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SYSTEMS OF KIRCHHOFF TYPE EQUATIONS WITH GRADIENT DEPENDENCE IN THE REACTION TERM VIA SUBSOLUTION-SUPERSOLUTION METHOD

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Abstract. We consider a Dirichlet problem for a system of equations involving Kirchhoff type p_i-Laplacian differential operators and exhibiting gradient dependence in the reaction term (convection). Using the subsolution-supersolution method, we establish the existence and localization of weak solutions into a suitable ordered rectangle. Following a unified approach, we also provide a comparison argument to obtain positive solutions of certain models.

1. **Introduction and main results.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2 boundary $\partial\Omega$. We study the Kirchhoff Dirichlet boundary value problem for the following system of elliptic equations

$$\begin{cases} -K(p_1,u_1)\Delta_{p_1} u_1 = f_1(x,u_1,u_2,\nabla u_1,\nabla u_2) & \text{in } \Omega, \\ -K(p_2,u_2)\Delta_{p_2} u_2 = f_2(x,u_1,u_2,\nabla u_1,\nabla u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

using the method of subsolution-supersolution. By $\Delta_{p_i} : W_0^{1,p_i}(\Omega) \to W^{-1,p_i'}(\Omega)$ for $1 < p_i < +\infty$ with $i = 1,2$, we mean the p_i-Laplacian differential operator defined by $\Delta_{p_i} u = \operatorname{div}(|\nabla u|^{p_i-2}\nabla u)$. Furthermore, the weight term $K(p_i,u)$ is assumed of Kirchhoff type and given as

$$K(p_i,u) = a_{p_i}(x) + b_{p_i}\int_\Omega \frac{1}{p_i}|\nabla u|^{p_i}\,dx, \tag{2}$$

where $a_{p_i} : \Omega \to \mathbb{R}$ is a measurable function and $b_{p_i} > 0$ is a positive constant. Such a nonlocal Kirchhoff term is linked to physical models of changes in lenght

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for a beam subject to transverse vibrations. In fact, Kirchhoff [11] generalized the classical D'Alembert wave equation of the form

$$\rho\frac{\partial^2}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2}{\partial x^2}=0,$$

where ρ, P_0, h, E, L are certain physical parameters (namely, mass density, initial tension, area of cross-section, Young modulus of material, length of beam).

A crucial key of our paper is the fact that the components f_i $(i=1,2)$ of the right-hand side lower order vector field in problem (1) exhibit dependence on both the solution and its gradient. This aspect known as convection is a source of difficulties to be overcome as it cannot be handled by variational methods. Moreover, in the system setting, we have to deal with the competing effects of different equations. Here the functions $f_i:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$, $i=1,2$, are of Carathéodory (that is, $x\mapsto f_i(x,s_1,s_2,\xi_1,\xi_2)$ is measurable for all $(s_1,s_2,\xi_1,\xi_2)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$ and $(s_1,s_2,\xi_1,\xi_2)\mapsto f_i(x,s_1,s_2,\xi_1,\xi_2)$ is continuous for a.a. $x\in\Omega$).

Referring to the existing literature, we mention that the existence, uniqueness and asymptotic behavior of solutions with respect to a couple of non-negative parameters (b_1,b_2) for the general Dirichlet system driven by $-\Delta_{p_i}-b_i\Delta q_i$ differential operator for $i=1,2$ (namely, the weighted sum of a p_i-Laplacian differential operator and of a q_i-Laplacian differential operator) have been studied by Motreanu et al. [14] (based on the theory of pseudomonotone operators but without using the subsolution-supersolution principle), while the same authors in [15] applied the subsolution-supersolution method to establish the existence and location of solutions for the similar systems as in [14] (using different sets of hypotheses). Problem (1) without the Kirchhoff terms (2) and hence only with the p_i-Laplacian differential operators (that is, taking $a_{p_i}(x)=constant=1$ for all $x\in\Omega$, and taking also $b_{p_i}=0$) was considered in Carl-Motreanu [6] using the method of subsolution-supersolution (we refer to the comprehensive book of Carl et al. [5] for more details and information). We also mention the recent works of Albalawi et al. [1] and of Vetro-Winkert [18] dealing with the equation setting in the case of variable exponents $-\Delta_{p(\cdot)}-b\Delta_{q(\cdot)}$ differential operator and exhibiting gradient dependence in the reaction term (hence, extending the topological approach based on the theory of pseudomonotone operators to the variable exponents setting). Furthermore, in [18] there is also parameter dependence of the reaction. Differently, a topological approach based on the Leray-Schauder alternative principle is used in Papageorgiou-Zhang [16] in establishing the existence of positive solutions to a p-Laplacian differential equation with a Robin boundary condition. Looking for positive solutions of Kirchhoff type equations, we refer to the works of Gasiński-Santos Júnior [9, 10] establishing the existence results as well as the nonexistence results. The developed approach in [9, 10] uses fixed point theorems and supposes that the Kirchhoff terms may vanish at different points; see also Boulaaras [4] for Kirchhoff elliptic system involving p-Laplacian differential operator (using the subsolution-supersolution method). In general, we note that the Kirchhoff term is a source of difficulties in establishing some comparison principles to construct subsolution-supersolution of Kirchhoff type equation. This issue was deeply discussed in the case of Laplacian differential operator (namely, setting $p_i=2$) in a recent work of Figueiredo-Suárez [7]; see also the references therein.

Here, we are interested to establish the existence of weak solutions to problem (1), namely we look to solutions of the form $(u_1,u_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ satisfying the equality

$$\begin{aligned}
&\left(a_{p_1}(x)+\frac{b_{p_1}}{p_1}\int_\Omega |\nabla u_1|^{p_1}dx\right)\int_\Omega |\nabla u_1|^{p_1-2}\nabla u_1\nabla v_1 dx\\
&=\int_\Omega f_1(x,u_1,u_2,\nabla u_1,\nabla u_2)v_1dx,\\
&\left(a_{p_2}(x)+\frac{b_{p_2}}{p_2}\int_\Omega |\nabla u_2|^{p_2}dx\right)\int_\Omega |\nabla u_2|^{p_2-2}\nabla u_2\nabla v_2 dx\\
&=\int_\Omega f_2(x,u_1,u_2,\nabla u_1,\nabla u_2)v_2dx
\end{aligned}$$

for all $(v_1,v_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$. On the other hand, we say that $(\underline{u}_1,\underline{u}_2),(\overline{u}_1,\overline{u}_2)\in W^{1,p_1}(\Omega)\times W^{1,p_2}(\Omega)$ is a subsolution-supersolution to problem (1) if $\underline{u}_i(x)\le\overline{u}_i(x)$ for a.a. $x\in\Omega$, $\underline{u}_i\le 0\le\overline{u}_i$ on $\partial\Omega$ for $i=1,2$, and the following is the case

$$\begin{aligned}
&\left(a_{p_1}(x)+\frac{b_{p_1}}{p_1}\int_\Omega |\nabla \underline{u}_1|^{p_1}dx\right)\int_\Omega |\nabla \underline{u}_1|^{p_1-2}\nabla \underline{u}_1\nabla v_1 dx\\
&-\int_\Omega f_1(x,\underline{u}_1,w_2,\nabla\underline{u}_1,\nabla w_2)v_1dx\\
&+\left(a_{p_2}(x)+\frac{b_{p_2}}{p_2}\int_\Omega |\nabla \underline{u}_2|^{p_2}dx\right)\int_\Omega |\nabla \underline{u}_2|^{p_2-2}\nabla \underline{u}_2\nabla v_2 dx\\
&-\int_\Omega f_2(x,w_1,\underline{u}_2,\nabla w_1,\nabla\underline{u}_2)v_2dx\le 0
\end{aligned}$$

