(2S_F^\alpha(t)) + \exp(5S_F^\alpha(t)) \\ -\exp(2S_F^\alpha(t)) + \exp(5S_F^\alpha(t)) \end{bmatrix}.$$

This approach demonstrates the advantage of employing the special fundamental matrix $\mathbf{\Psi}(t)$, which satisfies $\mathbf{\Psi}(t_0) = I$, to simplify the solution of initial value problems. By normalizing the fundamental matrix at the initial time t_0, the special fundamental matrix facilitates a more direct and efficient computation of the solution.

Corollary 1. *Let $\mathbf{A}$ be a constant $n \times n$ matrix and $\mathbf{x}_0$ be a given $n \times 1$ vector. The solution $\mathbf{x}(t)$ to the initial value problem*

$$D_F^{\alpha}\mathbf{x} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{17}$$

is given by:

$$\mathbf{x}(t) = \exp(\mathbf{A}S_F^{\alpha}(t))\mathbf{x}_0,$$

where $\exp(\mathbf{A}S_F^{\alpha}(t))$ denotes the matrix exponential defined by the series:

$$\exp(\mathbf{A}S_F^{\alpha}(t)) = \mathbf{I} + \mathbf{A}S_F^{\alpha}(t) + \frac{(\mathbf{A}S_F^{\alpha}(t))^2}{2!} + \frac{(\mathbf{A}S_F^{\alpha}(t))^3}{3!} + \cdots .$$

Proof. To prove this corollary, we begin by fractal integrating both sides of Eq.(17) with respect to t from 0 to t:

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t D_F^{\alpha}\mathbf{x}(s)\, d_F^{\alpha}s.$$

Given that $\mathbf{x}(0) = \mathbf{x}_0$, this simplifies to:

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t D_F^{\alpha}\mathbf{x}(s)\, d_F^{\alpha}s.$$

Substituting $D_F^{\alpha}\mathbf{x}(s) = \mathbf{A}\mathbf{x}(s)$, we obtain:

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{A}\mathbf{x}(s)\, d_F^{\alpha}s.$$

Let $\mathbf{x}(t) = \boldsymbol{\phi}(t)$, which leads to the fractal integral equation:

$$\boldsymbol{\phi}(t) = \mathbf{x}_0 + \int_0^t \mathbf{A}\boldsymbol{\phi}(s)\, d_F^{\alpha}s.$$

We define the initial approximation as $\boldsymbol{\phi}^{(0)}(t) = \mathbf{x}_0$. For each subsequent approximation, we use:

$$\boldsymbol{\phi}^{(n+1)}(t) = \mathbf{x}_0 + \int_0^t \mathbf{A}\boldsymbol{\phi}^{(n)}(s)\, d_F^{\alpha}s. \tag{18}$$

The first approximation using $\boldsymbol{\phi}^{(0)}(s) = \mathbf{x}_0$ is given by:

$$\boldsymbol{\phi}^{(1)}(t) = \mathbf{x}_0 + \int_0^t \mathbf{A}\mathbf{x}_0\, d_F^{\alpha}s = \mathbf{x}_0 + \mathbf{A}S_F^{\alpha}(t)\mathbf{x}_0 = (\mathbf{I} + \mathbf{A}S_F^{\alpha}(t))\mathbf{x}_0.$$

For the second approximation, using $\boldsymbol{\phi}^{(1)}(s) = (\mathbf{I} + \mathbf{A}S_F^{\alpha}(s))\mathbf{x}_0$, we obtain:

$$\boldsymbol{\phi}^{(2)}(t) = \mathbf{x}_0 + \int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}S_F^{\alpha}(s))\mathbf{x}_0\, d_F^{\alpha}s \tag{19}$$

$$= \mathbf{x}_0 + \mathbf{A}S_F^{\alpha}(t)\mathbf{x}_0 + \frac{\mathbf{A}^2 S_F^{\alpha}(t)^2}{2!}\mathbf{x}_0$$

$$= \left(\mathbf{I} + \mathbf{A}S_F^{\alpha}(t) + \frac{\mathbf{A}^2 S_F^{\alpha}(t)^2}{2!}\right)\mathbf{x}_0. \tag{20}$$

Assuming $\boldsymbol{\phi}^{(n)}(t) = \left(\mathbf{I} + \mathbf{A}S_F^\alpha(t) + \frac{\mathbf{A}^2 S_F^\alpha(t)^2}{2!} + \cdots + \frac{\mathbf{A}^n S_F^\alpha(t)^n}{n!}\right)\mathbf{x}_0$, and applying Eq.(18), we obtain:

$$\boldsymbol{\phi}^{(n+1)}(t) = \mathbf{x}_0 + \int_0^t \mathbf{A}\left(\mathbf{I} + \mathbf{A}S_F^\alpha(s) + \frac{\mathbf{A}^2 S_F^\alpha(s)^2}{2!} + \cdots + \frac{\mathbf{A}^n S_F^\alpha(s)^n}{n!}\right)\mathbf{x}_0 \, d_F^\alpha s.$$

Fractal integrating each term, we get:

$$\boldsymbol{\phi}^{(n+1)}(t) = \left(\mathbf{I} + \mathbf{A}S_F^\alpha(t) + \frac{\mathbf{A}^2 S_F^\alpha(t)^2}{2!} + \cdots + \frac{\mathbf{A}^{n+1} S_F^\alpha(t)^{n+1}}{(n+1)!}\right)\mathbf{x}_0.$$

