



Quantum simulations of complex systems

Oliver Morsch^{1,2} · G. Massimo Palma³ · Davide Rossini^{2,4}

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Abstract

In this review, we give a brief overview of quantum simulation as applied to the study of complex systems. In particular, we cover the basic ideas of quantum simulation, neuromorphic computation, the Sachdev–Ye–Kitaev model, as well as applications to quantum batteries.

Keywords Quantum simulations · Complex systems · Neural networks

1 Introduction

Quantum simulators are one of the leading characters of the modern quest for quantum information processing paradigms. Strongly related but at the same time different, both conceptually and experimentally, from quantum computers, they are receiving a stronger and stronger interest after the advent of the noisy intermediate scale quantum regime of quantum devices. The key reason for such interest resides in the fact that quantum simulators, often referred to as analogue quantum computers, are specialized quantum devices that mimic the behavior of complex quantum systems, while being easier to manipulate. Unlike universal quantum computers, these simulators are designed to address specific problems and follow the mathematical formalism of the quantum systems they are intended to emulate. Although they cannot perform general-purpose computation, quantum simulators are expected to solve suitable problems

✉ Oliver Morsch
morsch@df.unipi.it

G. Massimo Palma
massimo.palma@unipa.it

Davide Rossini
davide.rossini@unipi.it

¹ CNR-INO, via Moruzzi 1, Pisa 56124, Italy

² Dipartimento di Fisica, Università di Pisa, Largo B. Pontecorvo 3, Pisa 56127, Italy

³ Dipartimento di Fisica e Chimica - Emilio Segre', Università di Palermo, via Archirafi 34, Palermo 90123, Italy

⁴ INFN, Sezione di Pisa, Largo B. Pontecorvo 3, Pisa 56127, Italy

earlier than digital quantum computers, because they have less stringent requirements on the qubits quality.

This article is far from being a full review of the current state of the art in quantum simulation, for which an incredibly increasing amount of literature has been produced in the last decades [1–20]. In contrast, it has to be rather intended as an idiosyncratic introduction to the key features of quantum simulators and to their capability to address complex systems, reflecting our personal interests (and cultural bias). We have, however, tried to provide a logical sequence of topics to guide the reader through the main motivations behind quantum simulators, to their theoretical aspects and to their physical implementations.

To this end, the work is conceptually divided into three separate parts. We start, in Sect. 2, with some historical background and put quantum simulators in perspective, highlighting its similarities and its differences, and most importantly its advantages, with respect to “standard” quantum computing. In Sect. 3, we illustrate the links between quantum simulators and quantum neural networks, and in particular with quantum reservoir computing, a basic yet very effective quantum machine learning protocol. Then, in Sects. 4 and 5, we focus on a paradigmatic complex system, the Sachdev–Ye–Kitaev (SYK) model. As this is not meant as comprehensive review of all the possible complex systems which might implement a quantum simulators, we have singled out the SYK model for its reach features and cross-sectoral interest, from condensed-matter to high-energy physics communities. These make it particularly suited in the description of the key features of good quantum simulators and, more in general, of their relevance for quantum technologies. In the last part, in Sect. 6, we review the physical platforms which are currently more promising for the implementation of quantum simulators. Finally, in Sect. 7, we summarize and draw our final remarks.

2 Quantum simulators vs quantum computers

The history of the concepts of quantum computation and of quantum simulation are strongly intertwined. The scientific lore attributes the birth of quantum simulators to the seminal Richard Feynman 1982 paper [21]; however, the true beginning for a more complete and correct narrative would be the 1981 Physics of Computation Conference, jointly organized by MIT and IBM at MIT’s Endicott House. The conference picture is, for today community of quantum computation scientists, nearly as iconic as the Solvay 1927 conference picture is for quantum physicists. The list of participants included, among the others, Freeman Dyson, John Wheeler, Gregory Chaitin, Konrad Zuse, Rolf Landauer, Paul Benioff, Edward Friedkin, Lev Levitin, Daniel Greenberger, Charles Bennett (the picture is shot by him), and Richard Feynman. The hot topic then was logical vs thermodynamical reversibility, i.e., the intrinsic, unavoidable energetic cost of computation. Tom Toffoli and Edward Friedkin had developed the reversible gate model of computation, while Paul Benioff had developed a reversible quantum Turing machine. Following Rolf Landauer’s motto “information is physical”, the interest moved then toward atomic scale implementations of reversible computation. It should be stressed that these were atomic scale *classical* computers, e.g., atomic

scale computing devices performing classical computation. Any quantum effect was then considered as a source of noise [22–24]. Following this conference, Feynman, in his above-mentioned paper [21], pointed out that simulating quantum systems—especially large multipartite quantum systems—is hard on a classical computer, due to the exponential growth of their Hilbert space. He suggested that a better strategy to simulate a complex quantum systems would be to map their dynamics into the controlled dynamics of a different quantum system, thus introducing the concept of a *quantum simulator*.

Shortly after, David Deutsch gave birth to the quantum computation paradigm as we know it, with his seminal papers on universal quantum Turing machines and on universal quantum gates, introducing therefore the concept of *quantum algorithms* [25, 26]. Here, we are not going to discuss in detail the power of quantum algorithms, but let us stress that both quantum Turing machines and quantum gate-based computations imply a sequence of controlled, clocked, interactions between quantum subsystems, all of the same dimensions (the qubits). Quantum computers are universal, i.e., they can perform any computational task—any algorithm. Of course, a quantum computer can simulate any quantum system. However, on the one hand, this is different from Feynman’s concept of quantum simulators and, on the other hand, it can be demanding in terms of quantum gates. The reason being that often the individual subsystems of the multipartite system one wants to simulate are not necessarily bidimensional, i.e., they are not necessarily qubits; therefore, one would need several qubits to encode each single subsystem. Furthermore, the global evolution of the system must be decomposed in terms of a sequence of single and pairwise gates.

The problem of the resources (the number of quantum gates) needed to simulate quantum systems is nowadays particularly relevant, given the development, in an astonishingly short time, of present-day quantum computers, which have reached the so-called Noisy Intermediate Scale Quantum (NISQ) regime. Such regime is still far from being quantum fault tolerant and it is indeed noisy. Although great progress has been made in building quantum computers with a larger and larger number of qubits, the precise control of their quantum evolution required to implement complex quantum algorithms is still not satisfactory. This has revamped the interest for *quantum simulators*, which, in some of their versions, can be thought as quantum analogue computers. They are particularly convenient whenever the digital quantum simulation of a quantum system on a digital quantum computer—i.e., the simulation of a quantum system in which its state is encoded into a string of qubits and its dynamics is mapped into a sequence of gates—is demanding in terms of number of gates and of qubits. In fact, one of the advantages of quantum simulators is that they do not necessarily require a register of interacting qubits, but they can consist of interacting subsystems living in Hilbert spaces of arbitrary dimensions (qutrits, continuous variables, etc.). Furthermore, although they are not universal, their implementation is, in general, less demanding than building a full, universal, quantum computer. It should be also mentioned that they are typically less prone to the disruptive effect of quantum noise. Summing up, quantum simulators leverage on the intrinsic quantum complexity of multipartite interacting quantum system to achieve specific computational advantage with respect to both universal classical and quantum digital computers. The typical

scenario in which quantum simulators can prove to be a powerful tool is when the original system is difficult to manipulate, when one is interested in analyzing toy models with complex Hamiltonian and, in a more applied scenario, for a quantitative analysis of strongly interacting fermionic systems, like the ones encountered in quantum chemistry or in material science.

More recently, complex quantum systems have been used not only to simulate strongly interacting systems, but also to simulate and characterize open system dynamics. In particular, complex systems like bosonic quantum networks can be engineered to mimic and probe structured environments [27–30], often characterized by a non-Markovian open dynamics. The characterization of complex networks can be performed efficiently by random quantum walks [31]. An extensive discussion of quantum random walks [32] is outside the scope of this review.

2.1 Quantum annealing

So far, we have discussed the conceptual differences between quantum computing and quantum simulation. Let us now show how quantum simulators can be used to implement non-algorithmic (non-deterministic) computation, i.e., how they can be used to implement heuristic algorithms. During the same years in which the concepts of quantum simulation and quantum computation evolved, a strong interest for complex system in general and for spin glasses in particular has emerged in the field of statistical mechanics (the Nobel prize awarded to Giorgio Parisi in 2021 and to John Hopfield in 2024 are an acknowledgment of the relevance of the field). Spin glasses are disordered frustrated systems characterized by a complex fractal energy landscape. It became soon clear that the same techniques used to find the ground state of the system could be used to solve heuristically combinatorial or optimization problems. Such technique, known as *Simulated Annealing* [33], is a probabilistic algorithm to search for the global minimum in a complex energy landscape. The basic idea is map the function whose minimum one is looking for into the free energy of a suitable complex system. The system explores its energy landscape and, by slowly lowering the temperature (hence the name annealing), it will end, with high probability, into its lowest energy state. This technique finds useful application when one is searching for the lowest of many discrete minima in a large search space, a scenario encountered, e.g., in the traveling salesman problem, the Boolean satisfiability problem, protein structure prediction, and scheduling.

Shortly after, the quantum version of this process, namely *Quantum Annealing* (QA) [33–36], was put forward with the role of thermal fluctuations being played by quantum fluctuations. The basic idea is that, while, in Simulated Annealing, finite-temperature fluctuations prevent the system from being trapped into a local minimum, in QA, the same mechanism is played by tunneling. To illustrate the idea suppose that the solution of a combinatorial problem can be mapped into the ground state of a many body Hamiltonian H_c , typically a Ising-like Hamiltonian which can be expressed in terms of the Pauli $\{\sigma^z\}$ operators and which can be viewed as a potential energy term. To let the system evolve toward the ground state of H_c , one introduces a driving Hamiltonian H_d , i.e., a “kinetic” terms which, for an Ising model, corresponds to a

local transverse field term $-h \sum_i \sigma_i^x$. Such kinetic term does not commute with H_c and its eigenstates are like delocalized “plane waves” in the σ^z basis. The QA protocol relies on the adiabatic theorem—hence the name Adiabatic Quantum Computation (AQC) often used instead of QA—to drive the system from the ground state of H_d to the ground state of H_c . To do so, one engineers a time-dependent quantum adiabatic Hamiltonian

$$H_{qa}(t) = \alpha(t)H_c + \beta(t)H_d, \quad (1)$$

where $\alpha(t)$ and $\beta(t)$ are slowly varying function, such that $\alpha(0) \ll \beta(0)$ and $\alpha(T) \gg \beta(T)$, where $t = 0$ and $t = T$ are the initial and the final states of the protocol. If the process is done adiabatically and if the system is in the ground state of H_d at time $t = 0$, it will remain in the instantaneous ground state of H_{qa} and, at $t = T$, it will be in the ground state of H_c , which encodes the desired solution of the optimization problem. Let us recall that the adiabaticity condition holds as long as

$$\frac{\max \mathcal{R}(t)}{\min[\Delta(t)]^2} \ll 1, \quad (2)$$

where $\mathcal{R}(t) = \langle \varphi_1 | (dH_{qa}/dt) | g(t) \rangle$ is the rate of change of the H_{qa} between the instantaneous ground state $|g(t)\rangle$ and the first excited state $|\varphi_1\rangle$, and $\Delta(t)$ is the corresponding energy gap. The max and min being evaluated as a function of time.

There is evidence that classical simulation of QA is more efficient than simulated annealing [34, 35], with quantum tunneling being more efficient than thermal fluctuations in jumping across thin and high potential barriers. Its true power emerges; however, when, in adherence with the concept of quantum simulation, it is implemented on a true quantum hardware, where tunneling is not classically simulated but occurs intrinsically in the hardware dynamics. The advantage with respect to digital quantum computers is also evident, as no exact control of a sequence of gates and no error correction protocol need to be implemented.

3 From quantum simulators to quantum neuromorphic computation

The development of classical neuromorphic computation is prompting a strong interest for its quantum counterpart. This is a substantial change in the approach with which the dynamics of a complex system can be used to perform a computational task. Machine learning is a non-algorithmic form of computation. It can of course be run on a digital computer, but there is a growing interest toward its implementation on non-conventional dedicated hardware, ranging from mechanical systems to chemical reactions. Out of the several quantum machine learning schemes, in this review, we focus our attention to *quantum extreme learning machines* (QELM) and *quantum reservoir computing* (QRC) [37–42]. The reason for this choice is twofold. First of all, this is perhaps the simplest—both classical and quantum—non-algorithmic machine learning scheme. Second, it can be implemented into a broad range of different platforms and hardly requires any control. Reservoir computers and their memoryless counterparts, known as extreme learning machines (ELM), are a simplified version of recurrent neural networks with a training process only at the output stage.

