# ON NONCOERCIVE $(p,q)$-EQUATIONS

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Abstract. We consider a nonlinear Dirichlet problem driven by a $(p,q)$-Laplace differential operator $(1 < q < p)$. The reaction is $(p-1)$-linear near $\pm\infty$ and the problem is noncoercive. Using variational tools and truncation and comparison techniques together with critical groups, we produce five nontrivial smooth solutions all with sign information and ordered. In the particular case when $q = 2$, we produce a second nodal solution for a total of six nontrivial smooth solutions all with sign information.

## 1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^2$-boundary $\partial\Omega$. In this paper we study the following $(p,q)$-Dirichlet problem:

$$-\Delta_p u(z) - \Delta_q u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{1}$$

In this problem $1 < q < p < \infty$. For any $r \in (1,\infty)$ by $\Delta_r$ we denote the $r$-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

The reaction $f(z,x)$ is a Carathéodory function, that is, for all $x \in \mathbb{R}$ $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$ $x \to f(z,x)$ is continuous. We assume that $f(z,\cdot)$ exhibits $(p-1)$-linear growth at $\pm\infty$ (that is, $f(z,\cdot)$ is $(p-1)$-homogeneous at $\pm\infty$). However, the problem is noncoercive since asymptotically as $x \to \pm\infty$ the quotient $\dfrac{f(z,x)}{|x|^{p-2}x}$ stays above the principal eigenvalue $\widehat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$. Similarly asymptotically as $x \to 0$, the quotient $\dfrac{f(z,x)}{|x|^{q-2}x}$ stays above $\widehat{\lambda}_1(q) > 0$ of $(-\Delta_q, W_0^{1,q}(\Omega))$. Hence the origin can not be a local minimizer of the energy functional and this does not permit the use of the mountain pass theorem directly on the energy functional. Nevertheless by assuming an oscillatory behavior of $f(z,\cdot)$ near zero, and

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using variational methods based on the critical point theory together with suitable truncation and comparison techniques and with the use of critical groups (Morse theory), we prove two multiplicity theorems producing five and six nontrivial smooth solutions respectively, all with sign information. Coercive $(p,q)$-equations were studied by Papageorgiou-Rădulescu-Repovš [18], Papageorgiou-Vetro-Vetro [21] (with $q=2$), Marano-Papageorgiou [15] and Medeiros-Perera [16]. In these works the authors prove the existence of three or four nontrivial solutions, and nodal solutions (that is, sign changing solutions) were obtained only in [17], [20]. Noncoercive $(p-1)$-linear equations were investigated by Cingolani-Degiovanni [2] and Papageorgiou-Rădulescu-Repovš [18], [19]. In [2] we find only an existence result, while in [18], [19] $q=2$ and the equation is parametric. The authors produce up to four solutions for small values of the parameter. Our work complements that of Gasiński-Papageorgiou [7], where an analogous multiplicity theorem is proved for equations driven by the $p$-Laplacian only and with a reaction which satisfies more restrictive conditions and no nodal solutions are obtained. Finally we mention the recent works of He-Lei-Zhang-Sun [10] (with $q=2$ and $(p-1)$-superlinear reaction) and of Papageorgiou-Vetro-Vetro [22] (also with $q=2$, parametric concave-convex problems).

## 2. Mathematical Background

The main spaces in the analysis of (1) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega})=\{u\in C^1(\overline{\Omega}): u\big|_{\partial\Omega}=0\}$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\|=\|\nabla u\|_p \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is ordered with positive (order) cone $C_+=\{u\in C_0^1(\overline{\Omega}): u(z)\geq 0 \text{ for all } z\in\overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+=\left\{u\in C_+ : u(z)>0 \text{ for all } z\in\Omega,\quad \frac{\partial u}{\partial n}\bigg|_{\partial\Omega}<0\right\},$$

with $\dfrac{\partial u}{\partial n}=(\nabla u,n)_{\mathbb{R}}$ and $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

For $x\in\mathbb{R}$, we set $x^\pm=\max\{\pm x,0\}$. Then for $u\in W_0^{1,p}(\Omega)$, we define $u^\pm(z)=u(z)^\pm$ for all $z\in\Omega$. We know that $u^\pm\in W_0^{1,p}(\Omega)$ and $u=u^+-u^-$, $|u|=u^++u^-$. For $u,v\in W^{1,p}(\Omega)$ with $u\leq v$, we set

$$[u,v]=\left\{h\in W_0^{1,p}(\Omega): u(z)\leq h(z)\leq v(z) \text{ for a.a. } z\in\Omega\right\},$$
$$\operatorname{int}_{C_0^1(\overline{\Omega})}[u,v]=\text{the interior in } C_0^1(\overline{\Omega}) \text{ of } [u,v]\cap C_0^1(\overline{\Omega}),$$

$$[u) = \left\{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega\right\}.$$

Given $r \in (1,\infty)$, by $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^* \left(\frac{1}{r} + \frac{1}{r'} = 1\right)$, we denote the nonlinear map defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

This map has the following properties (see Gasiński-Papageorgiou [8], p. 279).

**Proposition 1.** $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ *is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type* $(S)_+$ *which means that the following property is true:*

$$u_n \xrightarrow{w} u, \ \limsup_{n\to+\infty}\langle A_r(u_n), u_n - u\rangle \leq 0 \quad \Rightarrow \quad u_n \to u \text{ in } W_0^{1,r}(\Omega).$$

We consider the following nonlinear eigenvalue problem

$$-\Delta_r u(z) = \widehat{\lambda}|u(z)|^{r-2}u(z) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0, \quad 1 < r < \infty. \tag{2}$$

It is well-known that (2) has a smallest eigenvalue $\widehat{\lambda}_1(r)$ which has the following properties (see Gasiński-Papageorgiou [5]):

$$(a) \quad 0 < \widehat{\lambda}_1(r) = \inf\left[\frac{\|\nabla u\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0\right], \tag{3}$$

$(b)$ $\widehat{\lambda}_1(r)$ is isolated (that is, if $\widehat{\sigma}(r)$ denotes the spectrum of (2), then we can find $\varepsilon > 0$ such that $(\widehat{\lambda}_1(r), \widehat{\lambda}_1(r) + \varepsilon) \cap \widehat{\sigma}(r) = \emptyset$).

$(c)$ $\widehat{\lambda}_1(r)$ is simple (that is, if $\widehat{u}, \widehat{v} \in W_0^{1,r}(\Omega)$ are eigenfunctions corresponding to $\widehat{\lambda}_1(r)$, then $\widehat{u} = \xi\widehat{v}$ with $\xi \in \mathbb{R}\setminus\{0\}$).

The infimum in (3) is realized on the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1(r)$ (see $(c)$). It is easy to see from $(a)$ and $(c)$ that the elements of this eigenspace have fixed sign. Let $\widehat{u}_1(r)$ be the positive, $L^r$-normalized (that is, $\|\widehat{u}_1(r)\|_r = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(r) > 0$. The nonlinear regularity theory (see Lieberman [14]), implies that $\widehat{u}_1(r) \in C_+\setminus\{0\}$. Moreover, the nonlinear maximum principle (see Pucci-Serrin [23], p. 120) implies that $\widehat{u}_1(r) \in \operatorname{int} C_+$. Using the Lusternik-Schnirelmann minimax scheme, in addition to $\widehat{\lambda}_1(r)$ we can have a whole sequence $\{\widehat{\lambda}_k(r)\}_{k\geq 1}$ of distinct eigenvalues of (2) such that $\widehat{\lambda}_k(r) \to +\infty$ as $k \to +\infty$. These are known as variational or

LS-eigenvalues. We do not know if they exhaust $\widehat{\sigma}(r)$. This is the case if $r = 2$ (linear eigenvalue problem) or if $N = 1$.

