SUPERALGEBRAS WITH INVOLUTION OR SUPERINVOLUTION AND ALMOST POLYNOMIAL GROWTH OF THE CODIMENSIONS

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Abstract. Let A be a superalgebra with graded involution or superinvolution $*$ and let $c_n^*(A)$, $n = 1, 2, \ldots$, be its sequence of $*$-codimensions. In case A is finite dimensional, in [6, 15] it was proved that such a sequence is polynomially bounded if and only if the variety generated by A does not contain the group algebra of $\mathbb{Z}_2$ and a 4-dimensional subalgebra of the 4×4 upper-triangular matrices with suitable graded involutions or superinvolutions.

In this paper we study the general case of $*$-superalgebras satisfying a polynomial identity. As a consequence we classify the varieties of $*$-superalgebras of almost polynomial growth, i.e., varieties of exponential growth such that any proper subvariety has polynomial growth, and we give a full classification of their subvarieties which was started in [18].

1. Introduction

Let A be an associative algebra satisfying a polynomial identity over a field F of characteristic zero. An important invariant of the identities of A is given by the growth of the sequence of the codimensions $c_n(A)$. Such a sequence was introduced by Regev in [33] who proved that if A is a PI-algebra, i.e., it satisfies a non-trivial polynomial identity, then $c_n(A)$, $n = 1, 2, \ldots$, is exponentially bounded.

A celebrated theorem of Kemer (see [20]) characterizes the varieties of algebras of polynomial growth, i.e., with a polynomially bounded codimension sequence, as follows. If G is the infinite dimensional Grassmann algebra over F and UT_2 is the algebra of 2×2 upper-triangular matrices over F then a variety of algebras $\mathcal{V}$ has polynomial growth if and only if $G, UT_2 \notin \mathcal{V}$. Hence $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety grows polynomially.

The varieties of polynomial growth were extensively studied in later years [8, 10, 11, 22, 23, 24] also in the setting of varieties of graded algebras, algebras with involution, graded involution and superinvolution [6, 13, 14, 15, 21].

The purpose of this paper is to study a similar phenomenon in the setting of algebras with superinvolution or graded involution, which have been extensively studied recently [1, 3, 5, 7, 17, 18, 19, 32, 34].

In analogy with the ordinary case, one defines the sequence of $*$-codimensions of a $*$-algebra A, i.e., an algebra endowed with a graded involution or a superinvolution $*$. It turns out that if A satisfies an ordinary identity, then its sequence of $*$-codimensions is exponentially bounded (see [6, 15]).

Recently, much interest has been devoted to the study of varieties of $*$-algebras of polynomial growth. More precisely in [6, 15] it was proved that a finite dimensional $*$-algebra has polynomial growth of the $*$-codimensions if and only if the corresponding variety does not contain the following algebras: the group algebra of a group of order 2 and a 4-dimensional subalgebra of UT_4, both algebras with suitable graded involutions or superinvolutions. Such algebras are the only finite dimensional $*$-algebras, up to T_2^*-equivalence, generating varieties of almost polynomial growth. Recall that, given two $*$-algebras A and B, we say that A is T_2^*-equivalent to B and we write $A \sim_{T_2^*} B$ in case A and B satisfy the same $*$-identities.

In this paper we study the general case with no restriction on the generating algebra of the variety.

We find out that in case $*$ is a graded involution the list of algebras, up to T_2^*-equivalence, generating varieties of almost polynomial growth does not change.

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In the setting of algebras with superinvolution, we find out that there are two more algebras to add to the list of the algebras generating different varieties of almost polynomial growth: the infinite dimensional Grassmann algebra with natural grading and suitable superinvolutions.

Also we complete the classification of all subvarieties of the varieties of almost polynomial growth started in [18] and we describe the $*$-algebras whose $*$-codimensions are bounded by a linear function.

2. Preliminaries and basic results

Throughout this paper F will denote a field of characteristic zero and $A = A_0 \oplus A_1$ an associative superalgebra (also called $\mathbb{Z}_2$-graded algebra) over F satisfying a non-trivial polynomial identity (PI-algebra). Recall that the elements of A_0 and A_1 are called homogeneous of degree zero (or even elements) and of degree one (or odd elements), respectively.

Now assume that the superalgebra A is endowed with a graded involution, i.e., an involution preserving the grading or with a superinvolution that is a graded linear map $* : A \longrightarrow A$ such that $(a^*)^* = a$ for all $a \in A$ and $(ab)^* = (-1)^{(\deg a)(\deg b)} b^* a^*$, for any homogeneous elements $a, b \in A$. Here $\deg c$ denotes the homogeneous degree of $c \in A_0 \cup A_1$.

Notice that if $A = A_0 \oplus A_1$ is a superalgebra such that $A_1^2 = 0$ then the superinvolutions on A coincide with the graded involutions on A and, in particular, with the involutions on A, if $A_1 = 0$.

In what follows we shall denote by $*$ a graded involution or a superinvolution on A and we shall say that A is a $*$-algebra. In case $A_1^2 = 0$ we shall call $*$ a gs-involution (i.e., a graded involution and also a superinvolution). We say that A is endowed with the trivial gs-involution if $A_1 = 0$ and $*$ is the trivial involution.

Notice that if $A = A_0 \oplus A_1$ is a $*$-algebra, then A_0 is just an algebra with involution.

Since $\mathrm{char} F = 0$, we can write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where for $i = 0, 1$, $A_i^+ = \{a \in A_i \mid a^* = a\}$ and $A_i^- = \{a \in A_i \mid a^* = -a\}$ denote the sets of symmetric and skew elements of A_i, respectively.

As in the case of graded algebras or of algebras with involution, one can define a graded involution or a superinvolution on the free algebra $F\langle X\rangle$ in a natural way. We write the set X as the union of two disjoint infinite sets Y and Z, requiring that their elements are of homogeneous degree 0 and 1, respectively. Then each set is written as the disjoint union of two other infinite sets of symmetric and skew elements, respectively. The free algebra with graded involution or superinvolution is denoted $F\langle Y \cup Z, *\rangle$ and we write

$$F\langle Y \cup Z, *\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \ldots\rangle,$$

where y_i^+ stands for a symmetric variable of even degree, y_i^- for a skew variable of even degree, z_i^+ for a symmetric variable of odd degree and z_i^- for a skew variable of odd degree.