and

$$\begin{aligned}
&\left(a_{p_1}(x)+\frac{b_{p_1}}{p_1}\int_\Omega |\nabla \overline{u}_1|^{p_1}dx\right)\int_\Omega |\nabla \overline{u}_1|^{p_1-2}\nabla \overline{u}_1\nabla v_1 dx\\
&-\int_\Omega f_1(x,\overline{u}_1,w_2,\nabla\overline{u}_1,\nabla w_2)v_1dx\\
&+\left(a_{p_2}(x)+\frac{b_{p_2}}{p_2}\int_\Omega |\nabla \overline{u}_2|^{p_2}dx\right)\int_\Omega |\nabla \overline{u}_2|^{p_2-2}\nabla \overline{u}_2\nabla v_2 dx\\
&-\int_\Omega f_2(x,w_1,\overline{u}_2,\nabla w_1,\nabla\overline{u}_2)v_2dx\ge 0
\end{aligned}$$

for all $(v_1,v_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ with $v_1(x),v_2(x)\ge 0$ for a.a. $x\in\Omega$, and $(w_1,w_2)\in W^{1,p_1}(\Omega)\times W^{1,p_2}(\Omega)$ such that $\underline{u}_i(x)\le w_i(x)\le\overline{u}_i(x)$ for a.a. $x\in\Omega$, $i=1,2$.

Our strategy works as follows. Under suitable growth conditions of the nonlinearities f_i for $i=1,2$ (see hypothesis (H_1) in Section 2), we ensure that the integrals involved in the definitions of weak solution and subsolution-supersolution to problem (1) are well-posed. Then, we introduce an auxiliary problem (see problem (7) below) associated to problem (1) by using appropriate truncation operators and cut-off functions, both related to the given subsolution-supersolution. Thus, we establish the following existence result for problem (7), see again [5, 6, 15] for a similar strategy.

Theorem 1.1. *Let $(\underline{u}_1,\underline{u}_2),(\overline{u}_1,\overline{u}_2)\in W^{1,p_1}(\Omega)\times W^{1,p_2}(\Omega)$ be a subsolution-supersolution of problem (1) satisfying hypotheses (H_1) and (H_2). Then, for all $\mu>0$ sufficiently large, problem (7) admits a weak solution $(u_1,u_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$.*

Based on Theorem 1.1 and on a judicious choice of cut-off functions (see (4)), which are involved in designing a comparison argument with the given subsolution-supersolution for problem (1), we obtain the main result of the paper in the following form (existence and location result).

Theorem 1.2. *Let $(\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)$ be a subsolution-supersolution of problem (1) satisfying hypotheses (H_1) and (H_2). Then problem (1) admits a weak solution $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ satisfying the enclosure property $\underline{u}_i(x) \le u_i(x) \le \overline{u}_i(x)$ for a.a. $x \in \Omega$, $i = 1, 2$.*

A major role in the proofs of above theorems is played by the properties of operators of monotone type. In particular, in the proof of Theorem 1.1 we use the surjectivity result of pseudomonotone, bounded and coercive operators. Furthermore, in the proof of Theorem 1.2 we use some classical inequalities of operators (with respect to monotonicity properties). In both the proofs, the appropriate main operators are defined involving suitable Nemytskij operators. In the last part of the paper, we obtain positive weak solutions for certain models, by using a suitable comparison argument with the classical Dirichlet p_i-Laplacian problem.

2. **Mathematical background and materials.** The appropriate setting to develop our study is that of constant exponents Lebesgue spaces $L^{p_i}(\Omega)$ and Sobolev spaces $W^{1,p_i}(\Omega), W_0^{1,p_i}(\Omega)$ (recall that $W_0^{1,p_i}(\Omega)$ means the completion of $C_0^\infty(\Omega)$ with respect to the W^{1,p_i}-norm, namely $\|u\| = \left(\int_\Omega |u|^{p_i} dx + \int_\Omega |\nabla u|^{p_i} dx\right)^{1/p_i}$). On account of the Poincaré inequality the norm of $W_0^{1,p_i}(\Omega)$ is defined by

$$\|u\| = \|\nabla u\|_{p_i} \quad \text{for all } u \in W_0^{1,p_i}(\Omega),$$

where $\|\nabla u\|_{p_i} = \left(\int_\Omega |\nabla u|^{p_i} dx\right)^{1/p_i}$. For $1 < p < +\infty$, we mention that $W^{1,p_i}(\Omega)$, $W_0^{1,p_i}(\Omega)$ are separable, uniformly convex (hence reflexive) Banach spaces. We recall that the Sobolev critical exponent related to p_i is defined by

$$p_i^* = \begin{cases} \frac{Np_i}{N-p_i} & \text{if } p_i < N, \\ +\infty & \text{if } N \le p_i, \end{cases} \quad i = 1, 2.$$

Here, we focus on the case where $\max\{p_1, p_2\} < N$. Hence, the Rellich-Kondrachov compactness theorem gives us that the embedding $W^{1,p_i}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $1 \le q < p_i^*$. Furthermore, we recall that the Hölder conjugate exponent related to p_i is defined by

$$p_i' = \frac{p_i}{p_i - 1}.$$

As mentioned in Section 1, we will use the main result on pseudomonotone operators (see also [5, Theorem 2.99]), in the form of the following surjectivity theorem.

Definition 2.1. For a reflexive Banach space X, let X^* the dual space of X and $\langle \cdot, \cdot \rangle$ the duality pairing. Let $A : X \to X^*$, then A is called

(i) to satisfy the (S_+)-property if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \le 0$ imply $u_n \to u$ in X;

(ii) pseudomonotone if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \le 0$ imply

$$\liminf_{n\to+\infty} \langle A(u_n), u_n - v\rangle \ge \langle A(u), u - v\rangle \text{ for all } v \in X;$$

(iii) coercive if

$$\lim_{\|u\|_X \to +\infty} \frac{\langle A(u), u\rangle}{\|u\|_X} = +\infty.$$

Theorem 2.2. *Let X be a real, reflexive Banach space, let $A : X \to X^*$ be a pseudomonotone, bounded, and coercive operator, and $h \in X^*$ Then, a solution of the equation $A(u) = h$ exists.*

Since we aim to apply the method of subsolution-supersolution, then we impose the following hypothesis (H_1) to control the growth of the components f_i $(i = 1, 2)$ of the right-hand side lower order vector field in problem (1). Namely, we assume the existence of a subsolution-supersolution $(\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)$ of problem (1) satisfying the following hypothesis:

(H_1) there exist constants $\beta_i \geq 0$, $\nu_i \in [0, \frac{p_i}{(p_i^*)'}[$ and functions $\alpha_i \in L^{\gamma_i'}(\Omega)$, with $1 \leq \gamma_i < p_i^*$, $i = 1, 2$, such that

$$|f_1(x, s_1, s_2, \xi_1, \xi_2)| \leq \alpha_1(x) + \beta_1(|\xi_1|^{\nu_1} + |\xi_2|^{\frac{\nu_1 p_2}{p_1}}),$$
$$|f_2(x, s_1, s_2, \xi_1, \xi_2)| \leq \alpha_2(x) + \beta_2(|\xi_1|^{\frac{\nu_2 p_1}{p_2}} + |\xi_2|^{\nu_2})$$

for a.a. $x \in \Omega$, all $s = (s_1, s_2) \in [\underline{u}_1(x), \overline{u}_1(x)] \times [\underline{u}_2(x), \overline{u}_2(x)]$, $\xi_1, \xi_2 \in \mathbb{R}^N$.

