ON ALMOST POLYNOMIAL GROWTH OF PROPER CENTRAL POLYNOMIALS

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Abstract. To any associative algebra A is associated a numerical sequence $c_n^\delta(A)$, $n \geq 1$, called the sequence of proper central codimensions of A. It gives information on the growth of the proper central polynomials of the algebra. If A is a PI-algebra over a field of characteristic zero it has been recently shown that such a sequence either grows exponentially or is polynomially bounded.

Here we classify, up to PI-equivalence, the algebras A for which the sequence $c_n^\delta(A)$, $n \geq 1$, has almost polynomial growth. Then we face a similar problem in the setting of group-graded algebras and we obtain a classification also in this case when the corresponding sequence of proper central codimensions has almost polynomial growth.

1. Introduction

The study of the growth of the central polynomials for an algebra A was started by Regev in [29]. It turns out that even if it is strictly related to the growth of the polynomial identities satisfied by A, still gives another invariant of the corresponding T-ideal of the free algebra.

Let A be an algebra over a field of characteristic zero and $\mathrm{Id}(A)$ the T-ideal of the free algebra $F\langle X\rangle$ consisting of the polynomial identities satisfied by A. Recall that a polynomial $f \in F\langle X\rangle$ is a central polynomial of A if for any evaluation $\varphi : F\langle X\rangle \to A$, $\varphi(f)$ lies in the center Z of A. If f takes a non-zero value in A, we say that f is a proper central polynomial of A, otherwise we say that f is a polynomial identity of A.

For instance it is known that $M_k(F)$, the algebra of $k \times k$ matrices over F, has proper central polynomials as shown by Formanek in [13] and Razmyslov in [27].

Now, one studies the growth of the identities and central polynomials as follows. Fix n variables and let P_n be the corresponding space of multilinear polynomials. Then define $P_n(A)$ as P_n modulo the identities of A and $P_n(A)^z$ as P_n modulo the central polynomials of A. The corresponding dimensions $c_n(A) = \dim P_n(A)$ and $c_n^z(A) = \dim P_n(A)^z$ are called the n-th codimension and the n-th central codimension of A, respectively. Clearly $c_n(A) - c_n^z(A) = c_n^\delta(A)$ is the dimension of the corresponding space of proper central polynomials.

The asymptotics of the codimensions of an algebra A have been extensively studied in the past, see for instance [28], [3], [8], [9] or [18] and [5] for a comprehensive account of the main results. It turns out that if A satisfies a non-trivial identity, then $\lim_{n\to\infty} \sqrt[n]{c_n(A)} = exp(A)$ exists, and is an integer called the PI-exponent of A ([16]). In recent years in [19] and [20] it was proved that also $\lim_{n\to\infty} \sqrt[n]{c_n^z(A)} = exp^z(A)$ and $\lim_{n\to\infty} \sqrt[n]{c_n^\delta(A)} = exp^\delta(A)$ exist, are integers called the central exponent and the proper central exponent of A, respectively.

When $exp(A) \geq 2$, $exp(A)$ and $exp^z(A)$ coincide, whereas it can be shown that the difference $exp(A) - exp^\delta(A)$ can be any positive integer ([19]). Hence $exp^\delta(A)$ is a second invariant of the T-ideal $\mathrm{Id}(A)$ that is worth investigating.

Actually the existence of the three exponents is shown by proving that the corresponding n-th codimensions are sandwiched between two functions of the form $Cn^t d^n$ where C is a nonzero constant and d is the corresponding exponent. This says that all three codimensions either grow exponentially or are polynomially bounded.