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle Y \cup Z, *\rangle \mid f \equiv 0 \text{ on } A\}$ the T_2^*-ideal of $*$-identities of A, i.e., $\mathrm{Id}^*(A)$ is an ideal of $F\langle Y \cup Z, *\rangle$ invariant under all graded endomorphisms of $F\langle Y \cup Z\rangle$ commuting with $*$.

We next state, in our language, the following results given in [13] for algebras with involution.

Lemma 2.1. [13, Lemma 2.4] *Let A be a $*$-algebra. If $(y^-)^d \in Id^*(A)$ for some $d \geq 1$, then there exists $t \geq 1$ such that $y_1^- \cdots y_t^- \in Id^*(A)$.*

As a consequence we get the even skew analogue of the Nagata-Higman theorem (see [4, Theorem 8.3.2]).

Theorem 2.1. [13, Theorem 2.5] *Let A be a $*$-algebra. If $(y^-)^d \in Id^*(A)$, then there exists $t \geq 1$ such that*

$$y_1^- w_1 y_2^- w_2 \cdots w_{t-1} y_t^- \in Id^*(A),$$

where $w_1, \ldots, w_{t-1}$ are (eventually empty) words in elements of Y.

As in the super case, it is easily seen that in characteristic zero, every $*$-identity is equivalent to a system of multilinear $*$-identities. Hence if we denote by

$$P_n^* = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i = y_i^+ \text{ or } w_i = y_i^- \text{ or } w_i = z_i^+ \text{ or } w_i = z_i^-,\ i = 1, \ldots, n\}$$

the space of multilinear polynomials of degree n in the variables $y_1^+, y_1^-, z_1^+, z_1^-, \ldots, y_n^+, y_n^-, z_n^+, z_n^-$, the study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^* \cap \mathrm{Id}^*(A)$, for all $n \geq 1$. The non-negative integer

$$c_n^*(A) = \dim_F \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)},\ n \geq 1,$$

is called the n-th $*$-codimension of A.

If A is a PI-algebra, then $c_n^*(A)$, $n = 1, 2, \dots$, is exponentially bounded (see [6], [15]).

Let $n \geq 1$ and write $n = n_1 + \cdots + n_4$ as a sum of non-negative integers. We denote by $P_{n_1,\dots,n_4} \subseteq P_n^*$ the vector space of multilinear $*$-polynomials in which the first n_1 variables are even symmetric, the next n_2 variables are even skew, the next n_3 variables are odd symmetric and the last n_4 variables are odd skew. The group $S_{n_1} \times \cdots \times S_{n_4}$ acts on the left on the vector space $P_{n_1,\dots,n_4}$ by permuting the variables of the same homogeneous degree which are all symmetric or all skew at the same time. Thus S_{n_1} permutes the variables $y_1^+, \dots, y_{n_1}^+$, S_{n_2} permutes the variables $y_{n_1+1}^-, \dots, y_{n_1+n_2}^-$, and so on. In this way $P_{n_1,\dots,n_4}$ becomes a module over the group algebra $F(S_{n_1} \times \cdots \times S_{n_4})$. Now $P_{n_1,\dots,n_4} \cap \mathrm{Id}^*(A)$ is invariant under this action and so the vector space

$$P_{n_1,\dots,n_4}(A) = \frac{P_{n_1,\dots,n_4}}{P_{n_1,\dots,n_4} \cap \mathrm{Id}^*(A)}$$

is an $(S_{n_1} \times \cdots \times S_{n_4})$-module with the induced action. It is immediate to see that

$$c_n^*(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1, \dots, n_4} \dim_F P_{n_1,\dots,n_4}(A), \tag{1}$$

where $\binom{n}{n_1, \dots, n_4} = \dfrac{n!}{n_1! \cdots n_4!}$ stands for the multinomial coefficient.

Given $\mathcal{V}$ a variety of $*$-algebras ($*$-variety) the growth of $\mathcal{V}$ is defined as the growth of the sequence of $*$-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^*(A)$ (in this case we write $\mathrm{Id}^*(\mathcal{V}) = \mathrm{Id}^*(A)$). Then we say that $\mathcal{V}$ has polynomial growth if $c_n^*(\mathcal{V})$ is polynomially bounded and we say that $\mathcal{V}$ has almost polynomial growth if $c_n^*(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

3. Finite dimensional $*$-algebras generating varieties of almost polynomial growth

In this section we shall describe some finite dimensional $*$-algebras generating varieties of almost polynomial growth and we shall recall the characterization of the varieties of $*$-algebras of polynomial growth given in [6, 15]. Given polynomials $f_1, \dots, f_n \in F\langle Y \cup Z, * \rangle$, we denote by $\langle f_1, \dots, f_n \rangle_{T_2^*}$ the T_2^*-ideal generated by $f_1, \dots, f_n$.

Let $F \oplus F$ be the two dimensional group algebra of $\mathbb{Z}_2$. We denote by D the algebra $F \oplus F$ with trivial grading and exchange gs-involution $*$ given by $(a, b)^* = (b, a)$, for all $(a, b) \in D$. Such an algebra generates a variety of almost polynomial growth and $\mathrm{Id}^*(D) = \langle [x_1, x_2], z^+, z^- \rangle_{T_2^*}$ (see [6, 12]).

Now we consider a non-trivial grading on $F \oplus F$. We denote by D^{sup} and $D^{sup,ex}$ the superalgebra $F \oplus F = F(1,1) \oplus F(1,-1)$ with trivial and exchange (graded) involution, respectively. The algebras D^{sup} and $D^{sup,ex}$ generate varieties of almost polynomial growth with $\mathrm{Id}^*(D^{sup}) = \langle [x_1, x_2], y^-, z^- \rangle_{T_2^*}$ and $\mathrm{Id}^*(D^{sup,ex}) = \langle [x_1, x_2], y^-, z^+ \rangle_{T_2^*}$ (see [12, 15]).

Let

$$M = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}) \oplus Fe_{12} \oplus Fe_{34},$$

be a subalgebra of UT_4, the algebra of 4×4 upper-triangular matrices, endowed with the reflection involution $*$, i.e., the involution obtained by reflecting a matrix along its secondary diagonal. Hence, if $a = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \gamma e_{12} + \delta e_{34}$ then

$$a^* = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \delta e_{12} + \gamma e_{34}.$$

If we regard M as endowed with trivial grading, then the above involution is a gs-involution. Such an algebra generates a variety of almost polynomial growth with T_2^*-ideal of identities $\mathrm{Id}^*(M) = \langle y_1^- y_2^-, z^+, z^- \rangle_{T_2^*}$ (see [6, 15, 31]).