Remark 1. Since the growth conditions in our paper are dictated by hypothesis (H_1), it follows that our results cannot be linked directly to the corresponding ones in [6] and [14], where the authors involve more restrictive growth conditions. For example in [6] the nonlinearities obey to the following conditions: there exist constants $\widehat{\beta}_i \geq 0$ and functions $\widehat{\alpha}_i \in L^{p_i'}(\Omega)$ such that

$$|f_1(x, s_1, s_2, \xi_1, \xi_2)| \leq \widehat{\alpha}_1(x) + \widehat{\beta}_1(|\xi_1|^{p_1 - 1} + |\xi_2|^{\frac{p_2}{p_1'}}),$$
$$|f_2(x, s_1, s_2, \xi_1, \xi_2)| \leq \widehat{\alpha}_2(x) + \widehat{\beta}_2(|\xi_1|^{\frac{p_1}{p_2'}} + |\xi_2|^{p_2 - 1})$$

for a.a. $x \in \Omega$ and all $s = (s_1, s_2) \in [\underline{u}_1(x), \overline{u}_1(x)] \times [\underline{u}_2(x), \overline{u}_2(x)]$, $\xi_1, \xi_2 \in \mathbb{R}^N$. We point out (see also [15]) that these growth conditions are stronger than (H_1) as we have

$$p_i - 1 = \frac{p_i}{p_i'} < \frac{p_i}{(p_i^*)'} \quad \text{for } i = 1, 2.$$

Example 1. The following functions satisfy hypothesis (H_1). For the sake of simplicity we drop the x-dependence:

$$f_i(s_1, s_2, \xi_1, \xi_2) = \begin{cases} s_i^{q_i - 1}|\xi_i|^{p_i - 1} & \text{if } 0 \leq s_i \leq 1, \\ \beta_i|\xi_i|^{p_i - 1} & \text{if } 1 < s_i, \end{cases} \text{all } \xi_i \in \mathbb{R}^N,$$

some $q_i > 1$, $\beta_i > 0$, $i = 1, 2$;

$$f_i(s_1, s_2, \xi_1, \xi_2) = \begin{cases} \omega_i s_i^{q_i}(1 + |\xi_i|^{p_i - 1}) & \text{if } 0 \leq s_i \leq 1, \\ \beta_i|\xi_j|^{\frac{\nu_i p_j}{p_i}} & \text{if } 1 < s_i, \end{cases} \text{all } \xi_i \in \mathbb{R}^N,$$

some $\beta_i, \omega_i, q_i > 0$, $0 < \nu_i < \frac{p_i}{(p_i^*)'}$, $i, j \in \{1, 2\}$, $i \neq j$;

$$f_i(s_1, s_2, \xi_1, \xi_2) = \begin{cases} |\xi_i|^{p_i - 1} & \text{if } |\xi_i| \leq 1, \\ |\xi_i|^{p_i - 2}[\ln|\xi_i| + 1] & \text{if } 1 < |\xi_i|, \end{cases} \text{all } \xi_i \in \mathbb{R}^N,\ i = 1, 2.$$

For the particular needs of our proofs, we impose the following hypothesis on the Kirchhoff term:

(H_2) $a_{p_i} : \Omega \to \mathbb{R}$ is a measurable function such that there exist $0 < a_0 \leq \widehat{a}_0$ satisfying $0 < a_0 \leq a_{p_i}(x) \leq \widehat{a}_0$ for a.a. $x \in \Omega$, $i = 1, 2$.

Hypothesis (H_2) is not so restrictive as it can be found in some other papers using different approaches (see for example the recent work of Boccardo-Orsina [3] dealing with a system of Kirchhoff-Schrödinger-Maxwell type and establishing the existence and non-existence of solutions). The classical Kirchhoff term $K(p_i, u_i)$ defined by

$$K(p_i, u_i) = a_0 + \|\nabla u_i\|_{p_i}^{p_i} \quad \text{for all } u_i \in W_0^{1,p_i}(\Omega), \text{ some } a_0 > 0,\ b_{p_i} = p_i$$

satisfies trivially hypothesis (H_2). As already mentioned in Section 1 our strategy requires the definition of an approximate problem corresponding to (1). Hence we appropriately introduce suitable truncation operators and cut-off functions.

Let $(\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)$ be a subsolution-supersolution of problem (1) as prescribed in hypothesis (H_1). For $i = 1, 2$, we consider the truncation operators $T_i : W^{1,p_i}(\Omega) \to W^{1,p_i}(\Omega)$ defined by

$$(T_i u)(x) = \begin{cases} \overline{u}_i(x) & \text{if } u(x) > \overline{u}_i(x), \\ u(x) & \text{if } \underline{u}_i(x) \leq u(x) \leq \overline{u}_i(x), \\ \underline{u}_i(x) & \text{if } u(x) < \underline{u}_i(x). \end{cases} \tag{3}$$

Of course, $T_i : W^{1,p_i}(\Omega) \to W^{1,p_i}(\Omega)$ for $i = 1, 2$ are continuous and bounded operators.

Furthermore, we will use the following notion of cut-off functions. So, for a.a. x in Ω and all $s \in \mathbb{R}$, we introduce the functions:

$$\varphi_i(x, s) = \begin{cases} (s - \overline{u}_i(x))^{\frac{\nu_i}{p_i - \nu_i}} & \text{if } s > \overline{u}_i(x), \\ 0 & \text{if } \underline{u}_i(x) \leq s \leq \overline{u}_i(x), \\ -(\underline{u}_i(x) - s)^{\frac{\nu_i}{p_i - \nu_i}} & \text{if } s < \underline{u}_i(x), \end{cases} \tag{4}$$

where ν_i for $i = 1, 2$ are the constants involved in hypothesis (H_1). We note that φ_i for $i = 1, 2$ are Carathéodory functions such that

$$|\varphi_i(x, s)| \leq \rho_i(x) + \widehat{c}_i |s|^{\frac{\nu_i}{p_i - \nu_i}} \tag{5}$$

for a.a. $x \in \Omega$, all $s \in \mathbb{R}$, with $\rho_i \in L^{\frac{p_i}{\nu_i}}(\Omega)$ and $\widehat{c}_i \geq 0$. We note that the function ρ_i leaves in $L^{\frac{p_i}{\nu_i}}(\Omega)$ because of $\underline{u}_i, \overline{u}_i \in W^{1,p_i}(\Omega)$, and hence $\underline{u}_i, \overline{u}_i \in L^{p_i^*}(\Omega)$ by Sobolev embedding theorem, recall also that $\nu_i < \frac{p_i}{(p_i^*)'}$ (by (H_1)). Following the similar arguments, we can conclude that there exist two constants $r_{1,i}, r_{2,i} > 0$ satisfying the inequality

$$\int_\Omega \varphi_i(x, u) u dx \geq r_{1,i} \|u\|_{\frac{p_i}{p_i - \nu_i}}^{\frac{p_i}{p_i - \nu_i}} - r_{2,i} \quad \text{for all } u \in W^{1,p_i}(\Omega),\ i = 1, 2. \tag{6}$$

