UNIVERSITÀ DEGLI STUDI DI PALERMO

DOCTORAL THESIS

Depictions and fiber products

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...alla mia Famiglia, nel ricordo
di ciò che sarà per sempre in noi...

Contents

Introduction

The use of fiber products in commutative algebra dates back for the first time to Prüfer and Krull's works about polynomial rings [21] and then had been studied for many years by several authors who discovered a lot of features and properties regarding this particular and universal algebraic structure, whose strong point is the versatility with which it is applied to produce examples and which allows it to adapt well to different algebraic and not-algebraic contexts. For this reason, several kinds of fiber products have been discovered, studied and formalized in order to be applied in the most suitable way to describe conveniently some geometrical and algebraic structures. Referring to the notation of the special type of fiber product R considered by Gabelli and Houston in [24]:

$$\begin{array}{ccc} \mathrm{R} & \longrightarrow & \mathrm{D} \\ \downarrow & & \downarrow \\ \mathrm{B} & \xrightarrow{\phi} & \mathbb{K} \end{array}$$

where D and B are domains, $\mathfrak{m}$ is a maximal ideal of B, $\mathbb{K} \cong \frac{\mathrm{B}}{\mathfrak{m}}$ and $\phi : \mathrm{B} \longrightarrow \mathbb{K}$ is the canonical map, let us collect chronologically the most important steps in the progresses in their use and applications [21].

1. **The classical** $\mathrm{D}+\mathrm{M}$ **construction**, with which Seidenberg [30, *1954*] studied the Krull's dimension of the polynomial rings and Nagata [28, *1962*] treated the composition of valuations and presented an example inspiring the whole fiber products' theory;

2. **A formalization of the classical** $\mathrm{D}+\mathrm{M}$ **construction**, where B is a valuation domain of the form $\mathbb{K} + \mathfrak{m}$ [25, Theorem A, 1968] by Gilmer; develops on overrings and divisorial ideals of the $\mathrm{D} + \mathrm{M}$ rings [4, 1973] by Bastida and Gilmer;

3. **The** CPI **extension** [10, *1977*] by Boisen and Sheldon;

4. **The** $\mathrm{D} + \mathrm{X}\mathrm{D}_{\mathrm{S}}[\mathrm{X}]$ **construction** [15, *1978*] by Costa, Mott and Zafrullah;

5. A topological approach to the pullbacks [19, *1980*] by Fontana;

6. Studies about pairs of rings with the same prime ideals and a characterization of pseudo-valuation domains in terms of fiber products [1, 1982] by Anderson and Dobbs;

7. **The** B**.** I**.** D**. construction** [14, *1988*, page 505] by Cahen, which consists of couple of rings which share a common ideal.

Among the several fiber products mentioned above, the one which best lends itself to dealing with the topics covered by this research work, is the ring R which appears in the B. I. D. construction [14, *1988*, page 505] by Cahen. Passing from Cahen's notation to Beil's one, the ring B will be denoted by S and the ring D will be an algebraically closed field $\mathbb{K}$ and we will adopt this set as follows:

$$\begin{array}{ccc} \mathrm{R} = \mathbb{K} + \mathrm{I} & \longrightarrow & \mathbb{K} \\ \big\downarrow & & \big\downarrow \\ \mathrm{S} & \longrightarrow & \frac{\mathrm{S}}{\mathrm{I}} \end{array}$$

Also, we will make S an integral domain first and additionally a finitely generated $\mathbb{K}$-algebra, with $\mathbb{K}$ an algebraically closed field then, with the aim to propose and adopt respectively two new constructions of fiber product which let us to obtain the results of Beil's article in a more direct, fast and simple way and to generalize some of them from a more algebraic point of view:

1. **Weak depiction-Weakly depicted fiber product:** Let S be an integral domain, $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$ any ring extension, with I a nonmaximal and nonzero ideal of S, then let us define S *a weak depiction* of R and R *a weakly depicted fiber product* of S.

2. **Almost depiction-Almost depicted fiber product**: Let $\mathbb{K}$ be an algebraically closed field, S be both a finitely generated $\mathbb{K}$ - algebra and an integral domain, $\mathrm{R} = \mathbb{K}+\mathrm{I} \subseteq \mathrm{S}$, with I a nonmaximal and nonzero ideal of S, then let us define S *an almost depiction* of R and R *an almost depicted fiber product* of S.

The idea to adapt the fiber products' theory to Beil's article [6, *Nonnoetherian Geometry*], treating almost all its examples as pullbacks, comes from the [6, Remark 3.4, page 12], in which he notices that one the most emblematic examples of his work, as well as the most classical example of depiction:

$$\mathbb{K} + x\mathbb{K}[x, y] = \mathbb{K}[x, xy, xy^2, \ldots] = \mathbb{K}[x, y] \times_{\mathbb{K}[y]} \mathbb{K} \subset \mathbb{K}[x, y]$$

is geometrically described in the article *Gluing schemes and a scheme without closed points* [29, Example 3.7] by Schwede as a fiber product and it is based on the gluing of schemes proposed by Fontana in [19].

So we realized that Beil is strongly moved by geometrical intents, in particular by the wish to give a geometrical explanation of a variety of a nonNoetherian subalgebra R of a polynomial ring in a finite number of variables, S, with which, sometimes, it shares one of its maximal ideal I. For this reason he realizes that the maximal spectrum $\mathrm{Max}(\mathrm{R})$ coincides with the algebraic variety $\mathrm{Max}(\mathrm{S})$, except that the zero locus $\mathrm{Z}(\mathrm{I}) \subset \mathrm{Max}(\mathrm{S})$ is identified as a single "smeared-out" point [8], as he clarifies in several examples.

Even for the *depictions* of nonNoetherian algebras R, a new geometric structure introduced by Beil in his work [6], he is directed to focus his point of view on the geometrical aspect, defining them as the finitely generated algebras that are as close as possible to R, in a suitable geometric sense ([6], [5] and [8]) and so they have enabled various notions in, for example, not commutative algebraic geometry, such as *not commutative crepant resolution* [31], *homological homogeneity* [12], *Azumaya loci* [8] and possible directions towards *a new theory of quantum gravity* [9], [7], most of them purely geometric. In contrast to this path followed by Beil, recognizing the algebraic power of the fiber products with which we have decided to deal with the topics of his articles, we set out to extrapolate from it new results of a more generically algebraic nature, extending the point of view from the pure geometrical aspect to the algebraic one, overcoming the correspondence between closed of Zariski's topology and radical ideals of the polynomial ring with a finite number of variables and coefficients in algebraically closed fields. For this reason, we demonstrated, for example, some results of Beil's article eliminating the hypothesis of radicality which is, in this case, specifically geometrical, obtaining the following:

1. **Generalization of Proposition 2.8**: *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$, *with* I *a nonmaximal and nonzero ideal of* S*, then* $\mathrm{U} = \mathrm{Z}(\mathrm{I})^c$*;*

2. **Generalization of Corollary 3.14**: *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$, *with* I *a nonmaximal and nonzero ideal of* S, R *is Noetherian if and only if* $dim\left(\frac{\mathrm{S}}{\mathrm{I}}\right) = 0$.

In particular we presented an innovative example of depiction of a fiber product of the $\mathrm{B.~I.~D.}$ construction [14], in which I is a nonradical ideal of B.

Moreover, the great innovation and the strength of this work is to have treated, for the first time, the depictions of Beil [6], using specifically the fiber products' theory, in order to deduce in an immediate way some known Beil's results and to determine some new properties of them. So, analyzing all the examples of depictions and not depictions presented in [6, Nonnoetherian geometry], and subsequent Beil's articles [5], [8], we realized that almost all of them are depictions of fiber products (and these are all polynomial rings in a finite number of indeterminates). For this reason we first conjectured that each finitely generated $\mathbb{K}$ - algebra which is a polynomial ring in a finite number of indeterminates, with $\mathbb{K}$ an algebraically closed field, is depiction of some fiber product. Later we came across an example of depiction of a nonfiber product.

The key fact will be to observe that R could not be a fiber product for S because S and R could not admit in any case a common ideal. This example invalidates our "conjecture" and led us to deal separately with the depictions of fiber and nonfiber products.

In this sense great relevance assume the almost depicted fiber products, for which several results have been obtained, the most important of those and of our whole work is a characterization of depictions of fiber products with prime ideals of the overring.

This thesis is structured as follows

In the first chapter we will give all the preliminaries notations, definitions and known propositions to approach appropriately to this work; in particular we will present the crucial definition of fiber product, that one which best describes Beil's examples and we will provide two possible settings to deal with his arguments. Thus, first we will give some known facts on fiber products used in the thesis as a support to prove the new results, then we will show some new facts on fiber products.

In the second chapter we will present the principal Beil's results, reorganized in a more general way (eliminating the hypothesis of radicality) which we obtained just adopting fiber products' theory, that represent the strength of this work, and some other ones immediately and, in some cases, more easily deducible using the well note fiber products.

In the third chapter we will focus on Beil's examples of depictions and not (collecting them in the first section); in particular we will present the emblematic example of a depiction that is not a fiber product which invalidates our "conjecture" that each finitely generated $\mathbb{K}$ - algebra which is a polynomial ring in a finite number of indeterminates, with $\mathbb{K}$ an algebraically closed field, is depiction of some fiber product. This fact will bring us to distinguish between and treat separately depictions of fiber and nonfiber products and will let us to discover that, in both cases, some properties of them are involved in depictions with a positive feedback; for this reason we will define respectively the weak/almost depicted fiber product and the depiction of weak/almost depicted fiber product, we will find some properties and characterizations of them and some new interesting examples of depictions and not depictions, like a depiction of a fiber product with a nonradical ideal of it.

At the end of this last chapter we will raise a new interesting and open question about fiber products and depictions treated with the fiber products' theory which we consider intuitively productive and subject of possible future studies.

Chapter 1

Preliminaries

1.1 Notations

In the following, unless otherwise specified, with the term *ring* we will mean always a commutative unitary ring, with the term *overring* we will mean a subring of a domain and containing his fraction field. Moreover, if R and S are rings and $f : \mathrm{R} \longrightarrow \mathrm{S}$ is a ring homomorphism, we will assume that f maps the identity of R into the identity of S.