Taking the limit as $n \to \infty$, the sequence of approximations $\boldsymbol{\phi}^{(n)}(t)$ converges to:

$$\boldsymbol{\phi}(t) = \sum_{k=0}^{\infty} \frac{(\mathbf{A}S_F^\alpha(t))^k}{k!}\mathbf{x}_0 = \exp(\mathbf{A}S_F^\alpha(t))\mathbf{x}_0.$$

This completes the proof. □

Example 3. Consider the system of fractal differential equations given by:

$$D_F^\alpha \mathbf{x} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{21}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{22}$$

To find the solution to Eq.(21), we use the matrix exponential series:

$$\exp(\mathbf{A}S_F^\alpha(t)) = \mathbf{I} + \mathbf{A}S_F^\alpha(t) + \frac{\mathbf{A}^2 S_F^\alpha(t)^2}{2!} + \frac{\mathbf{A}^3 S_F^\alpha(t)^3}{3!} + \cdots. \tag{23}$$

Substituting Eq.(22) into Eq.(23), we get:

$$\exp(\mathbf{A}S_F^\alpha(t)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + S_F^\alpha(t)\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + \frac{S_F^\alpha(t)^2}{2}\begin{bmatrix} 4 & 5 \\ 0 & 9 \end{bmatrix} + \frac{S_F^\alpha(t)^3}{6}\begin{bmatrix} 8 & 19 \\ 0 & 27 \end{bmatrix} + \cdots.$$

Taking the limit as the number of terms approaches infinity, the closed-form expression is:

$$\exp(\mathbf{A}S_F^\alpha(t)) = \begin{bmatrix} \exp(2S_F^\alpha(t)) & \exp(3S_F^\alpha(t)) - \exp(2S_F^\alpha(t)) \\ 0 & \exp(3S_F^\alpha(t)) \end{bmatrix}.$$

Using the initial value $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the solution is:

$$\mathbf{x}(t) = \exp(\mathbf{A}S_F^\alpha(t))\mathbf{x}_0 = \begin{bmatrix} \exp(2S_F^\alpha(t)) & \exp(3S_F^\alpha(t)) - \exp(2S_F^\alpha(t)) \\ 0 & \exp(3S_F^\alpha(t)) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \exp(2S_F^\alpha(t)) \\ 0 \end{bmatrix}.$$

Example 4. Consider a system of fractal homogeneous differential equations described by:

$$D_F^\alpha \mathbf{x}(t) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t), \tag{24}$$

where the 2×2 matrix $\mathbf{A}$ cannot be diagonalized due to a deficiency in the number of eigenvectors, typically caused by repeated eigenvalues. In such cases, $\mathbf{A}$ can be transformed into a nearly diagonal matrix known as its Jordan form. The Jordan form of $\mathbf{A}$ features its eigenvalues on the main diagonal, ones in specific positions immediately above the main diagonal, and zeros elsewhere.

To transform $\mathbf{A}$ into its Jordan form, we construct the transformation matrix $\mathbf{T}$ using the eigenvector and the generalized eigenvector. The transformation matrix $\mathbf{T}$ and its inverse are given by:

$$\mathbf{T}=\begin{bmatrix}1 & 0\\ -1 & -1\end{bmatrix},\quad \mathbf{T}^{-1}=\begin{bmatrix}1 & 0\\ -1 & -1\end{bmatrix}.$$

Upon verification, we find that:

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T}=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}=\mathbf{J},$$

where $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

Next, consider the differential equation given by Eq.(24). Applying the transformation $\mathbf{x}=\mathbf{T}\mathbf{y}$, we obtain:

$$D_F^{\alpha}\mathbf{y}(t)=\mathbf{J}\mathbf{y}(t), \tag{25}$$

In scalar form, Eq.(25) can be written as:

$$D_F^{\alpha}y_1=2y_1+y_2,\quad D_F^{\alpha}y_2=2y_2. \tag{26}$$

The independent solutions of Eq.(26) are:

$$\mathbf{y}^{(1)}(t)=\begin{bmatrix}1\\0\end{bmatrix}\exp(2S_F^{\alpha}(t)),\quad \mathbf{y}^{(2)}(t)=\begin{bmatrix}S_F^{\alpha}(t)\\1\end{bmatrix}\exp(2S_F^{\alpha}(t)).$$

The corresponding special fundamental matrix is:

$$\mathbf{\Psi}(t)=\begin{bmatrix}\exp(2S_F^{\alpha}(t)) & S_F^{\alpha}(t)\exp(2S_F^{\alpha}(t))\\ 0 & \exp(2S_F^{\alpha}(t))\end{bmatrix},$$

with $\mathbf{\Psi}(0)=\mathbf{I}$.

To find the fundamental matrix for Eq.(24), we compute:

$$\mathbf{\Psi}(t)=\mathbf{T}\exp(\mathbf{J}S_F^{\alpha}(t))=\begin{bmatrix}\exp(2S_F^{\alpha}(t)) & S_F^{\alpha}(t)\exp(2S_F^{\alpha}(t))\\ -\exp(2S_F^{\alpha}(t)) & -\exp(2S_F^{\alpha}(t))-S_F^{\alpha}(t)\exp(2S_F^{\alpha}(t))\end{bmatrix}.$$

Therefore, the solution to Eq.(24) is given by $\mathbf{x}(t)=\mathbf{\Psi}(t)\mathbf{x}_0$.

4 Fractal Nonhomogeneous Linear Systems

In this section, we generalize the study of nonhomogeneous linear systems on fractal sets within the framework of fractal calculus. This approach extends traditional differential equations to account for the complexities of fractal structures, which can model phenomena characterized by irregular or self-similar behavior often observed in nature. We also address solutions using the methods of undetermined coefficients and variation of parameters.