We will also deal with a weighted version of (2). So, let $m \in L^\infty(\Omega)$ such that $m(z) \geq 0$ for a.a. $z \in \Omega$, $m \not\equiv 0$. We consider the following eigenvalue problem:

$$-\Delta_r u(z) = \widetilde{\lambda} m(z)|u(z)|^{r-2}u(z) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{4}$$

The spectrum $\widetilde{\sigma}(m,r)$ of (4) has the same properties as $\widehat{\sigma}(r)$ and by $\widetilde{\lambda}(m,r)$ we denote the eigenvalues of (4). In this case the variational characterization of $\widetilde{\lambda}_1(m,r)$ has the following form

$$0 < \widetilde{\lambda}_1(m,r) = \inf \left[ \frac{\|\nabla u\|_r^r}{\int_\Omega m(z)|u|^r dz} : u \in W_0^{1,r}(\Omega), u \neq 0 \right]. \tag{5}$$

Using (5), we easily infer the following monotonicity property for the map $m \to \widetilde{\lambda}_1(m,r)$.

**Proposition 2.** *If $m, \widehat{m} \in L^\infty(\Omega)$, $0 \leq m(z) \leq \widehat{m}(z)$ for a.a. $z \in \Omega$, $m \not\equiv 0$, $\widehat{m} \not\equiv m$, $1 < r < \infty$, then $\widetilde{\lambda}_1(\widehat{m},r) < \widetilde{\lambda}_1(m,r)$.*

We mention that for both eigenvalue problems (2) and (4), only the first eigenvalue has eigenfunctions of constant sign. All the other eigenvalues have eigenfunctions which are nodal (sign changing).

Let $X$ be a Banach space and $\varphi \in C^1(X,\mathbb{R})$. By $K_\varphi$ we denote the critical set of $\varphi$, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Also, if $c \in \mathbb{R}$, then we set

$$\varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

We say that $\varphi \in C^1(X,\mathbb{R})$ satisfies the "$C$-condition", if the following holds:

"Every sequence $\{u_n\}_{n\geq1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|_X)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to +\infty,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the functional $\varphi(\cdot)$ which compensates for the fact that the ambient space $X$ is not locally compact (being in general infinite dimensional).

Let $(Y_1, Y_2)$ be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the $k^{th}$-relative singular homology group with integer coefficients for the pair $(Y_1, Y_2)$. Then the critical groups of $\varphi(\cdot)$ at an isolated $u \in K_\varphi$ with $c = \varphi(u)$, are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with $U$ being a neighborhood of $u$ such that

$$K_\varphi \cap \varphi^c \cap U = \{u\}.$$

The excision property of singular homology implies that the above definition of critical groups is independent of the isolating neighborhood $U$.

Suppose that $\varphi$ satisfies the $C$-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of $\varphi$ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see Papageorgiou-Rădulescu-Repovš [20], Theorem 5.3.12, p. 386) implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Assume that $K_\varphi$ is finite. We set

$$\begin{aligned} M(t,u) &= \sum_{k \in \mathbb{N}_0} \operatorname{rank} C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi, \\ P(t,\infty) &= \sum_{k \in \mathbb{N}_0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

Then the Morse relation says that

$$\sum_{u \in K_\varphi} M(t,u) = P(t,\infty) + (1+t)Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{6}$$

where $Q(t) = \sum_{k \in \mathbb{N}_0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients $\beta_k$.

Finally given $h_1, h_2 \in L^\infty(\Omega)$, we write $h_1 \prec h_2$ if for all $K \subseteq \Omega$ compact, there exists $c_K > 0$ such that

$$c_K \leq h_2(z) - h_1(z) \quad \text{for a.a. } z \in K.$$

If $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then clearly $h_1 \prec h_2$.

Also for $k, n \in \mathbb{N}_0$, by $\delta_{k,n}$ we denote the Kronecker symbol defined by

$$\delta_{k,n} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

## 3. Constant Sign Solutions

In this section we produce constant sign smooth solutions for problem (1). The hypotheses on the reaction $f(z,x)$ are the following:

$H_1$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$ and

$(i)$ $|f(z,x)|\leq a(z)[1+|x|^{p-1}]$ for a.a. $z\in\Omega$, all $x\in\mathbb{R}$, with $a\in L^\infty(\Omega)$;

$(ii)$ there exists a function $\eta\in L^\infty(\Omega)$ such that

$$\widehat{\lambda}_1(p)\leq\eta(z) \text{ for a.a. } z\in\Omega,\ \widehat{\lambda}_1(p)\not\equiv\eta,$$
$$\eta(z)\leq\liminf_{x\to\pm\infty}\frac{f(z,x)}{|x|^{p-2}x} \text{ uniformly for a.a. } z\in\Omega;$$

$(iii)$ there exists a function $\eta_0\in L^\infty(\Omega)$ such that

$$\widehat{\lambda}_1(q)\leq\eta_0(z) \text{ for a.a. } z\in\Omega,\ \widehat{\lambda}_1(q)\not\equiv\eta_0,$$
$$\eta_0(z)\leq\liminf_{x\to 0}\frac{f(z,x)}{|x|^{q-2}x} \text{ uniformly for a.a. } z\in\Omega;$$

$(iv)$ there exist $\vartheta_-<0<\vartheta_+$ such that $f(z,\vartheta_+)\leq\widehat{c}_0<0<\widehat{c}_1\leq f(z,\vartheta_-)$ for a.a. $z\in\Omega$;

$(v)$ for every $\rho>0$, there exists $\widehat{\xi}_\rho>0$ such that for a.a. $z\in\Omega$, the function

$$x\to f(z,x)+\widehat{\xi}_\rho|x|^{p-2}x$$

is nondecreasing on $[-\rho,\rho]$.

*Remark* 1. Hypotheses $H_1(iii),(iv)$ imply that $f(z,\cdot)$ has an oscillatory behavior near zero. Hypothesis $H_1\,(v)$ is a one-sided local Lipschitz condition and it is satisfied if for a.a. $z\in\Omega$, $f(z,\cdot)$ is differentiable and for every $\rho>0$, we can find $\widetilde{\xi}_\rho>0$ such that

$$f'_x(z,x)x\geq-\widetilde{\xi}_\rho|x|^p \quad \text{for a.a. } z\in\Omega, \text{ all } |x|\leq\rho.$$

First using only the growth condition $H_1(i)$ and the local conditions near zero $H_1(iii),(iv)$, we will produce two nontrivial constant sign smooth solutions.

**Proposition 3.** *If hypotheses $H_1(i),(iii),(iv)$ hold, then problem* (1) *has two constant sign solutions*

$$u_0\in\operatorname{int}C_+,\quad u_0(z)<\vartheta_+\quad \textit{for all } z\in\overline{\Omega},$$
$$v_0\in-\operatorname{int}C_+,\quad \vartheta_-<v_0(z)\quad \textit{for all } z\in\overline{\Omega}.$$

*Proof.* First we produce the positive solution. To this end, we introduce the Carathéodory function $\widehat{f}_+(z,x)$ defined by

$$\widehat{f}_+(z,x)=\begin{cases} f(z,x^+) & \text{if } x\leq \vartheta_+, \\ f(z,\vartheta_+) & \text{if } \vartheta_+<x. \end{cases} \tag{7}$$

We set $\widehat{F}_+(z,x)=\int_0^x \widehat{f}_+(z,s)ds$ and consider the $C^1$-functional $\widehat{\varphi}_+ : W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\varphi}_+(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{q}\|\nabla u\|_q^q-\int_\Omega \widehat{F}_+(z,u)dz \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

From (2) it is clear that $\widehat{\varphi}_+(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we see that $\widehat{\varphi}_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0\in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(u_0)=\inf\left[\widehat{\varphi}_+(u) : u\in W_0^{1,p}(\Omega)\right]. \tag{8}$$

On account of hypothesis $H_1(iii)$, given $\varepsilon>0$, we can find $\delta>0$ such that

$$F(z,x)\geq \frac{1}{q}\left[\eta_0(z)-\varepsilon\right]|x|^q \quad \text{for a.a. } z\in\Omega, \text{ all } |x|\leq\delta. \tag{9}$$

We choose $t\in(0,1)$ small such that $t\widehat{u}_1(q)(z)\in[0,\delta]$ for all $z\in\overline{\Omega}$ (recall that $\widehat{u}_1(q)\in \operatorname{int} C_+$). We have

$$\widehat{\varphi}_+(t\widehat{u}_1(q))\leq \frac{t^p}{p}\|\nabla \widehat{u}_1(q)\|_p^p+\frac{t^q}{q}\left[\widehat{\lambda}_1(q)-\int_\Omega \eta_0(z)\widehat{u}_1(q)^q dz+\varepsilon\right] \tag{10}$$