Every prime ideal of a ring is, in particular, a proper ideal. If R is a ring, we set:

$$\mathrm{Spec}(\mathrm{R}) := \{\mathfrak{p} \subseteq \mathrm{R} : \mathfrak{p} \text{ is a prime ideal of R}\},$$

$$\mathrm{Max}(\mathrm{R}) := \{\mathfrak{m} \subseteq \mathrm{R} : \mathfrak{m} \text{ is a maximal ideal of R}\},$$

$$\mathrm{J}(\mathrm{R}) := \bigcap_{\mathfrak{m}\in\mathrm{Max}(\mathrm{R})} \mathfrak{m}, \text{ the Jacobson radical of R},$$

$$\mathrm{N}(\mathrm{R}) := \bigcap_{\mathfrak{p}\in\mathrm{Spec}(\mathrm{R})} \mathfrak{p}, \text{ the nilradical of R}.$$

Unless otherwise specified, we shall consider the set $\mathrm{Spec}(\mathrm{R})$ endowed with the Zariski topology, i. e. the topology whose closed sets are the subsets of $\mathrm{Spec}(\mathrm{R})$ of the form $\mathrm{V}_{\mathrm{R}}(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(\mathrm{R}) : \mathfrak{p} \supseteq \mathfrak{a}\}$, for each ideal $\mathfrak{a}$ of R. When there is no danger of confusion, we denote $\mathrm{V}_{\mathrm{R}}(\mathfrak{a})$ simply by $\mathrm{V}(\mathfrak{a})$.

If $f : \mathrm{R} \longrightarrow \mathrm{S}$ is a ring homomorphism, let us denote by

$$\begin{array}{rcl} f^* : \mathrm{Spec}(\mathrm{S}) & \longrightarrow & \mathrm{Spec}(\mathrm{R}) \\ \mathfrak{q} & \mapsto & f^{-1}(\mathfrak{q}) \end{array}$$

the induced continuous mapping of spectra.

Let us recall the relevant notion of valuative dimension of an integral domain.

Definition 1.1. If D is an integral domain, we say that D has *valuative dimension* n, and we write $\dim_v \mathrm{D} = n$, if each valuation overring of D has dimension at most n and if there exists a valuation overring of D of dimension n. If no such integer n exists, we say that the valuative dimension of D is infinite.

Proposition 1.2 ([26, Theorem 30.9])**.** *Let* D *be an integral domain, and let* n *be a positive integer. The following conditions are equivalent:*

1. $\dim_v \mathrm{D} = n$*;*
2. *each overring of* D *has dimension at most* n*, and there exists an overring of* D *of dimension* n*.*

Proposition 1.3 ([26, Corollary 30.10])**.** *Let* D *be a* n*-dimensional Noetherian domain, and* n *be a positive integer, then* $\dim_v \mathrm{D} = n$*.*

Besides, according to Beil's positions adopted in his article [6, Nonnoetherian geometry], for the rest of this work and until it will not be specified in a different way, $\mathbb{K}$ will be an algebraically closed field, S an integral domain and a finitely generated $\mathbb{K}$ - algebra, R a subalgebra of S and I a maximal ideal of R which is not necessarily an ideal of S (when it happens, the maximality of I in R is insured by Proposition 1.16 of the article [3, *Sous-Anneaux de la forme* $\mathrm{D}+\mathrm{I}$ *d'une* $\mathbb{K}$*-algebre integre*] by Ayache, whose starting sets coincide with the ones of Beil's article [6] mentioned above).
Moreover, let us recall some Beil's notations:

- $\mathrm{U} := \mathrm{U}_{\frac{\mathrm{S}}{\mathrm{R}}} := \{\mathfrak{m} \in \mathrm{Max}(\mathrm{S}) : \mathrm{S}_{\mathfrak{m}} = \mathrm{R}_{\mathrm{R}\cap\mathfrak{m}}\}$
- $$\mathrm{Z}(\mathrm{I}) := \{\mathfrak{m} \in \mathrm{Max}(\mathrm{S}) : \mathfrak{m} \supseteq \mathrm{I}\}$$
- $$\dim(\mathrm{Z}(\mathrm{I})) := \dim\left(\frac{\mathrm{S}}{\mathrm{I}}\right)$$

and let us give the two following definitions:

Definition 1.4. [6, Definition 3.1] An integral domain and finitely generated $\mathbb{K}$ - algebra S is a *depiction* of a subalgebra $\mathrm{R} \subseteq \mathrm{S}$ if:

(D1) the mapping
$$\begin{array}{rcl} \mathrm{i}_{\frac{\mathrm{S}}{\mathrm{R}}} : \mathrm{Spec}(\mathrm{S}) & \rightarrow & \mathrm{Spec}(\mathrm{R}) \\ \mathfrak{q} & \mapsto & \mathfrak{q} \cap \mathrm{R} \end{array}$$
is surjective, or equivalently, the inclusion of R in S is LO (lying-over),

(D2) for each $\mathfrak{n} \in \mathrm{Max}(\mathrm{S})$, $\mathrm{R}_{\mathfrak{n}\cap\mathrm{R}}$ is Noetherian if and only if $\mathfrak{n} \in \mathrm{U}$, (D3) $\mathrm{U} \neq \emptyset$.

Definition 1.5. [6, page 19] A *depiction* of R is said to be *maximal* (resp. *minimal*) if it is not contained in (resp. does not contain) any other depiction of R.

Finally, let us recall the crucial definition of fiber product, on which all our research work is based.

Definition 1.6. [19, page 334] Let $u : \mathrm{A} \to \mathrm{C}$ be a ring-homomorphism and $v : \mathrm{B} \to \mathrm{C}$ a surjective ring-homomorphism. Then the subring $\mathrm{D} := \mathrm{A} \times_{\mathrm{C}} \mathrm{B} := u \times_{\mathrm{C}} v := \{(a, b) \in \mathrm{A} \times \mathrm{B} : u(a) = v(b)\}$ of $\mathrm{A} \times \mathrm{B}$ is called *the fiber product* of u and v (or *the pull-back of* A *and* B *over* C). Besides, let us denote by $u' : \mathrm{D} \to \mathrm{B}$ and $v' : \mathrm{D} \to \mathrm{A}$ the restriction to D of the canonical projections. Let $\mathrm{X} := \mathrm{Spec}(\mathrm{A})$, $\mathrm{Y} := \mathrm{Spec}(\mathrm{B})$, $\mathrm{Z} := \mathrm{Spec}(\mathrm{C})$, $\mathrm{W} := \mathrm{Spec}(\mathrm{D})$, and let us consider the induced ring-homomorphisms by maps on spectra $\alpha := u^* : \mathrm{Z} \longrightarrow \mathrm{X}$, $\beta := v^* : \mathrm{Z} \longrightarrow \mathrm{Y}$, $\alpha' := u'^* : \mathrm{Y} \longrightarrow \mathrm{W}$, $\beta' := v'^* : \mathrm{X} \longrightarrow \mathrm{W}$.

$$\begin{array}{ccc} \mathrm{D} & \xrightarrow{v'} & \mathrm{A} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ \mathrm{B} & \xrightarrow{v} & \mathrm{C} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{W} & \xleftarrow{\beta'} & \mathrm{X} \\ {\scriptstyle \alpha'}\uparrow & & \uparrow{\scriptstyle \alpha} \\ \mathrm{Y} & \xleftarrow{\beta} & \mathrm{Z} \end{array}$$

The map β, being a closed embedding, let us to identify Z with its image in Y, in order to simplify the notations.

Among the several fiber products which have been studied over the years, the one which best lends itself to deal with the topics of Beil's article (as we will see in the next section) is presented in the following definition:

Definition 1.7. [14, page 505] Let B be a ring, I an ideal of B, D a subring of the quotient ring $\frac{\mathrm{B}}{\mathrm{I}}$ and R the set of the elements of B of which the class module I is in D. We will say that R is *the ring of the* B. I. D. *construction*. In these hypotheses D is isomorphic to the quotient ring $\frac{\mathrm{R}}{\mathrm{I}}$ and the following Cartesian square (studied by Fontana) is determined:

$$\begin{array}{ccc} \mathrm{R} & \longrightarrow & \mathrm{D} \cong \frac{\mathrm{R}}{\mathrm{I}} \\ \downarrow & & \downarrow \\ \mathrm{B} & \longrightarrow & \frac{\mathrm{B}}{\mathrm{I}} \end{array}$$

1.2 Settings

In order to approach in a confidential way with the topics of Beil's article [6, *Non-noetherian geometry*], making the most of the fiber products' theory, we can use these two more general settings which derive from the [14, B. I. D. construction] by Cahen:

1) Let $\mathrm{A} \subseteq \mathrm{S}$ be a ring extension, I an ideal of S. Then $\mathrm{R} = \mathrm{A} + \mathrm{I}$ is a subring of S which contains A. Also R is the fiber product of the canonical projection π from S to $\frac{\mathrm{S}}{\mathrm{I}}$ and of the embedding i from $\frac{\mathrm{A}}{\mathrm{I}\cap\mathrm{A}}$ to $\frac{\mathrm{S}}{\mathrm{I}}$ as follows:

$$\begin{array}{ccc} \mathrm{R} = \mathrm{A} + \mathrm{I} & \longrightarrow & \frac{\mathrm{A}}{\mathrm{I}\cap\mathrm{A}} \\ \big\downarrow & & \big\downarrow i \\ \mathrm{S} & \xrightarrow{\pi} & \frac{\mathrm{S}}{\mathrm{I}} \end{array}$$

2) Let $\mathrm{R} \subseteq \mathrm{S}$ be a ring extension, I a common ideal of R and S, then R is the fiber product of canonical projection π from S to $\frac{\mathrm{S}}{\mathrm{I}}$ and of the embedding i from $\frac{\mathrm{R}}{\mathrm{I}}$ to $\frac{\mathrm{S}}{\mathrm{I}}$ as follows:

$$\begin{array}{ccc} \mathrm{R} & \longrightarrow & \frac{\mathrm{R}}{\mathrm{I}} \\ \big\downarrow & & \big\downarrow i \\ \mathrm{S} & \xrightarrow{\pi} & \frac{\mathrm{S}}{\mathrm{I}} \end{array}$$

Besides, considering the same hypotheses, if we also set $\mathbb{K}$ an algebraically closed field, R' a subalgebra of S and $\mathrm{R} = \mathbb{K}[\mathrm{R}', \mathrm{I}]$, we obtain a generalization of some Beil's results, like [6, Proposition 2.8].

Example 1.8. Using this last setting we can also deduce the [16, Remark 2.13.b, page 3] about amalgamations, seeing $0 \times \mathrm{I}$ as a common ideal of $\mathrm{R} = \mathrm{R}' \bowtie \mathrm{I}$ and of its extension $\mathrm{S} = \mathrm{R}' \times (\mathrm{R}' + \mathrm{I})$.