The fractal nonhomogeneous linear system of differential equations is given by:

$$D_F^{\alpha}\mathbf{x}(t)=\mathbf{P}(t)\mathbf{x}(t)+\mathbf{g}(t),\quad t\in F, \tag{27}$$

where $0<\alpha\leq 1$ represents the non-integer dimensional behavior associated with fractals. Here, $\mathbf{P}(t)$ is an $n\times n$ matrix that depends on t and is F-continuous on the interval $a<t<b$. $\mathbf{g}(t)$ is an $n\times 1$ vector function, representing the nonhomogeneous term, and is also F-continuous on the same interval.

The general solution to this fractal system can be expressed similarly to the standard nonhomogeneous system:

$$\mathbf{x}(t)=c_1\mathbf{x}^{(1)}(t)+\cdots+c_n\mathbf{x}^{(n)}(t)+\mathbf{v}(t),$$

where $c_1\mathbf{x}^{(1)}(t)+\cdots+c_n\mathbf{x}^{(n)}(t)$ represents the general solution of the associated homogeneous system:

$$D_F^\alpha \mathbf{x}(t)=\mathbf{P}(t)\mathbf{x}(t),$$

and $\mathbf{v}(t)$ is a particular solution of the fractal nonhomogeneous system described by Eq.(27). We employ the diagonalization method as a special case for solving this fractal nonhomogeneous system. When $\mathbf{P}(t)=\mathbf{A}$, where $\mathbf{A}$ is a constant, diagonalizable $n\times n$ matrix, the system simplifies to:

$$D_F^\alpha \mathbf{x}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{g}(t).$$

To diagonalize the matrix $\mathbf{A}$, proceed as follows:
1. Let $\mathbf{T}$ be the matrix whose columns consist of the eigenvectors $\boldsymbol{\xi}^{(1)},\ldots,\boldsymbol{\xi}^{(n)}$ of $\mathbf{A}$.
2. Express $\mathbf{A}$ as $\mathbf{A}=\mathbf{T}\mathbf{D}\mathbf{T}^{-1}$, where $\mathbf{D}$ is a diagonal matrix containing the eigenvalues $r_1,\ldots,r_n$ of $\mathbf{A}$ on its diagonal.

Define a new variable $\mathbf{y}$ by the transformation:

$$\mathbf{x}=\mathbf{T}\mathbf{y}.$$

Substituting this into Eq.(27), we obtain:

$$\mathbf{T}D_F^\alpha \mathbf{y}=\mathbf{A}\mathbf{T}\mathbf{y}+\mathbf{g}(t). \tag{28}$$

Multiplying Eq.(28) by $\mathbf{T}^{-1}$ results in:

$$D_F^\alpha \mathbf{y}=\mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y}+\mathbf{T}^{-1}\mathbf{g}(t)=\mathbf{D}\mathbf{y}+\mathbf{h}(t), \tag{29}$$

where $\mathbf{h}(t)=\mathbf{T}^{-1}\mathbf{g}(t)$. Since $\mathbf{D}$ is diagonal, the system $D_F^\alpha \mathbf{y}=\mathbf{D}\mathbf{y}+\mathbf{h}(t)$ consists of n uncoupled equations:

$$D_F^\alpha y_j(t)=r_j y_j(t)+h_j(t),\quad j=1,\ldots,n, \tag{30}$$

where $h_j(t)$ is the j-th component of the vector $\mathbf{h}(t)$. The general solution of each equation in Eq.(30) is:

$$y_j(t)=\exp(r_j S_F^\alpha(t))\left(\int_{t_0}^{t}\exp(-r_j S_F^\alpha(s))h_j(s)\,d_F^\alpha s+c_j\right),\quad j=1,\ldots,n,$$

where c_j are arbitrary constants determined by the initial conditions. Substituting back to the original variable $\mathbf{x}(t)$ using $\mathbf{x}=\mathbf{T}\mathbf{y}$:

$$\mathbf{x}(t)=\mathbf{T}\mathbf{y}(t)=\mathbf{T}\begin{bmatrix}y_1(t)\\ y_2(t)\\ \vdots\\ y_n(t)\end{bmatrix}.$$

The solution $\mathbf{x}(t)$ consists of two parts:
1. The first term in $y_j(t)$ provides a particular solution to the fractal nonhomogeneous system.
2. The second term, involving the constants c_j, represents the general solution to the fractal homogeneous system $D_F^\alpha \mathbf{x}=\mathbf{A}\mathbf{x}$.

Example 5. Consider the following fractal nonhomogeneous system:

$$D_F^\alpha \mathbf{x}=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\mathbf{x}+\begin{bmatrix}2\exp(-S_F^\alpha(t))\\ 3S_F^\alpha(t)\end{bmatrix}=\mathbf{A}\mathbf{x}+\mathbf{g}(t).$$

The coefficient matrix $\mathbf{A}$ is:

$$\mathbf{A}=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}.$$

The eigenvalues of $\mathbf{A}$ are $r_1 = -3$ and $r_2 = -1$. The corresponding eigenvectors are:

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The general solution of the fractal homogeneous system is given by:

$$\mathbf{x}_h = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \exp(-3S_F^\alpha(t)) + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(-S_F^\alpha(t)).$$

Considering Eq.(29) and the matrices:

$$\mathbf{T} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}, \tag{31}$$

we can express the system as:

$$D_F^\alpha \mathbf{y} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y} + \frac{1}{\sqrt{2}} \begin{bmatrix} 2\exp(-S_F^\alpha(t)) - 3S_F^\alpha(t) \\ 2\exp(-S_F^\alpha(t)) + 3S_F^\alpha(t) \end{bmatrix}.$$