Kemer in [22] proved that if an algebra A is such that $exp(A) \geq 2$, then either $\mathrm{Id}(E) \supseteq \mathrm{Id}(A)$ or $\mathrm{Id}(UT_2) \supseteq \mathrm{Id}(A)$. Here E is the infinite dimensional Grassmann algebra and UT_2 is the algebra of 2×2 upper triangular matrices and it is well-known that $exp(E) = exp(UT_2) = 2$ (see [18]). Hence if B is an algebra such that $\mathrm{Id}(B) \supsetneq \mathrm{Id}(E)$ or $\mathrm{Id}(B) \supsetneq \mathrm{Id}(UT_2)$ then the codimensions of B grow polynomially. In the language of varieties of algebras, this says that E and UT_2 generate the only two varieties of almost polynomial growth.

Here we address ourselves to the analogous question in the setting of proper central polynomials by asking the following: can one construct a finite list of PI-algebras $B_1, \dots, B_k$ with the following property: given any PI-algebra A, then $exp^\delta(A) \geq 2$ if and only if $\mathrm{Id}(B_i) \supseteq \mathrm{Id}(A)$, for some $i = 1, \dots, k$? These algebras would have the property

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that $exp^{\delta}(B_i) \geq 2$ and for any algebra B such that $\mathrm{Id}(B) \supsetneq \mathrm{Id}(B_i)$, the corresponding sequence of proper central codimensions grows polynomially.

Notice that since $exp^{\delta}(UT_2) = 0$ and $exp^{\delta}(E) = 2$, only E is a possible candidate.

Here we construct two finite dimensional algebras D and D_0 having exponential growth of the proper central codimensions and give a positive answer to the above question. It will follows that E, D and D_0 generate the only varieties of almost polynomial growth of the proper central polynomials.

The existence of the proper central exponent has been recently proved for algebras graded by a finite abelian group ([25]). Here we also ask the above question for this class of algebras and we get a complete answer by classifying the ideals of identities whose corresponding varieties have almost polynomial growth of the proper central codimensions.

2. Algebras of almost polynomial δ-growth

Throughout F is a field of characteristic zero and $F\langle X\rangle$ is the free algebra over F on a countable set $X = \{x_1, x_2, \ldots\}$. Recall that if A is an F-algebra, $\mathrm{Id}(A)$ is the T-ideal of polynomial identities of A and $\mathrm{Id}^z(A)$ is the T-space of central polynomials of A. Here "T-" stands for transformation and means that the corresponding ideal or space is invariant under all the endomorphisms of the free algebra. Moreover if $P_n \subset F\langle X\rangle$ is the space of multilinear polynomials in $x_1, \ldots, x_n$, then for $n = 1, 2, \ldots$, $c_n(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)}$, $c_n^z(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}^z(A)}$ and $c_n^{\delta}(A) = \dim \frac{P_n \cap \mathrm{Id}^z(A)}{P_n \cap \mathrm{Id}(A)}$ are the sequences of codimensions, central codimensions and proper central codimensions of A, respectively.

Since the above codimension sequences do not change by extension of the base field, throughout this section we assume that F is algebraically closed.

As shown in [15], [16], [19] and [20] for any PI-algebra A the following three limits

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}, \quad exp^z(A) = \lim_{n\to\infty} \sqrt[n]{c_n^z(A)}, \quad exp^{\delta}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{\delta}(A)}$$

exist and are called the (PI)-exponent, the central exponent and the proper central exponent of the algebra A, respectively. Clearly $exp^z(A), exp^{\delta}(A) \leq exp(A)$ and in [20, Theorem 3] it was shown that if $\exp(A) \geq 2$, then $exp^z(A) = exp(A)$. Another basic property of the above three numerical sequences is that they either grow exponentially or are polynomially bounded. Hence no intermediate growth is allowed.

There is an explicit way of computing the three limits and here we need to recall how to compute the exponent and the proper central exponent.

First we need to establish a well-known general setting as follows. Let E be the infinite dimensional Grassmann (or exterior) algebra. E is the algebra with 1 generated by countably many elements $e_1, e_2, \ldots$, subject to the condition $e_i e_j = -e_j e_i$. E has a natural structure of superalgebra $E = E^{(0)} \oplus E^{(1)}$ where every monomial in the e_i's of even (odd) length has homogeneous degree zero (one resp.).