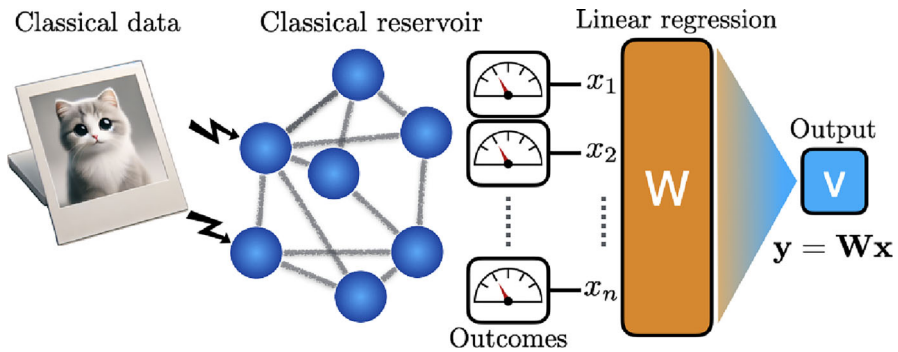


Fig. 1 Classical reservoir computing: classical data are fed into a reservoir consisting of a network of classical interacting subsystems. After the information about the input propagates into the reservoir, a (sub)set of nodes is measured. A linear regression is then made on the measurement outcomes to fit the desired function of the input. In the above example, the system is trained to tell if the input picture is a cat or not

Let us first describe the working of a classical reservoir computer. A set of input signals $\mathbf{x}_k$, where $k = 1, \dots, M_t$ labels the different training inputs, is fed into a reservoir, i.e., a system consisting of a large number of subsystems randomly coupled with each other (see Fig. 1). The strength of the couplings is fixed and it is not trained. The randomness of the coupling within the reservoir guarantees the absence of symmetries which would restrict the phase space of its internal states accessible during its evolution. Different input signals will trigger different reservoir dynamics. The role of the reservoir is twofold: it expands the phase space of the dynamics induced by the signal and it provides the nonlinearity required to perform arbitrary function evaluations. The reservoir subsystems can be anything, from mechanical systems to photonic platforms, to chemical elements. At a fixed time, the state of a subset of the reservoir subsystems is measured, the measurement outcome being a function $f(\mathbf{x}_k)$ of the input signal. The measurement outcomes are then linearly combined through suitable trained weights W , so that the $Wf(\mathbf{x}_k) = \mathbf{y}_k$ for all k where $\mathbf{y}_k$ is the desired output. Such scheme is much simpler than standard deep learning of recurrent networks, where the training takes place also at the level of the interlayer couplings. In general, reservoir computers will have a memory, which is useful when one is interested in tasks like time sequence recognition. If the state of the reservoir is reset after each input, we have an ELM, particularly useful in tasks like classification.

In QRC and in QELM, the information is “processed” by a quantum reservoir, again consisting of randomly coupled quantum subsystems (see Fig. 2). Also, in the quantum case, the randomness in the internal couplings guarantees that symmetries do not confine the reservoir dynamics prompted by external systems into a small subspace. The input signal of QRC and QELM can be either classical or quantum, i.e., one can encode the input information into a classical signal like a strong laser pulse or an impulse in a nanomechanical array, or couple a quantum state (i.e., a spin) in a given state to a multipartite quantum system. Classical input states can encode classical information, while quantum input states can be used to encode either classical or quantum information. In other words, QRC and QELM can either perform efficiently

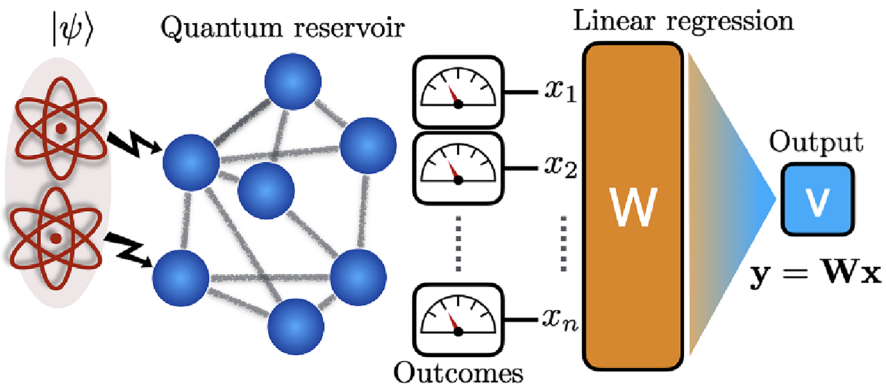


Fig. 2 In the quantum scenario, an input is fed into a reservoir consisting of a multipartite quantum system. A measurement is performed on some, or all, the reservoir subsystems. The training is performed on the measurement output probabilities. The task can be either purely quantum (e.g., to reconstruct the input state) or classical (e.g., to process some classical data encoded in the input quantum state)

classical tasks like quantitative predictions of some quantity, or even quantum tasks, like quantum state tomography or quantum entanglement witnessing.

In classical ELM, the training dataset is $\{(\mathbf{x}_k, \mathbf{y}_k)\}_k$, the model to train is $\mathbf{x} \mapsto Wf(\mathbf{x})$, and W is trained to minimize the cost function $\|\mathbf{y}_k - Wf(\mathbf{x}_k)\|^2$ for all k . In QELM, the input are the training states ϱ_k . Once the input state is coupled to the reservoir, they will jointly evolve according to a unitary evolution followed by a measurement. This amounts to evolving the input state with a completely positive trace preserving quantum map Λ and then performing a positive operator valued measurement (POVM) μ_b (here b runs over the possible measurement outcomes) on the evolved input state $\Lambda(\varrho_k)$, the probability measurement outcomes being $\mathbf{p}_{\Lambda, \mu}(\varrho)_b = \text{Tr}[\mu_b \Lambda(\varrho)]$. The QELM training datasets are then $\{(\varrho_k, \mathbf{y}_k)\}_k$, the model to train is $\varrho \mapsto W\mathbf{p}_{\Lambda, \mu}(\varrho)$, and again, W is trained to minimize the cost function $\|\mathbf{y}_k - W\mathbf{p}_{\Lambda, \mu}(\varrho_k)\|^2$ for all k . It is important to note that, while, in classical ELM, the function $f(\mathbf{x}_k)$ (i.e., the dependence of the measurement outcomes on the input) is non linear, in QELM, the measurement outcome probabilities $\mathbf{p}_{\Lambda, \mu}(\varrho)_b$ depend linearly on the input state ϱ_k . This implies that either the nonlinearity is in the encoding of some classical parameter in the input state or, if one is interested in learning some non linear functional of the input state, one resorts to consecutive multiple injections of ϱ_k . An interesting feature of QELM is that recovery of information about the input can be achieved even after the onset of scrambling [43] (more on scrambling in 4.5.4).

From the above discussion, it is clear how QELM can turn complex quantum systems into effective quantum computing devices, somehow interpolating between quantum simulators and quantum computers, with a clear advantage in terms of quantum resources needed, of noise resilience, and of fault tolerance, requiring very little quantum control. Below, we provide two different examples of implementation of QELM on two completely different physical platforms.

As first example, we illustrate how a QELM can predict molecular energy surfaces. The molecular energy surface depends on the molecular geometry, parametrized by generalized coordinates. For example, the geometry of a water molecule is fully determined by the two O-H bonds lengths r_1 and r_2 and by their relative angle φ . The QELM has been implemented on a true quantum computer [44], the IBM BRISBANE quantum processor, a device with 127 superconducting transmon qubits). The first step is to encode the input data in a quantum state. As this is done by suitable rotations along the qubit z axis, the generalized coordinates are mapped into angles by rescaling each bond length with respect to a reference one $\bar{r}$ (for water $\bar{r}_{\text{H}_2\text{O}} = 2\text{\AA}$). The bond angles are instead rescaled by a factor 2. The input data of water are thus, for example, of the following form:

$$\mathbf{x} = \left\{ \frac{r_1}{\bar{r}}\pi, \frac{r_2}{\bar{r}}\pi, \frac{\varphi}{2} \right\}. \quad (3)$$

The information scrambling of the input information into the reservoir is implemented by a random operator W acting on the N reservoir qubits. The overall encoding of the $x_1 \dots x_{\mathcal{X}}$ input parameters plus the mixing is, therefore, described by a product of local rotations by an angle x_i along the z axis of the N reservoir qubits interspersed by joint random unitaries on the N reservoir qubits

$$U(\mathbf{x}) = e^{-x_{\mathcal{X}}G} W^{(\mathcal{X})} \dots W^{(2)} e^{-x_1G} W^{(1)}, \quad (4)$$

where

$$G = \frac{1}{2} \sum_{i=1}^N \sigma_i^z. \quad (5)$$

All the N reservoir qubits are then measured in the σ^z basis and the training of probabilities is made as described above. The predicted energies have an accuracy of the order of $10^{-2} - 10^{-3}$ Hartree.

As a second task, which has been experimentally implemented on a QELM, we describe the state estimation of a single photon polarization. Here, the reservoir dynamics consists of a coined quantum walk (QW) in polarization and orbital angular momentum (OAM) of single photons [42].

The expectation values of observables on the input polarization states are obtained using the reservoir dynamics to transfer this information into the larger OAM space that is then measured. The joint polarization/OAM dynamics is a sequence of unitary operators $C_k = \mathcal{I}_o \otimes \mathcal{U}_k$ acting only on the polarization degrees of freedom ($\mathcal{U}_k$ is the so-called coin operator, in the quantum random walk jargon and $\mathcal{I}_o$ is the identity on the OAM) and of conditional transitions $\mathcal{T}$ on the OAM degrees of freedom with

$$\mathcal{T} = \sqrt{p} \mathcal{I}_o \otimes \mathcal{I}_p + \sqrt{1-p} (\mathcal{E}_+ \otimes |\downarrow\rangle\langle\uparrow| + \mathcal{E}_- \otimes |\uparrow\rangle\langle\downarrow|), \quad (6)$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the polarization states and $\mathcal{E}_{\pm}|j\rangle = |j \pm 1\rangle$, where the OAM states $|j\rangle$ play the role of the walker position state. The joint polarization /OAM state after n steps will be $|\Psi_n\rangle = (\Pi_k^n \mathcal{T} C_k)|0, \psi\rangle$ where $|\psi\rangle$ is the initial polarization state. After n steps, a projective measurement on the OAM degrees of freedom is performed

and the probability outcomes are trained to infer the expectation values of operators on the polarization state $|\psi\rangle$.

4 Quantum simulation of a paradigmatic complex system

As already mentioned in the introductory part of this review paper, while it is not our purpose to provide an exhaustive treatment of the field of quantum simulation, we feel that it would be instructive to focus on a specific, yet paradigmatic, example of a complex system where the above concepts can be applied. In fact, a large variety of quantum networks have been shown to exhibit complex features, such as the presence of entanglement, therefore they constitute an optimal testbed to study quantum communication and computation schemes, as well as genuine features of quantum dynamics [1, 45–48].

Below, we provide the reader with a general idea of what can be done in this context, focusing on a prototypical example of a complex quantum system, which exhibits a rich many-body phenomenology and is receiving considerable interest in the last decade: namely, we consider the SYK model [49, 50]. In this section, we briefly discuss why it is considered to be crucial to find a way to efficiently simulate it, and then provide some details on its most relevant physical properties. Then, in Sect. 5, we will address possible a technological application which could be able, in principle, to exploit the quantum advantage associated with the genuinely quantum features of its complex dynamics.