If we set $\mathrm{A} := \mathbb{K}$ an algebraically closed field (so I is a maximal ideal of R and not of S) and if we suppose that S is a finitely generated $\mathbb{K}$-algebra and an integral domain (so R is a $\mathbb{K}$-subalgebra of S), in both the cases just described, since $\frac{\mathrm{A}}{\mathrm{I}\cap\mathrm{A}}$ is isomorphic to $\frac{\mathrm{A}+\mathrm{I}}{\mathrm{I}}$, we have that these fiber products coincide in the following: $(*)$

$$\begin{array}{ccc} \mathrm{R} = \mathbb{K} + \mathrm{I} & \longrightarrow & \mathbb{K} \\ \big\downarrow & & \big\downarrow i \\ \mathrm{S} & \xrightarrow{\pi} & \frac{\mathrm{S}}{\mathrm{I}} \end{array}$$

which is just what we find in the most part of the examples in Beil's article and which represent the almost depicted fiber products we will present and formally define later.

1.3 Known facts on fiber products used in the thesis

In this section we collect some known results about fiber products, which will be used to reach our goals.

Proposition 1.9 ([19, Theorem 1.4])**.** *With the notation and hypotheses of Definition 1.6, let* $\mathrm{X} \cup_\alpha \mathrm{Y}$ *be the topological space obtained by attaching* X *to* Y*, over the closed set* Z*, by the continuous map* α *(where* $\mathrm{X} \cup_\alpha \mathrm{Y}$ *is the quotient space of the disjoint union of* X *and* Y*, modulo the equivalent relation generated by:* $\alpha(z) \sim z$*, for each* $z \in \mathrm{Z}$*). Then* $\mathrm{X} \cup_\alpha \mathrm{Y}$ *is a spectral space homeomorphic to* $\mathrm{Spec}(\mathrm{D})$*.*
From the definition of D *itself, it follows that:*

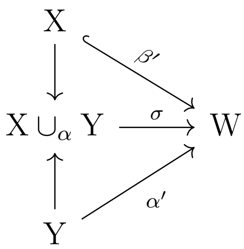
(a) v' *is a surjective homomorphism (and therefore,* β' *is a closed embedding; we identify for greater convenience* X *with its image in* W *under* β*).*

(b) Let $\mathfrak{b} = \ker(v)$ *and* $\mathfrak{d} = \ker(v')$*, then* $u'|_\mathfrak{d} : \mathfrak{d} \to \mathfrak{b}$ *is an isomorphism of modules (subordinate to* $u' : \mathrm{D} \to \mathrm{B}$*). Therefore, the conductor of* u' *contains* $\mathfrak{d}$ *and, hence, it is easily seen that, for every* $h \in \mathfrak{d}$*, the canonical homomorphism* $\mathrm{D}_h \to \mathrm{B}_{u'(h)}$ *is an isomorphism.*

(c) For every prime ideal $\mathfrak{p}$ *of* D*,* $\mathfrak{p} \not\supseteq \mathfrak{d}$*, if* $\mathfrak{q}$ *is the unique prime ideal of* B *such that* $u'^{-1}(\mathfrak{q}) = \mathfrak{p}$*, then* $\mathfrak{q} \not\supseteq \mathfrak{b}$ *and* $\mathrm{B}_\mathfrak{q} \cong \mathrm{D}_\mathfrak{p}$*.*

(d) The map $\alpha' : \mathrm{Y} \to \mathrm{W}$ *restricted to* $\mathrm{Y} \setminus \mathrm{Z} = \alpha'^{-1}(\mathrm{W} \setminus \mathrm{X})$ *establishes a scheme-isomorphism (and hence, in particular, an homeomorphism between topological spaces and an order-isomorphism between partially ordered sets) with* $\mathrm{W} \setminus \mathrm{X}$ *(we notice that* $\mathrm{X} \cong \mathrm{V}(\mathfrak{d})$ *and* $\alpha'^{-1}(\mathrm{X}) \cong \mathrm{V}(\mathfrak{b}) \cong \mathrm{Z}$*).*
The equality $\beta' \circ \alpha = \alpha' \circ \beta$ *allows us to affirm that:*

(e) There exists a unique continuous map $\sigma : \mathrm{X} \cup_\alpha \mathrm{Y} \to \mathrm{W}$ *which commutes the following diagram:*



From the statements (a) and (d) it follows that:

(f) $\sigma : \mathrm{X} \cup_\alpha \mathrm{Y} \to \mathrm{W}$ *is a bijective map; therefore, in particular* $\mathrm{W} = \mathrm{X} \cup \alpha'(\mathrm{Y})$*.*

Proposition 1.10 ([19, Corollary 1.5])**.** *We preserve the notations and hypotheses of the previous theorem.*

(1) The map $\mathfrak{a} \mapsto v'^{-1}(\mathfrak{a})$ *establishes an isomorphism between the lattice of all the ideals of* A *and that of all the ideals of* D *containing* $\mathfrak{d}$*. This map defines, by restriction, an isomorphism between* $\mathrm{Pospec}(\mathrm{A})$ *and the partially ordered*

subset of $\mathrm{Pospec(D)}$ *which consists of all prime ideals of* D *containing* $\mathfrak{d}$ *(this isomorphism, obviously, coincides with the one which can be deduced from the closed embedding* $\beta' : \mathrm{Spec(A)} \to \mathrm{Spec(D)}$*).*

(2) For every prime ideal $\mathfrak{q}$ *of* B, $\mathfrak{q} \not\supseteq \mathfrak{b}$, *the map* $\mathfrak{h} \mapsto u'^{-1}(\mathfrak{h})$ *establishes a bijection, which preserves the inclusion, between the set of all the ideals of* B *which are primary for* $\mathfrak{q}$ *and the set of all the ideals of* D *which are primary for* $\mathfrak{p} = u'^{-1}(\mathfrak{q})(\not\supseteq \mathfrak{d})$*.*

(3) The map defined in the statement (2), by restriction to the prime ideals, determines the isomorphism $\mathrm{Spec(B)} \setminus \mathrm{V}(\mathfrak{b}) \widetilde{\to} \mathrm{Spec(D)} \setminus \mathrm{V}(\mathfrak{d})$ *described formally in Theorem (1.4 (d)) above.*

(4) If $u : \mathrm{A} \to \mathrm{C}$ *is injective [resp. surjective, of finite type, integral, finite], then* $u' : \mathrm{D} \to \mathrm{B}$ *is injective [resp. surjective, of finite type, integral, finite].*

(5) If u *is an injective homomorphism and if* $\mathfrak{b}$ *is a regular ideal of* B, *then* $\mathrm{Tot(B)} \cong \mathrm{Tot(D)}$ *(where* $\mathrm{Tot}(-)$ *denote the total ring of fractions of the ring –).*

Proposition 1.11 ([19, Corollary 1.6])**.** *We preserve the notations and hypotheses of the previous theorem.* W *and* Z *are Noetherian spaces, if and only if,* X *and* Y *are Noetherian spaces.*

Proposition 1.12 ([19, Proposition 1.8])**.** *We preserve the notations and hypotheses of the previous theorem.* $\mathrm{A} \times_{\mathrm{C}} \mathrm{B}$ *and* C *are Noetherian rings,* u' *is a finite homomorphism if, and only if,* A *and* B *are Noetherian rings and* u *is a finite homomorphism.*

Proposition 1.13 ([19, Proposition 1.9])**.** *We preserve the notations and hypotheses of the previous theorem. If* S *is a multiplicatively closed set in the ring* D, *then indicating* $\mathrm{S_A} = v'(\mathrm{S}), \mathrm{S_B} = u'(\mathrm{S}), \mathrm{S_C} = u \circ v'(\mathrm{S}) = v \circ u'(\mathrm{S})$, *we obtain that*

$$\mathrm{S}^{-1}\mathrm{D} \cong \mathrm{S_A^{-1}A} \times_{\mathrm{S_C^{-1}C}} \mathrm{S_B^{-1}B}.$$

Conversely, if $\mathrm{S_A}$ *is a multiplicatively closed set of* A *and if* $\mathrm{S_B}$ *is a multiplicatively closed set of* B *and if* $u(\mathrm{S_A}) = v(\mathrm{S_B}) = \mathrm{S_C}$, *then*

$$\mathrm{S_A^{-1}A} \times_{\mathrm{S_C^{-1}C}} \mathrm{S_B^{-1}B} \cong (\mathrm{S_A} \times_{\mathrm{S_C}} \mathrm{S_B})^{-1}\mathrm{D}.$$

Proposition 1.14 ([20, Proposition 1.6])**.** *Let* $\mathbb{K}$ *be a field,* S *both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain and* I *a nonzero ideal of* S. *Let* $\mathrm{R} := \mathbb{K} + \mathrm{I}$ *and let us suppose that* $\dim \frac{\mathrm{S}}{\mathrm{I}} = 0$, *then* S *is a finite type* R *- module and, so,* R *is a Noetherian domain.*

Proposition 1.15 ([32, Proposition 1.1])**.** *Let* $\mathbb{K}$ *be a field,* $\mathrm{S} = \mathbb{K}[x_1, \ldots, x_t]$ *having Krull dimension* $n > 0$ *and* $\mathrm{R} := \mathrm{D} + \mathrm{I}$ *a subring of* S *(where* D *is a subring of* $\mathbb{K}$ *and* I *is a nonzero, proper ideal of* S*). Then the following conditions are equivalent:*

1. R *is Noetherian.*

2. D *is a field,* $\mathbb{K}$ *is a finite extension of* D*, and* I *is of height* n *in* S.

Proposition 1.16 ([3, Proposition 1.2])**.** *Let* $\mathbb{K}$ *be a field,* S *a Noetherian, integral domain, catenarian and coequidimensional of dimension* $n \in \mathbb{N}$ *finite,* I *a nonzero, proper ideal of* S *such that* $\frac{\mathrm{S}}{\mathrm{I}}$ *contains* $\mathbb{K}$ *and* R *is the ring of the* $(\mathrm{S}, \mathrm{I}, \mathbb{K})$ *type; then* R *is an integral, catenarian and coequidimensional domain of dimension* n *and* I *is a maximal ideal of* R *of height* n *for* R.