Now, we consider the Nemytskij operator $\Phi : W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega) \to L^{\frac{p_1}{\nu_1}}(\Omega) \times L^{\frac{p_2}{\nu_2}}(\Omega)$ defined by

$$\Phi(u) = (\Phi_1(u), \Phi_2(u)) = (\varphi_1(x, u), \varphi_2(x, u)).$$

By the estimate in (5) we deduce easily that the above Nemytskij operator is bounded and completely continuous (by the compact embedding of $W^{1,p_i}(\Omega)$ into $L^{p_i}(\Omega)$ for $i = 1, 2$). Finally for a positive parameter $\mu > 0$, we introduce the following auxiliary problem

$$\begin{cases} -K(p_1,T_1u_1)\Delta_{p_1}u_1+\mu\Phi_1(u_1)=f_1(x,T_1u_1,T_2u_2,\nabla(T_1u_1),\nabla(T_2u_2)) & \text{in } \Omega,\\ -K(p_2,T_2u_2)\Delta_{p_2}u_2+\mu\Phi_2(u_2)=f_2(x,T_1u_1,T_2u_2,\nabla(T_1u_1),\nabla(T_2u_2)) & \text{in } \Omega,\\ u_1=u_2=0 \quad \text{on } \partial\Omega. \end{cases} \tag{7}$$

3. **Proof of Theorems 1.1.**

Starting with a subsolution-supersolution of problem (1), $(\underline{u}_1,\underline{u}_2),(\overline{u}_1,\overline{u}_2)\in W^{1,p_1}(\Omega)\times W^{1,p_2}(\Omega)$ satisfying condition (H_1), we consider the ordered intervals given by

$$[\underline{u}_i,\overline{u}_i]=\{u\in W^{1,p_i}(\Omega):\underline{u}_i(x)\le u(x)\le \overline{u}_i(x) \ \text{ for a.a. } x\in\Omega\},\ i=1,2. \tag{8}$$

Furthermore, we introduce the Nemytskij operator

$$\begin{aligned} N:[\underline{u}_1,\overline{u}_1]\times[\underline{u}_2,\overline{u}_2]&\subset W^{1,p_1}(\Omega)\times W^{1,p_2}(\Omega)\\ &\to L^{\frac{p_1}{\nu_1}}(\Omega)\times L^{\frac{p_2}{\nu_2}}(\Omega)\hookrightarrow W^{-1,p_1'}(\Omega)\times W^{-1,p_2'}(\Omega) \end{aligned}$$

given as follows

$$N(u_1,u_2)=(f_1(x,u_1,u_2,\nabla u_1,\nabla u_2),f_2(x,u_1,u_2,\nabla u_1,\nabla u_2)). \tag{9}$$

Referring to the ordered intervals in (8), hypothesis (H_1) ensures that the above Nemytskij operator is bounded and completely continuous (it follows by the Rellich-Kondrachov compactness embedding theorem).

For a suitable positive parameter $\mu>0$ we consider the nonlinear operator $A:W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)\to W^{-1,p_1'}(\Omega)\times W^{-1,p_2'}(\Omega)$ defined by

$$\begin{aligned} A(u_1,u_2)&=(A_1(u_1,u_2),A_2(u_1,u_2)) \qquad (10)\\ &:=(-K(p_1,T_1u_1)\Delta_{p_1}u_1+\mu\Phi_1(u_1),-K(p_2,T_2u_2)\Delta_{p_2}u_2+\mu\Phi_2(u_2))\\ &\quad -N(T_1u_1,T_2u_2). \end{aligned}$$

Hipothesis (H_1) together with (5), (8), (9) give us that the operator A is well defined, bounded and continuous. Furthermore, we prove that $A:W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)\to W^{-1,p_1'}(\Omega)\times W^{-1,p_2'}(\Omega)$ possesses some regularity properties. The key property to establish is the pseudomonotonicity. Consistent with the definition of pseudomonotone operator (see Definition 2.1 (ii)), we assume the weak convergence $(u_{1,n},u_{2,n})\xrightarrow{w}(u_1,u_2)$ in $W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ and the limit condition

$$\limsup_{n\to+\infty}\langle A(u_{1,n},u_{2,n}),(u_{1,n}-u_1,u_{2,n}-u_2)\rangle\le 0. \tag{11}$$

Hypothesis (H_1) says us that the constants ν_i satisfy the inequality $\frac{p_i}{p_i-\nu_i}<p_i^*$ for $i=1,2$. Hence, we deduce the strong convergence

$$(u_{1,n},u_{2,n})\to(u_1,u_2) \text{ in } L^{\frac{p_1}{p_1-\nu_1}}(\Omega)\times L^{\frac{p_2}{p_2-\nu_2}}(\Omega).$$

Now, keeping in mind the estimate in (5) we obtain the following limit

$$\lim_{n\to+\infty}\int_\Omega \varphi_i(x,u_{i,n}(x))(u_{i,n}-u_i)dx=0,\ i=1,2. \tag{12}$$

Appealing again to the Rellich-Kondrachov compactness embedding theorem and involving Hölder inequality, then the weak convergence $(u_{1,n},u_{2,n})\xrightarrow{w}(u_1,u_2)$ in $W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ leads to the following convergence

$$\int_\Omega |\alpha_i|\,|u_{i,n}-u_i|dx\le\|\alpha_i\|_{\gamma_i'}\|u_{i,n}-u_i\|_{\gamma_i}\to 0 \text{ as } n\to+\infty. \tag{13}$$

Next step is to show that

$$\int_\Omega |\nabla(T_i u_{i,n})|^{\nu_i}|u_{i,n}-u_i|dx \to 0 \text{ as } n\to+\infty. \tag{14}$$

Referring to the definition of the truncation operator (see (3)), we deduce that

$$\begin{aligned}\int_\Omega |\nabla(T_i u_{i,n})|^{\nu_i}|u_{i,n}-u_i|dx &= \int_{\{u_{i,n}<\underline{u}_i\}} |\nabla \underline{u}_i|^{\nu_i}|u_{i,n}-u_i|dx \\ &+ \int_{\{\underline{u}_i\le u_{i,n}\le \overline{u}_i\}} |\nabla u_{i,n}|^{\nu_i}|u_{i,n}-u_i|dx \\ &+ \int_{\{u_{i,n}>\overline{u}_i\}} |\nabla \overline{u}_i|^{\nu_i}|u_{i,n}-u_i|dx.\end{aligned}$$

Using again the inequality $\frac{p_i}{p_i-\nu_i} < p_i^*$, we get the following convergences, via suitable estimates,

$$\int_{\{u_{i,n}<\underline{u}_i\}} |\nabla \underline{u}_i|^{\nu_i}|u_{i,n}-u_i|dx \le \|\nabla \underline{u}_i\|_{p_i}^{\nu_i}\|u_{i,n}-u_i\|_{\frac{p_i}{p_i-\nu_i}} \to 0,$$

$$\int_{\{\underline{u}_i\le u_{i,n}\le \overline{u}_i\}} |\nabla u_{i,n}|^{\nu_i}|u_{i,n}-u_i|dx \le \|\nabla u_{i,n}\|_{p_i}^{\nu_i}\|u_{i,n}-u_i\|_{\frac{p_i}{p_i-\nu_i}} \to 0,$$

$$\int_{\{u_{i,n}>\overline{u}_i\}} |\nabla \overline{u}_i|^{\nu_i}|u_{i,n}-u_i|dx \le \|\nabla \overline{u}_i\|_{p_i}^{\nu_i}\|u_{i,n}-u_i\|_{\frac{p_i}{p_i-\nu_i}} \to 0.$$