Next we consider a non-trivial grading on M: we denote by M^{sup} the algebra M with grading $M_0 = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33})$ and $M_1 = Fe_{12} \oplus Fe_{34}$. Notice that the reflection involution on M^{sup} is a gs-involution, since $M_1^2 = 0$. The algebra M^{sup} generates a variety of almost polynomial growth with $\mathrm{Id}^*(M^{sup}) = \langle y^-, z_1 z_2 \rangle_{T_2^*}$ (see [6, 15]).

The above algebras characterize the varieties of $*$-algebras of polynomial growth.

Theorem 3.1. [6, Theorem 5.1] *Let A be a finite dimensional algebra with superinvolution. Then $var^*(A)$ has polynomial growth if and only if $D, M, M^{sup} \notin var^*(A)$.*

Theorem 3.2. [15, Theorem 8.6] *Let A be a finite dimensional algebra with graded involution. Then $var^*(A)$ has polynomial growth if and only if $D, D^{sup}, D^{sup,ex}, M, M^{sup} \notin var^*(A)$.*

4. Infinite dimensional algebras with superinvolution generating varieties of almost polynomial growth

In this section we shall introduce and study two infinite dimensional algebras with superinvolution generating varieties of almost polynomial growth.

Let $G = \langle 1, e_1, e_2, \dots \mid e_ie_j = -e_je_i\rangle$ be the infinite dimensional Grassmann algebra over F with its natural grading $G = G_0 \oplus G_1$. Here G_0 is the span of all monomials in the e_i's of even length and G_1 is the span of all monomials in the e_i's of odd length.

We endow $G = G_0 \oplus G_1$ with two superinvolutions as follows.

1) We let $G^\sharp$ be the algebra G with natural grading and superinvolution $\sharp$ induced by setting $e_i^\sharp = e_i$. Hence $(G^\sharp)_0^+ = G_0$, $(G^\sharp)_1^+ = G_1$, $(G^\sharp)_0^- = (G^\sharp)_1^- = 0$ and it is immediate to see that $\mathrm{Id}^*(G^\sharp) = \langle [y, x], z_1z_2 + z_2z_1, y^-, z^-\rangle_{T_2^*}$.
2) We denote by $G^\star$ be the algebra G with natural grading and superinvolution $\star$ induced by setting $e_i^\star = -e_i$. In this case $(G^\star)_0^+ = G_0$, $(G^\star)_1^- = G_1$, $(G^\star)_0^- = (G^\star)_1^+ = 0$ and $\mathrm{Id}^*(G^\star) = \langle [y, x], z_1z_2 + z_2z_1, y^-, z^-\rangle_{T_2^*}$.

In the next lemma we characterize the proper subvarieties of $\mathrm{var}^*(G^\sharp)$ and $\mathrm{var}^*(G^\star)$, respectively.

Lemma 4.1. *Let $\mathcal{U}$ be a subvariety of $var^*(G^\sharp)$ (resp. $var^*(G^\star)$). Then $\mathcal{U}$ is a proper subvariety if and only if there exists $p \geq 1$ such that $z_1^+ \cdots z_p^+ \in Id^*(\mathcal{U})$ (resp. $z_1^- \cdots z_p^- \in Id^*(\mathcal{U})$).*

Proof. Since $z_1^+ \cdots z_p^+ \notin \mathrm{Id}^*(G^\sharp)$ for all $p \geq 1$, one direction is obvious. Now assume that $\mathcal{U}$ is a proper subvariety. Then there exists a multilinear polynomial f such that $f \in \mathrm{Id}^*(\mathcal{U})$ and $f \notin \mathrm{Id}^*(G^\sharp)$. Hence f must be of the type $f = f(y_1^+, \dots, y_r^+, z_1^+, \dots, z_{n-r}^+)$. Since $[y, x], z_1z_2 + z_2z_1 \in \mathrm{Id}^*(\mathcal{U})$, we get that $f \pmod{\mathrm{Id}^*(\mathcal{U})}$ is a monomial of the type

$$\alpha y_1^+ \cdots y_r^+ z_1^+ \cdots z_{n-r}^+.$$

With the substitution $y_i^+ = [z_{n-r+2i-1}^+, z_{n-r+2i}^+]$, $i = 1, \dots, r$, we get the desired conclusion since $[z_i^+, z_j^+] = 2z_i^+z_j^+$.

In a similar way we prove the other case. ☐

The following remark can be proved as in [14, Remark 1].

Remark 4.1. *If $g(z_1^+, \dots, z_p^+) \in Id^*(G^\sharp)$ (resp. $g(z_1^-, \dots, z_p^-) \in Id^*(G^\star)$) is a multilinear polynomial of degree $p \geq 1$ then, in the free algebra with superinvolution $F\langle Y \cup Z, *\rangle$, we have that*

$$\sum_{\sigma\in S_p} (sgn\sigma) g(z_{\sigma(1)}^+, \dots, z_{\sigma(p)}^+) = 0 \ (resp. \ \sum_{\sigma\in S_p} (sgn\sigma) g(z_{\sigma(1)}^-, \dots, z_{\sigma(p)}^-) = 0).$$

Now we are in a position to characterize the varieties not containing $G^\sharp$ (resp. $G^\star$) in terms of $*$-identities and we can prove that $\mathrm{var}^*(G^\sharp)$ and $\mathrm{var}^*(G^\star)$ have almost polynomial growth.

Recall that $St_r(x_1, \dots, x_r) = \sum_{\sigma\in S_r} (\mathrm{sgn}\sigma) x_{\sigma(1)} \cdots x_{\sigma(r)}$ is the standard polynomial of degree r.

Theorem 4.1. *Let $\mathcal{V}$ be a variety of algebras with superinvolution. Then $G^\sharp \notin \mathcal{V}$ if and only if $St_p(z_1^+, \dots, z_p^+) \in Id^*(\mathcal{V})$, for some $p \geq 1$.*

Proof. Let $St_p(z_1^+, \dots, z_p^+) \in \mathrm{Id}^*(\mathcal{V})$. Since $St_p(z_1^+, \dots, z_p^+) \notin \mathrm{Id}^*(G^\sharp)$, then $G^\sharp \notin \mathcal{V}$ and we are done.