If $B = B^{(0)} \oplus B^{(1)}$ is a superalgebra, one can construct the algebra $E(B) = (E^{(0)} \otimes B^{(0)}) \oplus (E^{(1)} \otimes B^{(1)})$ which is called the Grassmann envelope of B. By a well-known result of Kemer ([23]) if A is any PI-algebra, there exists a finite dimensional superalgebra B such that $\mathrm{Id}(A) = \mathrm{Id}(E(B))$. Hence throughout when establishing results about a T-ideal we may assume that we are dealing with the T-ideal of identities of the Grassmann envelope $E(B)$ of a finite dimensional superalgebra B.

It is worth noticing that if A is a finite dimensional algebra, then regarding A as an algebra with trivial grading we get $\mathrm{Id}(E(A)) = \mathrm{Id}(E^{(0)} \otimes A) = \mathrm{Id}(A)$ and we may work with A instead of $E(A)$.

Now let $B = \bar{B} + J$ be the Wedderburn-Malcev decomposition of the algebra B, where $\bar{B}$ is a maximal semisimple superalgebra inside B and J is the Jacobson radical of B (see [18, Theorem 3.4.3]). Write $\bar{B} = B_1 \oplus \cdots \oplus B_q$, a decomposition into simple superalgebras. Then we say that $C = B_{i_1} \oplus \cdots \oplus B_{i_k}$, where the B_{i_h}'s are distinct, is an admissible subalgebra of B if $B_{i_1} J \cdots J B_{i_k} \neq 0$. Moreover we say that C is a centrally admissible subalgebra of B, if there exists a proper central polynomial $f(x_1, \ldots, x_n)$ of $E(B)$ having a non-zero evaluation involving some elements of $E(B_{i_h})$, for all $h \in \{1, \ldots, k\}$.

Then $exp(E(B))$ is the maximal dimension of an admissible subalgebra of B and $exp^{\delta}(E(B))$ is the maximal dimension of a centrally admissible subalgebra of B.

Recall that if $\mathcal{V} = var(A)$ is the variety of algebras generated by an algebra A, then we write $exp(\mathcal{V}) = exp(A)$ and $exp^{\delta}(\mathcal{V}) = exp^{\delta}(A)$. Hence the growth of $\mathcal{V}$ is the growth of the codimensions of the algebra A.

Now, a variety $\mathcal{V}$ has almost polynomial growth if $\mathcal{V}$ grows exponentially but any proper subvariety grows polynomially. In [22] it was proved that the only varieties of almost polynomial growth are $var(E)$ and $var(UT_2)$, where UT_2 is the algebra of 2×2 upper triangular matrices.

Now by [20, Remark 1] if two algebras have the same identities, they also have the same proper central polynomials. It follows that the proper central exponent of an algebra A is an invariant of its T-ideal of identities and also of the corresponding variety of algebras. Hence the following definition makes sense.

Definition 2.1. *We say that a variety of algebras $\mathcal{V}$ has almost polynomial δ-growth if $exp^{\delta}(\mathcal{V}) \geq 2$ and for any proper subvariety $\mathcal{W} \subsetneq \mathcal{V}$ we have that $exp^{\delta}(\mathcal{W}) \leq 1$.*

By abuse of notation we also say that an algebra A has almost polynomial δ-growth if $exp^{\delta}(A) \geq 2$ and for any algebra B such that $\mathrm{Id}(B) \supsetneq \mathrm{Id}(A)$, we have that $exp^{\delta}(B) \leq 1$.

In search of possible candidates we define

$$D = \{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d, e \in F \}. \tag{1}$$

Let $St_4 = \sum_{\sigma \in S_4} (\mathrm{sgn}\ \sigma) x_{\sigma(1)} \cdots x_{\sigma(4)}$, be the standard polynomial of degree four. It can be easily checked that the algebra D satisfies St_4 and $[[x_1, x_2]^2, x_3]$.