The system reduces to two uncoupled fractal equations:

$$\begin{aligned} D_F^\alpha y_1 + 3y_1 &= \frac{1}{\sqrt{2}}(2\exp(-S_F^\alpha(t)) - 3S_F^\alpha(t)), \\ D_F^\alpha y_2 + y_2 &= \frac{1}{\sqrt{2}}(2\exp(-S_F^\alpha(t)) + 3S_F^\alpha(t)). \end{aligned} \tag{32}$$

Using the fractal integrating factor method [18], we obtain:

$$\begin{aligned} y_1(t) &= \exp(-3S_F^\alpha(t)) \left(\int \exp(3S_F^\alpha(t)) \frac{\sqrt{2}}{2}(2\exp(-S_F^\alpha(t)) - 3S_F^\alpha(t))\, d_F^\alpha t + c_1 \right), \\ y_2(t) &= \exp(-S_F^\alpha(t)) \left(\int \exp(S_F^\alpha(t)) \frac{\sqrt{2}}{2}(2\exp(-S_F^\alpha(t)) + 3S_F^\alpha(t))\, d_F^\alpha t + c_2 \right). \end{aligned} \tag{33}$$

Simplifying further, we have:

$$\begin{aligned} y_1(t) &= \frac{\sqrt{2}}{2}\exp(-S_F^\alpha(t)) - \frac{\sqrt{2}}{3} S_F^\alpha(t) - \frac{\sqrt{2}}{9} + c_1 \exp(-3S_F^\alpha(t)), \\ y_2(t) &= \sqrt{2} S_F^\alpha(t)\exp(-S_F^\alpha(t)) + 3\sqrt{2}(S_F^\alpha(t) - \chi_F) + c_2\exp(-S_F^\alpha(t)). \end{aligned} \tag{34}$$

The solution in terms of the original variable $\mathbf{x}$ is then given by:

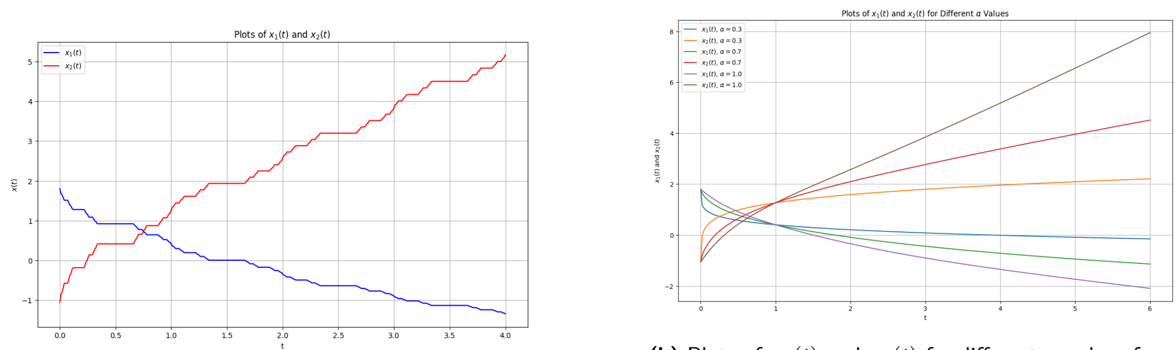
$$\mathbf{x} = \mathbf{T}\mathbf{y} = \frac{1}{\sqrt{2}} \begin{bmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{bmatrix}.$$

Thus, the complete solution for $\mathbf{x}(t)$ is:

$$\mathbf{x}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} c_1\exp(-3S_F^\alpha(t)) + \left(c_2 + \frac{\sqrt{2}}{2}\right)\exp(-S_F^\alpha(t)) - \frac{\sqrt{2}}{3} S_F^\alpha(t) \\ -\frac{\sqrt{2}}{9} + \sqrt{2} S_F^\alpha(t)\exp(-S_F^\alpha(t)) \\ -c_1\exp(-3S_F^\alpha(t)) + \left(c_2 - \frac{\sqrt{2}}{2}\right)\exp(-S_F^\alpha(t)) + 2S_F^\alpha(t) \\ -\frac{5\sqrt{2}}{9} + \sqrt{2} S_F^\alpha(t)\exp(-S_F^\alpha(t)) \end{bmatrix} \tag{35}$$

$$\approx \frac{1}{\sqrt{2}} \begin{bmatrix} c_1\exp(-3t^\alpha) + \left(c_2 + \frac{\sqrt{2}}{2}\right)\exp(-t^\alpha) - \frac{\sqrt{2}}{3} t^\alpha \\ -\frac{\sqrt{2}}{9} + \sqrt{2} t^\alpha \exp(-t^\alpha) \\ -c_1\exp(-3t^\alpha) + \left(c_2 - \frac{\sqrt{2}}{2}\right)\exp(-t^\alpha) + 2t^\alpha \\ -\frac{5\sqrt{2}}{9} + \sqrt{2} t^\alpha \exp(-t^\alpha) \end{bmatrix}. \tag{36}$$

This represents the complete solution for $\mathbf{x}(t)$ in terms of the initial fractal nonhomogeneous system.



(a) Components $x_1(t)$ and $x_2(t)$ of $\mathbf{x}(t)$ from Eq.(35). $x_1(t)$ is shown in blue, and $x_2(t)$ in red.

(b) Plots of $x_1(t)$ and $x_2(t)$ for different α values from Eq.(36). The functions are shown for $\alpha = 0.3$, $\alpha = 0.7$, and $\alpha = 1.0$.