Suppose now that $G^\sharp \notin \mathcal{V}$. Then $\mathcal{V} \cap \mathrm{var}^*(G^\sharp) \subsetneq \mathrm{var}^*(G^\sharp)$ and by Lemma 4.1, there exists $p \geq 1$ such that

$$z_1^+ \cdots z_p^+ \in \mathrm{Id}^*(\mathcal{V} \cap \mathrm{var}^*(G^\sharp)) = \mathrm{Id}^*(\mathcal{V}) + \mathrm{Id}^*(G^\sharp).$$

It follows that there exists $g \in \mathrm{Id}^*(G^\sharp)$ such that $z_1^+ \cdots z_p^+ + g \in \mathrm{Id}^*(\mathcal{V})$. Moreover, by the multihomogeneity of T_2^*-ideals, we may assume that $g = g(z_1^+, \ldots, z_p^+)$. Now by alternating $z_1^+ \cdots z_p^+ + g$ with respect to the variables $z_1^+, \ldots, z_p^+$ and by applying Remark 4.1, we get

$$\sum_{\sigma \in S_p} (\mathrm{sgn}\sigma) z_{\sigma(1)}^+ \cdots z_{\sigma(p)}^+ \in \mathrm{Id}^*(\mathcal{V}).$$

□

The following theorem is proved similarly.

Theorem 4.2. *Let $\mathcal{V}$ be a variety of algebras with superinvolution. Then $G^\star \notin \mathcal{V}$ if and only if $St_q(z_1^-, \ldots, z_q^-) \in Id^*(\mathcal{V})$, for some $q \geq 1$.*

Theorem 4.3. *The algebras $G^\sharp$ and $G^\star$ generate varieties of almost polynomial growth.*

Proof. We prove the result for $G^\sharp$. The proof concerning $G^\star$ is similar.

Let $n_1 + \cdots + n_4 = n$. Since $G^\sharp$ is a PI-algebra, we already know that $c_n^*(G^\sharp)$ is exponentially bounded. Since $\dim P_{n_1,\ldots,n_4}(G^\sharp) = 0$ if $n_2 \neq 0$ or $n_4 \neq 0$ and $\dim P_{n_1,\ldots,n_4}(G^\sharp) = 1$ in all other cases, we get

$$c_n^*(G^\sharp) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1, \ldots, n_4} \dim P_{n_1,\ldots,n_4}(G^\sharp) = \sum_{n_1+n_3=n} \binom{n}{n_1, n_3} \dim P_{n_1,0,n_3,0}(G^\sharp) = \sum_{n_1=0}^{n} \binom{n}{n_1} = 2^n.$$

Thus $\mathrm{var}^*(G^\sharp)$ has exponential growth and we are left to prove that any proper subvariety of $\mathrm{var}^*(G^\sharp)$ has polynomial growth. Let $\mathcal{U}$ be a proper subvariety of $\mathrm{var}^*(G^\sharp)$. By Lemma 4.1, we have that $z_1^+ \cdots z_p^+ \in \mathrm{Id}^*(\mathcal{U})$ for some $p \geq 1$ and, since $[y^+, z^+] \in \mathrm{Id}^*(\mathcal{U})$, we get that $P_{n_1,0,n_3,0} \subseteq \mathrm{Id}^*(\mathcal{U})$, as soon as $n_3 \geq p$. Moreover, since $y^-, z^- \in \mathrm{Id}^*(\mathcal{U})$, it follows that $P_{n_1,\ldots,n_4} \subseteq \mathrm{Id}^*(\mathcal{U})$ if $n_2 \neq 0$ or $n_4 \neq 0$. Then we have

$$c_n^*(\mathcal{U}) = \sum_{n_1+n_3=n} \binom{n}{n_1, n_3} \dim P_{n_1,0,n_3,0}(\mathcal{U}) \leq \sum_{n-n_1<p} \binom{n}{n_1} \leq \alpha n^p.$$

□

5. Varieties of polynomial growth

In this section we shall characterize the varieties of algebras with graded involution or superinvolution of polynomial growth. We start with the following lemma concerning D.

Lemma 5.1. *Let A be a $*$-algebra. Then $D \notin var^*(A)$ if and only if $(y^-)^d \in Id^*(A)$, $d \geq 1$.*

Proof. Since $(y^-)^d \notin \mathrm{Id}^*(D)$, one implication is obvious. Suppose now that $D \notin \mathrm{var}^*(A)$. Then $\mathrm{Id}^*(A) \nsubseteq \mathrm{Id}^*(D)$ and let $f \in \mathrm{Id}^*(A)$ be a multilinear polynomial such that $f \notin \mathrm{Id}^*(D)$. Since $z^+, z^- \in \mathrm{Id}^*(D)$, f is a polynomial of the type

$$f = f(y_1^+, \ldots, y_r^+, y_1^-, \ldots, y_{n-r}^-)$$

and it does not vanish on a basis of D. Since $\{a = (1, 1)\}$ and $\{b = (1, -1)\}$ are bases of D_0^+ and D_0^-, respectively, we get

$$0 \neq f(a, \ldots, a, b, \ldots, b) = f(b^2, \ldots, b^2, b, \ldots, b) = \alpha b^{n+r},$$

where $\alpha \neq 0$, is the sum of all the coefficients of f. But $(y^-)^2$ is an even symmetric variable and so it follows that $f((y^-)^2, \ldots, (y^-)^2, y^-, \ldots, y^-) = \alpha\,(y^-)^{n+r} \in \mathrm{Id}^*(A)$. Since $\alpha \neq 0$, we get $(y^-)^{n+r} \in \mathrm{Id}^*(A)$. □

Lemma 5.2. *If $D \notin var^*(A)$, then there exists $t \geq 1$ such that $[y_1, y_2] \cdots [y_{2t-1}, y_{2t}] \in Id^*(A)$.*

Proof. Since $D \notin \mathrm{var}^*(A)$, by Lemma 5.1, $(y^-)^d \in \mathrm{Id}^*(A)$ for some $d \geq 1$. Hence, by Theorem 2.1, there exists $t \geq 1$ such that any monomial in symmetric and skew variables of homogeneous degree 0 containing at least t even skew variables must lie in $\mathrm{Id}^*(A)$.