Combining the above results, we deduce immediately that the convergence (14) holds true. Adopting and adapting the similar arguments as above, we easily conclude that the following is the case: for $i\neq j$ we get

$$\int_\Omega |\nabla(T_j u_{j,n})|^{\frac{\nu_i p_j}{p_i}}|u_{i,n}-u_i|dx \le \|\nabla(T_j u_{j,n})\|_{p_j}^{\frac{\nu_i p_j}{p_i}}\|u_{i,n}-u_i\|_{\frac{p_i}{p_i-\nu_i}} \to 0. \tag{15}$$

For $i=1,2$, (H_1) together with the convergences (13), (14), (15) give us

$$\lim_{n\to+\infty}\int_\Omega f_i(x,T_1u_{1,n},T_2u_{2,n},\nabla(T_1u_{1,n}),\nabla(T_2u_{2,n}))(u_{i,n}-u_i)dx = 0. \tag{16}$$

From (11), using (12) and (16) we deduce the following limit condition

$$\begin{aligned}&\limsup_{n\to+\infty}[\langle -K(p_1,T_1u_{1,n})\Delta_{p_1}u_{1,n},u_{1,n}-u_1\rangle \\ &+\langle -K(p_2,T_2u_{2,n})\Delta_{p_2}u_{2,n},u_{2,n}-u_2\rangle]\le 0.\end{aligned} \tag{17}$$

Next, we establish that (17) leads to the following result

$$\limsup_{n\to+\infty}\langle -K(p_i,T_iu_{i,n})\Delta_{p_i}u_{i,n},u_{i,n}-u_i\rangle \le 0, \quad i=1,2. \tag{18}$$

To this aim, we argue by contradiction, hence we suppose that the following is the case

$$\lim_{n\to+\infty} K(p_1,T_1u_{1,n})\langle -\Delta_{p_1}u_{1,n},u_{1,n}-u_1\rangle > 0, \tag{19}$$

$$\lim_{n\to+\infty} K(p_2,T_2u_{2,n})\langle -\Delta_{p_2}u_{2,n},u_{2,n}-u_2\rangle < 0. \tag{20}$$

Since the Kirchhoff terms $K(p_1,T_1u_{1,n}), K(p_2,T_2u_{2,n})$ are positively bounded away from $a_0>0$ (by (H_2), see also (2)), then (19) and (20) lead to

$$\lim_{n\to+\infty}\langle -\Delta_{p_1}u_{1,n},u_{1,n}-u_1\rangle > 0,$$
$$\lim_{n\to+\infty}\langle -\Delta_{p_2}u_{2,n},u_{2,n}-u_2\rangle < 0.$$

The $(S)_+$-property of the p_2-Laplacian differential operator on the Dirichlet Sobolev space $W_0^{1,p_2}(\Omega)$ (see [13, p. 39] and recall Definition 2.1 (i)) together with the last inequality above, give us the convergence $u_{2,n} \to u_2$ in $W_0^{1,p_2}(\Omega)$, a contradiction to (17), and hence (18) is established.

From (18), appealing to the $(S)_+$-property of $-\Delta_{p_i}$ on $W_0^{1,p_i}(\Omega)$ we conclude that

$$(u_{1,n}, u_{2,n}) \to (u_1, u_2) \text{ in } W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega) \text{ as } n \to +\infty.$$

It follows that, for all $(v_1, v_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$, we get

$$\lim_{n\to+\infty} \langle A(u_{1,n}, u_{2,n}), (u_{1,n} - v_1, u_{2,n} - v_2)\rangle = \langle A(u_1, u_2), (u_1 - v_1, u_2 - v_2)\rangle,$$

and hence $A : W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega) \to W^{-1,p_1'}(\Omega)\times W^{-1,p_2'}(\Omega)$ is a pseudomonotone operator.

The second property possessed by our operator A is the coercivity. By Definition 2.1 (iii), we can say that $A : W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega) \to W^{-1,p_1'}(\Omega) \times W^{-1,p_2'}(\Omega)$ is coercive if for every sequence $(u_{1,n}, u_{2,n}) \subset W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ satisfying the norm condition $\|(u_{1,n}, u_{2,n})\| \to +\infty$, then the following is the case

$$\lim_{n\to+\infty} \frac{\langle A(u_{1,n}, u_{2,n}), (u_{1,n}, u_{2,n})\rangle}{\|(u_{1,n}, u_{2,n})\|} = +\infty. \tag{21}$$

Since $a_{p_i}(x) \geq a_0 > 0$ for a.a. $x \in \Omega$ (by hypothesis (H_2)) and in view of (H_1), using the estimate in (6) we obtain that

$$\begin{aligned}
&\langle A_1(u_{1,n}, u_{2,n}), u_{1,n}\rangle = a_{p_1}(x)\|\nabla u_{1,n}\|_{p_1}^{p_1} + \frac{b_{p_1}}{p_1}\|\nabla T u_{1,n}\|_{p_1}^{p_1}\|\nabla u_{1,n}\|_{p_1}^{p_1} \\
&+\mu \int_\Omega \varphi_1(x, u_{1,n})u_{1,n}dx - \int_\Omega f_1(x, T_1u_{1,n}, T_2u_{2,n}, \nabla(T_1u_{1,n}), \nabla(T_2u_{2,n}))u_{1,n}dx \\
&\geq a_0\|\nabla u_{1,n}\|_{p_1}^{p_1} + \mu\Big(r_{1,1}\|u_{1,n}\|_{\frac{p_1}{p_1-\nu_1}}^{\frac{p_1}{p_1-\nu_1}} - r_{2,1}\Big) - \int_\Omega \alpha_1(x)|u_{1,n}|dx \\
&-\beta_1 \int_\Omega |\nabla(T_1u_{1,n})|^{\nu_1}|u_{1,n}|dx - \beta_1 \int_\Omega |\nabla(T_2u_{2,n})|^{\frac{\nu_1 p_2}{p_1}}|u_{1,n}|dx.
\end{aligned}$$

Now, we establish useful estimates for each one of the three integrals involved above. To simplify the notation, we will denote by $C > 0$ any positive constant whose value may change from line to line, furthermore every relevant dependencies will be underlined by using round parentheses (as in the case of $C(\varepsilon)$ below, where we point out the dependence by ε). We note that the following is the case

$$\int_\Omega \alpha_1(x)|u_{1,n}|dx \leq \|\alpha_1\|_{\gamma_1'}\|u_{1,n}\|_{\gamma_1} \leq C\|\nabla u_{1,n}\|_{p_1}.$$

For the second integral, involving Young inequality with any $\varepsilon > 0$, we deduce that

$$\begin{aligned}
&\int_\Omega |\nabla(T_1u_{1,n})|^{\nu_1}|u_{1,n}|dx \leq \varepsilon\|\nabla(T_1u_{1,n})\|_{p_1}^{p_1} + C(\varepsilon)\|u_{1,n}\|_{\frac{p_1}{p_1-\nu_1}}^{\frac{p_1}{p_1-\nu_1}} \\
&\leq \varepsilon\|\nabla u_{1,n}\|_{p_1}^{p_1} + \varepsilon\|\nabla \underline{u}_1\|_{p_1}^{p_1} + \varepsilon\|\nabla \overline{u}_1\|_{p_1}^{p_1} + C(\varepsilon)\|u_{1,n}\|_{\frac{p_1}{p_1-\nu_1}}^{\frac{p_1}{p_1-\nu_1}}.
\end{aligned}$$