Fig. 1: Comparison of $x_1(t)$ and $x_2(t)$ for different parameter settings. (a) shows their behavior under the Cantor set influence, while (b) illustrates how varying α affects the functions.

Figure 2 compares $x_1(t)$ and $x_2(t)$ for different conditions: (a) on a fractal set and (b) for varying α values. This comparison reveals how the fractal domain and parameter α influence the functions' dynamics.

4.1 Undetermined Coefficients for Fractal Nonhomogeneous Differential Equations

In this section, we describe how to use the method of undetermined coefficients to solve fractal nonhomogeneous linear differential equations. To apply this method to the differential equation given by

$$D_F^{\alpha}\mathbf{x}(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t), \quad t \in F, \tag{37}$$

we assume a particular solution $\mathbf{x}_p(t)$ that matches the structure of the nonhomogeneous term $\mathbf{g}(t)$. This method is effective under the following conditions:

1. The coefficient matrix $\mathbf{P}(t)$ must be constant, i.e., $\mathbf{P}(t) = \mathbf{P}$.
2. The nonhomogeneous term $\mathbf{g}(t)$ should be a polynomial, fractal exponential, fractal sinusoidal function, or a combination of these types. This requirement allows us to correctly predict the form of $\mathbf{x}_p(t)$.

To apply the method, follow these steps:

- Based on the structure of $\mathbf{g}(t)$, assume a form for the particular solution $\mathbf{x}_p(t)$. The assumed form should include unknown coefficients that will be determined later.
- If $\mathbf{g}(t)$ is a polynomial of degree n, assume $\mathbf{x}_p(t)$ is a polynomial of the same degree.
- If $\mathbf{g}(t)$ is an exponential function $\exp(\lambda S_F^{\alpha}(t))$, assume $\mathbf{x}_p(t) = \mathbf{A}\exp(\lambda S_F^{\alpha}(t))$.
- If $\mathbf{g}(t)$ is a sinusoidal function $\sin(\omega S_F^{\alpha}(t))$ or $\cos(\omega S_F^{\alpha}(t))$, assume $\mathbf{x}_p(t)$ includes terms like $\mathbf{A}\cos(\omega S_F^{\alpha}(t)) + \mathbf{B}\sin(\omega S_F^{\alpha}(t))$.
- If $\mathbf{g}(t)$ consists of combinations (e.g., fractal polynomials multiplied by exponentials), then assume that $\mathbf{x}_p(t)$ includes a corresponding combined form.

1. Substitute the assumed $\mathbf{x}_p(t)$ into Eq.(37).
2. Solve for the unknown coefficients by equating terms from both sides of the fractal differential equation. This step involves simplifying the equation to match the assumed form of $\mathbf{x}_p(t)$ to the structure of $\mathbf{g}(t)$.
3. Combine the particular solution $\mathbf{x}_p(t)$ with the general solution $\mathbf{x}_h(t)$ of the associated fractal homogeneous differential equation $D_F^{\alpha}\mathbf{x}(t) = \mathbf{P}\mathbf{x}(t)$.

Example 6. Consider the fractal nonhomogeneous linear differential system:

$$D_F^\alpha \mathbf{x}(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2\exp(-S_F^\alpha(t)) \\ 3S_F^\alpha(t) \end{bmatrix} = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t). \tag{38}$$

This is the same system as in Example 5. To apply the method of undetermined coefficients, rewrite $\mathbf{g}(t)$ as:

$$\mathbf{g}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \exp(-S_F^\alpha(t)) + \begin{bmatrix} 0 \\ 3 \end{bmatrix} S_F^\alpha(t).$$

Assume the particular solution $\mathbf{x}_p(t)$ takes the form:

$$\mathbf{x}_p(t) = \mathbf{v}(t) = \mathbf{a}S_F^\alpha(t)\exp(-S_F^\alpha(t)) + \mathbf{b}\exp(-S_F^\alpha(t)) + \mathbf{c}S_F^\alpha(t) + \mathbf{d}, \tag{39}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ are vectors to be determined. Since $r = -1$ is an eigenvalue of the coefficient matrix $\mathbf{A}$, we include both $S_F^\alpha(t)\exp(-S_F^\alpha(t))$ and $\exp(-S_F^\alpha(t))$ terms in the assumed solution. Substituting Eq.(39) into Eq.(38) and simplifying, we derive the following algebraic equations:

$$\mathbf{A}\mathbf{a} = -\mathbf{a}, \quad \mathbf{A}\mathbf{b} = \mathbf{a} - \mathbf{b} - \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \mathbf{A}\mathbf{c} = -\begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{A}\mathbf{d} = \mathbf{c}.$$

From the first equation, $\mathbf{a}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $r = -1$. Therefore, $\mathbf{a}^T = (\nu, \nu)$ with ν being any nonzero constant. Solving the second equation, we find $\nu = 1$, which gives:

$$\mathbf{b} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

for any constant k. Choosing the simplest value $k = 0$, we get $\mathbf{b}^T = (0, -1)$. Solving the third and fourth equations, we find $\mathbf{c}^T = (1, 2)$ and $\mathbf{d}^T = \left(-\frac{4}{3}, -\frac{5}{3}\right)$, respectively. Thus, the particular solution is:

$$\mathbf{v}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} S_F^\alpha(t)\exp(-S_F^\alpha(t)) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \exp(-S_F^\alpha(t)) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} S_F^\alpha(t) - \frac{1}{3}\begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

4.2 Variation of Parameters for Fractal Nonhomogeneous Differential Equations

In this section, we introduce the method of Variation of Parameters for solving fractal nonhomogeneous linear differential equations, particularly when the coefficient matrix is non-constant or not diagonalizable.