(see (9) and recall that $\|\widehat{u}_1(q)\|_q=1$). Note that

$$\int_\Omega \eta_0(z)\widehat{u}_1(q)^q dz-\widehat{\lambda}_1(q)=\int_\Omega [\eta_0(z)-\widehat{\lambda}_1(q)]\widehat{u}_1(q)^q dz=\beta_0>0$$

(see hypothesis $H_1(iii)$ and recall $\widehat{u}_1(q)\in\operatorname{int} C_+$). So, if we choose $\varepsilon\in(0,\beta_0)$, from (10) we have

$$\widehat{\varphi}_+(t\widehat{u}_1(q))\leq c_1t^p-c_2t^q \quad \text{for some } c_1,c_2>0.$$

Since $1<q<p$, choosing $t\in(0,1)$ even smaller if necessary, we will have

$$\begin{aligned} & \widehat{\varphi}_+(t\widehat{u}_1(q))<0, \\ \Rightarrow\quad & \widehat{\varphi}_+(u_0)<0=\widehat{\varphi}_+(0) \quad \text{(see (8))}, \\ \Rightarrow\quad & u_0\neq 0. \end{aligned}$$

From (8) we have

$$\widehat{\varphi}'_+(u_0)<0,$$

$$\Rightarrow \quad \langle A_p(u_0), h\rangle + \langle A_q(u_0), h\rangle = \int_\Omega \widehat{f}_+(z, u_0) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{11}$$

In (11) we choose $h = -u_0^- \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} &\|u_0^-\|^p \leq 0 \quad \text{(see (7))}, \\ \Rightarrow \quad & u_0 \geq 0,\ u_0 \neq 0. \end{aligned}$$

Next in (11) we choose $h = (u_0 - \vartheta_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A_p(u_0), (u_0 - \vartheta_+)^+\rangle + \langle A_q(u_0), (u_0 - \vartheta_+)^+\rangle \\ & = \int_\Omega f(z, \vartheta_+)(u_0 - \vartheta_+)^+ dz \quad \text{(see (7))} \\ & \leq 0 \quad \text{(see hypothesis } H_1(iv)), \\ \Rightarrow \quad & u_0 \leq \vartheta_+. \end{aligned}$$

So, we have proved that

$$u_0 \in [0, \vartheta_+],\ u_0 \neq 0. \tag{12}$$

From (12), (7) and (11), we infer that $u_0$ is a positive solution of (1) and

$$-\Delta_p u_0(z) - \Delta_q u_0(z) = f(z, u_0(z)) \quad \text{for a.a. } z \in \Omega. \tag{13}$$

Invoking Theorem 7.1, p. 286, of Ladyzhenskaya-Ural′tseva [12] we have that $u_0 \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [14] implies that $u_0 \in C_+ \setminus \{0\}$. On account of hypotheses $H_1(i), (iii)$, given $\varepsilon > 0$, we can find $c_3 = c_3(\varepsilon) > 0$ such that

$$f(z, x) \geq [\eta_0(z) - \varepsilon] x^{q-1} - c_3 x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{14}$$

Since $q < p$, we see that for $\varepsilon, \delta > 0$ small, for a.a. $z \in \Omega$, the function

$$x \to [\eta_0(z) - \varepsilon] x^{q-1} - c_3 x^{p-1}$$

is nondecreasing on $[0, \delta]$. Then (13), (14) and Theorem 5.4.1, p. 111, of Pucci-Serrin [23] imply that

$$0 < u_0(z) \quad \text{for all } z \in \Omega.$$

Finally invoking the nonlinear boundary point theorem (see Pucci-Serrin [23], Theorem 5.5.1, p. 120), we have

$$u_0 \in \operatorname{int} C_+.$$

Let $\widehat{\xi}_{\vartheta_+} > 0$ be as postulated by hypothesis $H_1(v)$. We have

$$\begin{aligned} & -\Delta_p u_0(z) - \Delta_q u_0(z) + \widehat{\xi}_{\vartheta_+} u_0(z)^{p-1} \\ & = f(z, u_0(z)) + \widehat{\xi}_{\vartheta_+} u_0(z)^{p-1} \end{aligned}$$

$$\leq f(z,\vartheta_+) + \widehat{\xi}_{\vartheta_+}\vartheta_+^{p-1}$$
$$\leq -\Delta_p\vartheta_+ - \Delta_q\vartheta_+ + \widehat{\xi}_{\vartheta_+}\vartheta_+^{p-1}.$$

Invoking Proposition 3.2 of Gasiński-Papageorgiou [9] we obtain

$$u_0(z) < \vartheta_+ \quad \text{for all } z \in \overline{\Omega}.$$

For the negative solution, we introduce the Carathéodory function $\widehat{f}_-(z,x)$ defined by

$$\widehat{f}_-(z,x) = \begin{cases} f(z,\vartheta_-) & \text{if } x < \vartheta_-, \\ f(z,-x^-) & \text{if } \vartheta_- \leq x. \end{cases} \tag{15}$$

We set $\widehat{F}_-(z,x) = \int_0^x \widehat{f}_-(z,s)ds$ and consider the $C^1$-functional $\widehat{\varphi}_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_-(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \widehat{F}_-(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Working with $\widehat{\varphi}_-(\cdot)$ as above and using (15), we produce a negative solution $v_0$ such that

$$v_0 \in [\vartheta_-, 0] \cap (-\mathrm{int}\, C_+),\ \vartheta_- < v_0(z) \quad \text{for all } z \in \overline{\Omega}.$$

$\square$

Now using $u_0, v_0$ from the above proposition and making use also of hypothesis $H_1(ii)$ (the asymptotic condition as $x \to \pm\infty$), we will generate two more nontrivial constant sign smooth solutions of (1), which are localized with respect to $u_0$ and $v_0$.

**Proposition 4.** *If hypotheses $H_1(i)-(iv)$ hold, then problem (1) has two more constant sign solutions $\widehat{u} \in \mathrm{int}\, C_+$ and $\widehat{v} \in -\mathrm{int}\, C_+$ such that $u_0 \leq \widehat{u}$, $u_0 \neq \widehat{u}$ and $\widehat{v} \leq v_0$, $v_0 \neq \widehat{v}$.*

*Proof.* First we produce the second positive solution.

Let $u_0 \in \mathrm{int}\, C_+$ be the positive solution produced in Proposition 3. We introduce the Carathéodory function $g_+(z,x)$ defined by

$$g_+(z,x) = \begin{cases} f(z,u_0(z)) & \text{if } x \leq u_0(z), \\ f(z,x) & \text{if } u_0(z) < x. \end{cases} \tag{16}$$

We set $G_+(z,x) = \int_0^x g_+(z,s)ds$ and consider the $C^1$-functional $\psi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega G_+(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Claim 1: $\psi_+(\cdot)$ satisfies the $C$-condition.

We consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that $\{\psi_+(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1+\|u_n\|)\psi_+'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \text{ as } n \to +\infty \quad (17)$$
$$\text{(recall that } \tfrac{1}{p}+\tfrac{1}{p'}=1\text{)}.$$

From (17) we have

$$\left|\langle A_p(u_n),h\rangle + \langle A_q(u_n),h\rangle - \int_\Omega g_+(z,u_n)hdz\right| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad (18)$$

for all $h \in W_0^{1,p}(\Omega)$, with $\varepsilon_n \to 0^+$. In (18) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ and obtain

$$\|\nabla u_n^-\|_p^p \leq c_4\left[1+\|u_n^-\|\right] \quad \text{for some } c_4>0, \text{ all } n\in\mathbb{N} \text{ (see (16))},$$
$$\Rightarrow \quad \{u_n^-\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \quad (19)$$

Suppose that $\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is not bounded. We may assume that

$$\|u_n^+\| \to +\infty \quad \text{as } n \to +\infty. \quad (20)$$

We set $y_n = \dfrac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega),\ y_n \to y \text{ in } L^p(\Omega),\ y \geq 0. \quad (21)$$