Proposition 1.17 ([3, Proposition 1.7])**.** *Let* $\mathbb{K}$ *be a field,* S *a finitely generated* $\mathbb{K}$ *- algebra and* I *a nonzero, proper ideal of* S*. Let* $\mathrm{R} := \mathbb{K} + \mathrm{I}$*. Then the following conditions are equivalent:*

i) I *has maximal height over* S.

ii) $\frac{\mathrm{S}}{\mathrm{I}}$ *is a finite* $\mathbb{K}$ *- algebra.*

iii) S *is integral over* R.

Proposition 1.18 ([3, Théorème 3.1])**.** *Let* $\mathbb{K}$ *be a field,* S *a finitely generated* $\mathbb{K}$*-algebra and an integral domain, and let* I *be a nonzero, proper ideal of* S*. Let* $\mathrm{R} := \mathrm{D} + \mathrm{I}$*, where* D *is a subring of* $\mathbb{K}$ *with fraction field* k*. Then the following properties hold:*

(i) $\dim \mathrm{R} = \dim \mathrm{D} + \dim \mathrm{S}$.

(ii) R *is Noetherian if and only if* D *is a field,* $\mathbb{K}$ *is a finite extension of* k *and* I *is of maximal height over* S.

Proposition 1.19 ([14, Proposition 5])**.** *Let* R *and* S *be distinct rings,* $\mathrm{R} \subseteq \mathrm{S}$ *such those they share a common nonzero, proper ideal* I*, then* $\mathrm{ht}_{\mathrm{S}}(\mathrm{I}) \leq \mathrm{ht}_{\mathrm{R}}(\mathrm{I}) \leq \dim \mathrm{S}$.

1.4 Known relevant facts on depictions used in the thesis

Let us collect here some significant results of the depictions by Beil which have been frequently mentioned for their relevance in this work.

Proposition 1.20 ([6, Theorem 3.12])**.** *Let* S *be a depiction of* R*. Then the following conditions are equivalent:*

1. R *is Noetherian;*

2. $\mathrm{U}_{\frac{\mathrm{S}}{\mathrm{R}}} = \mathrm{Max}(\mathrm{S})$*;*

3. $\mathrm{S} = \mathrm{R}$.

In particular, if R *is Noetherian, then its only depiction is itself.*

Proposition 1.21 ([8, Corollary B]). *Let* $\mathbb{K}$ *be an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal, nonzero and radical ideal of* S*, then the following conditions are equivalent:*

1. R *is nonNoetherian;*
2. $\dim \frac{\mathrm{S}}{\mathrm{I}} \geq 1$*;*
3. R *is depicted by* S.

1.5 Useful new facts on fiber products

The new obtained results on fiber products, collected in this section, are preparatory to present some examples, theorems and remarks of the next chapter. The following one describes two variants of Proposition 1.21, because it is obtained replacing the hypothesis of radicality for I, with two different particular conditions.

Proposition 1.22. *Let* $\mathbb{K}$ *be an algebraically closed field,* S *be both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S*. Suppose at least one of the following conditions is satisfied.*

1. I *is strictly contained in a prime, nonmaximal ideal of* S*;*
2. I *is a prime, nonmaximal ideal of* S.

Then R *is nonNoetherian.*

Proof. In both cases, there exist maximal ideals $\mathfrak{m}$ of S such that $\mathrm{I} \subsetneq \mathfrak{m}$. Intersecting both of the terms of this last inclusion with R and considering the fact that both I (by Proposition 1.16) and $\mathfrak{m} \cap \mathrm{R}$ are maximal ideals of R, we obtain:

$$\mathrm{I} \cap \mathrm{R} = \mathrm{I} = \mathfrak{m} \cap \mathrm{R},$$

for every maximal ideal $\mathfrak{m}$ of $\mathrm{Z}(\mathrm{I}) = \mathrm{U}^c$, which is nonempty because I is a nonmaximal ideal of S (this last equality is guaranteed by the generalization of Proposition 2.8, above introduced); so $\mathrm{S}_\mathfrak{m}$ properly contains R_I and $\dim \mathrm{S}_\mathfrak{m}$ is strictly bigger than $\dim \mathrm{R}_\mathrm{I} := n$, with $n \in \mathbb{N}$ finite, (by both the hypotheses for I, since $\dim \mathrm{S}_\mathfrak{m} = \mathrm{ht}(\mathfrak{m})$ and $\dim \mathrm{R}_\mathrm{I} = \mathrm{ht}(\mathrm{I})$. Moreover, we are sure of the fact that there exist no prime ideals $\mathfrak{p}$ of R contained in I different from the corresponding prime ideals $\mathfrak{q}$ of S by Proposition 1.10. However, if R was Noetherian by a contradiction, then R_I is a Noetherian domain and by Proposition 1.3 $\dim_v \mathrm{R}_\mathrm{I} = \dim \mathrm{R}_\mathrm{I} = n$.

This fact, for Proposition 1.2, is equivalent to assert that $\mathrm{S}_{\mathfrak{m}}$, overring of R_{I}, has dimension at most n; for this reason the following chain of inequality holds:

$$n = \dim \mathrm{R}_{\mathrm{I}} < \dim \mathrm{S}_{\mathfrak{m}} \leq n,$$

and we obtain a contradiction. □

It is important to make a preliminary clarification about $\mathrm{R} \subseteq \mathrm{S}$ when R has a structure of fiber product.

Proposition 1.23. *Let $\mathbb{K}$ be an algebraically closed field, S be both a finitely generated $\mathbb{K}$ - algebra and an integral domain, $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$; I is a nonmaximal and nonzero ideal of S if and only if $\mathrm{R} \neq \mathrm{S}$.*

Proof. ($\Rightarrow$) If I is a nonmaximal ideal of S, it exists a maximal ideal $\mathfrak{m}$ of S such that $\mathrm{I} \subsetneq \mathfrak{m}$. Besides, I is a maximal ideal of R for the fiber products' theory and $\mathrm{I} = \mathfrak{m} \cap \mathrm{R}$. If by a contradiction $\mathrm{R} = \mathrm{S}$, then $\mathrm{I} = \mathfrak{m} \cap \mathrm{R} = \mathfrak{m} \cap \mathrm{S} = \mathfrak{m}$, a contradiction.

($\Leftarrow$) Let $\mathrm{R} \neq \mathrm{S}$ and let us suppose, by a contradiction that I is a maximal ideal of S. Then, since S is a finitely generated $\mathbb{K}$ - algebra, then $\mathrm{S} = \mathbb{K} + \mathrm{I} = \mathrm{R}$; thus $\mathrm{R} = \mathrm{S}$, a contradiction. □

Let us study a way to create nontrivial fiber products in the polynomial ring characterized by some prime ideals of a certain fixed height:

Proposition 1.24. *Let $\mathbb{K}$ be an algebraically closed field and consider the polynomial ring $\mathrm{T} = \mathbb{K}[x_1, \ldots, x_n]$. Take a nonmaximal proper ideal I of T such that there is a maximal ideal $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)\mathrm{T}$ satisfying*

1. *$I \subseteq \mathfrak{m}$;*
2. *$I \not\subseteq \mathfrak{p} := (x_1 - a_1, \ldots, x_{n-1} - a_{n-1})\mathrm{T}$,*

where $a_1, \ldots, a_n \in \mathbb{K}$.
Consider the ring $\mathrm{R} = \mathbb{K} + \mathrm{I}$ and notice that $\mathrm{I} \in \mathrm{Max}(\mathrm{R})$. Then $\mathrm{ht}_{\mathrm{R}}(\mathrm{I}) = n$.

Proof. Let us show that:

- $\mathfrak{p} \cap \mathrm{R} \subsetneq \mathrm{I}$, that is $\mathrm{ht}_{\mathrm{R}}(\mathrm{I}) > 1$.
  As a matter of fact, take an element $\phi \in \mathrm{R}$, i. e. $\phi = k + i(x_1, \ldots, x_n)$, where $k \in \mathbb{K}$ and $i(x_1, \ldots, x_n) \in \mathrm{I}$. Since $\mathrm{T} = \mathbb{K}[x_1, \ldots, x_{n-1}][x_n - a_n]$, we can write
  $$i(x_1, \ldots, x_n) = \sum_{h=0}^{r} \gamma_h(x_1, \ldots, x_{n-1})(x_n - a_n)^h,$$
  for some $\gamma_h(x_1, \ldots, x_{n-1}) \in \mathbb{K}[x_1, \ldots, x_{n-1}]$.
  Since $\mathrm{I} \subseteq \mathfrak{m}$, then $\gamma_0(a_1, \ldots, a_{n-1}) = 0$. Then $\phi \in \mathfrak{p}$ if and only if
  $$\phi(a_1, \ldots, a_{n-1}, x_n) = 0$$

if and only if

$$k+\sum_{h=1}^{r}\gamma_h(a_1,\dots,a_{n-1})(x_n-a_n)^h=0$$

and then we infer that $k=0$ and that $\phi\in\mathrm{I}$. Finally the inclusion $\mathfrak{p}\cap\mathrm{R}\subsetneq\mathrm{I}$ is strict, otherwise $\mathfrak{p}\cap\mathrm{R}=\mathrm{I}$ implies that $\mathrm{I}\subseteq\mathfrak{p}$, which is a contradiction.