In order to get $[y_1, y_2] \cdots [y_{2t-1}, y_{2t}] \in \mathrm{Id}^*(A)$, it is enough to prove that $[w_1, w_2] \cdots [w_{2t-1}, w_{2t}] \equiv 0$, where the w_i's are either symmetric or skew variables of homogeneous degree zero. But each commutator $[w_{2i-1}, w_{2i}]$ either evaluates to an even skew element (if both w_{2i-1} and w_{2i} are even symmetric variables) or contains at least one even skew variable. In any case $g=[w_1,w_2]\cdots[w_{2t-1},w_{2t}]$ evaluates to a linear combination of monomials each containing at least t skew elements of degree zero and the proof is complete. $\square$

Next we shall prove that a variety not containing D satisfies a Capelli identity in even variables. Recall that, if $x_1,\dots x_m,x'_1,\dots x'_{m+1}$ are variables in X, then the Capelli polynomial of rank m is

$$Cap_m\left(x_1,\dots,x_m;x'_1,\dots,x'_{m+1}\right)=\sum_{\sigma\in S_m}(\mathrm{sgn}\sigma)x'_1x_{\sigma(1)}x'_2x_{\sigma(2)}\cdots x'_mx_{\sigma(m)}x'_{m+1}.$$

We say that an algebra A satisfies the Capelli identity of rank m if it satisfies all polynomials obtained from $Cap_m\left(x_1,\dots,x_m;x'_1,\dots,x'_{m+1}\right)$ by eventually setting the variables x'_i equal to 1 in all possible ways.

Proposition 5.1. *Let $A=A_0\oplus A_1$ be a $*$-algebra. If $D\notin var^*(A)$ then A_0 satisfies a Capelli identity.*

Proof. Since $D\notin \mathrm{var}^*(A)$, by Lemma 5.2, there exists $t\geq 1$ such that $[y_1,y_2]\cdots[y_{2t-1},y_{2t}]\in \mathrm{Id}^*(A)$. Since

$$St_{2t}\left(y_1,\dots,y_{2t}\right)=\frac{1}{2t}\sum_{\sigma\in S_{2t}}(\mathrm{sgn}\sigma)[y_{\sigma(1)},y_{\sigma(2)}]\cdots[y_{\sigma(2t-1)},y_{\sigma(2t)}]$$

we get $St_{2t}\left(y_1,\dots,y_{2t}\right)\in \mathrm{Id}^*(A)$ and the proof follows by [16, Theorem 7.1.4]. $\square$

Our next goal is to find conditions which ensure that the standard polynomial in odd variables is an identity for a $*$-algebra A.

Remark 5.1. *Let A be a $*$-algebra. If $St_n\left(w_1,\dots,w_n\right)\in Id^*(A)$ for some $n\geq 1$, then $St_{n+1}\left(w_1,\dots,w_n,w_{n+1}\right)\in Id^*(A)$, for all $w_1,\dots,w_{n+1}\in Y\cup Z$.*

Proof. Let n be even. Since $St_n\left(w_1,\dots,w_n\right)\in \mathrm{Id}^*(A)$ then $f=St_n\left(w_1,\dots,w_n\right)w_{n+1}+w_{n+1}St_n\left(w_1,\dots,w_n\right)\in \mathrm{Id}^*(A)$. If we now alternate f with respect to the variables $w_1,\dots,w_{n+1}$ we get that $2n!\,St_{n+1}\left(w_1,\dots,w_n,w_{n+1}\right)\in \mathrm{Id}^*(A)$ and we are done in this case.

If n is odd the proof is similar by considering $f=[St_n\left(w_1,\dots,w_n\right),w_{n+1}]$. $\square$

Now we are in a position to prove the following lemma.

Lemma 5.3. *Let A be a $*$-algebra. If $St_p\left(z_1^+,\dots,z_p^+\right)$ and $St_q\left(z_1^-,\dots,z_q^-\right)\in Id^*(A)$ for some $p,q\geq 1$, then*

$$St_{p+q}\left(z_1,\dots,z_{p+q}\right)\in Id^*(A).$$

Proof. Notice that

$$St_{p+q}\left(z_1,\dots,z_{p+q}\right)=\sum_{a_i\in\{+,-\}}St_{p+q}\left(z_1^{a_1},\dots,z_{p+q}^{a_{p+q}}\right).$$

Since $St_p\left(z_1^+,\dots,z_p^+\right)\in \mathrm{Id}^*(A)$, by Remark 5.1 we get that $St_{p+q}\left(z_1^+,\dots,z_r^+,z_1^-,\dots,z_{p+q-r}^-\right)\in \mathrm{Id}^*(A)$, for all $r=p,\dots,p+q$. Similarly, $St_q\left(z_1^-,\dots,z_q^-\right)\in \mathrm{Id}^*(A)$ implies $St_{p+q}\left(z_1^+,\dots,z_{p+q-s}^+,z_1^-,\dots,z_s^-\right)\in \mathrm{Id}^*(A)$, for all $s=q,\dots,p+q$. In this way $St_{p+q}\left(z_1^{a_1},\dots,z_{p+q}^{a_{p+q}}\right)\in \mathrm{Id}^*(A)$ for all $a_i\in\{+,-\}$, $i=1,\dots,p+q$ and the proof is complete. $\square$

In the following two propositions we find conditions ensuring that the standard polynomial in odd variables is an identity for a $*$-algebra A.

Proposition 5.2. *Let $A=A_0\oplus A_1$ be an algebra with superinvolution. If $G^\sharp,G^\star\notin var^*(A)$ then A_1 satisfies a standard identity.*

Proof. Since $G^\sharp,G^\star\notin \mathrm{var}^*(A)$, by Theorems 4.1 and 4.2 we obtain that $St_p\left(z_1^+,\dots,z_p^+\right)$ and $St_q\left(z_1^-,\dots,z_q^-\right)$ are identities of A for some $p,q\geq 1$. Hence, by Lemma 5.3, A_1 satisfies a standard identity. $\square$

Proposition 5.3. *Let $A=A_0\oplus A_1$ be an algebra with graded involution. If $D\notin var^*(A)$ then A_1 satisfies a standard identity.*

Proof. Since $D \notin \mathrm{var}^*(A)$, by Lemma 5.1, we have that $(y^-)^d \in \mathrm{Id}^*(A)$. Then by Lemma 2.1, there exists $t \geq 1$ such that $y_1^- \cdots y_t^- \in \mathrm{Id}^*(A)$. Since $[z_1^+, z_2^+]$ and $[z_1^-, z_2^-]$ are even skew variables, we get

$$[z_1^+, z_2^+] \cdots [z_{2t-1}^+, z_{2t}^+], [z_1^-, z_2^-] \cdots [z_{2t-1}^-, z_{2t}^-] \in \mathrm{Id}^*(A).$$

Hence we obtain that $St_{2t}\left(z_1^+, \ldots, z_{2t}^+\right)$ and $St_{2t}\left(z_1^-, \ldots, z_{2t}^-\right)$ are identities of A and so the proof follows by Lemma 5.3. □

Now we are ready to prove the following.