The similar arguments as above lead to the following estimate of the third integral

$$\begin{aligned}
&\int_\Omega |\nabla(T_2u_{2,n})|^{\frac{\nu_1 p_2}{p_1}}|u_{1,n}|dx \leq \varepsilon\|\nabla(T_2u_{2,n})\|_{p_2}^{p_2} + C(\varepsilon)\|u_{1,n}\|_{\frac{p_1}{p_1-\nu_1}}^{\frac{p_1}{p_1-\nu_1}} \\
&\leq \varepsilon\|\nabla u_{2,n}\|_{p_2}^{p_2} + \varepsilon\|\nabla \underline{u}_2\|_{p_2}^{p_2} + \varepsilon\|\nabla \overline{u}_2\|_{p_2}^{p_2} + C(\varepsilon)\|u_{1,n}\|_{\frac{p_1}{p_1-\nu_1}}^{\frac{p_1}{p_1-\nu_1}}.
\end{aligned}$$

Combining the obtained estimates, we conclude that

$$\begin{aligned}\langle A_1(u_{1,n},u_{2,n}),u_{1,n}\rangle \geq& (a_0-\varepsilon)\|\nabla u_{1,n}\|_{p_1}^{p_1} - C\|\nabla u_{1,n}\|_{p_1} \\ &+(\mu r_{1,1}-C(\varepsilon))\|u_{1,n}\|_{\frac{p_1}{p_1-\nu_1}}^{\frac{p_1}{p_1-\nu_1}} - \varepsilon\|\nabla u_{2,n}\|_{p_2}^{p_2} - \widehat{C}(\varepsilon),\end{aligned} \tag{22}$$

for some constant $\widehat{C}(\varepsilon)>0$. Following the similar calculations with respect to A_2, we obtain the following estimate

$$\begin{aligned}\langle A_2(u_{1,n},u_{2,n}),u_{2,n}\rangle \geq& (a_0-\varepsilon)\|\nabla u_{2,n}\|_{p_2}^{p_2} - C\|\nabla u_{2,n}\|_{p_2} \\ &+(\mu r_{1,2}-C(\varepsilon))\|u_{2,n}\|_{\frac{p_2}{p_2-\nu_2}}^{\frac{p_2}{p_2-\nu_2}} - \varepsilon\|\nabla u_{1,n}\|_{p_1}^{p_1} - \widehat{C}(\varepsilon),\end{aligned} \tag{23}$$

for some constant $\widehat{C}(\varepsilon)>0$.

Since $p_1,p_2>1$, taking $\varepsilon>0$ sufficiently small and $\mu>0$ sufficiently large in (22) and (23) we deduce that the limit in (21) is reached. Hence, we established that $A: W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)\to W^{-1,p_1'}(\Omega)\times W^{-1,p_2'}(\Omega)$ is a coercive operator too.

This allows us to invoke Theorem 2.2. So, corresponding to the pseudomonotone, bounded, and coercive operator A, we can find $(u_1,u_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ solving the equation

$$A(u_1,u_2)=0.$$

It follows that such $(u_1,u_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ is a solution to the auxiliary problem (7) (see (10)).

4. **Proof of Theorem 1.2.** By the proof of Theorem 1.1, we can deduce the existence of a solution

$$(u_1,u_2)=(u_1(\mu),u_2(\mu))$$

to the auxiliary problem (7) for sufficiently large values of the parameter $\mu>0$. By using an appropriate choice of the cut-off functions (recall (4)) we will establish the existence of a solution to problem (1) within the ordered rectangle determined by a subsolution-supersolution provided that hypothesis (H_1) is verified. Namely, we consider a solution $(u_1,u_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ to problem (7) as established by Theorem 1.1, then we are going to prove that $(u_1,u_2)\in[\underline{u}_1,\overline{u}_1]\times[\underline{u}_2,\overline{u}_2]$ (see (8)).

The first step of the proof is in verifying that

$$u_1(x)\leq \overline{u}_1(x) \text{ for a.a. } x\in\Omega.$$

Starting from the definitions of solution to problem (7) and of supersolution to problem (1), we consider the test function

$$v=(u_1-\overline{u}_1)^+=\max\{u_1-\overline{u}_1,0\}\in W_0^{1,p_1}(\Omega),$$

so that we have

$$\begin{aligned}&\langle -K(p_1,T_1u_1)\Delta_{p_1}u_1,(u_1-\overline{u}_1)^+\rangle + \mu\int_\Omega \varphi_1(x,u_1)(u_1-\overline{u}_1)^+dx \\ &= \int_\Omega f_1(x,T_1u_1,T_2u_2,\nabla(T_1u_1),\nabla(T_2u_2))(u_1-\overline{u}_1)^+dx\end{aligned} \tag{24}$$

and

$$\langle -K(p_1,\overline{u}_1)\Delta_{p_1}\overline{u}_1,(u_1-\overline{u}_1)^+\rangle \geq \int_\Omega f_1(x,\overline{u}_1,w_2,\nabla\overline{u}_1,\nabla w_2)(u_1-\overline{u}_1)^+dx \tag{25}$$

whenever $w_2 \in W^{1,p_2}(\Omega)$ with $\underline{u}_2(x) \le w_2(x) \le \overline{u}_2(x)$ a.a. $x \in \Omega$. Referring to the definition of the truncation operator T_2 we note that

$$\underline{u}_2 \le T_2 u_2 \le \overline{u}_2$$

1 and hence we can use $w_2 = T_2 u_2$ in the inequality (25) to deduce that

$$\begin{aligned} &\langle -K(p_1, \overline{u}_1)\Delta_{p_1}\overline{u}_1, (u_1 - \overline{u}_1)^+\rangle \\ &\ge \int_\Omega f_1(x, \overline{u}_1, T_2u_2, \nabla\overline{u}_1, \nabla(T_2u_2))(u_1 - \overline{u}_1)^+ dx. \end{aligned} \tag{26}$$

2 Subtracting (26) from (24) we obtain the following non-positivity condition

$$\begin{aligned} &\langle -K(p_1, T_1u_1)\Delta_{p_1}u_1 - (-K(p_1, \overline{u}_1)\Delta_{p_1}\overline{u}_1), (u_1 - \overline{u}_1)^+\rangle \\ &+ \mu \int_\Omega \varphi_1(x, u_1)(u_1 - \overline{u}_1)^+ dx \\ &\le \int_\Omega [f_1(x, T_1u_1, T_2u_2, \nabla(T_1u_1), \nabla(T_2u_2)) \\ &- f_1(x, \overline{u}_1, T_2u_2, \nabla\overline{u}_1, \nabla(T_2u_2))](u_1 - \overline{u}_1)^+ dx \\ &= 0 \quad \text{(recall that } T_1u_1 = \overline{u}_1 \text{ on the set } \{u_1 > \overline{u}_1\}). \end{aligned}$$