- $\mathrm{ht}_{\mathrm{R}}(\mathrm{I})=n$.
  Indeed, consider the chain of prime ideals of T:

$$(0)\subsetneq\mathfrak{p}_1=(x_1-a_1)\subsetneq\mathfrak{p}_2=(x_1-a_1,x_2-a_2)\subsetneq,\dots,\subsetneq\mathfrak{p}_{n-1}=$$

$$=(x_1-a_1,x_2-a_2,\dots,x_{n-1}-a_{n-1})=\mathfrak{p}.$$

Since R is a fiber product and $\mathfrak{p}_j\supsetneq\mathrm{I}$ for every j, we get the following chain of prime ideals of R:

$$(0)\subsetneq\mathfrak{p}_1\cap\mathrm{R}\subsetneq\mathfrak{p}_2\cap\mathrm{R}\subsetneq,\dots,\subsetneq\mathfrak{p}_{n-1}\cap\mathrm{R}=\mathfrak{p}\cap\mathrm{R}\subsetneq\mathrm{I},$$

for the first part. Suppose by a contradiction, that $\mathrm{ht}_{\mathrm{R}}(\mathrm{I})>n$. Thus we can pick a chain in $\mathrm{Spec(R)}$ of length $n+1$;

$$(0)\subsetneq\mathfrak{q}_1\subsetneq\mathfrak{q}_2\subsetneq,\dots,\subsetneq\mathfrak{q}_n\subsetneq\mathrm{I}.$$

Clearly $\mathfrak{q}_j\supsetneq\mathrm{I}$ for every j. Since the contraction mapping:

$$\mathrm{Spec(T)}\backslash\mathrm{V(I)}\longrightarrow\mathrm{Spec(R)}\backslash\mathrm{V(I)}$$

$$\mathfrak{h}\longmapsto\mathfrak{h}\cap\mathrm{R}$$

is an isomorphism of partially ordered sets, there is a chain:

$$(0)\subsetneq\mathfrak{h}_1\subsetneq\mathfrak{h}_2\subsetneq,\dots,\subsetneq\mathfrak{h}_n$$

in $\mathrm{Spec(T)}\backslash\mathrm{V(I)}$ such that $\mathfrak{h}_j\cap\mathrm{R}=\mathfrak{q}_j$, for every j. Moreover $\mathfrak{h}_n\in\mathrm{Max(T)}$ and thus

$$\mathfrak{h}_n=\{f\in\mathrm{T}:f(\alpha_1,\dots,\alpha_n)=0,\text{for every }(\alpha_1,\dots,\alpha_n)\in\mathbb{K}^n\}.$$

Choose a polynomial $g\in\mathrm{I}\backslash\mathfrak{q}_n$, i. e. $\lambda:=g(\alpha_1,\dots,\alpha_n)\neq0$, and let $\tilde{g}(x_1,\dots,x_n):=\lambda-g(x_1,\dots,x_n)\in\mathbb{K}+\mathrm{I}=\mathrm{R}$. Since $\tilde{g}(\alpha_1,\dots,\alpha_n):=\lambda-\lambda=0$, we infer that $\tilde{g}(x_1,\dots,x_n)\in\mathfrak{h}_n\cap\mathrm{R}=\mathfrak{q}_n\subsetneq\mathrm{I}$. It follows that $0\neq\lambda\in\mathrm{I}$, a contradiction.

□

Chapter 2

Fiber products involved in Beil's results

2.1 Principal results

Let us present the two most relevant Beil's results we dealt with fiber products. The innovation in using fiber products consists in the possibility to obtain valid results for a more general category of ideals, those ones not specifically radical. In order to do this, let us recall (see Section 1.2) the fact that if $\mathrm{R} \subseteq \mathrm{S}$ is a ring extension and I is a common ideal of R and S, then R is the fiber product of the canonical projection π from S to $\frac{\mathrm{S}}{\mathrm{I}}$ and of the embedding i from $\frac{\mathrm{R}}{\mathrm{I}}$ to $\frac{\mathrm{S}}{\mathrm{I}}$ as follows:

$$\begin{array}{ccc} \mathrm{R} & \longrightarrow & \frac{\mathrm{R}}{\mathrm{I}} \\ \big\downarrow & & \big\downarrow i \\ \mathrm{S} & \xrightarrow{\pi} & \frac{\mathrm{S}}{\mathrm{I}} \end{array}$$

Besides, considering the same hypotheses, if we also set $\mathbb{K}$ an algebraically closed field, R' a subalgebra of S and $\mathrm{R} = \mathbb{K}[\mathrm{R}', \mathrm{I}]$ we obtain a generalization of some Beil's results, like the following one:

1) [6, Proposition 2.8]. *Let* $\mathbb{K}$ *be an algebraically closed field,* S *an integral domain and a Noetherian* $\mathbb{K}$ *- algebra. Consider a subalgebra* R' *of* S*, an ideal* I *of* S*, and form the algebra* $\mathrm{R} = \mathbb{K}[\mathrm{R}', \mathrm{I}]$*. Then* U *contains the open subset* $\mathrm{Z}(\mathrm{I})^c$ *of* $\mathrm{Max}(\mathrm{S})$*. Furthermore, if* $\mathrm{I} \subset \mathrm{S}$ *is a nonmaximal radical ideal and* $\mathrm{R} = \mathbb{K}[\mathrm{I}] = \mathbb{K} + \mathrm{I}$*, then* $\mathrm{U} = \mathrm{Z}(\mathrm{I})^c$.

Proposition 2.1 (Generalization of [6, Proposition 2.8])**.** *If* $\mathbb{K}$ *is a field,* S *is an integral domain and* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S *and* $\mathbb{K}$ *is a proper subring of the quotient ring* $\frac{\mathrm{S}}{\mathrm{I}}$*, then* $\mathrm{U} = \mathrm{Z}(\mathrm{I})^c$.

Proof. ($\supseteq$) For every $\mathfrak{m} \in \mathrm{Z(I)}^c$, $\mathfrak{m} \in \mathrm{U}$, by Proposition 1.9 and Proposition 1.10.

($\subseteq$) For every $\mathfrak{m} \notin \mathrm{Z(I)}^c$, $\mathfrak{m} \in \mathrm{Z(I)}$, then $\mathfrak{m} \supseteq \mathrm{I}$ and $\mathfrak{m} \cap \mathrm{R} \supseteq \mathrm{I} \cap \mathrm{R} = \mathrm{I}$. Since I is a maximal ideal of R it follows that $\mathfrak{m} \cap \mathrm{R} = \mathrm{I}$. So we can write

$$\mathrm{R}_{\mathrm{R}\cap\mathfrak{m}} = (\mathbb{K} + \mathrm{I})_{\mathrm{I}} = \left\{ \frac{k+i}{1+j} \middle| k \in \mathbb{K}, i, j \in \mathrm{I} \right\},$$

because every element of this set has the form $\frac{k_1+i_1}{k_2+j_1}$, with $k_1 \in \mathbb{K}, k_2 \in \mathbb{K}^*, i_1, j_1 \in \mathrm{I}$, so it can be returned to the form $\frac{k+i}{1+j}$, dividing both numerator and denominator by k_2 thanks to the fact that it is in $\mathbb{K}^*$. Thus, if $\mathfrak{m} \in \mathrm{U}$ then $\mathrm{S} \subseteq \mathrm{S}_{\mathfrak{m}} = \mathrm{R}_{\mathrm{R}\cap\mathfrak{m}}$, so for every $s \in \mathrm{S}$ exist $k \in \mathbb{K}, i, j \in \mathrm{I}$ such those $s = \frac{k+i}{1+j}$. From this last equality it follows that $s + sj = k + i$, that $s = k + i - sj$ and that $s = k + i'$ for some $i' \in \mathrm{I}$. Then $s \in \mathbb{K} + \mathrm{I} = \mathrm{R}$, from that it derives that $\mathrm{S} \subseteq \mathrm{R}$ and that $\mathrm{S} = \mathrm{R}$, a contradiction. So $\mathfrak{m} \in \mathrm{U}^c$ and this fact implies that $\mathrm{Z(I)} \subseteq \mathrm{U}^c$ and then $\mathrm{U} \subseteq \mathrm{Z(I)}^c$. □

In the previous proof, unlike Beil did in his article, we did not use this following hypotheses:

1. $\mathbb{K}$ is integrally closed;
2. I is a radical ideal;
3. S is a finitely generated $\mathbb{K}$ - algebra.

Befor proceeding with the following Proposition, let us recall that if S is a ring and I is an ideal of S, then $\dim(\mathrm{Z(I)}) := \dim\left(\frac{\mathrm{S}}{\mathrm{I}}\right)$.

2) [6, Corollary 3.14]. *Let* $\mathbb{K}$ *be an algebraically closed field,* I *a radical ideal of a finitely generated* $\mathbb{K}$ *- algebra* S*. Then the ring* $\mathrm{R} = \mathbb{K} + \mathrm{I}$ *is nonnoetherian if and only if* $\dim \mathrm{Z(I)} \geq 1$.

Proposition 2.2 (Generalization of [6, Corollary 3.14])**.** *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S, R *is Noetherian if and only if* $\dim\left(\frac{\mathrm{S}}{\mathrm{I}}\right) = 0$.

Proof. ($\Leftarrow$) Since $\mathbb{K}$ is a field and S is a finitely generated $\mathbb{K}$ - algebra, S is Noetherian and $\frac{\mathrm{S}}{\mathrm{I}}$ is, so for the hypotheses and the characterization of the Artinian rings, $\frac{\mathrm{S}}{\mathrm{I}}$ is Artinian. Moreover $\frac{\mathrm{S}}{\mathrm{I}}$ is a finitely generated $\mathbb{K}$ - algebra, since S is, so the embedding i from $\mathbb{K}$ to $\frac{\mathrm{S}}{\mathrm{I}}$ is finite by [2, Chapter 8, Exercise 3] and it follows that R is Noetherian, since both $\mathbb{K}$ and $\frac{\mathrm{S}}{\mathrm{I}}$ are too, by Proposition 1.10.

($\Rightarrow$) It follows immediately from Proposition 1.14 or from Proposition 1.15 or from Proposition 1.18. In particular, the second condition of this last one is equivalent to assert that $\dim \frac{\mathrm{S}}{\mathrm{I}} = \dim \mathrm{S} - \mathbf{ht}\,\mathrm{I} = \dim \mathrm{S} - n = \dim \mathrm{S} - \dim \mathrm{S} = 0$. □

Remark 2.3. Even in this case we don't use the hypothesis of radicality for the ideal I, as opposed to the direction followed by Beil in his article, showing that this result is valid in a more general sense.

2.2 Other Beil's results immediately deduced from fiber products

Two trivial corollaries obtained by the results of the previous section are:

Corollary 2.4. *If* $\mathbb{K}$ *is a field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonzero, radical ideal of* S*, then* R *is Noetherian if and only if* I *is intersection of a finite number of maximal ideals of* S.

Proof. ($\Rightarrow$) If R is Noetherian, then by Proposition 2.2 it follows that $\dim(\frac{\mathrm{S}}{\mathrm{I}}) = 0$ and this happens if I is intersection of a finite number of maximal ideals of S.

($\Leftarrow$) If I is intersection of a finite number of maximal ideals of S, then $\dim(\frac{\mathrm{S}}{\mathrm{I}}) = 0$ and it follows that R is Noetherian by Proposition 2.2. □

Corollary 2.5. *If* $\mathbb{K}$ *is a field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonzero and radical ideal of* S*, then*

$$\mathrm{I} = \bigcap_{\mathfrak{m} \in \mathrm{U}^c} \mathfrak{m}.$$

Proof. Since S is a Jacobson ring, then every radical ideal of S is intersection of the maximal ideals of S which contain it. So

$$\mathrm{I} = \bigcap_{\mathfrak{m} \in \mathrm{Z}(\mathrm{I})} \mathfrak{m} = \bigcap_{\mathfrak{m} \in \mathrm{U}^c} \mathfrak{m}$$

by Proposition 2.1. □

Besides, the advantage to use fiber products is to derive immediately some Beil's results from authors who already proved them a long time ago dealing with fiber products' theory, often with a more light proof. Let us present the most relevant ones.