4.1 The Sachdev–Ye–Kitaev model

The SYK model was introduced in condensed-matter physics as a prototype of strange metal, with a non-Fermi liquid behavior. Namely, it describes Q strongly coupled spinless fermions, moving on a N -site lattice, and characterized by random and uncorrelated two-particle interactions of arbitrary strength [49, 50]. One of its basic features is the lack of quasiparticle excitations, resulting in an exponentially dense low-energy spectrum (Sect. 4.3). Besides that, at large N , it is possible to obtain an exact formal treatment (Sect. 4.4), which reveals that the SYK model is maximally chaotic: more precisely, it displays fast scrambling, with a non-zero entropy density at vanishing temperature, and exhibits a volume-law bipartite entanglement entropy for all the eigenstates at any energy scale (even for the ground state). These facts, together with the emergence of a conformal symmetry at low temperatures, suggest that it has an interesting holographic dual which describes a theory of quantum gravity [51–53].

The Hamiltonian reads

$$H_{\text{SYK}} = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,l=1}^N \tilde{J}_{ijkl} c_i^\dagger c_j^\dagger c_k c_l - \mu \sum_{i=1}^N c_i^\dagger c_i, \quad (7)$$

where $c_i^{(\dagger)}$ are anticommuting operators in the second quantization formalism, which denote annihilation (creation) operators for spinless fermions moving between N different sites of a lattice, labeled by $i = 1, \dots, N$. The couplings $\tilde{J}_{ijkl}$ are independent Gaussian distributed complex variables with zero average $\langle\langle \tilde{J}_{ijkl} \rangle\rangle = 0$ and variance $\langle\langle |\tilde{J}_{ijkl}|^2 \rangle\rangle = J^2$ ($J \in \mathbb{R}$), which satisfy $\tilde{J}_{ijkl} = -\tilde{J}_{jikl} = -\tilde{J}_{ijlk} = \tilde{J}_{klij}^*$, while μ acts as a chemical potential. The $1/\sqrt{N^3}$ scaling in front of the interaction strength guarantees that the system bandwidth is of the order of N , in the $N \rightarrow \infty$ thermodynamic limit, such that extensivity of thermodynamic quantities as the energy is preserved. Hereafter, we redefine $J_{ijkl} = \tilde{J}_{ijkl}/(2N)^{3/2}$.

The SYK model conserves the total number of fermions, as the Hamiltonian H_{SYK} commutes with the operator

$$\mathcal{Q} = \sum_{i=1}^N c_i^\dagger c_i; \quad (8)$$

the second contribution in Eq. (7) ensures that, in the grand canonical formalism, the average density of fermions on each site is $Q/N \equiv \langle\mathcal{Q}\rangle/N \in [0, 1]$. As we shall see below, the situation for which $Q/N = 1/2$ is of particular importance and is referred to as the *half-filling* condition. In the following, we will be mostly interested in quantities or observables $\mathcal{O}$ averaged over the distribution of the couplings $\{J_{ijkl}\}$. Thus, we adopt the notation

$$\langle\langle \mathcal{O} \rangle\rangle \equiv \int P(\{J_{ijkl}\}) \mathcal{O}(\{J_{ijkl}\}) d\{J_{ijkl}\}, \quad (9)$$

where $P(\{J_{ijkl}\})$ denotes the corresponding probability distribution of the couplings.

Finding a way to efficiently simulate Eq. (7) is of the uttermost importance, as currently the SYK model lacks of a full analytic treatment and several issues related to its emerging physics are still unsolved. The main obstacle for classical simulations is the exponential growth of the Hilbert space with the number of sites, a fact that drastically limits the application of exact-diagonalization (ED) methods to systems with $N \sim \mathcal{O}(10)$. As an alternative, several quantum hardware platforms have been proposed and thoroughly discussed, as clarified below (Sect. 4.2).

4.2 Simulations of the SYK model

In general, any numerical simulation of a quantum many-body system on a lattice with arbitrary geometry has to face with the fact that the global Hilbert space $\mathcal{H}$ has an exponentially growing dimension, $\dim(\mathcal{H}) = 2^N$. This is true also for the SYK model, which lacks of any particular symmetry (apart from the conservation of the number of fermions) and for which no efficient classical numerical method is known yet. In fact, for the H_{SYK} Hamiltonian in Eq. (7), $\mathcal{H}$ can be decomposed as a direct sum of subspaces with a fixed number $Q = 1, 2, \dots, N$ of fermions

$$\mathcal{H} = \bigoplus_{Q=1}^N \mathcal{H}_Q, \quad \text{with } \dim(\mathcal{H}_Q) = \binom{N}{Q}. \quad (10)$$

In the thermodynamic limit $N \rightarrow \infty$, the only relevant subspace is the one corresponding to the half-filling condition $Q = N/2$, whose cardinality is the largest and keeps scaling exponentially with N . It is thus useful to concentrate on $\mathcal{H}_{N/2}$: its dimension keeps growing exponentially with N , although, for $N \sim 10$, its computational complexity is one order of magnitude smaller than that for $\mathcal{H}$.

In practice, it is convenient to transform the SYK model into a system of interacting spin-1/2 particles, through a Jordan–Wigner (JW) transformation [54]. This enables us to map creation/annihilation operators for fermions $c_j^{(\dagger)}$ into spin-1/2 Pauli matrices σ_j^α ($\alpha = x, y, z$) and the corresponding raising/lowering operators $\sigma_j^\pm = \frac{1}{2}(\sigma_j^x \pm \sigma_j^y)$. To correctly transform fermionic anticommutation relations into commutation relations for the Pauli matrices, one needs to exploit the so-called JW string, in such a way that

$$c_i^\dagger = \sigma_i^+ \left(\prod_{j < i} \sigma_j^z \right). \quad (11)$$

The JW string delocalizes the single excitation on the whole spin network. In the case (i) where all four indexes i, j, k, l in Eq. (7) are different ($i \neq j \neq k \neq l$), the two-body interaction term in H_{SYK} is mapped into

$$c_i^\dagger c_j^\dagger c_k c_l = \text{sgn}(i - j) \text{sgn}(k - l) \prod_{\zeta=\zeta_1+1}^{\zeta_2-1} \sigma_\zeta^z \prod_{\zeta'=\zeta_3+1}^{\zeta_4-1} \sigma_{\zeta'}^z \sigma_i^+ \sigma_j^+ \sigma_k^- \sigma_l^-, \quad \text{case (i),} \quad (12a)$$

where $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ correspond to the four indexes $\{i, j, k, l\}$ ordered from the smallest to the largest one. The remaining non-zero cases are those with (ii) $i = k, j = l$, $i = k, j \neq l$ and (iii) $i \neq k, j = l$ [the latter one being traceable back to case (ii)], which can be written as

$$c_i^\dagger c_j^\dagger c_i c_j = -\sigma_i^+ \sigma_j^- \sigma_i^+ \sigma_j^-, \quad \text{case (ii),} \quad (12b)$$

$$c_i^\dagger c_j^\dagger c_i c_l = \prod_{\zeta=\zeta_1+1}^{\zeta_2-1} \sigma_\zeta^z \sigma_i^+ \sigma_j^+ \sigma_i^- \sigma_l^-, \quad \text{case (iii).} \quad (12c)$$

Conventional ED approaches to classically simulate the SYK Hamiltonian, although being highly inefficient, make use of such mapped non-local spin-1/2 network. To overcome these limitations, proposals involving analog quantum simulations with ultracold gases [55, 56], cavity QED [57, 58], as well as solid-state systems [59–62] have been recently put forward. Unfortunately, from an experimental point of view, it is very difficult to simulate strong both randomness and fully non-local interactions. Moreover, even the initialization of the system into specific states at a given temperature and the measurement of the dynamical properties poses some conceptual issues. For all these reasons, the most promising strategy seems to be the use of a digital quantum simulator [54, 63, 64] on a real quantum computer hardware, with individual and high-fidelity controllability. Viable platforms in this respect are superconducting qubits or nuclear spin systems.

4.3 Failure of the quasiparticle picture

Coming back to the reasons why the physics of the SYK model is so special in the quantum many-body scenario, we start from recalling the concept of *quasiparticle*, one of the cornerstones in modern condensed-matter physics, being crucial to describe a variety of situations in quantum matter. For example, metals and semiconductors can be seen as free-Fermi systems, where each single electron is surrounded by a cloud of other electrons, effectively renormalizing its characteristics; this leads to the so-called “dressed fermion” (i.e., quasiparticle) that behaves exactly as a free electron, although with modified mass, magnetic moment, etc. Other paradigmatic situations rely on analogous grounds, such as the theory of superconductivity (pairing of quasiparticles), disordered metals and insulators (physics of localization), one-dimensional materials (collective modes are seen as quasiparticles), and fractional quantum Hall effect (where quasiparticles are seen as “fractions” of an electron). The SYK model lies outside this conventional scenario, since it cannot be described through the paradigm of quasiparticles. For this reason, it is often referred to as a model for a strange metal [52].

At low energies, the physics of a weakly interacting fermionic system is faithfully characterized by collective excitations that constitute the normal modes of the associated Hamiltonian. Let us suppose that $\{\varepsilon_\alpha\}$ and $\{n_\alpha\}$ are the eigenenergies and the occupation numbers associated with quasiparticles of a given theory. The energy of the system can be cast in the following simple form:

$$E = \sum_{\alpha} n_{\alpha} \varepsilon_{\alpha} + \sum_{\alpha, \beta} n_{\alpha} n_{\beta} F_{\alpha\beta} + \dots, \quad (13)$$

where the coupling constants $F_{\alpha\beta}$ between two quasiparticles are small and can be treated perturbatively, while further terms involving interactions between more than two particles are suppressed. Systems undergoing this paradigm are characterized by a many-body energy spectrum that obeys some simple and universal rules: assuming that there are Q weakly interacting quasiparticles, one can put forward a general scaling behavior for the gap Δ_0 between the ground state and the first excited state, as well as for the typical energy gap Δ between two generic eigenstates in the middle of the spectrum. In the thermodynamic limit, Δ_0 is expected to close polynomially with Q , while Δ should be exponentially suppressed with Q

$$\Delta_0 \sim Q^{-r} \quad \text{with } r = O(1); \quad (14a)$$

$$\Delta \sim e^{-\gamma Q} \quad \text{with } \gamma \sim \log 2. \quad (14b)$$

The above behaviors can be intuitively understood by counting the number of possible combinations of quasiparticles (excitations), starting from their ground-state vacuum. In fact, energies of quasiparticles are typically spaced polynomially with Q , while the total number of many-body energy states should scale exponentially with Q . One thus expects a polynomial spacing of low-lying quasiparticle excitations, in contrast with an exponential spacing for excitations in the middle of the spectrum.

A simple model of a free-fermion theory in a disordered N -site lattice, falling into this paradigm, is provided by the Hamiltonian

$$H = \frac{1}{\sqrt{N}} \sum_{i,j} t_{ij} c_i^\dagger c_j, \quad (15)$$

where the prefactor $1/\sqrt{N}$ ensures extensivity and the hopping strength t_{ij} is a Gaussian variable, such that $\langle t_{ij} \rangle = 0$ and $\langle |t_{ij}|^2 \rangle = t^2$, analogously as for the couplings J_{ijkl} in the SYK Hamiltonian (7). Any given realization of the tunneling amplitudes t_{ij} corresponds to different eigenenergies, such that a diagonalization of the corresponding $N \times N$ hopping matrix maps the problem in noninteracting normal modes [i.e., $F_{\alpha\beta} = 0$ in Eq. (13)] which, of course, cannot thermalize. Switching on a weak interaction term between such quasiparticles eventually leads to equilibration, although with a long thermalization time τ_{eq} (quasiparticles are still almost eigenstates of the system). In fact, for a Fermi liquid close to the Fermi energy E_F , one finds the behavior

$$\tau_{\text{eq}} \sim \frac{\hbar E_F}{k_B^2 T^2}, \quad \text{for } T \rightarrow 0, \quad (16)$$

which is typical of normal metals (see, e.g., Refs. [65, 66]). In the following, we will work in natural units, where the Boltzmann constant k_B , as well as the reduced Planck constant $\hbar$, are set equal to one.