3 Hence, from $a_{p_i}(x) \ge a_0 > 0$ for a.a. $x \in \Omega$ (by (H_2)) and (3) for $i = 1$ we have

$$\begin{aligned} &a_0 \int_{\{u_1 > \overline{u}_1\}} (|\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla\overline{u}_1|^{p_1-2}\nabla\overline{u}_1)(\nabla u_1 - \nabla\overline{u}_1)dx \\ &+ \frac{b_{p_1}}{p_1}\int_{\{u_1 > u_1\}} |\nabla T u_1|^{p_1} dx \int_{\{u_1 > \overline{u}_1\}} |\nabla u_1|^{p_1-2}\nabla u_1(\nabla u_1 - \nabla\overline{u}_1)dx \\ &- \frac{b_{p_1}}{p_1}\int_{\{u_1 > \overline{u}_1\}} |\nabla\overline{u}_1|^{p_1} dx \int_{\{u_1 > \overline{u}_1\}} |\nabla\overline{u}_1|^{p_1-2}\nabla\overline{u}_1(\nabla u_1 - \nabla\overline{u}_1)dx \\ &+ \mu \int_{\{u_1 > \overline{u}_1\}} (u_1 - \overline{u}_1)^{\frac{p_1}{p_1-\nu_1}} dx \le 0. \end{aligned}$$

4 Again, since $T_1u_1 = \overline{u}_1$ on the set $\{u_1 > \overline{u}_1\}$, we deduce that

$$\begin{aligned} &a_0 \int_{\{u_1 > \overline{u}_1\}} (|\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla\overline{u}_1|^{p_1-2}\nabla\overline{u}_1)(\nabla u_1 - \nabla\overline{u}_1)dx \\ &+ \frac{b_{p_1}}{p_1}\int_{\{u_1 > \overline{u}_1\}} |\nabla\overline{u}_1|^{p_1} dx \\ &\times \left(\int_{\{u_1 > \overline{u}_1\}} (|\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla\overline{u}_1|^{p_1-2}\nabla\overline{u}_1)(\nabla u_1 - \nabla\overline{u}_1)dx\right) \\ &+ \mu \int_{\{u_1 > \overline{u}_1\}} (u_1 - \overline{u}_1)^{\frac{p_1}{p_1-\nu_1}} dx \le 0. \end{aligned}$$

From the classical theory of monotone operators (see also [12] for more information about the p_i-Laplacian differential operator), for all $\zeta, \eta \in \mathbb{R}^N$ with $\zeta \ne \eta$, we recall the following inequality

$$(|\zeta|^{p_1-2}\zeta - |\eta|^{p_1-2}\eta)(\zeta - \eta) > 0,$$

5 which gives us that $u_1(x) \le \overline{u}_1(x)$ for a.a. $x \in \Omega$.

The second step of the proof consists in obtaining the boundedness from below, that is, we have to show that

$$\underline{u}_1(x) \le u_1(x) \text{ for a.a. } x \in \Omega.$$

Since this result can be easily written following the similar arguments as above, we omit the details to avoid repetitions. Finally a judicious choice of the test functions for needed comparison and analogous calculations to the ones developed above, lead us to obtain that

$$\underline{u}_2(x) \le u_2(x) \le \overline{u}_2(x) \text{ for a.a. } x \in \Omega.$$

Summing up, we conclude that the solution (u_1, u_2) to the auxiliary problem (7) leaves into the rectangle $[\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2]$. Consequently, by (3) we deduce that $T_i u_i = u_i$ and by (4) we get that $\Phi_i u_i = 0$ for $i = 1, 2$. This means that the auxiliary problem (7) reduces to the main problem (1), equivalently we can say that $(u_1, u_2) \in [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2]$ is a weak solution to (1).

The following example satisfies the hypotheses of Theorem 1.2

Example 2. Let $(\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)$ be a subsolution-supersolution of the following problem

$$\begin{cases} -(a_0 + \|\nabla u_1\|_{p_1}^{p_1}) \Delta_{p_1} u_1 = g_1(x, u_1, u_2) + |\nabla u_1|^{p_1 - 1} + |\nabla u_2|^{\frac{p_2}{p_1'}} & \text{in } \Omega, \\ -(a_0 + \|\nabla u_2\|_{p_2}^{p_2}) \Delta_{p_2} u_2 = g_2(x, u_1, u_2) + |\nabla u_1|^{\frac{p_1}{p_2'}} + |\nabla u_2|^{p_2 - 1} & \text{in } \Omega, \\ u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{27}$$

where for $i = 1, 2$, $K(p_i, u_i) = a_0 + \|\nabla u_i\|_{p_i}^{p_i}$ for all $u_i \in W_0^{1,p_i}(\Omega)$ with $a_0 > 0$, the functions $g_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, are of Carathéodory and there exist $\alpha_i \in L^{p_i'}(\Omega)$ satisfying the following conditions

$$|g_i(x, s_1, s_2)| \le \alpha_i(x)$$

for a.a. $x \in \Omega$, all $s = (s_1, s_2) \in [\underline{u}_1(x), \overline{u}_1(x)] \times [\underline{u}_2(x), \overline{u}_2(x)]$.

Referring to Remark 1, we recall that

$$p_i - 1 = \frac{p_i}{p_i'} < \frac{p_i}{(p_i^*)'} \quad \text{for } i = 1, 2,$$

and hence the right-hand side lower order vector field in problem (27) satisfies hypothesis (H_1) for constants $\gamma_i = p_i$, $\beta_i = 1$ and $\nu_i = \frac{p_i}{p_i'} \in [0, \frac{p_i}{(p_i^*)'}[$, $i = 1, 2$. As hypothesis (H_2) is also trivially satisfied, then we can apply Theorem 1.2 to conclude that problem (27) admits a weak solution $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ satisfying the enclosure property $\underline{u}_i(x) \le u_i(x) \le \overline{u}_i(x)$ for a.a. $x \in \Omega$, $i = 1, 2$.

5. **Positive solutions.** An interesting byproduct of the involved strategy is the fact that we can provide precise information about the sign of solutions. To illustrate this approach, we consider the following problem

$$\begin{cases} -(a_0 + \|\nabla u_1\|_{p_1}^{p_1}) \Delta_{p_1} u_1 = \widehat{\beta}_1 [u_1^{q_1} + |\nabla u_1|^{q_1}] & \text{in } \Omega, \\ -(a_0 + \|\nabla u_2\|_{p_2}^{p_2}) \Delta_{p_2} u_2 = \widehat{\beta}_2 [u_2^{q_2} + |\nabla u_2|^{q_2}] & \text{in } \Omega, \\ u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{28}$$

where for $i = 1, 2$, $\widehat{\beta}_i > 0$, $1 < q_i < +\infty$ and, according to (2), we involve again the Kirchhoff terms $K(p_i, u_i)$ defined by

$$K(p_i, u_i) = a_0 + \|\nabla u_i\|_{p_i}^{p_i} \quad \text{for all } u_i \in W_0^{1,p_i}(\Omega), \text{ some } a_0 > 0.$$

Now, to establish the existence of weak solutions to problem (28) whose components are both positive (namely, $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ such that $u_i(x) > 0$ for a.a. $x \in \Omega$, $i = 1, 2$), we develop a comparison argument with the non-Kirchhoff Dirichlet boundary value problem for the following system of elliptic equations

$$\begin{cases} -\Delta_{p_1} w_1 = \widehat{\beta}_1[w_1^{q_1} + |\nabla w_1|^{q_1}] & \text{in } \Omega, \\ -\Delta_{p_2} w_2 = \widehat{\beta}_2[w_2^{q_2} + |\nabla w_2|^{q_2}] & \text{in } \Omega, \\ w_1 = w_2 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{29}$$