Proposition 2.6. *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonzero ideal of* S*, then* $\mathrm{U} \neq \emptyset$.

Proof. If U is empty, then $\mathrm{Z(I)}^c$ is empty, by Proposition 2.1, so it does not exist any maximal ideal $\mathfrak{m}$ of S such that I is not contained in $\mathfrak{m}$, or, in other words, every maximal ideal $\mathfrak{m}$ of S contains I, that is $\mathrm{Z(I)} = \mathrm{Max(S)}$. Thus I is contained in the Jacobson radical of S which coincides with the nilradical of S because S is a finitely generated $\mathbb{K}$ - algebra. However, S is an integral domain and the nilradical of every integral domain is zero. So we can conclude that I is the zero ideal, a contradiction. □

This basic proposition, in case of fiber products, is essential to deduce the following results of Beil's works [6], [8], just simply using some known ones of Fontana and their derivations.

Proposition 2.7 ([6, Proposition 2.4.1])**.** *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonzero ideal of* S*, then* S *is an overring of the integral domain* R*.*

Proof. It follows by Proposition 2.6 using Proposition 1.10. □

Proposition 2.8 ([8, Lemma 4.3 and Proposition 4.4.1])**.** *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S*, then the morphism*

$$\mathrm{i}_{\frac{\mathrm{S}_\alpha}{\mathrm{R}_\alpha}} : \mathrm{Spec}(\mathrm{S}_\alpha) \to \mathrm{Spec}(\mathrm{R}_\alpha)$$

$$\mathfrak{q} \mapsto \mathfrak{q} \cap \mathrm{R}_\alpha$$

is surjective for every element $\alpha \in \mathrm{I}$ *and if* S *is depiction of* R*, then* S_α *is a depiction of* R_α*.*

Proof. It follows by Proposition 1.9. □

Proposition 2.9 ([6, Theorem 2.5.3])**.** *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S*, then* $\mathrm{Max(S)}$ *and* $\mathrm{Max(R)}$ *are homeomorphic on an open dense subset and thus birationally equivalent.*

Proof. It follows by Proposition 2.6 using Proposition 1.9 and Proposition 1.10. □

Proposition 2.10 ([6, Theorem 2.5.4])**.** *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S*, then* $\dim \mathrm{S} = \dim \mathrm{R}$*.*

Proof. It follows by Proposition 2.6 using Proposition 1.18. □

Proposition 2.11 ([6, Lemma 3.7.1])**.** *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a* *nonmaximal and nonzero ideal of* S*, if* $\mathfrak{p}$ *is a prime ideal of* R *and* $\mathfrak{q}$ *is a prime ideal of* S *lying over* $\mathfrak{p}$ *and*

$$\begin{array}{rcl} \mathrm{i}_{\frac{\mathrm{S}}{\mathrm{R}}} : \mathrm{Spec}\,(\mathrm{S}) & \to & \mathrm{Spec}\,(\mathrm{R}) \\ \mathfrak{q} & \mapsto & \mathfrak{q} \cap \mathrm{R} = \mathfrak{p} \end{array}$$

is surjective, then $\mathrm{ht}(\mathfrak{q}) \leq \mathrm{ht}(\mathfrak{p})$.

Proof. It follows by Proposition 2.6 using Proposition 1.19 and repeating Beil's proof without using the hypothesis S is depiction of R, in particular avoiding to use (D2). □

Proposition 2.12 ([6, Generalization of Lemma 3.7.2])**.** *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S*, then, if* $\mathfrak{m} \in \mathrm{Max}(\mathrm{R})$ *and*

$$\begin{array}{rcl} \mathrm{i}_{\frac{\mathrm{S}}{\mathrm{R}}} : \mathrm{Spec}\,(\mathrm{S}) & \to & \mathrm{Spec}\,(\mathrm{R}) \\ \mathfrak{q} & \mapsto & \mathfrak{q} \cap \mathrm{R} \end{array}$$

is surjective, then $\mathrm{ht}(\mathfrak{m}) = \dim \mathrm{R}$.

Proof. It follows by Proposition 2.6 using Proposition 1.17 and repeating Beil's proof without using the hypothesis S is depiction of R, in particular avoiding to use (D2). □

Chapter 3

Depictions and fiber products

3.1 Known examples of depictions and not depictions

We choose to add this section, which could be considered the most important of this work, because it collects all the ideas that gave us the intuition to produce this work, the starting point of our research, the inspiration of all our results. In the following examples $\mathbb{K}$ is an algebraically closed field.

1. *Nonpolynomial case of depiction-Nonuniqueness of maximal depictions* [6, Proposition 3.19.2]
   $\mathrm{T} = \mathbb{K}[x, y, z]$
   $\mathrm{R} = \mathbb{K} + xy\mathrm{T}$
   T is a depiction of R.
   $\mathrm{S} := \mathrm{T}[x^{-1}]$ is a depiction of R
   $\mathrm{S}' := \mathrm{T}[y^{-1}]$ is a depiction of R
   $\mathrm{S}'' := \mathrm{T}[x^{-1}, y^{-1}]$ is not a depiction of R.

   Remark 3.1. This example is really relevant because it shows that exist depictions of subalgebras which cannot share any common ideal with them and consequently which are not depictions of any fiber product, as it happens in this case for S. which is depiction of R. We will see in detail the insights of this fact below, in the following section.

2. [6, Remark 3.3]
   $\mathrm{S} = \mathbb{K}[x, y, z]$ is a depiction of $\mathrm{R} = \mathbb{K} + x(y, z)\mathrm{S}$
   $\mathrm{S}[x^{-1}]$ is a depiction of R.

3. [6, Example 3.15]
   Let S be a finitely generated $\mathbb{K}$-algebra and let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots \mathfrak{m}_l$ be a finite set of maximal ideals of S. Then S is not a depiction of $\mathrm{R} = \mathbb{K} + \sqrt{\mathfrak{m}_1 \cdots \mathfrak{m}_l}$.

4. *Nonexistence of minimal depictions* [6, Proposition 3.19.1]
   $\mathrm{S} = \mathbb{K}[x, y]$
   $\mathrm{R} = \mathbb{K} + x\mathrm{S}$
   $\mathrm{S}_l = \mathrm{R}[y^l, y^{l+1}, y^{l+2}, \ldots] = \mathbb{K}[x, xy, xy^2, \ldots, xy^{l-1}, y^l, y^{l+1}, \ldots, y^{2l-1}]$ is a depiction of R and $\mathrm{S}_{l+1} \subsetneq \mathrm{S}_l$, for every $l \in \mathbb{N}$.

   Remark 3.2. This example shows that, in general, the intersection of some depictions is not a depiction because

$$\mathrm{R} = \bigcap_{\ell \geq 1} \mathrm{S}_\ell,$$

   so it cannot happen that R is a depiction because the depictions are Noetherian but R is not.

5. [5, Example 5.2]
   $\mathrm{S} = \mathbb{K}[x, y]$ is a depiction of $\mathrm{R} = \mathbb{K} + x(x-1, y)\mathrm{S}$
   $\mathrm{S}[x^{-1}]$ is the only maximal depiction of R.

6. [5, Example 5.3]
   $\mathrm{S} = \mathbb{K}[x, y]$ is the only maximal depiction of
   $\mathrm{R} = \mathbb{K} + x\mathrm{S} = \mathbb{K}[x, xy, xy^2, \ldots] = \mathbb{K}[x, y] \times_{\mathbb{K}[y]} \mathbb{K}$.

7. [5, Example 5.1]
   $\mathrm{S}_j = \mathbb{K}[x, y, xz, yz, xz^2, yz^2, \ldots, xz^{j-1}, yz^{j-1}, z^j]$, for every $j \geq 1$
   $\mathrm{S}_1 = \mathbb{K}[x, y, z]$
   $\mathrm{R} = \mathbb{K} + (x, y)\mathrm{S}_1$
   S_1 is the only maximal depiction of R
   $\mathrm{S}_j \subseteq \mathrm{S}_1$ is a depiction of R, for every $j \geq 1$.

3.2 An example of a depiction of a ring that is not a fiber product of a quotient of its depiction

Investigating Beil's examples of depictions we found depictions of subalgebras which are fiber products and not.

Given an algebraically closed field $\mathbb{K}$, let

- $\mathrm{R} = \mathbb{K} + xy\mathrm{T}$,
- $\mathrm{T} = \mathbb{K}[x, y, z]$ and
- $\mathrm{S} := \mathrm{T}[x^{-1}]$ a depiction of R.

It is trivial to notice that R could not be a fiber product because S and R could not admit in any case a common ideal. In order to prove this, if we show that the conductor of the extension $\mathrm{R} \subseteq \mathrm{S}$ is the zero ideal, we reach our goal, since the conductor of an extension is the biggest ideal of the subring shared with the bigger ring. So, let

$$\mathrm{J} := \mathrm{R} : \mathrm{S} := \{0\} \cup \{r \in \mathrm{R} \setminus \{0\} : r\mathrm{S} \subseteq \mathrm{R}\}$$

be the conductor of the extension $\mathrm{R} \subseteq \mathrm{S}$, and let us suppose by a contradiction that $r = k + i$, with $k \in \mathbb{K}$ and $i \in \mathrm{I}$ is a nonzero element of J then:

1. if $k = 0$ then $r = i \in \mathrm{I} \setminus \{0\}$, thus $r = x^m y^l f(x, y, z)$, for some $f(x, y, z) \in \mathrm{T} \setminus x\mathrm{T}$ and for some m and $l \in \mathbb{N}^*$. So $r \cdot \frac{1}{x^m} = y^l f(x, y, z)$, with $f(x, y, z) \in \mathrm{T} \setminus x\mathrm{T}$, which is not in R, a contradiction;

2. if $i = 0$ then $r = k \in \mathbb{K}^*$ and the ideal J of R would explode in the whole R, a contradiction;

3. if $r = k + i$, with $k \in \mathbb{K}^*$ and $i \in \mathrm{I} \setminus \{0\}$, and we multiply it for $z \in \mathrm{S}$, we obtain an element which is not in R because $r \cdot z = kz + iz$ and the first addend of this sum surely is not in R because $k \neq 0$, a contradiction.