On the other hand, the SYK model for a strange metal violates the above paradigm. In particular, it is no longer true that low-lying excitations are polynomially spaced in energy [cf. Eq. (14a)], nor that the thermalization time diverges as $\tau_{\text{eq}} \sim T^{-2}$ [cf. Eq. (16)]. The reason eventually resides in the fact that two-body interactions dominate, so that any perturbative treatment is impossible and the quasiparticle picture inevitably fails. In principle, one could diagonalize the full Hamiltonian model in Eq. (7), thus obtaining 2^N normal modes (and not simply N normal-mode quasiparticles). In fact, numerical simulations for finite N have shown that the gap between the ground state and the first excited state at half-filling is

$$\Delta_0 \sim e^{-S_0 N}, \quad \text{with } S_0 \approx 0.465. \quad (17)$$

4.4 Large- N limit

Although finding the exact form of eigenvalues and eigenvectors of the SYK model is a hard task, a formally exact treatment can be derived in the limit of large N . In particular, one can obtain closed analytic expressions for the fermionic Green's function and for the free energy, which can then be processed numerically. Following Refs. [49, 67], below we outline how to employ a path integral approach on fermionic fields in imaginary time [66], to calculate the disorder-averaged partition function, and obtain the saddle-point equations for the Euclidean action in the $N \rightarrow \infty$ limit.

4.4.1 Euclidean action

We use the formalism of coherent states. Such states $|\psi\rangle$ are eigenstates of the fermionic annihilation operators c_i , such that $c_i|\psi\rangle = \psi_i|\psi\rangle$ and $\langle\psi|c_i^\dagger = \langle\psi|\bar{\psi}_i$. In the fermionic case, the fields ψ_i are anticommuting generators of a Grassmann algebra: $\psi_i\psi_j = -\psi_j\psi_i$, where ψ and $\bar{\psi}$ have to be treated as independent variables. A useful property of coherent states is that any normal ordered operator written in their basis can be cast as a Grassmann number, by substituting $c_i \rightarrow \psi_i$ and $c_j^\dagger \rightarrow \bar{\psi}_j$. The partition function Z of model (7) can be thus written as a field integral over such Grassmann variables

$$Z = \text{Tr} \left[e^{-\beta(H_{\text{SYK}} - \mu N)} \right] = \int \mathcal{D}(\bar{\psi}, \psi) e^{-S[\bar{\psi}, \psi]}, \quad (18)$$

where $\beta = 1/T$ is the inverse temperature. Here, we use the replica trick (n counts the number of replicas), and introduce the Feynman differential

$$\mathcal{D}(\bar{\psi}, \psi) \equiv \lim_{N \rightarrow \infty} \prod_{i=1}^N \prod_{a=1}^n d\bar{\psi}_i^{(a)} d\psi_i^{(a)} \quad (19)$$

and the imaginary-time (τ) Euclidean action

$$\begin{aligned} S[\bar{\psi}, \psi] = & \int_0^\beta d\tau \sum_a \left\{ \sum_i \bar{\psi}_i^{(a)}(\tau) \left(\frac{\partial}{\partial \tau} - \mu \right) \psi_i^{(a)}(\tau) \right. \\ & \left. + \sum_{i,j,k,l=1}^N J_{ijkl} \bar{\psi}_i^{(a)} \bar{\psi}_j^{(a)} \psi_k^{(a)} \psi_l^{(a)} \Big|_\tau \right\}. \end{aligned} \quad (20)$$

The action and the partition function depend on the specific realization of the coupling constants J_{ijkl} ; it is thus useful to consider averages over the disorder in such a way to obtain the following modified average partition function:

$$\begin{aligned} \langle\langle Z \rangle\rangle &= \int d(J_{ijkl}) \rho(\{J_{ijkl}\}) \mathcal{Z}(\{J_{ijkl}\}) \\ &= \mathcal{N} \int d(J_{ijkl}) \exp \left\{ -\frac{N^3}{2J^2} \sum_{i,j,k,l} J_{ijkl}^2 \right\} \int \mathcal{D}(\bar{\psi}, \psi) e^{-S[\bar{\psi}, \psi, \{J_{ijkl}\}]} \\ &= \mathcal{N} \int d(J_{ijkl}) \int \mathcal{D}(\bar{\psi}, \psi) e^{-\tilde{S}[\bar{\psi}, \psi, \{J_{ijkl}\}]}, \end{aligned} \quad (21)$$

where $\mathcal{N}$ is an irrelevant normalization constant and the modified Euclidean action $\tilde{S}$ absorbs the functional form of the Gaussian distribution for the J_{ijkl} 's. Completing

the square in the action, we can write

$$\begin{aligned} \tilde{S} = & \sum_{i,a} \int_0^\beta d\tau \bar{\psi}_i^{(a)}(\tau) \left(\frac{\partial}{\partial \tau} - \mu \right) \psi_i^{(a)}(\tau) \\ & + \frac{1}{2} \sum_{i,j,k,l} \left\{ \frac{N^{3/2} J_{ijkl}}{J} + \frac{J}{N^{3/2}} \sum_a \int_0^\beta d\tau \bar{\psi}_i^{(a)} \bar{\psi}_j^{(a)} \psi_k^{(a)} \psi_l^{(a)} \Big|_\tau \right\}^2 \\ & - \frac{J^2}{2N^3} \sum_{i,j,k,l} \sum_{a,b} \iint_0^\beta d\tau d\tau' \bar{\psi}_i^{(a)} \bar{\psi}_j^{(a)} \psi_k^{(a)} \psi_l^{(a)} \Big|_\tau \bar{\psi}_i^{(b)} \bar{\psi}_j^{(b)} \psi_k^{(b)} \psi_l^{(b)} \Big|_{\tau'} \end{aligned} \quad (22)$$

Now, performing the Gaussian integral over the coupling constants $\{J_{ijkl}\}$, one reabsorbs the second line of Eq. (22) in the measure of the functional integral, as well as washes out the normalization constant $\mathcal{N}$. Moreover, in the large- N limit, the normal-ordering corrections in the third line can be neglected, to finally obtain

$$\begin{aligned} \tilde{S} = & \sum_{i,a} \int_0^\beta d\tau \bar{\psi}_i^{(a)}(\tau) \left(\frac{\partial}{\partial \tau} - \mu \right) \psi_i^{(a)}(\tau) \\ & - \frac{J^2}{2N^3} \sum_{a,b} \iint_0^\beta d\tau d\tau' \left| \sum_i \bar{\psi}_i^{(a)}(\tau) \psi_i^{(b)}(\tau') \right|^4. \end{aligned} \quad (23)$$

4.4.2 Hubbard–Stratonovich transformation

To decouple the interaction term containing eight Grassmann fields, one can perform two subsequent Hubbard–Stratonovich transformations. This can be done by introducing two auxiliary scalar fields G and Σ , which couple to the Grassmann fields, such that the residual interaction is quadratic in the fields themselves. We only consider the diagonal solution in the replica space and omit the corresponding replica index. It is thus possible to write

$$\langle\langle Z \rangle\rangle = \iint dG d\Sigma \int \mathcal{D}(\bar{\psi}, \psi) e^{-S[\bar{\psi}, \psi, G, \Sigma]}, \quad (24)$$

with

$$\begin{aligned} S[\bar{\psi}, \psi, G, \Sigma] = & \sum_i \int_0^\beta d\tau \bar{\psi}_i(\tau) \left(\frac{\partial}{\partial \tau} - \mu \right) \psi_i(\tau) \\ & + \iint_0^\beta d\tau d\tau' \Sigma(\tau, \tau') \left[N G(\tau, \tau') - \sum_i \bar{\psi}_i(\tau) \psi_i(\tau') \right] \\ & + \frac{NJ^2}{2} \iint_0^\beta d\tau d\tau' G^2(\tau, \tau') G^{*2}(\tau, \tau'). \end{aligned} \quad (25)$$

If one now writes the saddle-point equations $\frac{\delta S}{\delta \Sigma} = \frac{\delta S}{\delta G^*} = 0$, it is possible to recognize that G coincides with the imaginary-time-ordered retarded Green's function

$$G(\tau, \tau') \equiv -\left\langle \mathcal{T}_\tau \left[c(\tau) c^\dagger(\tau') \right] \right\rangle. \quad (26)$$

Assuming time-translation invariance and defining the fermionic Matsubara frequencies $\omega_n = (2n + 1)\pi/\beta$, ($n \in \mathbb{Z}$), the saddle-point equations finally read

$$G(i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)}, \quad (27a)$$

$$\Sigma(\tau) = -J^2 G^2(\tau) G(-\tau). \quad (27b)$$

Although these are algebraic equations, rigorously exact in the thermodynamic limit, their solution is neither obvious nor immediate to be found. In fact, they are connected non locally through an imaginary-time Fourier transform and they need to be solved numerically. As reported in Sect. 4.5.3, their solution gives a behavior $\text{Im}\{G(\omega)\} \sim \omega^{-1/2}$ for $\omega \rightarrow 0$, in stark contrast with a normal Fermi-liquid behavior.

4.5 Main properties of the SYK model

We are now in the position to summarize some important features of the SYK Hamiltonian, emphasizing the properties that lead it to a prototypical nonperturbative model of strongly correlated fermions. All the results discussed below have been obtained either through the large- N limit solution, or by classical ED simulations for small system sizes. Possible forthcoming quantum simulations could pave the way for a further and deeper understanding of the physics of this model.

4.5.1 Particle-hole symmetry

First of all, the Hamiltonian defined in Eq. (7) is not symmetric under particle-hole (PH) exchange. In fact, given the operator

$$\mathcal{P} \equiv \prod_i (c_i^\dagger + c_i) \mathcal{K}, \quad \text{with } \mathcal{K}\mathcal{O} = \mathcal{O}^*, \quad (28)$$

which exchanges particles into holes and vice versa, one finds $[\mathcal{P}, H_{\text{SYK}}] \neq 0$. The reason comes from terms with two equal indexes. It is, however, possible to construct a modified SYK model which exploits the PH symmetry, by explicitly subtracting terms in Eq. (7) associated with two equal indexes

$$H_{\text{SYK}}^{(\text{PH})} = H_{\text{SYK}} + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,l} \frac{\tilde{J}_{ijkl}}{2} \left(\delta_{ik} c_j^\dagger c_l - \delta_{il} c_j^\dagger c_k - \delta_{jk} c_i^\dagger c_l + \delta_{jl} c_i^\dagger c_k \right). \quad (29)$$

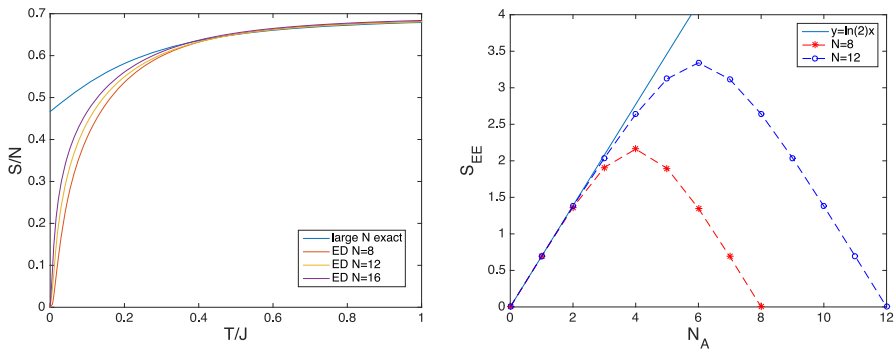


Fig. 3 Some results for the entropy behavior in the fermionic-SYK model (7), obtained either from ED results (with N up to 16) or semi-analytically in the large- N limit. (left) The thermal entropy as a function of the temperature. (right) The zero-temperature bipartite entanglement entropy as a function of the subsystem size N_A , for a given system size N . Adapted from Ref. [67]

We point out that, in any case, in the thermodynamic limit $N \rightarrow \infty$, contributions with two equal indexes are suppressed as $\mathcal{O}(N^{-1})$; thus, the original model restores the PH symmetry: $H_{\text{SYK}}^{(\text{PH})} = H_{\text{SYK}}$ for $N \rightarrow \infty$.