Theorem 5.1. *Let $\mathcal{V}$ be a $*$-variety.*

- *If $*$ is a graded involution and $D \notin \mathcal{V}$ then A satisfies a Capelli identity.*
- *If $*$ is a superinvolution and $D, G^\sharp, G^\star \notin \mathcal{V}$ then A satisfies a Capelli identity.*

Proof. Let $A = A_0 \oplus A_1$ be a generating $*$-algebra of $\mathcal{V}$. First we assume that $*$ is a graded involution. Since $D \notin \mathrm{var}^*(A)$, by Propositions 5.1 and 5.3 we have that A_0 satisfies a Capelli identity and A_1 satisfies a standard identity. In case $*$ is a superinvolution, by Propositions 5.1 and 5.2 we get that A_0 satisfies a Capelli identity and A_1 satisfies a standard identity.

The conclusion now follows by applying [16, Lemma 11.4.1]. □

The following theorem follows by the proof of [16, Theorem 11.4.3].

Theorem 5.2. *Let $\mathcal{V}$ be a $*$-variety. If $\mathcal{V}$ satisfies a Capelli identity of some rank, then $\mathcal{V} = var^*(B)$, for some finitely generated $*$-algebra B.*

By putting together Theorems 5.1 and 5.2 we get the following corollary.

Corollary 5.1. *Let $\mathcal{V}$ be a $*$-variety. Then $\mathcal{V} = var^*(B)$, for some finitely generated $*$-algebra B, if*

1. *$*$ is a graded involution and $D \notin \mathcal{V}$,*
2. *$*$ is a superinvolution and $D, G^\sharp, G^\star \notin \mathcal{V}$.*

In order to characterize the $*$-varieties of polynomial growth we need to apply the following result, proved in [1] in the setting of algebras with superinvolution. Here we remark that a similar proof holds also in the case of algebras with graded involution.

Theorem 5.3. [1]. *Let $\mathcal{V}$ be a $*$-variety generated by a finitely generated $*$-algebra B over an algebraically closed field F of characteristic zero. Then $\mathcal{V} = var^*(C)$, for some finite dimensional $*$-algebra C over F.*

From now on, we assume that F is an algebraically closed field of characteristic zero.

The following theorems characterize the varieties of $*$-algebras of polynomial growth.

Theorem 5.4. *Let $\mathcal{V}$ be a variety of algebras with superinvolution. Then $\mathcal{V}$ has polynomial growth if and only if $M, M^{sup}, D, G^\sharp, G^\star \notin \mathcal{V}$.*

Proof. Since $M, M^{sup}, D, G^\sharp, G^\star$ generate varieties of exponential growth one direction is obvious.

On the other hand, since $D, G^\sharp, G^\star \notin \mathcal{V}$, by Corollary 5.1 and Theorem 5.3, we get that $\mathcal{V} = \mathrm{var}^*(C)$, for some finite dimensional $*$-algebra C. Finally the result follows by Theorem 3.1. □

Theorem 5.5. *Let $\mathcal{V}$ be a variety of algebras with graded involution. Then $\mathcal{V}$ has polynomial growth if and only if $M, M^{sup}, D, D^{sup}, D^{sup,ex} \notin \mathcal{V}$.*

Proof. Since $M, M^{sup}, D, D^{sup}, D^{sup,ex}$ generate varieties of exponential growth one direction is obvious.

On the other hand, since $D \notin \mathcal{V}$, by Corollary 5.1 and Theorem 5.3, we get that $\mathcal{V} = \mathrm{var}^*(C)$, for some finite dimensional $*$-algebra C and the result follows by Theorem 3.2. □

As an immediate consequence we obtain the following corollaries.

Corollary 5.2. *There is no $*$-variety of intermediate growth between polynomial and exponential.*

Corollary 5.3. *The varieties of algebras with superinvolution $var^*(D)$, $var^*(M)$, $var^*(M^{sup})$, $var^*(G^\sharp)$ and $var^*(G^\star)$ are the only ones of almost polynomial growth.*

Corollary 5.4. *The varieties of algebras with graded involution* $var^*(D)$, $var^*(D^{sup})$, $var^*(D^{sup,ex})$, $var^*(M)$ *and* $var^*(M^{sup})$ *are the only ones of almost polynomial growth.*

As a consequence of Theorem 5.3 and [18] it is possible to get a classification, up to T_2^*-equivalence, of the $*$-algebras generating varieties of at most linear growth. Such a classification for $*$-algebras with trivial grading was given in [30].

Theorem 5.6. *Let A be an algebra with graded involution or superinvolution $*$ such that $c_n^*(A) \leq an$, for some constant a. Then*

$$A \sim_{T_2^*} B_1 \oplus \cdots \oplus B_m \oplus N,$$

where $B_i \in var^*(M)$ or $B_i \in var^*(M^{sup})$, for all $i = 1, \ldots, m$ and N is a nilpotent $*$-algebra.

Proof. Let $*$ be a superinvolution (resp. graded involution). Since $c_n^*(A)$ is polynomially bounded, by Theorem 5.4 (resp. Theorem 5.5), we get that $M, M^{sup}, D, G^\sharp, G^\star \notin \mathrm{var}^*(A)$ (resp. $M, M^{sup}, D, D^{sup}, D^{sup,ex} \notin \mathrm{var}^*(A)$). Hence, by Corollary 5.1 and Theorem 5.3, we may assume that A is a finite dimensional $*$-algebra and the result follows by [18, Theorem 7.1]. ☐

A finer classification, up to T_2^*-equivalence, of the $*$-algebras of at most linear growth is given in [18].

6. CLASSIFYING THE SUBVARIETIES OF VAR$^*(G^\sharp)$, VAR$^*(G^\star)$ AND VAR$^*(D^{sup,ex})$

In this section we complete the classification of the subvarieties of the $*$-varieties of almost polynomial growth started in [18]. First we recall some basic results.