From (18) and (19) we have

$$\left|\langle A_p(u_n^+),h\rangle + \langle A_q(u_n^+),h\rangle - \int_\Omega g_+(z,u_n^+)hdz\right| \leq c_5\|h\|$$
$$\text{for some } c_5>0, \text{ all } h \in W_0^{1,p}(\Omega) \text{ (see (16))},$$
$$\Rightarrow \quad \left|\langle A_p(y_n),h\rangle + \frac{1}{\|u_n^+\|^{p-q}}\langle A_q(y_n),h\rangle - \int_\Omega \frac{g_+(z,u_n^+)}{\|u_n^+\|^{p-1}}hdz\right| \leq \frac{c_5\|h\|}{\|u_n^+\|^{p-1}} \quad (22)$$
$$\text{for all } h \in W_0^{1,p}(\Omega).$$

From (16) and hypothesis $H_1(i)$, we have

$$|g_+(z,x)| \leq c_6[1+|x|^{p-1}] \quad \text{for a.a. } z\in\Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_6>0.$$

Hence we have

$$\left|\frac{g_+(z,u_n^+)}{\|u_n^+\|^{p-1}}\right| \leq c_6\left[\frac{1}{\|u_n^+\|^{p-1}} + y_n^{p-1}\right] \quad \text{for all } n \in \mathbb{N},$$
$$\Rightarrow \quad \left\{\frac{g_+(\cdot,u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(\Omega) \quad \text{is bounded (see (20)).} \quad (23)$$

From (23), hypothesis $H_1(ii)$ and by passing to a subsequence if necessary we have

$$\frac{g_+(\cdot,u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \widehat{\eta}(\cdot)y^{p-1} \text{ in } L^{p'}(\Omega) \text{ as } n\to+\infty, \tag{24}$$

where $\widehat{\eta}\in L^\infty(\Omega)$, $\eta(z)\le\widehat{\eta}(z)$ for a.a. $z\in\Omega$ (see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29).

In (22) we choose $h=y_n-y\in W_0^{1,p}(\Omega)$, pass to the limit as $n\to+\infty$ and use (21), (24) and (20). We obtain

$$\lim_{n\to+\infty}\langle A_p(y_n),y_n-y\rangle=0,$$

$$\Rightarrow \quad y_n\to y \text{ in } W_0^{1,p}(\Omega) \text{ and so } \|y\|=1,\ y\ge 0 \text{ (see Proposition 1).} \tag{25}$$

If in (22) we pass to the limit as $n\to+\infty$ and use (24), (25) and (20) (recall $q<p$), then we have

$$\langle A_p(y),h\rangle=\int_\Omega \widehat{\eta}(z)y^{p-1}hdz \quad \text{for all } h\in W_0^{1,p}(\Omega),$$

$$\Rightarrow \quad -\Delta_p y(z)=\widehat{\eta}(z)y(z)^{p-1} \quad \text{for a.a. } z\in\Omega, \quad y\Big|_{\partial\Omega}=0. \tag{26}$$

Recall that $\eta(z)\le\widehat{\eta}(z)$ for a.a. $z\in\Omega$. So, on account of hypothesis $H_1\,(ii)$ and Proposition 2, we have

$$\widetilde{\lambda}_1(\widehat{\eta},p)<\widetilde{\lambda}_1(\widehat{\lambda}_1(p),p)=1. \tag{27}$$

From (26) and (27) it follows that $y$ must be nodal, a contradiction (see (25)). This means that

$$\{u_n^+\}_{n\ge1}\subseteq W_0^{1,p}(\Omega) \text{ is bounded,}$$

$$\Rightarrow \quad \{u_n\}_{n\ge1}\subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (19)).}$$

So, we may assume that

$$u_n\xrightarrow{w}u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n\to u \text{ in } L^p(\Omega). \tag{28}$$

In (18) we choose $h=u_n-u\in W_0^{1,p}(\Omega)$, pass to the limit as $n\to+\infty$ and use (22). Then

$$\lim_{n\to+\infty}[\langle A_p(u_n),u_n-u\rangle+\langle A_q(u_n),u_n-u\rangle]=0,$$

$$\Rightarrow \quad \limsup_{n\to+\infty}[\langle A_p(u_n),u_n-u\rangle+\langle A_q(u),u_n-u\rangle]\le 0 \quad \text{(since } A_q(\cdot) \text{ is monotone),}$$

$$\Rightarrow \quad \limsup_{n\to+\infty}\langle A_p(u_n),u_n-u\rangle\le 0 \quad \text{(see (28)),}$$

$$\Rightarrow \quad u_n\to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 1).}$$

So $\psi_+(\cdot)$ satisfies the $C$-condition and this proves Claim 1.

Using (16) and the nonlinear regularity theory (see Lieberman [14]), we obtain that

$$K_{\psi_+} \subseteq [u_0) \cap \operatorname{int} C_+. \tag{29}$$

Without any loss of generality, we may assume that

$$K_{\psi_+} \cap [u_0, \vartheta_+] = \{u_0\}. \tag{30}$$

Otherwise we already have a second positive smooth solution bigger than $u_0$ and so we are done.

Claim 2: $u_0$ is a local minimizer of the functional $\psi_+(\cdot)$.

Consider the following truncation of $g_+(z,\cdot)$:

$$\widehat{g}_+(z,x) = \begin{cases} g_+(z,x) & \text{if } x \leq \vartheta_+, \\ g_+(z,\vartheta_+) & \text{if } \vartheta_+ < x. \end{cases} \tag{31}$$

This is a Carathéodory function. We set $\widehat{G}_+(z,x) = \int_0^x \widehat{g}_+(z,s)ds$ and consider the $C^1$-functional $\widehat{\psi}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \widehat{G}_+(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (31) we see that $\widehat{\psi}_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \widehat{\psi}_+(u_0) = \min\left[\widehat{\psi}_+(u) : u \in W_0^{1,p}(\Omega)\right], \\ \Rightarrow \quad & \widehat{u}_0 \in K_{\widehat{\psi}_+}. \end{aligned} \tag{32}$$

Using (31) we see that

$$K_{\widehat{\psi}_+} \subseteq [u_0, \vartheta_+] \cap \operatorname{int} C_+, \tag{33}$$

$$\widehat{\psi}_+\Big|_{[0,\vartheta_+]} = \psi_+\Big|_{[0,\vartheta_+]}, \quad \widehat{\psi}'_+\Big|_{[0,\vartheta_+]} = \psi'_+\Big|_{[0,\vartheta_+]}. \tag{34}$$

Then from (32), (33), (34) and (30), we infer that

$$\widehat{u}_0 = u_0 \in \operatorname{int} C_+.$$

Recall that

$$u_0(z) < \vartheta_+ \quad \text{for all } z \in \overline{\Omega} \text{ (see Proposition 3).}$$

Then from (34) we have that

$$\begin{aligned} & u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \psi_+(\cdot), \\ \Rightarrow \quad & u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \psi_+(\cdot) \text{ (see Gasiński-Papageorgiou [6]).} \end{aligned}$$

This proves Claim 2.

From (29) it is clear that we may assume that

$$K_{\psi_+} \text{ is finite.} \tag{35}$$

Otherwise we already have an infinity of positive smooth solutions of (1) which are bigger than $u_0$.