4.5.2 Entropy

The finite-temperature thermodynamic entropy can be obtained from the free energy F . In fact, according to the rules of thermodynamics, one has

$$\frac{S}{N} = -\frac{1}{N} \frac{\partial F}{\partial T}, \quad \text{with} \quad \frac{F}{N} = -\frac{\log Z}{N\beta}. \quad (30)$$

The partition function $Z = \sum_n e^{-\beta E_n}$ can be obtained either by explicitly computing the full Hamiltonian spectrum through exact-diagonalization (ED) methods (details are provided in Sect. 4.2), or in a semi-analytic way, from the modified action approach (see Sect. 4.4). In fact, one can show that [67]

$$\frac{F}{N} = \frac{1}{\beta} \sum_n \log [-\beta G(i\omega_n)] - \frac{3}{4\beta} \sum_n \Sigma(i\omega_n) G(i\omega_n). \quad (31)$$

Numerical data displayed in Fig. 3(left) show that, at high temperatures, the entropy per site tends to the maximum admissible value $S_\infty/N = \ln 2 \approx 0.693$, while, in the zero-temperature limit and for finite system sizes, it approaches the finite value $S_0/N \approx 0.465$. The latter corresponds to the rate at which the gap between the lowest energy levels closes [see Eq. (17)]. Interestingly, at low temperatures, the ED results at finite N show a marked discrepancy with the exact solution in the large- N limit: this is due to the non-commutativity of the thermodynamic limit ($N \rightarrow \infty$) with the low-temperature ($T \rightarrow 0$). In fact, the non-zero entropy for $T \rightarrow 0$ can be obtained only after taking $N \rightarrow \infty$ first [67].

One can also compute the ground-state entanglement entropy for a bipartition $A|B$ of the whole system $A + B$, through the von Neumann entropy of the reduced density matrix ρ_A of subsystem A

$$S_{EE} = -\text{Tr}[\rho_A \ln \rho_A], \quad (32)$$

where $\rho_A = \text{Tr}_B[|0\rangle\langle 0|]$ is obtained by tracing over the sites corresponding to the complementary subsystem B . Here, $|0\rangle$ denotes the system global ground state. Let us suppose that subsystem A contains $N_A < N$ sites. Numerical results displayed in Fig. 3(right) show that, for $N_A < N/2$, the entanglement entropy is proportional to N_A , thus obeying a volume-law behavior. Therefore, one expects thermalization to occur even in the ground state. This is a remarkable feature of the SYK model, in sharp contrast with typical low-dimensional systems where an emerging area-law behavior is expected at low energies [68]. It is worth pointing out that, while the ratio S_{EE}/N_A should correspond to the zero-temperature limit of the entropy per site S_0/N , numerical results appear to be closer to the opposite infinite-temperature limit S_∞/N . Such qualitative discrepancy has been ascribed to finite-size effects [67].

4.5.3 Green's function

The on-site retarded Green's function

$$G_i^R(t, t') = -i\Theta(t - t') \left\langle \left\{ c_i^\dagger(t), c_i(t') \right\} \right\rangle \quad (33)$$

can be faithfully studied within the Lehmann representation

$$G_i^R(\omega) = \frac{1}{Z} \sum_{n,m} \frac{\langle n|c_i|m\rangle\langle m|c_i^\dagger|n\rangle}{\omega + E_n - E_m + i\eta} \left(e^{-\beta E_n} + e^{-\beta E_m} \right), \quad (34)$$

where $\{|n\rangle\}$ are the eigenstates of H_{SYK} corresponding to the eigenenergies $\{E_n\}$ and $\eta > 0$ is a small regularization parameter. In the zero-temperature limit, we can cast the latter expression as

$$G_i^R(\omega) = \sum_m \frac{\langle 0|c_i|m\rangle\langle m|c_i^\dagger|0\rangle}{\omega + E_0 - E_m + i\eta} + \frac{\langle 0|c_i|m\rangle\langle m|c_i^\dagger|0\rangle}{\omega - E_0 + E_m + i\eta}. \quad (35)$$

It is now useful to compare the results for the imaginary part of the Green's function $G(\omega)$ in the large- N limit, which can be obtained from Eqs. (27), with those at finite N , numerically computable from Eq. (35), given the fact that $G_i^R(\omega) = G(i\omega_n \rightarrow \omega + i\eta)$. Using the fact that $(\omega + i\eta)^{-1} = \mathcal{P}\frac{1}{\omega} - i\pi\delta(\omega)$, one can see that the imaginary part of $G_i^R(\omega)$ contains a Dirac delta, which can be numerically estimated through a Lorentzian, using the well-known limit

$$\delta(E_0 - E_m + \omega) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \frac{\eta}{(E_0 - E_m + \omega)^2 + \eta^2}. \quad (36)$$

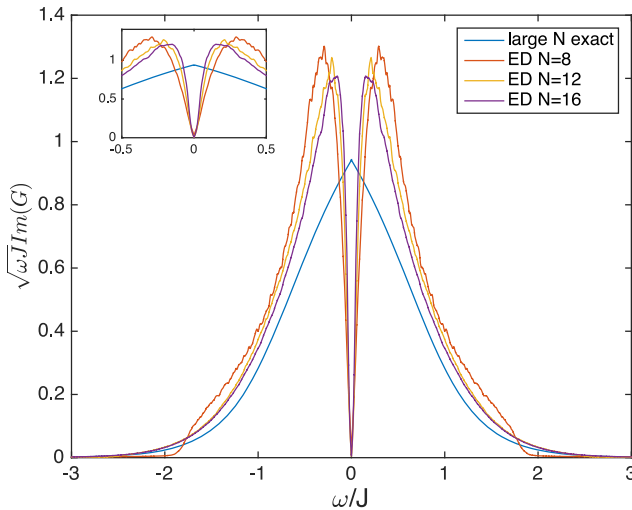


Fig. 4 Behavior in frequency space of the imaginary part of the retarded Green's function for the SYK model (7). The inset is a zoom close to $\omega = 0$. From Ref. [67]

We also recall that, for $\mu = 0$, one has to keep into account the anti-unitary PH symmetry, which gives rise to double degenerate ground states for certain system sizes. In such cases, one needs to sum over both ground states $|0\rangle$ and $|0'\rangle$, in the expression for the propagator. Figure 4 compares the imaginary part of the retarded Green's function as obtained either in the large- N limit (blue curve) or at finite size (other colored curves) [67]. In the latter cases, ED results do not faithfully reproduce the peculiar $\sim \omega^{-1/2}$ divergence at low frequencies, typical of strange metals (to be contrasted with the usual $\sim \omega^{-2}$ behavior for normal metals), which is clearly seen in the thermodynamic limit.

4.5.4 Quantum chaos and scrambling

A prominent feature of the SYK model is that it exhibits a quantum chaotic behavior [69]. To better highlight its meaning, we start from the heuristic definition of *chaos* in the classical realm. Let us consider a classical dynamical system and the corresponding phase space Γ , in which we can define a notion of measure. Chaos sets in whenever, for two generic initial conditions $x_1(0), x_2(0) \in \Gamma$ close together, the distance $\delta x(t) \equiv |x_1(t) - x_2(t)|$ between the two originating trajectories in the phase space grows exponentially in time: $\delta x(t) \sim \delta x(0)e^{\lambda_L t}$, where $\lambda_L > 0$ is the largest classical Lyapunov exponent of the classical system. More precisely, the latter is defined as

$$\lambda_L = \lim_{t \rightarrow \infty} \lim_{\delta x(0) \rightarrow 0} \frac{1}{t} \ln \frac{\delta x(t)}{\delta x(0)}. \quad (37)$$

To generalize this notion to quantum systems, it is useful to rewrite the above sensitivity in terms of a Poisson bracket

$$\frac{\delta x(t)}{\delta x(0)} = \{x(t), p(0)\}_{\text{cl}}, \quad (38)$$

where $p(0)$ is the initial momentum. We can then proceed with the canonical quantization prescription, promoting the variables x and p to operators and substituting the Poisson brackets with the commutator: $\{\cdot, \cdot\}_{\text{cl}} \rightarrow -i[\cdot, \cdot]$. When performing expectation values of this kind of observable, cancellations may set in, due to terms with opposite sign. In general, it is thus better to take the norm squared of this operator

$$\left\{ -i[x(t), p(0)] \right\} \left\{ i[x(t), p(0)]^\dagger \right\} = -[x(t), p(0)]^2. \quad (39)$$

This is an example of an out-of-time-order correlator (OTOC), whose expectation value is believed to provide insight and characterize quantum chaos [70, 71].

In a chaotic quantum system at thermal equilibrium and at early times, the expectation value of the above OTOC is expected to grow exponentially as

$$\langle [x(t), p(0)]^2 \rangle \sim c_0 + c_1 e^{\lambda_L t}. \quad (40)$$

The time scale at which such correlator becomes significant is often referred to as the “scrambling time”, since it roughly defines the onset of quantum information spreading (or “scrambling”) over the system. Furthermore, it can be shown that there is an upper thermal limit for this growth, provided by $\lambda_L \lesssim 2\pi T$ [69], which is expected to saturate in the strong-coupling limit ($\beta J \gg 1$) of the SYK model [50], as verified by extensive numerical simulations [67, 72, 73]. Such Lyapunov exponent represents the equilibration rate for the system, for which one can define a thermalization time scale behaving as

$$\tau_{\text{eq}} \sim T^{-1}. \quad (41)$$

This result qualitatively differs from the normal metal behavior of Eq. (16), which presents a much slower equilibration $\propto T^{-2}$.

Explicit numerical simulations of the time behavior for the OTOC $\langle x(t)p(0)x(t)p(0) \rangle$ at infinite-temperature evidence an exponential decay in time of such correlator, which translates into an exponential growth of $\langle [x(t), p(0)]^2 \rangle$ [67, 72, 73].

5 Case study of a complex quantum dynamics

At this stage, we are ready to discuss a physical example where the peculiar dynamic properties of a complex quantum network emerge and can be exploited for quantum technological purposes. In particular, we refer to the possibility to build up a concept of quantum battery (QB) [74] that could take advantage of the fast thermalization time scales and scrambling properties achieved by the dynamics of a complex quantum

network over more traditional quantum materials. The goal is to outperform conventional classical schemes, where the charging efficiency of the battery typically scales linearly with the number of its components.

5.1 The quantum battery paradigm

The concept of QB has been put forward a dozen of years ago [75, 76], in the context of work extraction from quantum systems that are used to temporarily store energy [77]. In its basic formulation, a quantum battery (QB) can be identified by a quantum system that is capable to accumulate energy, when undergoing a double quantum quench [78]. Specifically, the typical charging protocol of a QB performs as follows. We consider a system described by a time-independent Hamiltonian

$$H_0 = \sum_{j=1}^N h_j, \quad (42)$$

where N denotes the number of quantum cells, each of them being initially decoupled from the rest of the system and governed by a local Hamiltonian term h_j . In the hypothesis that the global system is in a generic (mixed) state $\rho_N(t)$, the total internal energy can be calculated as

$$E_N(t) = \text{Tr}[\rho_N(t) H_0] = \sum_{j=1}^N \text{Tr}[\rho_N(t) h_j]. \quad (43)$$

At a given reference time, say $t = 0$, the Hamiltonian is instantaneously changed into H_I , which includes interactions (I) between the various quantum cells, so that the system evolves unitarily under the new Hamiltonian. Subsequently, after a given time τ (the so-called charging time), a second quench is performed and the system comes back to be described by the original Hamiltonian H_0 in Eq. (42). Thus, the Hamiltonian governing the system along the whole protocol can be cast in the form

$$H(t) = [1 - \lambda(t)]H_0 + \lambda(t)H_I, \quad (44)$$

where $\lambda(t)$ is a double step function which can take either the value 1 for $0 < t < \tau$, or be zero otherwise. For the sake of simplicity, in the following, we neglect any dissipative mechanism and suppose that the QB is initialized in the (nondegenerate) ground state $|\psi_N(0)\rangle$ of H_0 , so that, under the above protocol, the system remains in a pure state.

Let us now introduce the quantities in which one might be interested. Denoting with $|\psi_N(t)\rangle$ the evolved state of the system at time t , one finds

$$|\psi_N(t)\rangle = e^{-iH_I t} |\psi_N(0)\rangle. \quad (45)$$

It is clear that the energy of the QB is constant at any time, with the exception of the quenching points $t = 0$ and $t = \tau$, and is given by Eq. (43) with $\rho_N(t) = |\psi_N(t)\rangle\langle\psi_N(t)|$. The total energy $E_N(\tau)$ injected into the N quantum cells at the end of the protocol can be thus expressed in terms of the mean local energy. It is also possible to define the corresponding power of the QB as $P_N(\tau) = E_N(\tau)/\tau$. To ensure a non-trivial unitary evolution in the time $t \in [0, \tau]$ between the two quenches (and thus to effectively charge the system), one must require that $[H_0, H_I] \neq 0$.