By the Wedderburn-Malcev theorem for $*$-algebras (see [6], [15]), if B is a finite dimensional $*$-algebra over an algebraically closed field, we can write $B = B' + J$, where B' is a semisimple $*$-subalgebra of B and $J = J(B)$ is its Jacobson radical. Moreover $B' = B_1 \oplus \cdots \oplus B_k$, where $B_1, \ldots, B_k$ are simple $*$-algebras and J can be decomposed into the direct sum of graded B'-bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11},$$

where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively and for $i, k, l, m \in \{0, 1\}$, $J_{ik}J_{lm} \subseteq \delta_{kl}J_{im}$, where δ_{kl} is the Kronecker delta.

Theorem 6.1. [18, Theorem 2.4] *Let A be a finite dimensional $*$-algebra over a field F of characteristic zero. Then $c_n^*(A)$, $n = 1, 2, \ldots,$ is polynomially bounded if and only if $A \sim_{T_2^*} B$, where $B = B_1 \oplus \cdots \oplus B_m$ with $B_1, \ldots, B_m$ finite dimensional $*$-algebras over F and* $\dim B_i/J(B_i) \leq 1$, *for all $i = 1, \ldots, m$.*

In order to study $*$-identities of algebras A with 1 we define the proper $*$-polynomials. We say that a polynomial $f \in P_n^*$ is a proper $*$-polynomial if it is a linear combination of elements of the type

$$y_{i_1}^- \cdots y_{i_s}^- z_{j_1}^+ \cdots z_{j_t}^+ z_{l_1}^- \cdots z_{l_r}^- w_1 \cdots w_m,$$

where $w_1, \ldots, w_m$ are left normed (long) Lie commutators in the variables of $Y \cup Z$ (here the symmetric even variables appear only inside the commutators). We denote by Γ_n^* the subspace of P_n^* of proper $*$-polynomials and $\Gamma_0^* = \mathrm{span}\{1\}$. The sequence of proper $*$-codimensions is defined as

$$\gamma_n^*(A) = \dim \frac{\Gamma_n^*}{\Gamma_n^* \cap \mathrm{Id}^*(A)}, \; n = 0, 1, 2, \ldots.$$

If for some $k \geq 2$, $\gamma_k^*(A) = 0$ then $\gamma_m^*(A) = 0$ for all $m \geq k$ (see [18]).

For an unitary $*$-algebra A, the relation between $*$-codimensions and proper $*$-codimensions (see [18]), is given by the following:

$$c_n^*(A) = \sum_{i=0}^{n} \binom{n}{i} \gamma_i^*(A), \; n = 0, 1, 2 \ldots. \tag{2}$$

Let now focus our attention on the subvarieties of $\mathrm{var}^*(G^\sharp)$. For $k \geq 1$, let $G_k^\sharp$ denote the Grassmann algebra with 1 on a k-dimensional vector space over F, i.e., $G_k^\sharp = \langle 1, e_1, \ldots, e_k \mid e_ie_j = -e_je_i \rangle$, with superinvolution induced by $G^\sharp$. Next we describe explicitly the identities of $G_k^\sharp$, for any $k \geq 1$.

Theorem 6.2. *Let* $k \geq 1$. *Then*

1) $Id^*(G_k^\sharp) = \langle [y,x], z_1z_2 + z_2z_1, z_1 \cdots z_{k+1}, y^-, z^- \rangle_{T_2^*}$.

2) $c_n^*(G_k^\sharp) = \sum_{j=0}^{k} \binom{n}{j} \approx \frac{1}{k!} n^k$.

Proof. Let $I = \langle [y,x], z_1z_2 + z_2z_1, z_1 \cdots z_{k+1}, y^-, z^- \rangle_{T_2^*}$. It is easily checked that $I \subseteq \mathrm{Id}^*(G_k^\sharp)$. In order to prove the opposite inclusion, let f be a $*$-identity of $G_k^\sharp$ of degree t. We may assume that f is multilinear and, since $G_k^\sharp$ is an algebra with 1, we may take f proper. After reducing the polynomial f modulo I we obtain that f is the zero polynomial if $t \geq k+1$ and $f = \alpha z_1^+ \cdots z_t^+$ if $t < k+1$. If $\alpha \neq 0$, evaluating $z_i^+ = e_i$, $i = 1, \ldots, t$, we get $f = \alpha e_1 \cdots e_t \neq 0$, a contradiction. Thus we get that $\mathrm{Id}^*(G^\sharp) = I$.

The argument above also proves that $\gamma_t^*(G_k^\sharp) = 1$ for $t < k+1$ and $\gamma_t^*(G_k^\sharp) = 0$ otherwise and 2) follows by (2). □

A variety of $*$-algebras $\mathcal{V}$ is minimal of polynomial growth if $c_n^*(\mathcal{V}) \approx qn^k$ for some $k \geq 1$, $q > 0$, and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$ we have that $c_n^*(\mathcal{U}) \approx q' n^t$ with $t < k$.

Theorem 6.3. *For any* $k \geq 1$, $G_k^\sharp$ *generates a minimal variety of polynomial growth.*

Proof. Let $A \in \mathrm{var}^*(G_k^\sharp)$ and suppose that $c_n^*(A) \approx qn^k$, for some $q > 0$. We shall prove that $A \sim_{T_2^*} G_k^\sharp$. Since $c_n^*(A)$ is polynomially bounded, by Theorem 5.4 we have that $M, M^{sup}, D, G^\sharp, G^\star \notin \mathrm{var}^*(A)$. Hence, by Corollary 5.1 and Theorem 5.3, we get that A satisfies the same $*$-identities as a finite dimensional algebra. Thus, by Theorem 6.1, we may assume that

$$A = B_1 \oplus \cdots \oplus B_m,$$

where $B_1, \ldots, B_m$ are finite dimensional $*$-algebras such that $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \ldots, m$. This implies that either $B_i \cong F + J(B_i)$ or $B_i = J(B_i)$ is a nilpotent $*$-algebra. Since $c_n^*(A) \leq c_n^*(B_1) + \cdots + c_n^*(B_m)$, then there exists B_i such that $c_n^*(B_i) \approx bn^k$, for some $b > 0$. Hence

$$\mathrm{var}^*(G_k^\sharp) \supseteq \mathrm{var}^*(A) \supseteq \mathrm{var}^*(F + J(B_i)) \supseteq \mathrm{var}^*(F + J_{11}(B_i)).$$

In order to complete the proof it is enough to show that $F + J_{11}(B_i) \sim_{T_2^*} G_k^\sharp$ and so, without loss of generality, we may assume that A is an unitary $*$-algebra. Hence

$$c_n^*(A) = \sum_{i=0}^{k} \binom{n}{i} \gamma_i^*(A),$$

and $\gamma_i^*(A) \neq 0$ for all $i = 2, \ldots, k$. Now, since $A \in \mathrm{var}^*(G_k^\sharp)$, we have that $\gamma_i^*(A) \leq \gamma_i^*(G_k^\sharp) = 1$. It follows that $c_n^*(A) = c_n^*(G_k^\sharp)$ for all n and so $A \sim_{T_2^*} G_k^\sharp$. □

Now we are in a position to classify all the subvarieties of $\mathrm{var}^*(G^\sharp)$.