So, we can conclude that S and R cannot admit any kind of common ideal.

Then, for this reasons we took care of separately analyzing the depictions of fiber and nonfiber products as follows.

3.3 Depictions of fiber products

3.3.1 Subalgebras of fiber products: the almost depictions!

Focusing our point of view on depictions of fiber products, we realized that some of them are well described by fiber products properties and we are able to present these useful remarks, which let us to give (see below) a new definition of depiction:

Remark 3.3. Let $\mathbb{K}$ be an algebraically closed field, S be both a finitely generated $\mathbb{K}$ - algebra and an integral domain, $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$, with I a nonmaximal and nonzero ideal of S:

- (D3) is guaranteed by Proposition 2.6;
- for (D1), since I is a finitely generated ideal of S, according to the Proposition 1.9, all the prime ideals of R are I and $\mathfrak{p} \cap \mathrm{R}$, where $\mathfrak{p}$ is any prime ideal of S which does not contain all together the generators of I. Moreover, by Proposition 1.9, the mapping

$$\begin{array}{rcl} \mathrm{Spec(S)} \setminus \mathrm{V(I)} & \to & \mathrm{Spec(R)} \setminus \mathrm{V(I)} \\ \mathfrak{q} & \mapsto & \mathfrak{q} \cap \mathrm{R} \end{array}$$

is a homeomorphism of topological spaces and, in particular, an isomorphism of partially ordered sets. This fact let us to conclude that in this case (D1) is always verified because every prime ideal of R is the contraction of some prime ideal of S to R.

For the previous remark, the following definition naturally arises:

Definition 3.4. Let S be an integral domain, $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$ any ring extension, with I a nonmaximal and nonzero ideal of S, then let us define S *a weak depiction* of R and R *a weakly depicted fiber product* of S.

Let us posing in the particular case of Beil's set, we can rearrange the previous definition in this new following one:

Definition 3.5. Let $\mathbb{K}$ be an algebraically closed field, S be both a finitely generated $\mathbb{K}$ - algebra and an integral domain, $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$, with I a nonmaximal and nonzero ideal of S, then let us define S *an almost depiction* of R and R *an almost depicted fiber product* of S.

Dealing with almost depicted fiber products, in order to show that they are also depictions in suitable cases, it is necessary to make the following premise:

Remark 3.6. Let $\mathbb{K}$ be an algebraically closed field, S be both a finitely generated $\mathbb{K}$ - algebra and an integral domain, $\mathrm{R} = \mathbb{K}+\mathrm{I} \subseteq \mathrm{S}$ and I is a nonmaximal and nonzero ideal of S. In every theorem in which we want to prove that S is a depiction of R, we have to suppose that R is nonNoetherian because otherwise, by Proposition 1.20, if S is depiction of R, then $\mathrm{R} = \mathrm{S}$ and it happens by Proposition 1.23 if and only if I is a maximal ideal of S, a contradiction.

Remark 3.7. Considering the previous remarks, we can conclude that, dealing with overrings of fiber products which are integral domains and finitely generated $\mathbb{K}$ - algebras too, with $\mathbb{K}$ an algebraically closed field, (D1) and (D3) are immediately verified and in order to discover if we are in presence of a depiction it will be sufficient only to prove that (D2) holds.

By the previous remark, let us give the following proposition, whose proof is trivial:

Proposition 3.8. *Let* $\mathbb{K}$ *be an algebraically closed field,* S *be both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S *(*S *an almost depiction of* R*) which satisfies (D2), then* S *is a depiction of* R.

Remark 3.9. Each depiction S (which is both a finitely generated $\mathbb{K}$ - algebra and an integral domain, with $\mathbb{K}$ an algebraically closed field) of a fiber product $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$, with I a nonmaximal and nonzero ideal of S is an almost depiction.

The converse is not true. Let us give a counterexample of this fact:

Example 3.10. Let $\mathbb{K}$ be an algebraically closed field, S be both a finitely generated $\mathbb{K}$ - algebra and an integral domain, $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$, with I a nonmaximal and nonzero ideal of S (S an almost depiction of R) such that R is Noetherian, then the localization R_I is Noetherian and S is not a depiction of R, for the next theorem. For instance, it is sufficient to take I radical as an intersection of a finite number of maximal ideals of S, by Corollary 2.4.

Example 3.11. There exist depictions of a subring R which are not almost depictions of it because they cannot share any ideal with R, like the example in [6, Proposition 3.19.2].

Let us give the most relevant result of our work for the almost depictions:

Theorem 3.12. *Let $\mathbb{K}$ be an algebraically closed field, S be both a finitely generated $\mathbb{K}$ - algebra and an integral domain, $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$, with I a nonmaximal and nonzero ideal of S (S is almost depiction of R), then R_I is Noetherian if and only if S is not a depiction of R.*

Proof. ($\Rightarrow$) By Proposition 2.1, we know that I is equal to $\mathfrak{m} \cap \mathrm{R}$ for every maximal ideal $\mathfrak{m}$ of $\mathrm{Z(I)} = \mathrm{U}^c$ (which is nonempty because I is a nonmaximal ideal of S), so exists $\mathfrak{m} \in \mathrm{U}^c$ such that $\mathrm{R}_{\mathrm{R}\cap\mathfrak{m}}$ is Noetherian, fact that contradicts (D2).

($\Leftarrow$) We prove that if R_I is nonNoetherian, then S is a depiction of R. (D1) and (D3) are immediately verified by Proposition 3.8; so, since I is a maximal ideal of R by Proposition 1.16, which is nonmaximal in S, then the only maximal ideals $\mathfrak{m}$ of S which contract to the ideal I are the ones that contain I, by Proposition 1.9. So it follows that $\mathrm{I} = \mathfrak{m} \cap \mathrm{R}$. Thus, by the choice of the maximal ideals $\mathfrak{m}$ which are in $\mathrm{Z(I)} = \mathrm{U}^c$ (by Proposition 2.1) and by the hypothesis we have that $\mathrm{R}_\mathrm{I} = \mathrm{R}_{\mathfrak{m}\cap\mathrm{R}}$ is nonNoetherian for every maximal ideal $\mathfrak{m}$ of U^c and these are the only ones of S accepted, according to fiber products' theroy; thus, passing to the complements, we obtain that $\mathrm{R}_{\mathfrak{m}\cap\mathrm{R}}$ is Noetherian if and only if $\mathfrak{m} \in \mathrm{U}$, which is (D2). Then, since all the conditions required for being a depiction are verified, we can conclude that S is a depiction of R. □

So, in case of a radical ideal of almost depicted fiber products, it is possible to deduce the following interesting characterization:

Corollary 3.13. *Let $\mathbb{K}$ be an algebraically closed field, S is both a finitely generated $\mathbb{K}$ - algebra and an integral domain, $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$, with I a nonmaximal, nonzero and radical ideal of S, then R is Noetherian if and only if R_I is Noetherian.*

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) If, by a contradiction, R_I is nonNoetherian, then S is a depiction of R by Theorem 3.12, hence R is nonNoetherian, by Proposition 1.21, that is a contradiction. □

Let us concentrate now just on a note result of Beil, the Proposition 1.21, which can be seen like an extension of Proposition 2.2 and which follows just from Theorem 3.12. We choose to treat it replacing the hypothesis of radicality for the ideal I with the reasonable hypothesis I has not any embedded prime ideal of S in order to reach the aim to regardless of the pure geometrical aspect of radicality, preferring instead a more algebraic focus on the Beil's results. For this reason, let us give the following variant of Proposition 1.21.

Corollary 3.14. *Let* $\mathbb{K}$ *be an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S *which has no embedded prime ideal of* S*, then the following conditions are equivalent:*

1. R *is nonNoetherian;*
2. $\dim \frac{\mathrm{S}}{\mathrm{I}} \geq 1$*;*
3. R *is depicted by* S.

Proof. $(1) \Leftrightarrow (2)$ It is just Proposition 2.2.

$(3) \Rightarrow (1)$ It is a trivial consequence of Proposition 1.20, since $\mathrm{S} \neq \mathrm{R}$.

$(1) \Rightarrow (3)$ It is sufficient to observe that, under the hypothesis I has not any embedded prime ideal of S, R_{I} is nonNoetherian. This fact happens because if we replace in the proof of Proposition 1.21 the hypothesis of radicality for I, with I has not any embedded prime ideal of S, then the existence of the h which appears in the proof is guaranteed, otherwise the ideal $\mathfrak{n}$ of S would be contained in the union of all the $\mathfrak{p}_i$, for $i = 1, 2, ..., n$ and for the primary avoidance theorem $\mathfrak{n}$ would be contained in some of the $\mathfrak{p}_i$, for $i = 1, 2, ..., n$, which is a contradiction. So S is depiction of R by Theorem 3.12. □

Let us consider now the possibility to intercept other intermediate depictions of depictions containing as subring an almost depicted fiber product, obtaining under an appropriate hypothesis a positive feedback.