From Claim 2, (35) and Theorem 5.7.6, p. 449, of Papageorgiou-Rădulescu-Repovš [20], we know that we can find $\rho \in (0,1)$ small such that

$$\psi_+(u_0) < \inf\,[\psi_+(u) : \|u-u_0\| = \rho] = m_+. \tag{36}$$

On account of hypotheses $H_1(i),(ii)$ and (16), we see that given $\varepsilon > 0$ we can find $c_7 = c_7(\varepsilon) > 0$ such that

$$G_+(z,x) \geq \frac{1}{p}[\eta(z)-\varepsilon]x^p - c_7 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{37}$$

Then we have

$$\begin{aligned}\psi_+(t\widehat{u}_1(p)) &\leq \frac{t^p}{p}\widehat{\lambda}_1(p) + \frac{t^q}{q}\|\nabla(t\widehat{u}_1(p))\|_q^q - \frac{t^p}{p}\int_\Omega \eta(z)\widehat{u}_1(p)^p dz + \frac{\varepsilon t^p}{p} + c_8 \\ &\qquad \text{for some } c_8 > 0 \text{ (see (37) and recall that } \|\widehat{u}_1(p)\|_p = 1) \\ &= \frac{t^p}{p}\left[\int_\Omega [\widehat{\lambda}_1(p) - \eta(z)]\widehat{u}_1(p)^p dz + \varepsilon\right] + \frac{t^q}{q}\|\nabla(t\widehat{u}_1(p))\|_q^q + c_8.\end{aligned}$$

Since $\widehat{u}_1(p) \in \operatorname{int} C_+$, using hypothesis $H_1(ii)$ we see that

$$\widehat{\beta} = \int_\Omega [\eta(z) - \widehat{\lambda}_1(p)]\widehat{u}_1(p)^p dz > 0.$$

Choosing $\varepsilon \in (0,\widehat{\beta})$, we obtain

$$\begin{aligned}&\psi_+(t\widehat{u}_1(p)) \leq \frac{t^q}{q}\|\nabla(t\widehat{u}_1(p))\|_q^q - c_9 t^p + c_8 \quad \text{for some } c_9 > 0, \text{ all } t > 0, \\ \Rightarrow \quad &\psi_+(t\widehat{u}_1(p)) \to -\infty \quad \text{as } t \to +\infty \text{ (recall that } q < p).\end{aligned} \tag{38}$$

Then Claim 1, (36) and (38) permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\psi_+} \text{ and } m_+ \leq \psi_+(\widehat{u}). \tag{39}$$

From (39), (36) and (29), we infer that

$$\begin{aligned}&\widehat{u} \in [u_0) \cap \operatorname{int} C_+,\ \widehat{u} \neq u_0, \\ \Rightarrow \quad &\widehat{u} \in \operatorname{int} C_+ \text{ is a positive solution of (1) (see (16))}, u_0 \leq \widehat{u},\ u_0 \neq \widehat{u}.\end{aligned}$$

To produce a second negative solution, we introduce the Carathéodory function $g_-(z,x)$ defined by

$$g_-(z,x) = \begin{cases} f(z,x) & \text{if } x < v_0(z), \\ f(z,v_0(z)) & \text{if } v_0(z) \le x \end{cases} \tag{40}$$

($v_0 \in -\operatorname{int} C_+$ is the negative solution from Proposition 3).

We set $G_-(z,x) = \int_0^x g_-(z,s)ds$ and consider the $C^1$-functional $\psi_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_-(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega G_-(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Working as above, using this time $\psi_-(\cdot)$, (40) and $\vartheta_- < 0$, we produce a second negative solution $\widehat{v} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{v} \in -\operatorname{int} C_+, \quad \widehat{v} \le v_0, \quad v_0 \ne \widehat{v}.$$

□

Next we will show that problem (1) admits extremal constant sign solutions, that is, a smallest positive solution $u_* \in \operatorname{int} C_+$ and a biggest negative solution $v_* \in -\operatorname{int} C_+$. In Section 4 we will use these extremal constant sign solutions in order to produce a nodal solution for problem (1).

To produce the extremal constant sign solutions, we need to do some preparatory work. Hypotheses $H_1(i),(ii)$ imply that given $\varepsilon > 0$, we can find $c_{10} = c_{10}(\varepsilon) > 0$ such that

$$f(z,x)x \ge [\eta_0(z) - \varepsilon]|x|^q - c_{10}|x|^p \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{41}$$

Motivated by this unilateral growth condition on the reaction $f(z,\cdot)$, we introduce the following auxiliary Dirichlet $(p,q)$-problem

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = [\eta_0(z) - \varepsilon]|u(z)|^{q-2}u(z) - c_{10}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0,\ 1 < q < p,\ \varepsilon > 0. \end{cases} \tag{42}$$

**Proposition 5.** *For all $\varepsilon > 0$ small, problem* (42) *admits a unique positive solution $\overline{u} \in \operatorname{int} C_+$ and since the problem is odd, $\overline{v} = -\overline{u} \in -\operatorname{int} C_+$ is the unique negative solution of* (42).

*Proof.* First we prove the existence of a positive solution for problem (42) when $\varepsilon > 0$ is small.

To this end, let $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the $C^1$-functional defined by

$$\sigma_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{1}{q}\int_\Omega [\eta_0(z) - \varepsilon](u^+)^q dz + \frac{c_{10}}{p}\|u^+\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $q<p$, it is clear that $\sigma_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u}\in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\overline{u})=\min\left[\sigma_+(u):u\in W_0^{1,p}(\Omega)\right]. \tag{43}$$

Let $t\in(0,1)$. We have

$$\begin{aligned}\sigma_+(t\widehat{u}_1(q)) &\leq c_{11}t^p+\frac{t^q}{q}\left[\widehat{\lambda}_1(q)-\int_\Omega \eta_0(z)\widehat{u}_1(q)^q dz+\varepsilon\right]\\ &\qquad \text{for some } c_{11}>0 \text{ (recall that } \|\widehat{u}_1(q)\|_q=1)\\ &=c_{11}t^p-\frac{t^q}{q}\left[\int_\Omega[\eta_0(z)-\widehat{\lambda}_1(q)]\widehat{u}_1(q)^q dz-\varepsilon\right].\end{aligned}$$

Note that

$$\gamma_0=\int_\Omega[\eta_0(z)-\widehat{\lambda}_1(q)]\widehat{u}_1(q)^q dz>0$$

(see hypothesis $H_1(iii)$ and recall that $\widehat{u}_1(q)\in \operatorname{int}C_+$). So, if we let $\varepsilon\in(0,\gamma_0)$, then

$$\sigma_+(t\widehat{u}_1(q))\leq c_{11}t^p-c_{12}t^q \quad \text{for some } c_{12}>0.$$

Since $q<p$, choosing $t\in(0,1)$ small, we have

$$\begin{aligned}&\sigma_+(t\widehat{u}_1(q))<0,\\ \Rightarrow\quad &\sigma_+(\overline{u})<0=\sigma_+(0)\quad \text{(see (43))},\\ \Rightarrow\quad &\overline{u}\neq 0.\end{aligned}$$

From (43) we have

$$\begin{aligned}&\sigma_+'(\overline{u})=0,\\ \Rightarrow\quad &\langle A_p(\overline{u}),h\rangle+\langle A_q(\overline{u}),h\rangle=\int_\Omega\left[\eta_0(z)-\varepsilon\right](\overline{u}^+)^{q-1}hdz-c_{10}\int_\Omega(\overline{u}^+)^{p-1}hdz\\ &\qquad\qquad \text{for all } h\in W_0^{1,p}(\Omega).\end{aligned}$$

Let $h=-\overline{u}^-\in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}&\|\nabla\overline{u}^-\|_p^p+\|\nabla\overline{u}^-\|_q^q=0,\\ \Rightarrow\quad &\overline{u}\geq 0,\ \overline{u}\neq 0,\\ \Rightarrow\quad &\overline{u} \text{ is a positive solution of (42).}\end{aligned}$$

The nonlinear regularity theory implies that $\overline{u}\in C_+\setminus\{0\}$. Also, we have

$$\begin{aligned}&\Delta_p\overline{u}(z)+\Delta_q\overline{u}(z)\leq c_{10}\overline{u}(z)^{p-1}\quad \text{for a.a. } z\in\Omega,\\ \Rightarrow\quad &\overline{u}\in \operatorname{int}C_+\quad \text{(see Pucci-Serrin [23], pp. 111, 120).}\end{aligned}$$

Next we show the uniqueness of this positive solution. To this end, let $\widetilde{u} \in W_0^{1,p}(\Omega)$ be another positive solution of (42). Again the nonlinear regularity theory implies that $\widetilde{u} \in \operatorname{int} C_+$. We consider the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\|\nabla u^{1/q}\|_p^p + \frac{1}{q}\|\nabla u^{1/q}\|_q^q & \text{if } u \geq 0,\ u^{1/q} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Lemma 1 of Díaz-Saa [3], we have that $j(\cdot)$ is convex. Let $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of $j(\cdot)$). Let $h = \overline{u}^q - \widetilde{u}^q \in C_0^1(\overline{\Omega})$. Then for $|t| \leq 1$, we have

$$\overline{u}^q + th \in \operatorname{dom} j \quad \text{and} \quad \widetilde{u}^q + th \in \operatorname{dom} j.$$