It is worth mentioning that the energy fraction which can be effectively extracted for thermodynamic purposes, when acting unitarily on $M \leq N$ quantum cells and without having access to the full system, does not necessarily coincide with the total energy $E_N(\tau)$. Indeed, part of the energy injected into the QB is in the form of correlations between the quantum cells that develop in time, and is thus useless for practical purposes. A rigorous measure of such useful energy fraction is provided by the so-called “ergotropy” [77] of the local state $\rho_M(t)$ of M cells, defined as the difference

$$\mathcal{E}_M(t) = E[\rho_M(t)] - E[\tilde{\rho}_M(t)] \quad (46)$$

between the energy $E[\rho_M(t)] = \text{Tr}[\rho_M(t) H_M]$ of the state ρ , where $H_M = \sum_{j=1}^M h_j$ is the local Hamiltonian of M cells, and that of the passive counterpart $\tilde{\rho}_M(t)$ of $\rho_M(t)$, on the same local Hamiltonian. The latter state $\tilde{\rho}_M(t)$ is defined as a density matrix which is diagonal on the eigenbasis of $\mathcal{H}_M$ and whose eigenvalues correspond to a proper reordering of those of ρ_M

$$\tilde{\rho}_M = \sum_n r_n |\epsilon_n\rangle\langle\epsilon_n|, \quad \text{where} \quad \begin{array}{l} \rho_M = \sum_n r_n |r_n\rangle\langle r_n|, \quad r_0 \geq r_1 \geq r_2 \geq \dots, \\ H_M = \sum_n \epsilon_n |\epsilon_n\rangle\langle\epsilon_n|, \quad \epsilon_0 \leq \epsilon_1 \leq \epsilon_2 \leq \dots, \end{array} \quad (47)$$

so that $E[\tilde{\rho}_M] = \sum_n r_n \epsilon_n$. Note that, if $\epsilon_0 = 0$ and the state ρ_M is pure, then $E(\tilde{\rho}_M) = 0$ and, thus, the ergotropy coincides with the energy of ρ_M . On the other hand, if the state ρ_M is mixed, one generally finds $\mathcal{E}_M(t) < E[\rho_M(t)]$ and, thus, the extractable work is smaller than the actual energy of ρ_M .

5.2 The Dicke quantum battery

We first consider the probably most iconic example of a QB, provided by the Dicke model, which describes an array of N two-level atoms inside a resonant cavity [79]. For simplicity, we focus on the case in which the laser in the cavity is resonant with the relevant atomic transition of the atoms (zero-detuning condition); we also ignore all the other modes in the cavity. Here, the atoms will play the role of the quantum cells and their interaction with the light will constitute the charging mechanism of the battery. The system of interest is described by a time-dependent Hamiltonian of the same form as in Eq. (44)

$$H(t) = \omega \left[a^\dagger a + J^z + 2\lambda(t)(J^+ + J^-)(a^\dagger + a) \right], \quad (48)$$

where $a^{(\dagger)}$ are the canonical annihilation (creation) operators for photons in the cavity, while $J^\alpha = \frac{1}{2} \sum_j \sigma_j^\alpha$ (with $\alpha = x, y, z, +, -$) is the total spin of the atomic component. The first term in Eq. (48) is the energy of the resonating mode and the second one denotes the energy of the atoms, while the third one quantifies the dipolar light–matter interaction. In the rotating wave approximation, valid for a weak light–matter coupling, all the interaction terms that do not conserve the number of excitations (as J^+a and $Ja^\dagger$) can be neglected. In that case, during the whole dynamics, the system maintains the same number excitations of the initial state and the Dicke model reduces to the well known, and much easier to solve, Jaynes–Cummings model.

Before discussing the proper QB Dicke model, as described by Eq. (48), we focus on a simplified scenario in which, instead of taking a single cavity with all the atoms inside, one puts each of the N atoms in a separate cavity. All the cavities are treated independently, so that one realizes a so-called “parallel charging” ($\parallel$) protocol. In this situation, the j -th atom ($j = 1, \dots, N$) is described through the reduced local Hamiltonian

$$h_j = \omega \left[a_j^\dagger a_j + \frac{1}{2} \sigma_j^z + \lambda(t) (\sigma_j^+ + \sigma_j^-) (a_j^\dagger + a_j) \right]. \quad (49)$$

The total energy of N atoms in N cavities is thus equal to N times the one of a single atom and its corresponding cavity, so that $E_N^\parallel(\tau) \propto N$. Likewise, the power is linear in the system size: $P_N^\parallel(\tau) \propto N$.

Coming back to the QB Dicke model of Eq. (48), photons inside the only cavity are able to mediate interactions between the N atoms, so that one realized a so-called “collective charging” ($\perp$) protocol. Starting from an initial factorized state of the form $|\psi_N(0)\rangle = |g, \dots, g\rangle_N \otimes |N\rangle$, where $|g\rangle$ denotes the ground state of one atom and $|N\rangle$ is the Fock state of N photons in the cavity, one finds again an extensive energy $E_N^\perp(\tau) \propto N$, but the scaling of the power is superextensive: $P_N^\perp(\tau) \propto N^{3/2}$. A more accurate analysis shows that the latter result is an artifact due to the improper dependence of the light–matter coupling term in Eq. (48) with the system size. In fact, to ensure extensivity of the system, one should rescale the coupling constant $\lambda(t)$ by a $1/\sqrt{N}$ factor, so that the interaction strength decreases with the number of atoms. In such case, the power $P_N^\perp(\tau)$ would again scale linearly with N , so that the collective charging would behave analogously as the parallel charging, apart from a prefactor independent of the size [80].

Several other QB toy models have been proposed and tested, using different interaction Hamiltonians to inject energy into the quantum cells (see, e.g., the review [78]). Unfortunately, the typical local interactions of quantum many-body systems do not provide a charging speedup that is genuinely induced by the entanglement developed along the dynamics; on the other hand, this is rather given by the lack of a well-defined thermodynamic limit, as for the ill-defined Dicke QB. The SYK interaction falls beyond this paradigm and presents qualitative differences, as we discuss below [81, 82].

5.3 The SYK model as a quantum battery

The key point to promote the SYK model to a toy system realizing a conceptual QB that outperforms more conventional schemes resides in adopting its random two-particle interaction term [cf. Eq. (7)] as the charging Hamiltonian

$$H_I = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,l=1}^N \tilde{J}_{ijkl} c_i^\dagger c_j^\dagger c_k c_l. \quad (50)$$

After performing the mapping of Sec. 4.2 from the original fermionic formulation into a spin-1/2 network, one can identify each cell of the QB with a single mapped spin [81]. The battery, when not in charge, should be described by a time-independent spin Hamiltonian which does not commute with $\mathcal{H}_I$. By convention, we consider

$$H_0 = \omega J^y = \frac{\omega}{2} \sum_{j=1}^N \sigma_j^y, \quad (51)$$

so that $[H_0, H_I] \neq 0$. As for the initial state, we take the ground state of H_0 , $|\psi_N(0)\rangle = \bigotimes_{j=1}^N |\leftarrow\rangle_j$, with $\sigma_j^y |\leftarrow\rangle_j = -|\leftarrow\rangle_j$. Extensive numerical calculations with up to $N = 16$ spins have shown that a QB based on such SYK charging scheme displays a genuine superextensive scaling of its power output with N [81]. This property is directly associated with the chaoticity of the model and its extremely fast thermalization time.

Focusing on the spectrum of H_0 , one has $N + 1$ equally spaced energy levels, with a degeneracy associated to the k th level that is equal to $g_k = \binom{N}{k}$. The spectral decomposition of the Hamiltonian can be cast as a simple summation over the k levels and their degeneracy index i , which labels g_k -degenerate states

$$H_0 = \sum_k \varepsilon_k \sum_i |k, i\rangle \langle k, i|. \quad (52)$$

The populations of the various states along the charging process are expressed as

$$p_k(\tau) = \sum_i |\langle \psi(\tau) | k, i \rangle|^2, \quad \text{with} \quad |\psi(\tau)\rangle = e^{-iH_I\tau} |\psi(0)\rangle. \quad (53)$$

Numerical results for the $p_k(\tau)$ probabilities displayed in Fig. 5(a) show that the occupation of the various states becomes uniform after a short transient time, in which a collective charging rapidly sets in. The levels are populated in a homogeneous and random way, as is typical for a chaotic and scrambling model like the SYK one. In fact, it can be shown that the population p_k of each single energy level is proportional to its degeneracy, so that $p_k(\tau) \approx \frac{1}{2^N} \binom{N}{k}$. In conclusion, the energy of the fermionic-SYK QB model preserves extensivity with N , such that $E^\#(\tau) \propto N$, while the power behaves superextensively as

$$P_N^\#(\tau) \propto N^{3/2}. \quad (54)$$

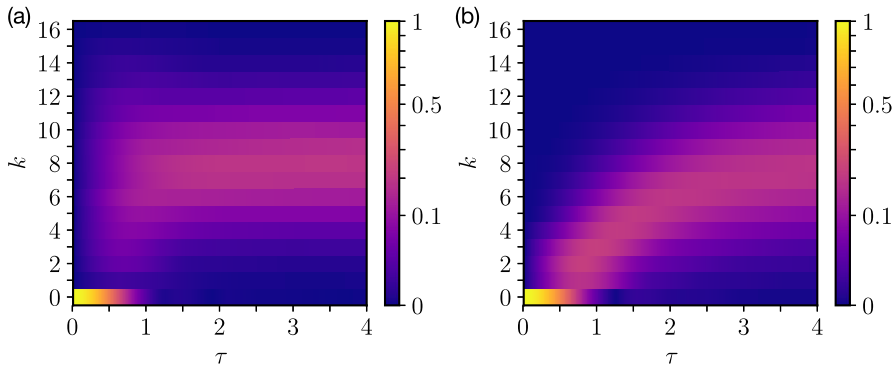


Fig. 5 The population p_k of each energy level of the QB as a function of the index k and of the charging time τ . Panel (a) is for a fermionic-SYK charging scheme, while panel (b) shows the same result for the bosonic-SYK counterpart. The reported data are for QBs with $N = 16$ cells and correspond to a single disorder realization in the couplings. Adapted from Ref. [81]

We conclude by noticing that the enhanced performance of the SYK charging protocol over parallel charging, or over more conventional QB models exhibiting extensivity in the power, is lost if fermions in Eq. (7) are replaced by bosons ($c_j \rightarrow b_j$). In fact, the charging protocol for a modified bosonic version of the SYK interaction Hamiltonian generates a dynamics that is clearly local in energy space, as reported in Fig. 5(b). More specifically, its poor performance with respect to the fermionic-SYK counterpart suggests that random pair hopping is not enough to warrant superextensivity. On the other hand, non-local JW strings for fermions are crucial, as they maximize entanglement production during the time evolution and thus correlations between the various quantum cells.

6 Experimental platforms

The quantum simulations presented in the previous sections can, in principle, be executed on any suitable experimental platform that can realize the required Hamiltonian evolution. In this section, we will briefly review what it takes to build such a quantum simulator or quantum computer.

When Richard Feynman, at the Physics of Computation conference in 1981 mentioned in Sect. 2, first asked the question whether it was possible to simulate (quantum) nature on a classical computer, his answer was a clear “no” [21]. Feynman realized that because of the exponential scaling of the size of the Hilbert space with the number of constituents due to entanglement, simulating or calculating the quantum evolution of large systems ultimately make it necessary to use computers that also exploit the laws of quantum physics. Since then, building such computers has become not just an academic challenge, but also a commercial quest for “supercomputers” that could revolutionize many aspects of science and technology.

The ultimate test of a quantum computer or simulator is the so-called quantum advantage (or quantum supremacy) [83]: a proof that a quantum computer can solve

a problem for which there is no efficient classical algorithm or, more simply, that a quantum computer can calculate something (that may or may not be practically useful) that a classical computer cannot calculate in a reasonable time and with reasonable resources (also called “practical quantum advantage” [84]). At the time of writing of this review, no convincing claim of academically or commercially relevant quantum advantage has been made, and the jury is still out on the question of the ideal physical platform for quantum computation and simulation. A number of comprehensive overviews have been published in the past few years in which all the physical realizations tried to date are explained in great detail, and we refer the interested reader to those reviews for a detailed picture of the state-of-the art and future perspectives, both from a practical and fundamental point of view [14, 15, 19, 20].