Consider the fractal nonhomogeneous linear differential system given by:

$$D_F^\alpha \mathbf{x}(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t), \tag{40}$$

where $\mathbf{P}(t)$ is a matrix function. To find a solution to Eq.(40), we replace the fractal constant vector $\mathbf{c}$ with a vector function $\mathbf{u}(t)$ in Eq.(7), leading to:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{u}(t), \tag{41}$$

where $\mathbf{u}(t)$ is the fractal vector function that we need to determine. To find $\mathbf{u}(t)$, we differentiate $\mathbf{x}(t)$ fractally as expressed in Eq.(41) and require that it satisfies Eq.(40). This yields:

$$D_F^\alpha \mathbf{\Phi}(t)\mathbf{u}(t) + \mathbf{\Phi}(t)D_F^\alpha \mathbf{u}(t) = \mathbf{P}(t)\mathbf{\Phi}(t)\mathbf{u}(t) + \mathbf{g}(t). \tag{42}$$

Since $\mathbf{\Phi}(t)$ is a fundamental matrix, it satisfies $D_F^\alpha \mathbf{\Phi}(t) = \mathbf{P}(t)\mathbf{\Phi}(t)$. Substituting this into Eq.(42), we simplify it to:

$$\mathbf{\Phi}(t)D_F^\alpha \mathbf{u}(t) = \mathbf{g}(t). \tag{43}$$

Because $\mathbf{\Phi}(t)$ is nonsingular on any interval where $\mathbf{P}(t)$ is continuous, its inverse $\mathbf{\Phi}^{-1}(t)$ exists. Thus, Eq.(43) simplifies to:

$$D_F^\alpha \mathbf{u}(t) = \mathbf{\Phi}^{-1}(t)\mathbf{g}(t). \tag{44}$$

To find $\mathbf{u}(t)$, we integrate Eq.(44), resulting in:

$$\mathbf{u}(t) = \int \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)\, d_F^\alpha s + \mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector. Substituting $\mathbf{u}(t)$ back into Eq.(41), we obtain the solution to Eq.(40):

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)\, d_F^\alpha s. \tag{45}$$

Therefore, Eq.(45) represents the general solution of the nonhomogeneous linear differential system described by Eq.(40). The first term on the right side of Eq.(45) corresponds to the general solution of the associated homogeneous system, while the second term provides a particular solution to the nonhomogeneous system.

Now, consider the initial value problem defined by Eq.(40) with the initial condition:

$$\mathbf{x}(t_0) = \mathbf{x}_0. \tag{46}$$

To conveniently express the solution, we choose a particular solution of Eq.(40) that equals the zero vector at $t = t_0$. This can be achieved by setting the lower limit of integration in Eq.(45) to t_0, resulting in:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{\Phi}(t) \int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)\, d_F^\alpha s. \tag{47}$$

The initial condition given by Eq.(46) is satisfied if:

$$\mathbf{c} = \mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0. \tag{48}$$

Substituting $\mathbf{c}$ from Eq.(48) into Eq.(47) yields the solution for the initial value problem:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \mathbf{\Phi}(t) \int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)\, d_F^\alpha s. \tag{49}$$

While expressing solutions using the inverse of $\mathbf{\Phi}$, as seen in Eqs.(45) and (49), can be useful, in many cases, it is more practical to use row reduction techniques rather than directly calculating $\mathbf{\Phi}^{-1}$.

The solution in Eq.(49) can be further simplified if we employ a special fundamental matrix $\mathbf{\Psi}(t)$ that satisfies $\mathbf{\Psi}(t_0) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. With this choice, the solution takes a simpler form:

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{x}_0 + \mathbf{\Psi}(t) \int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)\, d_F^\alpha s.$$

This expression offers a streamlined method for solving initial value problems involving nonhomogeneous systems, especially when the coefficient matrix $\mathbf{P}(t)$ is constant. In such cases, the fundamental matrix $\mathbf{\Phi}(t)$ simplifies significantly, making it easier to compute its inverse and the overall solution.

When $\mathbf{P}$ is a constant matrix, the fundamental matrix $\mathbf{\Phi}(t)$ can be represented as $\exp(\mathbf{P}S_F^\alpha(t))$. Substituting this form into Eq.(49), the solution simplifies to:

$$\mathbf{x}(t) = \exp(\mathbf{P}(S_F^\alpha(t) - S_F^\alpha(t_0)))\mathbf{x}_0 + \exp(\mathbf{P}S_F^\alpha(t)) \int_{t_0}^{t} \exp(-\mathbf{P}S_F^\alpha(s))\mathbf{g}(s)\, d_F^\alpha s.$$

This simplified expression effectively captures the behavior of the system by leveraging the specific structure of the fundamental matrix for constant coefficient matrices. It combines the homogeneous solution and a particular solution, providing a more efficient path to solving the initial value problem.

Remark 1. The method of variation of parameters is particularly useful for cases where the coefficient matrix $\mathbf{A}$ is not constant or diagonalizable.