Exploiting the convexity of $j(\cdot)$, we see that the Gateaux derivative of $j(\cdot)$ at $\overline{u}^q$ and at $\widetilde{u}^q$ in the direction $h$ exists and via the nonlinear Green's identity we have

$$\begin{aligned} j'(\overline{u}^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p \overline{u} - \Delta_q \overline{u}}{\overline{u}^{q-1}} h dz, \\ j'(\widetilde{u}^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p \widetilde{u} - \Delta_q \widetilde{u}}{\widetilde{u}^{q-1}} h dz. \end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence we have

$$\begin{aligned} & 0 \leq \frac{c_{10}}{q}\int_\Omega \left[\widetilde{u}^{p-q} - \overline{u}^{p-q}\right](\overline{u}^q - \widetilde{u}^q) dz, \\ \Rightarrow \quad & \widetilde{u} = \overline{u} \quad (\text{since } q < p). \end{aligned}$$

This proves the uniqueness of the positive solution $\overline{u} \in \operatorname{int} C_+$ of problem (42). Since the problem is odd, $\overline{v} = -\overline{u} \in -\operatorname{int} C_+$ is the unique negative solution of (42). ☐

Let $S_+$ (resp. $S_-$) be the set of positive (resp. negative) solutions of problem (1). We know that

$$\emptyset \neq S_+ \subseteq \operatorname{int} C_+ \text{ and } \emptyset \neq S_- \subseteq -\operatorname{int} C_+ \quad (\text{see Proposition 3}).$$

**Proposition 6.** *If hypotheses $H_1$ hold, then $\overline{u} \leq u$ for all $u \in S_+$ and $v \leq \overline{v}$ for all $v \in S_-$.*

*Proof.* Let $u \in S_+$ and consider the Carathéodory function $k_+(z,x)$ defined by

$$k_+(z,x) = \begin{cases} [\eta_0(z) - \varepsilon](x^+)^{q-1} - c_{10}(x^+)^{p-1} & \text{if } x \leq u(z), \\ [\eta_0(z) - \varepsilon]u(z)^{q-1} - c_{10}u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{44}$$

We set $K_+(z,x) = \int_0^x k_+(z,s)ds$ and consider the $C^1$-functional $\widehat{\sigma}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\sigma}_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega K_+(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (44) it is clear that $\widehat{\sigma}_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\sigma}_+(\widetilde{u}) = \min[\widehat{\sigma}_+(u) : u \in W_0^{1,p}(\Omega)]. \tag{45}$$

As before (see the proof of Proposition 5), since $q < p$, we have that

$$\begin{aligned} & \widehat{\sigma}_+(\widetilde{u}) < 0 = \widehat{\sigma}_+(0), \\ \Rightarrow \quad & \widetilde{u} \neq 0. \end{aligned}$$

From (45) we have

$$\begin{aligned} & \widehat{\sigma}'_+(\widetilde{u}) = 0, \\ \Rightarrow \quad & \langle A_p(\widetilde{u}), h\rangle + \langle A_q(\widetilde{u}), h\rangle = \int_\Omega k_+(z,\widetilde{u})hdz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{46}$$

In (46), we choose $h = -\widetilde{u}^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \|\nabla \widetilde{u}^-\|_p^p + \|\nabla \widetilde{u}^-\|_q^q = 0 \quad \text{(see (44))}, \\ \Rightarrow \quad & \widetilde{u} \geq 0,\, \widetilde{u} \neq 0. \end{aligned}$$

Next in (46) we choose $h = (\widetilde{u} - u)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A_p(\widetilde{u}), (\widetilde{u}-u)^+\rangle + \langle A_q(\widetilde{u}), (\widetilde{u}-u)^+\rangle \\ & = \int_\Omega \left([\eta_0(z) - \varepsilon]u^{q-1} - c_{10}u^{p-1}\right)(\widetilde{u}-u)^+dz \quad \text{(see (44))} \\ & \leq \int_\Omega f(z,u)(\widetilde{u}-u)^+dz \quad \text{(see (41))} \\ & = \langle A_p(u), (\widetilde{u}-u)^+\rangle + \langle A_q(u), (\widetilde{u}-u)^+\rangle \quad \text{(since } u \in S_+), \\ \Rightarrow \quad & \widetilde{u} \leq u. \end{aligned}$$

We have proved that

$$\widetilde{u} \in [0,u],\, \widetilde{u} \neq 0. \tag{47}$$

Then from (47), (44), (46), we infer that $\widetilde{u}$ is a positive solution of problem (42). Therefore $\widetilde{u} = \overline{u} \in \operatorname{int} C_+$ (see Proposition 5). So, we have

$$\overline{u} \leq u \quad \text{for all } u \in S_+.$$

Similarly we show that $v \leq \overline{v}$ for all $v \in S_-$. $\square$

Now we are ready to produce the extremal constant sign solutions of problem (1). As we already mentioned, in Section 4 using these solutions, we will be able to produce a nodal solution.

**Proposition 7.** *If hypotheses $H_1$ hold, then problem* (1) *admits extremal constant sign solutions, that is,*

- *there exists $u_* \in S_+$ such that $u_* \leq u$ for all $u \in S_+$;*
- *there exists $v_* \in S_-$ such that $v \leq v_*$ for all $v \in S_-$.*

*Proof.* From Filippakis-Papageorgiou [4] we know that $S_+$ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leq u_1$, $u \leq u_2$). Invoking Lemma 3.10, p. 178, of Hu-Papageorgiou [11], we can find $\{u_n\}_{n\geq 1} \subseteq S_+$ decreasing such that

$$\inf_{n\geq 1} u_n = \inf S_+.$$

We have

$$\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle = \int_\Omega f(z, u_n)hdz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \tag{48}$$

$$\overline{u} \leq u_n \leq u_1 \quad \text{for all } n \in \mathbb{N}. \tag{49}$$

From (48) with $h = u_n \in W_0^{1,p}(\Omega)$ and (49), it follows that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^p(\Omega). \tag{50}$$

In (48) we choose $h = u_n - u_* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (50). Then as in the proof of Proposition 4 (see Claim 1), we obtain

$$u_n \to u_* \text{ in } W_0^{1,p}(\Omega). \tag{51}$$

So, if in (48) we pass to the limit as $n \to +\infty$ and use (51), we obtain

$$\langle A_p(u_*), h\rangle + \langle A_q(u_*), h\rangle = \int_\Omega f(z, u_*)hdz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

Since $\overline{u} \leq u_*$ (see (49)), we infer that

$$u_* \in S_+ \text{ and } u_* = \inf S_+.$$

The set of negative solutions $S_-$ is upward directed (that is, if $v_1, v_2 \in S_-$, then we can find $v \in S_-$ such that $v_1 \leq v$, $v_2 \leq v$). So we can find $\{v_n\}_{n\geq 1} \subseteq S_-$ increasing such that

$$\sup_{n\geq 1} v_n = \sup S_-.$$

Reasoning as above, we obtain that

$$v_n \to v_* \text{ in } W_0^{1,p}(\Omega), \quad v_* \in S_-, \quad v_* = \sup S_-, \quad v_* \leq \overline{v}.$$

$\square$

## 4. Nodal Solutions

In this section using the extremal constant sign solutions produced in Proposition 6 and by strengthening the condition on $f(z,\cdot)$ near zero, we produce a nodal solution.

The new hypotheses on the reaction $f(z,x)$ are the following:

$H_2$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, hypotheses $H_2(i), (ii), (iv), (v)$ are the same as the corresponding hypotheses $H_1(i), (ii), (iv), (v)$ and

(iii) $\lim\limits_{x\to 0} \dfrac{f(z,x)}{|x|^{q-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$,
there exists $1 < \tau < q$ such that

$$\lim_{x\to 0} \frac{f(z,x)}{|x|^{\tau-2}x} = 0 \quad \text{uniformly for a.a. } z \in \Omega,$$

$$0 \leq \liminf_{x\to 0} \frac{\tau F(z,x) - f(z,x)x}{|x|^p} \quad \text{uniformly for a.a. } z \in \Omega,$$

with $F(z,x) = \int_0^x f(z,s)ds$.