Generally speaking, to build a quantum computer (in the following by “quantum computer” we will always mean “quantum computer or simulator”), one first needs to have a physical system in which one can clearly identify two (out of possibly many) quantum states that can be assigned the quantum-logical values $|0\rangle$ and $|1\rangle$ —the quantum bit or qubit. This is the first of the well-known DiVincenzo criteria [85], which also includes the need for scalability—it must be feasible to have not just one, but many qubits in a quantum computer to perform complex tasks. Furthermore, it must be possible to initialize the qubits in some well-defined initial state, and also to implement a set of universal one- and two-qubit quantum gates in a time that is much shorter than the coherence time of the qubits. Finally, there needs to be a protocol for the read-out of the final state of the system.

By now, several physical systems have been shown to satisfy the DiVincenzo criteria. These systems fall into two broad categories: natural and man-made. The natural qubit systems are atoms (or ions), molecules and photons, whereas the man-made ones include superconducting (Josephson junction) qubits, quantum dots, and exotic materials that are believed to host Majorana fermions which feature topological protection against decoherence.

Historically, the first demonstrations of quantum gates and quantum algorithms were achieved using trapped ions [86] and molecules manipulated by NMR techniques [87]. While NMR-based quantum computation has essentially been abandoned because of the obvious problems regarding scalability, trapped ions continue to be intensively investigated as possible candidates for large-scale quantum computers, along with neutral Rydberg atoms, superconducting qubits, and photons. In the following, we will give a concise overview of these four approaches.

6.1 Trapped ions and Rydberg atoms

The idea behind using atoms as physical systems for realizing qubits is simple: one identifies some suitable energy levels of the atoms that are used to represent $|0\rangle$ and $|1\rangle$, and finds ways to induce a coupling between them to control the state of the qubit and to perform the read-out. Also, there has to be some interaction between the atoms that allows for the realization of two-qubit gates, and one has to be able hold the atoms in place for a sufficiently long time during which the gate operations or quantum evolution take place.

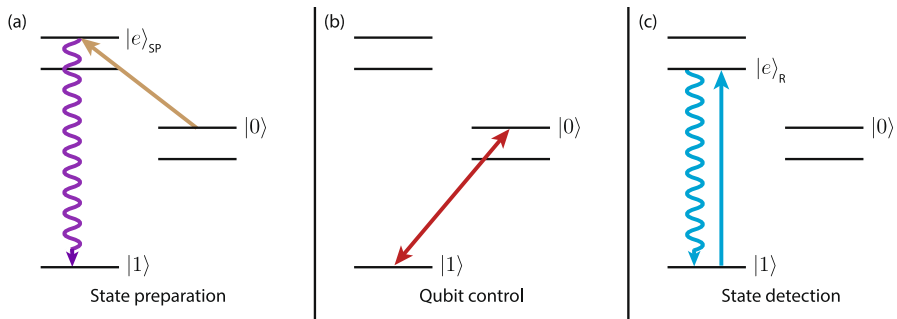


Fig. 6 In trapped ion quantum computing, qubits can be initialized by optical pumping from a long-lived state via a fast-decaying state (a). The state of the qubits can be manipulated via laser coupling (b), and read-out can be achieved by selectively coupling one of the qubit states to an excited state and observing the fluorescence from that state. From Ref. [13]

One practical solution that satisfies these requirements is the use of ions trapped in electric traps [88], either of the Paul [89] or, more recently, the Penning type [90]. Once the ions are trapped, they are cooled down to micro-Kelvin temperatures using laser-cooling techniques. The electric charge on the ions naturally leads to a Coulomb interaction between them. This makes it possible to implement, for instance, a controlled-NOT quantum gate. The first demonstration of this gate was done by the group of David Wineland in 1995 [86], where a single beryllium ion was used: the two qubits were represented by two internal states of the ion and its two lowest vibrational states. Gate operations based on the shared motional modes of two or more ions mediated by the Coulomb interaction, leading to entanglement between the ions, were demonstrated a few years later [13].

Depending on the energy levels chosen to define the qubit—hyperfine states Zeeman sublevels, or states separated by energies corresponding to optical wavelengths—the coherence times of ion qubits range from hundreds of seconds to a fraction of a second. Even at the lower end, though, the coherence times are much longer than the times required to execute a one- or two-qubit gate, which are between a few microseconds and around a hundred microseconds [13].

Initially, the $|0\rangle$ or $|1\rangle$ states of the qubit are prepared via optical pumping, e.g., from some long-lived state to a short-lived state from which the ion can rapidly decay into the desired initial state (see Fig. 6). Read-out of a trapped ion qubit is achieved by driving a transition that is resonant, e.g., with the $|1\rangle$ state of the qubit but not with the $|0\rangle$ state and observing the fluorescence induced by the driving. When fluorescence is detected, one can infer that the qubit is in the $|1\rangle$ state.

Some recent advances in ion trap quantum computing include the realization of a scalable version of Shor's factoring algorithm [91] using 11 qubits and the demonstration of a racetrack ion trap architecture in which entangled states of 32 qubits could be realized [92]. Quantum simulations have also been carried out on trapped ion devices, such as the digital simulation of the Schwinger model (1+1 dimensional quantum electrodynamics) [93], and quantum simulations of high-energy physics are also envisioned on such devices [18].

A more recent approach involving atomic qubits is based on Rydberg atoms [5, 94] trapped in arrays of optical dipole traps, also called optical tweezers [95]. In a Rydberg atom, an outer electron is excited to a high-lying energy state with a principal quantum number $n > 10$. Such Rydberg states have much longer lifetimes than low-lying energy states. In alkali atoms, for example, these lifetimes can be several hundreds of microseconds as opposed to tens of nanoseconds. Rydberg atoms are (in general) charge-neutral, so to realize two-qubit gates, one needs an interaction that replaces the Coulomb repulsion of ions. That interaction is the van der Waals interaction between Rydberg atoms, which is much stronger than that between ground-state atoms and scales as n^{11} . Again, taking alkali atoms as an example, the van der Waals interaction between Rydberg atoms with a principal quantum number around 70 at a distance of $10\text{ }\mu\text{m}$ can be tens of MHz, compared to a few milli-Hertz for the ground state.

Neutral atoms can be trapped in free space using the ac-Stark shift (or light shift) of their energy levels induced by, e.g., a laser beam that is detuned with respect to an atomic transition [96]. If the intensity of the laser varies in space, the resulting energy gradient gives rise to a force which, depending on the detuning, can either attract the atom to regions of higher intensity (negative or red detuning) or repel them from those regions (positive or blue detuning). For a focused, red-detuned Gaussian beam atoms are trapped around the focus of the beam.

Arrays of red-detuned optical dipole traps can be created using spatial light modulators, micromirror devices, or acousto-optical deflectors (or a combination thereof) [97]. The distance between the traps is typically a few micrometers. The traps are loaded from a cloud of cold atoms, typically a magneto-optical trap. This loading process is random, and there will be a distribution of atom numbers in the individual traps depending on the density of the atomic cloud. If the waists of the dipole trap beams are on the order of $1\text{ }\mu\text{m}$, light-assisted collisions between atoms can lead to the occupation of a dipole trap by exactly one atom. Initially empty traps can be eliminated by re-arranging the traps in the array (the empty traps are identified by fluorescence imaging).

Once a defect-free array of trapped atoms has been created, quantum operations can be carried out using a combination of (possibly detuned) excitation lasers that couple the ground state of the atoms to a Rydberg state, additional focused laser beams that induce local ac-Stark shifts, and microwave radiation. During this process, the dipole traps are temporarily switched off.

At the end of a computation, read-out is achieved by switching the dipole traps back on, thus re-capturing those atoms that are in the ground state. Atoms in the Rydberg state, by contrast, will not be re-captured as the dipole trap lasers do not induce an appropriate ac-Stark shift on them (except in the special case of so-called magic wavelengths, which are typically not used in the experiments referred to here). After a few milliseconds, those atoms that are not trapped will fall down due to gravity and leave an empty trap behind. A resonant laser is now briefly flashed on, and the resulting fluorescence can be captured on a CCD to identify which traps are full and which ones are empty.

Arrays of trapped Rydberg atoms lend themselves particularly well to the realization of analogue quantum simulators. In such simulators, the evolution of a quantum system is studied by implementing a desired (possibly time-dependent) Hamiltonian $H(t)$

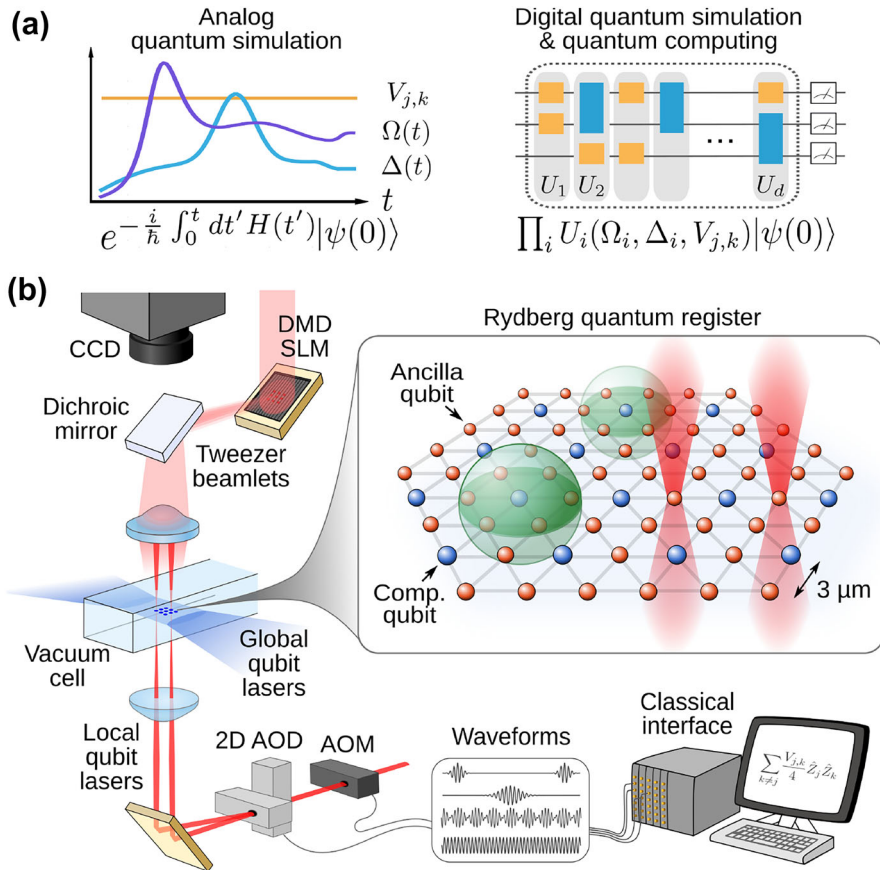


Fig. 7 (a) An array of trapped Rydberg atoms can be used to carry out analogue or digital quantum simulations. (b) In a typical experimental setup, atoms are held in an array of optical dipole traps. Read-out is achieved by imaging the fluorescence of re-trapped atoms on a CCD camera. From Ref. [97]

on the Rydberg array, choosing the geometry of the dipole traps and appropriate parameters for the driving. These parameters can also be varied in time, for instance by varying the driving strength or the detuning of the driving lasers from resonance. The Rydberg array effectively calculates the time evolution operator $\exp[-\frac{i}{\hbar} \int_0^t dx H(t')]$. After some evolution time, information on the system can be read out as described above. An example of a quantum simulation carried out on such a system is the recent work by the Lukin group [98], in which a Rydberg quantum simulator with more than 200 qubits was used to probe quantum spin liquid states in a kagome lattice.

Figure 7 shows a schematic view of a typical experimental setup for Rydberg quantum simulation. In such simulators, it is also possible to implement digital quantum simulation, i.e., using the Lie–Trotter–Suzuki decomposition to break down the time evolution of the system into small steps, each represented by a unitary operator. This approach is equivalent to the execution of a quantum algorithm on a quantum computer.