Example 7. Consider the system of fractal linear differential equations:

$$D_F^\alpha \mathbf{x} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2\exp(-S_F^\alpha(t)) \\ 3S_F^\alpha(t) \end{bmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \tag{50}$$

where the fundamental matrix $\mathbf{\Phi}(t)$ is given by:

$$\mathbf{\Phi}(t) = \begin{bmatrix} \exp(-3S_F^\alpha(t)) & \exp(-S_F^\alpha(t)) \\ -\exp(-3S_F^\alpha(t)) & \exp(-S_F^\alpha(t)) \end{bmatrix}.$$

The solution to the fractal nonhomogeneous system is $\mathbf{x} = \mathbf{\Phi}(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ satisfies:

$$\mathbf{\Phi}(t) D_F^\alpha \mathbf{u}(t) = \mathbf{g}(t).$$

Substituting $\mathbf{g}(t)$:

$$\begin{bmatrix} \exp(-3S_F^\alpha(t)) & \exp(-S_F^\alpha(t)) \\ -\exp(-3S_F^\alpha(t)) & \exp(-S_F^\alpha(t)) \end{bmatrix} \begin{bmatrix} D_F^\alpha u_1 \\ D_F^\alpha u_2 \end{bmatrix} = \begin{bmatrix} 2\exp(-S_F^\alpha(t)) \\ 3S_F^\alpha(t) \end{bmatrix}.$$

Solving this system by row reduction, we find:

$$D_F^\alpha u_1 = \exp(2S_F^\alpha(t)) - \frac{3}{2} S_F^\alpha(t) \exp(3S_F^\alpha(t)), \quad D_F^\alpha u_2 = 1 + \frac{3}{2} S_F^\alpha(t) \exp(S_F^\alpha(t)).$$

Fractal integrating these equations, we get:

$$u_1(t) = \frac{1}{2}\exp(2S_F^\alpha(t)) - \frac{1}{2} S_F^\alpha(t) \exp(3S_F^\alpha(t)) + \frac{1}{6}\exp(3S_F^\alpha(t)) + c_1, \tag{51}$$

$$u_2(t) = S_F^\alpha(t) + \frac{3}{2} S_F^\alpha(t) \exp(S_F^\alpha(t)) - \frac{3}{2}\exp(S_F^\alpha(t)) + c_2. \tag{52}$$

Substituting $\mathbf{u}(t)$ into $\mathbf{x} = \mathbf{\Phi}(t)\mathbf{u}(t)$, we obtain:

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \exp(-3S_F^\alpha(t)) + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(-S_F^\alpha(t)) + \tag{53}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} S_F^\alpha(t) \exp(-S_F^\alpha(t)) + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \exp(-S_F^\alpha(t)) + \tag{54}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} S_F^\alpha(t) - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}. \tag{55}$$

This solution matches the one obtained by other methods, confirming the consistency and validity of variation of parameters.

5 Application of Fractal Differential Equations to RLC Circuits

In this section, we explore the application of fractal nonhomogeneous differential equations to model electrical circuits with external sources, focusing specifically on RLC circuits composed of resistors, inductors, and capacitors. By incorporating fractal derivatives, we extend the classical model to account for fractal time, thus providing a more generalized framework that captures the effects of fractal geometry on circuit behavior. Traditionally, an RLC circuit is governed by differential equations derived from Kirchhoff's laws. However, by modifying these equations with fractal derivatives, which address the complex nature of fractal time, we extend the classical approach to a fractal context. This extension is crucial for modeling systems operating in fractal spaces, where temporal evolution exhibits non-integer dimensions and scale invariance.

The fractal differential equations for an RLC circuit with an external voltage source $V(t)$ are given by:

$$D_F^{\alpha} I(t) = \frac{1}{L}\left(V(t) - RI(t) - V_C(t)\right),$$

where $D_F^{\alpha} I(t)$ denotes the fractal derivative of the current with respect to time of order α, and $V_C(t) = \frac{1}{C}\int_0^t I(\tau)\, d_F^{\alpha}\tau$ represents the voltage across the capacitor, modified by fractal integration. Combining the component equations and differentiating with respect to fractal time yields:

$$LD_F^{2\alpha} I(t) + RD_F^{\alpha} I(t) + \frac{1}{C} I(t) = D_F^{\alpha} V(t).$$

To convert this into a system of first-order fractal differential equations, introduce state variables:

$$x_1(t) = I(t) \quad \text{(fractal current through the inductor)}, \tag{56}$$

$$x_2(t) = V_C(t) \quad \text{(fractal voltage across the capacitor)}. \tag{57}$$

We obtain:

$$D_F^{\alpha} x_1(t) = \frac{1}{L}\left(V(t) - Rx_1(t) - x_2(t)\right),$$

and

$$D_F^{\alpha} x_2(t) = \frac{x_1(t)}{C}.$$

This system can be written in matrix form as:

$$D_F^{\alpha}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V(t). \tag{58}$$

To find an analytical solution, assume a specific form for the input function, such as $V(t) = V_0 \exp(\kappa S_F^{\alpha}(t))$, where V_0 is a constant and κ is a complex frequency parameter, with $R = 4\,\Omega$, $L = 2\,\mathrm{H}$, and $C = 0.5\,\mathrm{F}$. This simplifies Eq. (58) to:

$$D_F^{\alpha}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2 & -0.5 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} V_0 \exp(\kappa S_F^{\alpha}(t)) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}V_0 \exp(\kappa S_F^{\alpha}(t)). \tag{59}$$

Using Example 1, the fractal homogeneous differential equation

$$D_F^{\alpha}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2 & -0.5 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{A}\mathbf{x}(t) \tag{60}$$

has the solution:

$$\mathbf{x}_h(t) = c_1 \exp\left((-1+\sqrt{2})S_F^{\alpha}(t)\right)\mathbf{v}_1 + c_2 \exp\left((-1-\sqrt{2})S_F^{\alpha}(t)\right)\mathbf{v}_2,$$

where

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}-1} \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \frac{1}{-\sqrt{2}-1} \\ 1 \end{bmatrix}.$$