*Remark* 2. Evidently hypothesis $H_2(iii)$ is more restrictive than hypothesis $H_1(iii)$. Note that $H_1(iii)$ allowed nonlinearities with $(p-1)$-linear growth near zero. Under hypothesis $H_2(iii)$ this is no longer possible.

**Example 1.** *The following function satisfies hypotheses $H_2$ (for the sake of simplicity we drop the $z$-dependence):*

$$f(x) = \begin{cases} \eta[|x|^{r-2}x - |x|^{s-2}x] & \text{if } |x| \leq 1, \\ \eta[|x|^{p-2}x - |x|^{\eta-2}x] & \text{if } 1 < |x|, \end{cases}$$

*with $\eta > \widehat{\lambda}_1(p)$ and $1 < s < r < q$, $\eta < p$.*

**Proposition 8.** *If hypotheses $H_2$ hold, then problem* (1) *admits a nodal solution $\widehat{y} \in [v_*, u_*] \cap C_0^1(\overline{\Omega})$ and if $q = 2$, then $\widehat{y} \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]$.*

*Proof.* Let $u_* \in \mathrm{int}\, C_+$ and $v_* \in -\mathrm{int}\, C_+$ be the two extremal constant sign solutions of (1) produced in Proposition 7. We introduce the Carathéodory function $w(z,x)$ defined by

$$w(z,x)=\begin{cases} f(z,v_*(z)) & \text{if } x<v_*(z),\\ f(z,x) & \text{if } v_*(z)\le x\le u_*(z),\\ f(z,u_*(z)) & \text{if } u_*(z)<x. \end{cases} \tag{52}$$

Also we consider the positive and negative truncations of $w(z,\cdot)$, namely the Carathéodory functions

$$w_\pm(z,x)=w(z,\pm x^\pm) \quad \text{for all } (z,x)\in\Omega\times\mathbb{R}. \tag{53}$$

We set $W(z,x)=\int_0^x w(z,s)ds$ and $W_\pm(z,x)=\int_0^x w_\pm(z,s)ds$ and consider the $C^1$-functionals $\widehat{\beta},\widehat{\beta}_\pm : W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\beta}(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{q}\|\nabla u\|_q^q-\int_\Omega W(z,u)dz,$$
$$\widehat{\beta}_\pm(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{q}\|\nabla u\|_q^q-\int_\Omega W_\pm(z,u)dz \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

Using (52) and (53) we can easily check that

$$K_{\widehat{\beta}}\subseteq[v_*,u_*]\cap C_0^1(\overline{\Omega}),\quad K_{\widehat{\beta}_+}\subseteq[0,u_*]\cap C_+,\quad K_{\widehat{\beta}_-}\subseteq[v_*,0]\cap(-C_+).$$

The extremality of $v_*$ and $u_*$ implies that

$$K_{\widehat{\beta}}\subseteq[v_*,u_*]\cap C_0^1(\overline{\Omega}),\quad K_{\widehat{\beta}_+}=\{0,u_*\},\quad K_{\widehat{\beta}_-}=\{v_*,0\}. \tag{54}$$

From (52) and (53) it is clear that $\widehat{\beta}_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_*\in W_0^{1,p}(\Omega)$ such that

$$\widehat{\beta}_+(\widehat{u}_*)=\min\left[\widehat{\beta}_+(u): u\in W_0^{1,p}(\Omega)\right]. \tag{55}$$

On account of hypothesis $H_2(iii)$ we have that

$$\begin{aligned} &\widehat{\beta}_+(\widehat{u}_*)<0=\widehat{\beta}_+(0),\\ \Rightarrow\quad &\widehat{u}_*\neq 0. \end{aligned} \tag{56}$$

From (55) we have that $\widehat{u}_*\in K_{\widehat{\beta}_+}$, hence $\widehat{u}_*=u_*$ (see (54), (56)).

Note that $\widehat{\beta}\big|_{C_+}=\widehat{\beta}_+\big|_{C_+}$. So, it follows that

$$\begin{aligned} &u_* \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \widehat{\beta},\\ \Rightarrow\quad &u_* \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \widehat{\beta} \quad \text{(see Gasiński-Papageorgiou [6]).} \end{aligned} \tag{57}$$

Similarly we show that

$$v_* \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \widehat{\beta}. \tag{58}$$

We assume that $K_{\widehat{\beta}}$ is finite or otherwise on account of (54) and the extremality of $u_*$ and $v_*$, we already have an infinity of nodal solutions. Then (57), (58) and the mountain pass theorem imply that we can find $\widehat{y} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{y} \in K_{\widehat{\beta}} \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}) \text{ (see (54))}, \ \widehat{y} \notin \{v_*, u_*\}.$$

So, if we can show that $\widehat{y} \neq 0$, then $\widehat{y}$ will be a nodal solution of problem (1).

Hypothesis $H_2(iii)$ and Proposition 6 of Leonardi-Papageorgiou [13] imply that

$$C_k(\widehat{\beta}, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{59}$$

On the other hand, we know that $\widehat{y} \in K_{\widehat{\beta}}$ is a critical point of mountain pass type. Therefore

$$C_1(\widehat{\beta}, \widehat{y}) \neq 0 \tag{60}$$

(see Papageorgiou-Rădulescu-Repovš [20], Theorem 6.5.8, p. 527). From (59) and (60) it follows that $\widehat{y} \neq 0$ and so it is a nodal solution of (1).

Finally we show that $\widehat{y} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]$, when $q = 2$.

As in the proof of Proposition 3, using the tangency principle of Pucci-Serrin [23] (Theorem 2.5.2, p. 35), we have

$$v_*(z) < \widehat{y}(z) < u_*(z) \quad \text{for all } z \in \Omega. \tag{61}$$

Set $\rho = \max\{\|v_*\|_\infty, \|u_*\|_\infty\}$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H_2(v) = H_1(v)$. Let $\widetilde{\xi}_\rho > \widehat{\xi}_\rho$. We have

$$\begin{aligned}
& -\Delta_p \widehat{y}(z) - \Delta \widehat{y}(z) + \widetilde{\xi}_\rho |\widehat{y}(z)|^{p-2} \widehat{y}(z) \\
&= f(z, \widehat{y}(z)) + \widehat{\xi}_\rho |\widehat{y}(z)|^{p-2} \widehat{y}(z) + [\widetilde{\xi}_\rho - \widehat{\xi}_\rho] |\widehat{y}(z)|^{p-2} \widehat{y}(z) \\
&\leq f(z, u_*(z)) + \widehat{\xi}_\rho u_*(z)^{p-1} + [\widetilde{\xi}_\rho - \widehat{\xi}_\rho] u_*(z)^{p-1} \\
&\qquad \text{(see (61) and hypothesis } H_2(v) = H_1(v)) \\
&= -\Delta_p u_*(z) - \Delta u_*(z) + \widetilde{\xi}_\rho u_*(z)^{p-1} \quad \text{for a.a. } z \in \Omega.
\end{aligned} \tag{62}$$

Note that

$$f(z, \widehat{y}(z)) + \widehat{\xi}_\rho |\widehat{y}(z)|^{p-2} \widehat{y}(z) \leq f(z, u_*(z)) + \widehat{\xi}_\rho u_*(z)^{p-1} \quad \text{for a.a. } z \in \Omega,$$

$$\Rightarrow \quad [\widetilde{\xi}_\rho - \widehat{\xi}_\rho] |\widehat{y}|^{p-2} \widehat{y} \prec [\widetilde{\xi}_\rho - \widehat{\xi}_\rho] u_*^{p-1} \quad \text{(see Section 2).}$$

So, from (62) and Proposition 3.2 of Gasiński-Papageorgiou [9], we have

$$u_* - \widehat{y} \in \operatorname{int} C_+.$$

In a similar fashion, we show that

$$\widehat{y} - v_* \in \operatorname{int} C_+.$$

We conclude that
$$\widehat{y} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_*, u_*].$$
□

We can state the following multiplicity theorem for problem (1).