Theorem 6.4. *Let* $A \in var^*(G^\sharp)$. *Then either* $A \sim_{T_2^*} G^\sharp$ *or* $A \sim_{T_2^*} N$ *or* $A \sim_{T_2^*} C \oplus N$ *or* $A \sim_{T_2^*} G_k^\sharp \oplus N$, *for some* $k \geq 1$, *where* N *is a nilpotent algebra with superinvolution and* C *is a commutative algebra with trivial superinvolution.*

Proof. If $A \sim_{T_2^*} G^\sharp$ there is nothing to prove. Let now A generate a proper subvariety of $\mathrm{var}^*(G^\sharp)$. Since $\mathrm{var}^*(G^\sharp)$ has almost polynomial growth, $\mathrm{var}^*(A)$ has polynomial growth and let $c_n^*(A) \approx qn^r$ for some $r \geq 0$. If $r = 0$ then either $A \sim_{T_2^*} C \oplus N$ or $A \sim_{T_2^*} N$. Let now $r > 0$. As before we may assume that $A = B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are finite dimensional $*$-algebras such that either B_i is a nilpotent $*$-algebra or $B_i \cong (F + J_{11}) \oplus J_{00}$, since $[y,x]$ is an identity of A (see [26, Lemma 5.1]). Hence

$$A = B_1 \oplus \cdots \oplus B_m = B \oplus N,$$

where B is an unitary $*$-algebra, N is a nilpotent $*$-algebra and, for n large enough,

$$c_n^*(A) = c_n^*(B) = \sum_{i=0}^{r} \binom{n}{i} \gamma_i^*(B).$$

In particular we get that $\Gamma^*_{r+1} \subseteq \mathrm{Id}^*(B)$. This implies that $B \in \mathrm{var}^*(G^\sharp_r)$ and, since $G^\sharp_r$ generates a minimal variety and $c^*_n(G^\sharp_r) \approx q'n^r$, we obtain that $B \sim_{T^*_2} G^\sharp_r$, and so $A \sim_{T^*_2} G^\sharp_r \oplus N$. □

As a consequence we get the following.

Corollary 6.1. *An algebra with superinvolution $A \in var^*(G^\sharp)$ generates a minimal variety of polynomial growth if and only if $A \sim_{T^*_2} G^\sharp_k$, for some $k \geq 1$.*

Next we shall classify the subvarieties of $\mathrm{var}^*(G^\star)$. For $k \geq 1$, let $G^\star_k$ denote the Grassmann algebra with 1 on a k-dimensional vector space over F with superinvolution induced by $G^\star$. The proof of the following results can be obtained as the previous ones.

Theorem 6.5. *Let $k \geq 1$. Then*

1) $Id^*(G^\star_k) = \langle [y, x], z_1z_2 + z_2z_1, z_1 \cdots z_{k+1}, y^-, z^+\rangle_{T^*_2}$.
2) $c^*_n(G^\star_k) = \sum_{j=0}^{k} \binom{n}{j} \approx \frac{1}{k!}n^k$.

Theorem 6.6. *Let $A \in var^*(G^\star)$. Then either $A \sim_{T^*_2} G^\star$ or $A \sim_{T^*_2} N$ or $A \sim_{T^*_2} C \oplus N$ or $A \sim_{T^*_2} G^\star_k \oplus N$, for some $k \geq 1$, where N is a nilpotent algebra with superinvolution and C is a commutative algebra with trivial superinvolution.*

As a consequence we get the following.

Corollary 6.2. *An algebra with superinvolution $A \in var^*(G^\star)$ generates a minimal variety of polynomial growth if and only if $A \sim_{T^*_2} G^\star_k$, for some $k \geq 1$.*

Next we classify, up to T^*_2-equivalence, all the algebras with graded involution contained in the variety generated by $D^{sup,ex}$, the algebra $F \oplus F$ with grading $(F(1,1), F(1,-1))$ and exchange (graded) involution.

For $k \geq 2$, let I_k be the $k \times k$ identity matrix and $E_1 = \sum_{i=1}^{k-1} e_{i,i+1}$, where the e_{ij}'s denote the usual matrix units. We denote by $C^{sup,ex}_k$ the commutative subalgebra of UT_k

$$C^{sup,ex}_k = \{\alpha I_k + \sum_{1 \leq i < k} \alpha_i E^i_1 \mid \alpha, \alpha_i \in F\} \subseteq UT_k,$$

with elementary grading (see [2, 26]) induced by $g = (0, 1, 0, 1, \dots) \in \mathbb{Z}^k_2$ and (graded) involution given by

$$(\alpha I_k + \sum_{1 \leq i < k} \alpha_i E^i_1)^* = \alpha I_k + \sum_{1 \leq i < k} (-1)^i \alpha_i E^i_1.$$

The following results can be obtained easily from [27, 28, 29].

Theorem 6.7. *Let $k \geq 2$. Then*

1) $Id^*(C^{sup,ex}_k) = \langle [x_1, x_2], z^-_1 \cdots z^-_k, y^-, z^+\rangle_{T^*_2}$.
2) $c^*_n(C^{sup,ex}_k) = \sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!}n^{k-1}$.

Theorem 6.8. *Let $A \in var^*(D^{sup,ex})$. Then either $A \sim_{T^*_2} D^{sup,ex}$ or $A \sim_{T^*_2} N$ or $A \sim_{T^*_2} C \oplus N$ or $A \sim_{T^*_2} C^{sup,ex}_k \oplus N$, for some $k \geq 2$, where N is a nilpotent algebra with graded involution and C is a commutative algebra with trivial involution.*

Corollary 6.3. *An algebra with graded involution $A \in var^*(D^{sup,ex})$ generates a minimal variety of polynomial growth if and only if $A \sim_{T^*_2} C^{sup,ex}_k$, for some $k \geq 2$.*

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