Let $w_i > 0$ be a positive solution of the Dirichlet problem

$$\begin{cases} -\Delta_{p_i} w = \widehat{\beta}_i[w^{q_i} + |\nabla w|^{q_i}] & \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega \end{cases} \tag{30}$$

and set

$$u_i = \tau_i \frac{w_i}{\|w_i\|}. \tag{31}$$

For the sake of simplicity, since $\|u_i\| = \|\nabla u_i\|_{p_i}$ (by Poincaré inequality), we observe that the following is the case

$$\begin{aligned} -(a_0 + \|\nabla u_i\|_{p_i}^{p_i})\Delta_{p_i} u_i &= -(a_0 + \|u_i\|^{p_i})\Delta_{p_i} u_i \\ &= -\left(a_0 + \frac{\|\tau_i w_i\|^{p_i}}{\|w_i\|^{p_i}}\right)\left(\frac{\tau_i}{\|w_i\|}\right)^{p_i - 1} \Delta_{p_i} w_i \\ &= -\frac{(a_0 + \tau_i^{p_i})\tau_i^{p_i-1}}{\|w_i\|^{p_i-1}} \Delta_{p_i} w_i. \end{aligned}$$

Since $|\nabla u_i|^{q_i} = \left(\frac{\tau_i}{\|w_i\|}\right)^{q_i} |\nabla w_i|^{q_i}$ (by (31)), we deduce easily that

$$\widehat{\beta}_i[u_i^{q_i} + |\nabla u_i|^{q_i}] = \widehat{\beta}_i \frac{\tau_i^{q_i}}{\|w_i\|^{q_i}}[w_i^{q_i} + |\nabla w_i|^{q_i}].$$

Thus, $u_i > 0$ is a positive solution of the problem

$$\begin{cases} -(a_0 + \|\nabla u\|_{p_i}^{p_i})\Delta_{p_i} u = \widehat{\beta}_i[u^{q_i} + |\nabla u|^{q_i}] & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{32}$$

provided that $\tau_i > 0$ is a solution of the equation

$$\frac{(a_0 + \tau_i^{p_i})\tau_i^{p_i-1}}{\|w_i\|^{p_i-1}} = \frac{\tau_i^{q_i}}{\|w_i\|^{q_i}},$$

that is,

$$(a_0 + \tau_i^{p_i})\tau_i^{p_i-q_i-1} = \|w_i\|^{p_i-q_i-1}. \tag{33}$$

A similar comparison argument was considered in the case of Kirchhoff type Laplacian differential operator (but without convection) by Alves et al. [2]; see also the references therein. On this basis we have the following general existence and multiplicity type result.

Proposition 1. *If problem (29) has a weak positive solution $(w_1, w_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ such that*

$$0 < w_i(x) \quad \textit{for a.a. } x \in \Omega,\ i = 1, 2,$$

then problem (28) *admits as many positive weak solutions* $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ *such that*

$$0 < u_i(x) \quad \text{for a.a. } x \in \Omega,\ i = 1, 2,$$

as well as the equation (33) *admits solutions* $\tau_i > 0$, $i = 1, 2$.

We note that the existence of solution (as well as the construction of subsolution-supersolution) for the type problem (30) can be obtained following the usual approach based on the first eigenvalue of the p_i-Laplacian differential operator (see also [6]). Let us recall some basic facts about the spectrum of $(-\Delta_{p_i}, W_0^{1,p_i}(\Omega))$ (see Gasiński-Papageorgiou [8]). Hence, we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_{p_i} w = \widehat{\lambda}|w|^{p_i-2}w & \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega \end{cases}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an "eigenvalue" of $(-\Delta_{p_i}, W_0^{1,p_i}(\Omega))$, if the above problem admits a nontrivial solution $\widehat{w} \in W_0^{1,p_i}(\Omega)$ known as an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$. By $\widehat{\sigma}(p_i)$ we denote the set of eigenvalues of $(-\Delta_{p_i}, W_0^{1,p_i}(\Omega))$. This set is closed and has a smallest element $\widehat{\lambda}_1(p_i)$ such that the following is the case:

- $\widehat{\lambda}_1(p_i) > 0$;
- $\widehat{\lambda}_1(p_i)$ is isolated (namely, we can find $\varepsilon > 0$ such that $(\widehat{\lambda}_1(p_i), \widehat{\lambda}_1(p_i) + \varepsilon) \cap \widehat{\sigma}(p_i) = \emptyset$);
- $\widehat{\lambda}_1(p_i)$ is simple (namely, if w, v are eigenfunctions corresponding to $\widehat{\lambda}_1(p_i)$, then $w = \chi v$ for some $\chi \in \mathbb{R} \setminus \{0\}$).

We have the following variational characterization of this eigenvalue

$$\widehat{\lambda}_1(p_i) = \inf\left\{ \frac{\|\nabla w\|_{p_i}^{p_i}}{\|w\|_{p_i}^{p_i}} \,:\, w \in W_0^{1,p_i}(\Omega),\ w \neq 0 \right\}. \tag{34}$$

The infimum in (34) is realized on the corresponding one dimensional eigenspace. By (34) we get that the elements of this eigenspace have fixed sign. In fact $\widehat{\lambda}_1(p_i) > 0$ is the only eigenvalue with eigenfunctions of fixed sign. All other eigenvalues have eigenfunctions which are sign changing. By $\widehat{w}_1 = \widehat{w}_1(p_i)$ we denote the positive L^{p_i}-normalized (namely, $\|\widehat{w}_1\|_{p_i} = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(p_i) > 0$. The nonlinear regularity theory and the nonlinear maximum principle (see Pucci-Serrin [17]) imply that $\widehat{w}_1(p_i) \in \operatorname{int} C_+^0$, where C_+^0 is the positive order cone of $C_0^1(\overline{\Omega}) = \{w \in C^1(\overline{\Omega}) \,:\, w\big|_{\partial\Omega} = 0\}$. Precisely, we get

$$C_+^0 = \left\{ w \in C_0^1(\overline{\Omega}) \,:\, w(x) \geq 0 \text{ for all } x \in \overline{\Omega} \right\},$$

and

$$\operatorname{int} C_+^0 = \left\{ w \in C_+^0 \,:\, w(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial w}{\partial n}\Big|_{\partial\Omega} < 0 \right\},$$

where $n(\cdot)$ is the outward unit normal on $\partial\Omega$.

Now, starting from a weak solution $(w_1, w_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ to problem (29) such that

$$0 < w_i(x) \quad \text{for a.a. } x \in \Omega,\ i = 1, 2,$$

and using the comparison arguments developed at the beginning of this section, we deduce that

$$(u_1, u_2) := \left(\tau_1 \frac{w_1}{\|w_1\|}, \tau_2 \frac{w_2}{\|w_2\|}\right)$$

is a weak positive solution to problem (28) for any suitable values $\tau_i > 0$ solving the equation (33), $i = 1, 2$.

Remark 2. For some precise results about the existence of at least a positive weak solution to certain Dirichlet boundary value problems with gradient dependence in the reaction term and satisfying suitable growth conditions, the reader can refer to [6, Theorem 3.1] and [15, Theorem 4.1].

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