**Theorem 1.** *If hypotheses $H_2$ hold, then*

(*a*) *problem* (1) *admits at least five nontrivial solutions*
$u_0, \widehat{u} \in \operatorname{int} C_+$, $u_0 \leq \widehat{u}$, $u_0 \neq \widehat{u}$,
$v_0, \widehat{v} \in -\operatorname{int} C_+$, $\widehat{v} \leq v_0$, $v_0 \neq \widehat{v}$,
$\widehat{y} \in [v_0, u_0] \cap C_0^1(\overline{\Omega})$ *and* $\widehat{y} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0]$ *if* $q = 2$;

(*b*) *problem* (1) *admits extremal constant sign solutions*
$u_* \in \operatorname{int} C_+$ *and* $v_* \in -\operatorname{int} C_+$
*(that is,* $u_* \leq u$ *for all* $u \in S_+$ *= set of positive solutions of* (1) *and* $v \leq v_*$ *for all* $v \in S_-$ *= set of negative solutions of* (1)*).*

*Remark* 3. We point out that in the above theorem, not only we provide sign information for all the solutions produced, but the solutions are also ordered (that is, $\widehat{v} \leq v_0 \leq \widehat{y} \leq u_0 \leq \widehat{u}$). In the above theorem the nodal solution was obtained at the expense of requiring that $f(z, \cdot)$ is strictly $(q-1)$-sublinear near zero (presence of a concave term near zero, see hypothesis $H_2(iii)$). If $q = 2$, then we can treat also the case of linear growth near zero. This is done in the next section using critical groups.

## 5. The $(p, 2)$-Equation

In this section we deal with the following particular case of problem (1):
$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0, \quad 2 < p < \infty. \tag{63}$$

The hypotheses on the reaction $f(z, x)$ are the following:

$H_3$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$ $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z, x)| \leq a(z)[1 + |x|^{r-2}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$,
$$p \leq r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases};$$

(ii) there exist a function $\eta \in L^\infty(\Omega)$ and $c_\infty > \|\eta\|_\infty$ such that
$$\widehat{\lambda}_1(p) \leq \eta(z) \text{ for a.a. } z \in \Omega, \ \eta \not\equiv \widehat{\lambda}_1(p),$$
$$\eta(z) \leq \liminf_{x \to \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \limsup_{x \to \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq c_\infty \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exists $m \in \mathbb{N}$, $m \geq 2$ such that

$$\widehat{\lambda}_m(2) \leq f'_x(z,0) \leq \widehat{\lambda}_{m+1}(2) \quad \text{for a.a. } z \in \Omega,$$
$$f'_x(\cdot,0) \not\equiv \widehat{\lambda}_m(2), \quad f'_x(\cdot,0) \not\equiv \widehat{\lambda}_{m+1}(2);$$

(iv) there exist $\vartheta_- < 0 < \vartheta_+$ such that $f(z,\vartheta_+) \leq \widehat{c}_0 < 0 < \widehat{c}_1 \leq f(z,\vartheta_-)$ for a.a. $z \in \Omega$;

(v) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \to f(z,x) + \widehat{\xi}_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho,\rho]$.

*Remark* 4. Hypothesis $H_3(iii)$ dictates a linear growth for $f(z,\cdot)$ near zero. This is in contrast to hypothesis $H_2(iii)$. In that hypothesis we required that $f(z,\cdot)$ is strictly $(q-1)$-sublinear near zero.

**Proposition 9.** *If hypotheses $H_3$ hold, then problem* (63) *has at least two nodal solutions $\widehat{y}, \widetilde{y} \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]$.*

*Proof.* Reasoning as in the proof of Proposition 8 and since $m \geq 2$, we produce a solution

$$\widehat{y} \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]. \tag{64}$$

This solution is obtained via an application of the mountain pass theorem (see the proof of Proposition 8). Therefore

$$C_1(\widehat{\beta}, \widehat{y}) \neq 0 \quad \text{(see (60))}. \tag{65}$$

Let $\varphi : W_0^{1,p} \to \mathbb{R}$ be the energy functional of problem (63) defined by

$$\varphi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Note that $\varphi \in C^2(W_0^{1,p}(\Omega), \mathbb{R})$. Using (64) and a standard homotopy invariance argument (see Papageorgiou-Rădulescu-Repovš [20], Theorem 6.3.6, p. 505), we obtain that

$$\begin{aligned}
& C_k(\varphi, \widehat{y}) = C_k(\widehat{\beta}, \widehat{y}) \quad \text{for all } k \in \mathbb{N}_0, \\
\Rightarrow\quad & C_1(\varphi, \widehat{y}) \neq 0 \quad \text{(see (65))}, \\
\Rightarrow\quad & C_k(\varphi, \widehat{y}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0,
\end{aligned}$$

since $\varphi \in C^2(W_0^{1,p}(\Omega), \mathbb{R})$, see Papageorgiou-Rădulescu [17], Claim 3, p. 412. Therefore

$$C_k(\widehat{\beta}, \widehat{y}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{66}$$

Using $f'_x(\cdot,0) \in L^\infty(\Omega)$, we introduce the $C^2$-functional $\widehat{\gamma} : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\gamma}(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_\Omega f'_x(z,0)u^2 dz \quad \text{for all } u \in H_0^1(\Omega).$$

On account of hypothesis $H_3(iii)$, we have

$$C_k(\widehat{\gamma},0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \text{ with } d_m = \dim \textstyle\bigoplus_{i=1}^m E(\widehat{\lambda}_i(2)), \tag{67}$$

(see Papageorgiou-Rădulescu-Repovš [20], Proposition 6.2.6, p. 479). Here by $E(\widehat{\lambda}_i(2))$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_i(2)$.

Let $\gamma = \widehat{\gamma}\big|_{W_0^{1,p}(\Omega)}$. Since $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$ densely, using Theorem 6.6.26, p. 545, of Papageorgiou-Rădulescu-Repovš [20], we have

$$\begin{aligned} & C_k(\gamma,0) = C_k(\widehat{\gamma},0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow \quad & C_k(\gamma,0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (67))}. \end{aligned} \tag{68}$$

The norm continuity of critical groups (see Papageorgiou-Rădulescu-Repovš [20], Theorem 6.3.4, p. 503), implies that

$$\begin{aligned} & C_k(\widehat{\beta},0) = C_k(\gamma,0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow \quad & C_k(\widehat{\beta},0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (68))}. \end{aligned} \tag{69}$$

Since $d_m \geq 2$, from (66) and (69) it follows that

$$\begin{aligned} & \widehat{y} \neq 0, \\ \Rightarrow \quad & \widehat{y} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_*, u_*] \text{ is a nodal solution of (63)}. \end{aligned}$$

Recall that $u_*$ and $v_*$ are local minimizers of $\widehat{\beta}$ (see (68), (69)). Hence we have

$$C_k(\widehat{\beta},u_*) = C_k(\widehat{\beta},v_*) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{70}$$

Recall that $\widehat{\beta}$ is coercive. Therefore we have

$$C_k(\widehat{\beta},\infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{71}$$

Suppose that $K_{\widehat{\beta}} = \{\widehat{y}, 0, u_*, v_*\}$. Then from (66), (69), (70), (71) and the Morse relation with $t = -1$ (see (6)), we have

$$\begin{aligned} & (-1)^1 + (-1)^{\mathrm{d_m}} + 2(-1)^0 = (-1)^0, \\ \Rightarrow \quad & (-1)^{\mathrm{d_m}} = 0, \text{ a contradiction.} \end{aligned}$$

So, there exists $\widetilde{y} \in K_{\widehat{\beta}}$, $\widetilde{y} \notin \{\widehat{y}, 0, u_*, v_*\}$. From (54) it follows that $\widetilde{y} \in \operatorname{int} C_+$ is nodal and as in the last part of the proof of Proposition 8, we have that $\widetilde{y} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]$. $\square$

Therefore for problem (63) we can state the following multiplicity theorem.

**Theorem 2.** *If hypotheses $H_3$ hold, then*

($a$) *problem* (63) *admits at least six nontrivial solutions*
$u_0, \widehat{u} \in \operatorname{int} C_+$, $u_0 \leq \widehat{u}$, $u_0 \neq \widehat{u}$,
$v_0, \widehat{v} \in -\operatorname{int} C_+$, $\widehat{v} \leq v_0$, $v_0 \neq \widehat{v}$,
$\widehat{y}, \widetilde{y} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0]$ *nodal;*

($b$) *problem* (63) *admits extremal constant sign solutions*
$u_* \in \operatorname{int} C_+$ *and* $v_* \in -\operatorname{int} C_+$.

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