To find the particular solution, assume the solution has the form:

$$\mathbf{x}_p(t) = \mathbf{C}\exp(\kappa S_F^\alpha(t)). \tag{61}$$

Substituting Eq. (61) into Eq. (59), we get:

$$\kappa\mathbf{C}\exp(\kappa S_F^\alpha(t)) = \mathbf{AC}\exp(\kappa S_F^\alpha(t)) + \mathbf{B}V_0\exp(\kappa S_F^\alpha(t)).$$

Solving for $\mathbf{C}$, we obtain:

$$\mathbf{C} = (\kappa\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}V_0.$$

Thus,

$$\mathbf{C} = \frac{V_0}{\kappa^2 + 2\kappa + 1}\begin{bmatrix}\kappa\\2\end{bmatrix}.$$

The particular solution is:

$$\mathbf{x}_p(t) = \frac{V_0\exp(\kappa S_F^\alpha(t))}{\kappa^2 + 2\kappa + 1}\begin{bmatrix}\kappa\\2\end{bmatrix}.$$

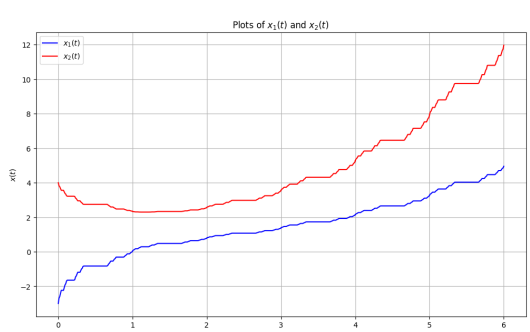
The general solution is:

$$\mathbf{x}(t) = c_1\exp\left((-1+\sqrt{2})S_F^\alpha(t)\right)\mathbf{v}_1 + c_2\exp\left((-1-\sqrt{2})S_F^\alpha(t)\right)\mathbf{v}_2 + \frac{V_0\exp(\kappa S_F^\alpha(t))}{\kappa^2 + 2\kappa + 1}\begin{bmatrix}\kappa\\2\end{bmatrix}.$$

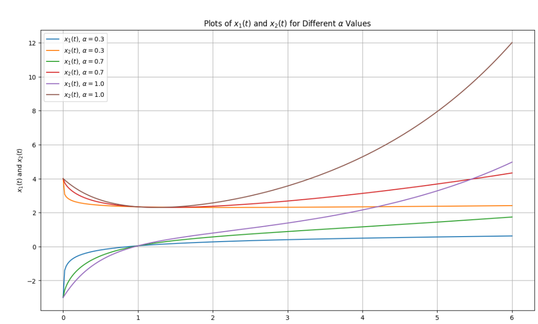
For a simple exponential decay, let $\kappa = -1$:

$$\begin{aligned} x_1(t) &= c_1(\sqrt{2}-1)\exp((-1+\sqrt{2})S_F^\alpha(t)) + c_2(-\sqrt{2}-1)\exp((-1-\sqrt{2})S_F^\alpha(t)) \\ &\quad - V_0\exp(-S_F^\alpha(t)), \\ x_2(t) &= c_1\exp((-1+\sqrt{2})S_F^\alpha(t)) + c_2\exp((-1-\sqrt{2})S_F^\alpha(t)) + 2V_0\exp(-S_F^\alpha(t)), \end{aligned} \tag{62}$$

$$\begin{aligned} x_1(t) &\approx c_1(\sqrt{2}-1)\exp((-1+\sqrt{2})t^\alpha) + c_2(-\sqrt{2}-1)\exp((-1-\sqrt{2})t^\alpha) \\ &\quad - V_0\exp(-t^\alpha), \\ x_2(t) &\approx c_1\exp((-1+\sqrt{2})t^\alpha) + c_2\exp((-1-\sqrt{2})t^\alpha) + 2V_0\exp(-t^\alpha). \end{aligned} \tag{63}$$



(a) $x_1(t)$ and $x_2(t)$ on a fractal set from Eq.(62). $x_1(t)$ is blue; $x_2(t)$ is red dashed.



(b) $x_1(t)$ and $x_2(t)$ on $\mathbb{R}$ for different α values from Eq.(63).

Fig. 2: Comparison of $x_1(t)$ and $x_2(t)$. (a) shows the fractal set; (b) shows effects of α on $\mathbb{R}$.

Figure 2 compares $x_1(t)$ and $x_2(t)$ on a fractal set (subfigure a) and $\mathbb{R}$ with varying α (subfigure b).

6 Conclusion

In this paper, we have tackled the solution of non-homogeneous systems of fractal differential equations, providing a comprehensive exploration of fractal homogeneous linear systems. We have introduced and detailed the method of undetermined coefficients, demonstrating its application for solving fractal non-homogeneous linear differential equations. This method has proven to be effective in deriving solutions for a range of fractal differential equations, extending classical approaches to accommodate the complexities introduced by fractal time.

Furthermore, we have proposed and developed the method of variation of parameters as a powerful technique for addressing fractal non-homogeneous linear differential equations. This approach has allowed us to tackle more intricate systems where traditional methods might fall short, highlighting its versatility and utility in the context of fractal calculus.

To illustrate the practical application of these methods, we have applied them to the differential equations governing RLC circuits. By incorporating fractal derivatives into the classical RLC circuit model, we have successfully solved the corresponding system of fractal differential equations. This application not only demonstrates the effectiveness of our methods but also showcases how fractal calculus can provide deeper insights into electrical circuit behavior in fractal time domains.

In summary, our work has expanded the theoretical framework of fractal differential equations and has introduced practical methodologies that enhance our understanding and capability to solve complex systems within fractal spaces.

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