Proposition 3.15. *Let* $\mathbb{K}$ *be an algebraically closed field,* S *be both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S *(so* S *is an almost depiction of* R*), then each finitely generated* $\mathbb{K}$ *- subalgebra of* S *containing as subring* R *is an almost depiction.*

Proof. Since each intermediate subring H of S containing as subring the fiber product R shares with this last one each common ideal to R and S, even the rings R and H share the common ideal I. So R is a fiber product for each intermediate ring H, too. Thus, the thesis immediately follows for H by Proposition 3.8 by the hypotesis H finitely generated $\mathbb{K}$ - algebra. Besides, H is a integral domain, because it is a subring of an integral domain. □

Corollary 3.16. *Let* $\mathbb{K}$ *be an algebraically closed field,* S *be both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S*,* S *depiction of* R *and* H *a finitely generated* $\mathbb{K}$ *- subalgebra of* S *containing as subring* R*; if*

$$\begin{array}{rcl} \mathrm{i}_{\frac{\mathrm{S}}{\mathrm{H}}} : \mathrm{Spec}\,(\mathrm{S}) & \rightarrow & \mathrm{Spec}\,(\mathrm{H}) \\ \mathfrak{q} & \mapsto & \mathfrak{q} \cap \mathrm{H} \end{array}$$

is surjective or $\mathrm{H} \subseteq \mathrm{S}$ *is an integral ring extension, then* H *is a depiction of* R*.*

Proof. (D1) and (D3) follow from the previous proposition. In order to try (D2), we can observe that since H is a finitely generated $\mathbb{K}$ - algebra, it becomes Noetherian and at the same way each its localization for every prime ideal is. So, if we consider:

$$\mathrm{R} \subseteq \mathrm{H} \subseteq \mathrm{S},$$

for every maximal ideal $\mathfrak{m}$ of S, we can pass to the localizations and we obtain:

$$\mathrm{R}_{\mathfrak{m} \cap \mathrm{R}} \subseteq \mathrm{H}_{\mathfrak{m} \cap \mathrm{H}} \subseteq \mathrm{S}_{\mathfrak{m}}.$$

By (D2), since S is depiction of R, if $\mathrm{R}_{\mathfrak{m} \cap \mathrm{R}}$ is Noetherian then $\mathrm{R}_{\mathfrak{m} \cap \mathrm{R}} = \mathrm{S}_{\mathfrak{m}}$; in this case, by the previous chain of inclusions it follows that $\mathrm{R}_{(\mathfrak{m} \cap \mathrm{H}) \cap \mathrm{R}} = \mathrm{R}_{\mathfrak{m} \cap \mathrm{R}} = \mathrm{H}_{\mathfrak{m} \cap \mathrm{H}} = \mathrm{S}_{\mathfrak{m}}$, that is $\mathrm{R}_{\mathfrak{h} \cap \mathrm{R}} = \mathrm{H}_{\mathfrak{h}}$, for every maximal ideal $\mathfrak{h}$ of H. Conversely, if $\mathrm{R}_{\mathfrak{h} \cap \mathrm{R}}$ is equal to $\mathrm{H}_{\mathfrak{h}}$ for every maximal ideal $\mathfrak{h}$ of H, then $\mathrm{R}_{\mathfrak{h} \cap \mathrm{R}}$ is Noetherian, because $\mathrm{H}_{\mathfrak{h}}$ is. □

Giving the sufficient condition of minimality for a principal ideal of an almost depiction, even in this case we obtain a depiction:

Proposition 3.17. *Let* $\mathbb{K}$ *be an algebraically closed field,* S *be both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$ *(*S *is almost depiction of* R*), with* I *a nonmaximal and nonzero ideal of* S *and a minimal principal prime ideal of* S *with* $\mathrm{ht}_{\mathrm{R}}(\mathrm{I}) > 1$*, then* S *is a depiction of* R*.*

Proof. In order to show (D2), it will be sufficient to apply the Krull's height theorem to the ideal I of R. The rest of the proof follows from Proposition 3.8. □

An exemplification of that is just the following extension of the example in [6, Proposition 3.19.2].

Example 3.18. Let $\mathbb{K}$ be an algebraically closed field,
$\mathrm{T} := \mathbb{K}[x, y, z, w]$
$\mathrm{R} := \mathbb{K} + xyw\mathrm{T} := \mathbb{K} + \mathrm{I}$
$\mathrm{S} := \mathrm{T}[x^{-1}, y^{-1}]$ is a depiction of R, because the ideal I is principal and minimal in T and in S and has height equal to 4.

Let us concentrate, now, on the depiction as a sufficient condition of finiteness:

Proposition 3.19. *If* $\mathbb{K}$ *is an algebraically closed field,* S *is both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} = \mathbb{K} + \mathrm{I} \subseteq \mathrm{S}$*, with* I *a nonmaximal and nonzero ideal of* S *(*S *is almost depiction of* R*) and* S *is depiction of* R*, then the morphism* $\mathrm{h} : \mathbb{K} \to \frac{\mathrm{S}}{\mathrm{I}}$ *defined in the following commutative square (*$*$*), representative of the fiber product* R *is not finite.*

$$\begin{array}{ccc} \mathrm{R} = \mathbb{K} + \mathrm{I} & \longrightarrow & \mathbb{K} \\ {\scriptstyle \mathrm{u}}\downarrow & & \downarrow{\scriptstyle \mathrm{h}} \\ \mathrm{S} & \xrightarrow{\pi} & \frac{\mathrm{S}}{\mathrm{I}} \end{array}$$

Proof. It immediately follows applying Proposition 1.12 to ($*$), thanks to which we have:

$$\left.\begin{array}{ll} \mathrm{R} & \text{Noetherian} \\ \frac{\mathrm{S}}{\mathrm{I}} & \text{Noetherian} \\ \mathrm{u} & \text{finite} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{ll} \mathrm{S} & \text{Noetherian} \\ \mathbb{K} & \text{Noetherian} \\ \mathrm{h} & \text{finite} \end{array}\right.$$

If, by a contradiction, h was finite, then R is Noetherian, a contradiction with the hypothesis R is depicted by S. □

3.3.2 New relevant examples of depictions and not depictions of fiber products

The following example give credit to our starting insight to obtain some Beil's results without using the hypothesis of radicality for the ideal of an (almost) depicted fiber product and represents the starting point which inspired us to write this thesis.

Example 3.20. Let $\mathbb{K}$ be an algebraically closed field,

$$\mathrm{S} := \mathbb{K}[x, y, z]$$

$$\mathrm{R} := \mathbb{K} + x^2y^2\mathbb{K}[x, y, z] := \mathbb{K} + \mathrm{I},$$

then S is depiction of R thanks to Proposition 3.17 because I is a minimal and principal prime ideal of S such that $\mathrm{ht}_{\mathrm{R}}(\mathrm{I}) = 3$ by Proposition 1.17. Besides, I is a nonradical ideal.

The following example is obtained as a particular case of Proposition 1.24 and represents a generalization of the one in [6, Proposition 3.19.2].

Example 3.21. Let $\mathbb{K}$ be an algebraically closed field and consider the polynomial ring $\mathrm{T} = \mathbb{K}[x_1, \ldots, x_n]$, with n greater or equal to 2. Take a nonmaximal, principal, proper ideal I of T such that there is a maximal ideal $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)\mathrm{T}$ satisfying

1. $\mathrm{I} \subseteq \mathfrak{m}$;

2. $\mathrm{I} \not\subseteq \mathfrak{p} := (x_1 - a_1, \ldots, x_{n-1} - a_{n-1})\mathrm{T}$,

where $a_1, \ldots, a_n \in \mathbb{K}$.
Consider the ring $\mathrm{R} = \mathbb{K} + \mathrm{I}$, then T is a depiction of R, by the Krull's height theorem.

Let us give now an extension of the [6, Example 3.18].

Example 3.22. Let $\mathbb{K}$ be an algebraically closed field, $\mathrm{S} := \mathbb{K}[x, y, z, w]$ and $\mathrm{R} := \mathbb{K}[x, xy, xy^2, \ldots, z, zw, zw^2, \ldots]$, then R is depicted by S by Proposition 1.21 because:

1. R and S share the common ideal $\mathrm{I} := xz\mathbb{K}[x, y, z, w]$; in order to show this fact, let us try that each generator of I is in R and it can be written as a product of two elements of R:

   (a) $xz = x \cdot z \in \mathrm{R}$;

   (b) $x^2z = x \cdot xz \in \mathrm{R}$;

   (c) $xyz = xy \cdot z \in \mathrm{R}$;

   (d) $xz^2 = xz \cdot z \in \mathrm{R}$;

   (e) $xzw = x \cdot zw \in \mathrm{R}$.

   So we can deduce that each element of I is a polynomial which can be written as algebraic sum of elements of R and thus that the ideal I of S is an ideal of R, too.

2. $\mathrm{R} \neq \mathrm{T} := \mathbb{K} + \mathrm{I}$ because $x \in \mathrm{R} \setminus \mathbb{K} + \mathrm{I}$.

3. I is a radical ideal of S because it is intersection of two prime ideals of S, $x\mathbb{K}[x, y, z, w]$ and $z\mathbb{K}[x, y, z, w]$.

4. T is nonNoetherian by Proposition 2.2 because $\dim \frac{\mathrm{S}}{\mathrm{I}} \neq 0$.

5. S is depiction of T by Proposition 1.21.

3.4 Depictions of subrings which are not fiber products

Replacing the hypothesis of almost depiction with (D1) and (D3) (which, as we know, are guaranteed from this structure just by Proposition 3.8), some results of the previous section clearly keep to hold in a more general situation like the following one:

Proposition 3.23. *Let* $\mathbb{K}$ *be an algebraically closed field,* S *be both a finitely generated* $\mathbb{K}$ *- algebra and an integral domain,* $\mathrm{R} \subseteq \mathrm{S}$ *and* S *is a depiction of* R*; if* H *is a finitely generated* $\mathbb{K}$ *- subalgebra of* S *containing as subring* R *such that:*

$$\begin{array}{rcl} \mathrm{i}_{\frac{\mathrm{H}}{\mathrm{R}}} : \mathrm{Spec}\,(\mathrm{H}) & \rightarrow & \mathrm{Spec}\,(\mathrm{R}) \\ \mathfrak{q} & \mapsto & \mathfrak{q} \cap \mathrm{R} \end{array}$$

is a surjective map or $\mathrm{R} \subseteq \mathrm{H}$ *is an integral ring extension and*

$$\mathrm{U}_{\frac{\mathrm{H}}{\mathrm{R}}} \neq \emptyset,$$

then H *is a depiction of* R.

Proof. (D1) and (D3) are verified by the hypotheses. (D2) is identically showed by the proof of Corollary 3.16, since even in that one such condition was independent of the other two. □

3.5 One open question about depictions

Let us recall now the example described in Section 3.2 of the present chapter. Take an algebraically closed field $\mathbb{K}$, consider the polynomial ring $\mathrm{T} := \mathbb{K}[x, y, z]$, and let $\mathrm{R} := \mathbb{K} + xy\mathrm{T}$, $\mathrm{S} := \mathrm{T}[x^{-1}]$. Then it is observed that S is a depiction of R and the unique common ideal of R and S is 0. In particular R cannot be presented in the form $\mathbb{K} + \mathrm{I}$, for every ideal I of S. Thus the following question naturally arises.

Question. Let $\mathbb{K}$ be an algebraically closed field, let S be a finitely generated $\mathbb{K}$-algebra which is also an integral domain and let R be a $\mathbb{K}$-subalgebra of S such that S is a depiction of R. Under what conditions on the ring extension $\mathrm{R} \subseteq \mathrm{S}$ does an ideal I of S such that $\mathrm{R} = \mathbb{K} + \mathrm{I}$ exist?

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