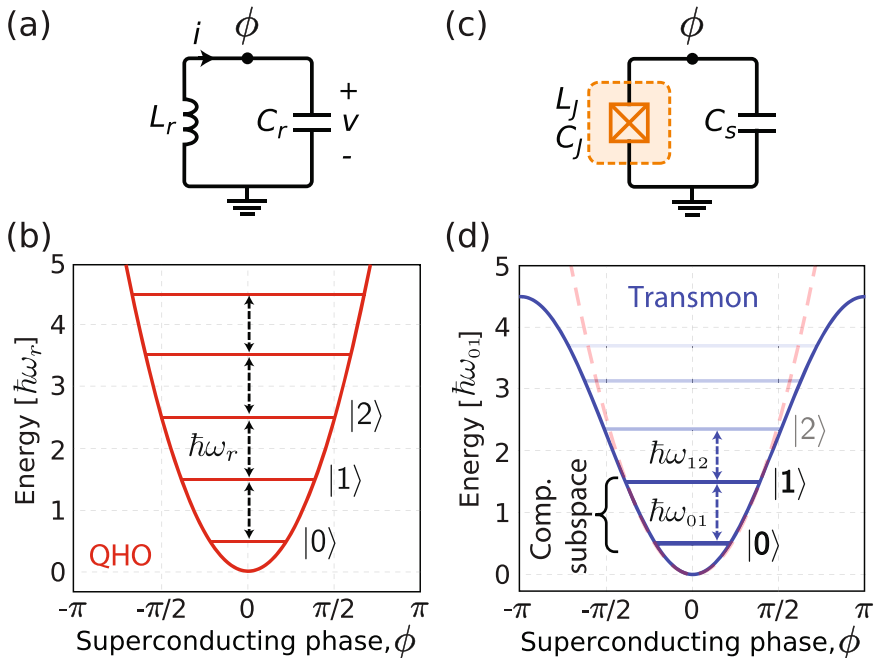


Fig. 8 The energy spectrum of a superconducting LC circuit consists of equally spaced levels (a). To be able to use such a circuit as a qubit, it is necessary to introduce a nonlinear element—the Josephson junction—into the circuit (b). This nonlinearity allows one to selectively drive transitions between the two lowest energy levels. From Ref. [99]

6.2 Superconducting platforms

When an electric LC circuit is cooled down sufficiently so that it becomes superconducting, the current oscillations in it can be quantized. The corresponding energy levels are those of a quantum harmonic oscillator and, therefore, evenly spaced (see Fig. 8). Such an energy spectrum does not lend itself immediately to the realization of a qubit, since any external driving needed to manipulate the qubit states would act on all pairs of adjacent energy levels in the same way. If the qubit is initially prepared in the lowest oscillatory state, an external drive can excite it not just to the first excited level, but successively to any higher energy level.

To make superconducting circuits useful for quantum computation, it is necessary to introduce a nonlinear element into the circuit to make the energy spacing different between adjacent levels [99]. This nonlinear element is the Josephson junction, which consists of two superconductors separated by a thin insulating layer. Using a combination of inductances, capacitances, and Josephson junctions, it is possible to make superconducting qubits tailored specifically to certain applications, such as the transmon qubit which is now widely used [14]. Single-qubit gates can be executed by coupling microwaves with appropriate frequencies and phases to the qubits, thus inducing Rabi oscillations (i.e., rotations on the Bloch sphere) around a desired axis.

Two-qubit gates can be realized using a number of different techniques, such as parametric driving of a coupling element between two adjacent superconducting qubits that induces a tunable coupling between the qubits.

Read-out of superconducting qubits is achieved by coupling the qubit to a resonator whose resonant frequency is detuned with respect to that of the qubit. In this way, there is no energy exchange between the qubit and the resonator, but the state of the qubit can be inferred from the shift in the resonator resonance induced by it. This is also referred to as dispersive read-out.

Superconducting qubits can be manufactured using techniques that have been developed for decades by the semiconductor industry, such as lithographic patterning, deposition, etching, and controlled oxidation [14]. Several companies have adopted superconducting qubits as their platform of choice, such as IBM, Google, Rigetti, and D-Wave. Since 2016, IBM offers a cloud service that allows users to run quantum algorithms on processors with a handful of qubits [100].

In general, quantum computing platforms based on superconducting qubits are designed to run quantum algorithms in terms of one- and two-qubit gates executed sequentially. This includes digital quantum simulation (see above), in which the evolution of a quantum system is realized by breaking the time evolution operator down into small parts. However, superconducting qubits can also be used for analogue quantum simulation. For instance, one-dimensional Bose–Hubbard chains have been experimentally studied [101] using a 12-qubit quantum processor. The Bose–Hubbard model was realized through quantum walks of photons on a one-dimensional array of 12 coupled superconducting transmon qubits [102].

Another analogue approach to quantum computation is quantum annealing, in which the qubits and the couplings between them are tuned in such a way as to continuously vary the parameters of the Hamiltonian implemented by the qubits. The quantum computation then consists in performing that variation without exciting the system, and thus eventually finding its ground state. This approach has been followed mainly by D-Wave, which used quantum annealing on a 2000-qubit device to measure the phase diagram of a spin glass [103] and observe the Kosterlitz–Thouless phase transition on a frustrated Ising model [104].

6.3 Photons

Photons have been used extensively in quantum communication over the past forty years, for instance in quantum key exchange (quantum cryptography) and quantum teleportation. More recently, they have also been considered for applications in quantum simulation and computing [105]. Several schemes have been developed to encode quantum information in photons: through their path of propagation, polarization, orbital angular momentum, or time (via time-binning) (see Fig. 9).

Path encoding amounts to assigning the qubit values $|0\rangle$ and $|1\rangle$ to specific directions of propagation, for instance through waveguides on a chip. Manipulation of the qubits is achieved by coupling the waveguides to each other, thus realizing beam-splitters.

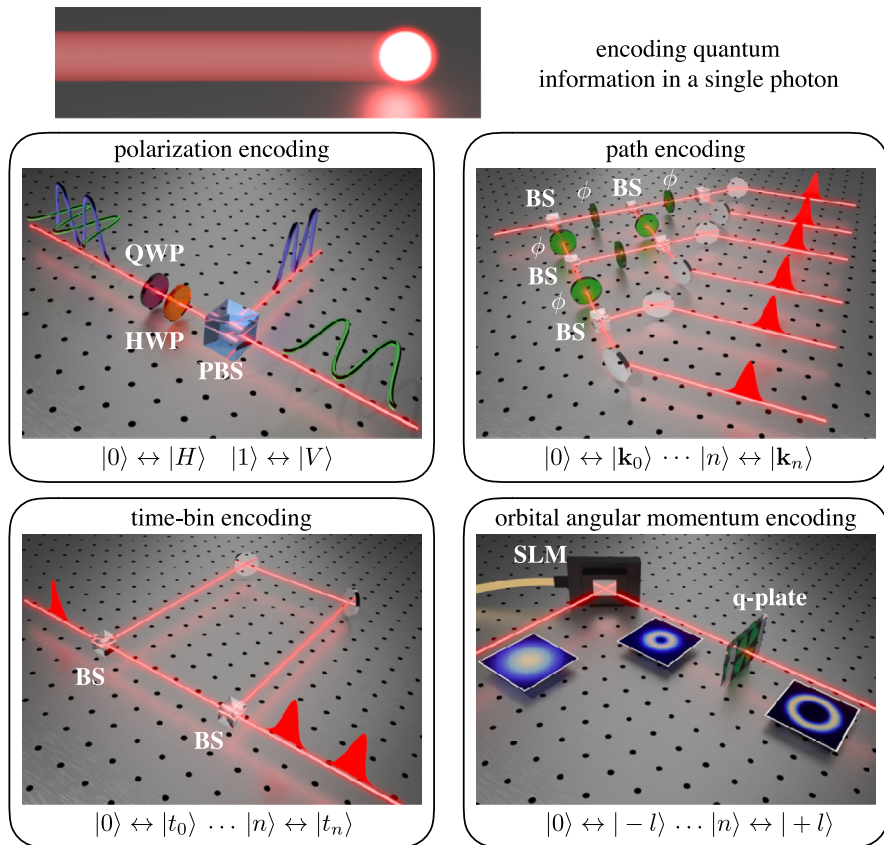


Fig. 9 Photons can be used as qubits using different types of encoding: polarization, path, time-bin, and orbital angular momentum. The state of the photon qubit can be controlled with waveplates (quarter-wave plate QWP, half-wave plate HWP), polarizing beam-splitters (PBS), beam-splitters (BS), and q-plates (which couple the polarization of a photon to its orbital angular momentum). From Ref. [12]

Polarization encoding associates the qubit values with, for instance, horizontal or vertical polarization of the photons $|H\rangle$ or $|V\rangle$. Equivalently, a rotated linear polarization basis can be chosen, such as $|\pm\rangle = \frac{1}{\sqrt{2}}(|H\rangle \pm |V\rangle)$, or a basis $|L/R\rangle = \frac{1}{\sqrt{2}}(|H\rangle \pm i|V\rangle)$ of left- or right-handed circularly polarized photons. The qubits in this representation can be manipulated using waveplates.

Orbital angular momentum (OAM) encoding is based on the spatial structure of the wavefront of the light, such as in Laguerre–Gauss beams, which have a phase profile corresponding to an optical vortex (or helical mode) with a phase term $e^{-iq\theta}$. Here, θ is the angular coordinate and q is an integer describing the phase winding. A single photon has an OAM given by $L = q\hbar$.

Finally, time-bin encoding is realized using a Mach–Zehnder interferometer with two arms of different lengths. The two qubit states then correspond to the photon having taken either the long or the short arm.

An obvious difference between photonic qubits and the other physical realizations presented above is that photons do not interact with each other—unless, for instance, they propagate inside a nonlinear material. At first glance, this might seem to violate one of the DiVincenzo criteria, namely that a two-qubit gate is needed to have a universal set of quantum gates. However, in 2001, it was shown by Knill, Laflamme, and Milburn [106] that universal quantum computing is possible using only linear operations involving beam-splitters, phase shifters, single photon sources, and photo-detectors. A few years later, Childs [107] demonstrated that any quantum computation can be recast in terms of a quantum walk [108].

6.4 Prospects for the future

As mentioned in the introduction to this section, despite considerable progress has been made in quantum computing and quantum simulation over the past 30 years, there is no clear “winner” among the physical platforms described above. All of them have their pros and cons: for instance, superconducting qubits achieve high gate fidelities but need sophisticated cryogenics. Trapped ions also achieve high gate fidelities but are not straightforward to scale to large numbers (hundreds or thousands) of qubits. Rydberg-atom-based systems, on the other hand, are not currently suitable for gate-based quantum computing due to lower gate fidelity, but are being successfully used as quantum simulators with good scaling properties.

In terms of technological maturity and market-readiness, one might argue that quantum computers based on superconducting qubits and trapped ions as well as those based on Rydberg atoms are currently the frontrunners, with companies already selling products based on these technologies (as opposed to photonic or quantum dot approaches, which have not reached that level yet). However, the field is still very much evolving, as witnessed, for instance, by the recent announcement (February 2025) by Microsoft that they have realized a chip containing eight topological qubits [109, 110]. Such qubits, based on quasiparticles also known as Majorana fermions, are topologically protected from decoherence and could, therefore, be used to implement quantum algorithms without the need for error correction. In other platforms, error correction requires a considerable overhead of qubits with tens or hundreds of physical qubits per logical qubit. If the number of topological qubits could be scaled to the hundreds of thousands, this would give topological quantum computing a huge advantage over all other approaches. In the meantime, research on superconductors, ions, etc. will no doubt continue and enable the realization on small-scale quantum simulators of some of the models discussed in this review.

7 Summary

Guided by our feelings and interests, we have given a perspective on a few modern aspects of quantum simulation, involving the possibility to exploit peculiarities of complex quantum many-body systems for quantum technologies. We first contextualized our dissertation in the history of conventional quantum information processing and

quantum computing, whose origins can be dated back to the early 1980s, emphasizing some modern approaches, such as quantum neural networks and reservoir computing. As a concrete example of a simulatable complex system, we focused in some detail on the Sachdev–Ye–Kitaev model, a paradigm for understanding a variety of physical situations, ranging from the behavior of strange metals (in condensed-matter physics) to that of black holes (in high-energy physics). We showed how the fast scrambling chaotic dynamics of the SYK model can be exploited to boost the performances of the so-called quantum batteries, i.e., quantum devices that might be capable to store energy more efficiently than with classical logic. Finally, we provided a description of some promising experimental platforms to be realized in the lab.

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Code availability Not applicable.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose. The authors have no conflict of interest to declare that are relevant to the content of this article.

Ethics approval and consent to participate Not applicable.

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