Since by [11] these two polynomials generate the T-ideal of identities of $M_2(F)$, the algebra of 2×2 matrices over F, we have the following.

Remark 2.1. *D satisfies all the identites of 2×2 matrices.*

In [21] Gordienko studied the identities of the algebra D and here we summarize his results in the following theorem.

First notice that the center of D is $Z(D) = F(e_{11} + e_{22} + e_{33}) + Fe_{13}$. Also, since $F(e_{11} + e_{33})Fe_{12}Fe_{22} \neq 0$, then $F(e_{11} + e_{33}) \oplus Fe_{22}$ is a maximal admissible subalgebra and, so, $exp(D) = 2$. By [19] then also $exp^{z}(D) = 2$.

Since $[D, D][D, D] \subseteq Fe_{13}$ and $[e_{11}+e_{33}, e_{12}][e_{22}, e_{23}] = e_{13}$ we get that $[x_1, x_2][x_3, x_4]$ is a proper central polynomial of D and $F(e_{11} + e_{33}) \oplus Fe_{22}$ is a maximal central admissible subalgebra. Hence $exp^{\delta}(D) = 2$.

Let $\langle f_1, \ldots, f_r \rangle_T$ denote the T-ideal of the free algebra generated by the polynomials $f_1, \ldots, f_r$.

We have.

Theorem 2.1.

1) $exp(D) = exp^{z}(D) = exp^{\delta}(D) = 2$;
2) $\mathrm{Id}(D) = \langle [[x_1, x_2][x_3, x_4], x_5], St_4, [x_1, x_2][x_3, x_4][x_5, x_6] \rangle_T$;
3) *A basis of P_n modulo $\mathrm{Id}(D)$ is*
   a) $x_1 x_2 \cdots x_n$,
   b) $x_{i_1} \cdots x_{i_k} [x_{i_{k+1}}, x_{i_{k+2}}] x_{i_{k+3}} \cdots x_{i_n}$,
   $i_1 < i_2 < \cdots < i_k < i_{k+1},\ i_{k+2} < i_{k+1},\ i_{k+3} < \cdots < i_n$,
   c) $x_{i_1} \cdots x_{i_k} [x_{i_{k+1}}, x_{i_{k+2}}] x_{i_{k+3}} \cdots x_{i_{n-2}} [x_{i_{n-1}}, x_{i_n}]$,
   $i_1 < i_2 < \cdots < i_k,\ i_{k+2} < i_{k+1},\ i_{k+2} < i_{k+3} < \cdots < i_{n-2} < i_{n-1},\ i_n < i_{n-1}$;
4) $c_n(D) = 2^{n-2}(n^2 - 3n)$, *for* $n \geq 4$.

Another possible candidate is the following algebra.

$$D_0 = \{ \begin{pmatrix} 0 & c & d & e \\ 0 & a & g & h \\ 0 & 0 & b & i \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a, b, c, d, e, g, h, i \in F \}. \tag{2}$$

We have.

Theorem 2.2.

1) $exp(D_0) = exp^{z}(D_0) = exp^{\delta}(D_0) = 2$;
2) $\mathrm{Id}(D_0) = \langle x_1 [x_2, x_3][x_4, x_5] x_6 \rangle_T$;
3) $x_a x_{i_1} \cdots x_{i_{k-1}} [x_{j_{k+1}}, \ldots, x_{j_{n-1}}] x_b$, *with* $i_1 < \cdots < i_{k-1},\ j_{k+1} > j_{k+2} < \cdots < j_{n-1}$, *is a basis of P_n modulo $P_n \cap \mathrm{Id}(D_0)$*;
4) $c_n(D_0) = 2^{n-3} n(n-1)(n-4) + 2n(n-1)$, *for* $n \geq 7$.

Proof. Since $Fe_{22}Fe_{23}Fe_{33} \neq 0$, then $Fe_{22} \oplus Fe_{33}$ is a maximal admissible subalgebra and, so, $exp(D_0) = 2$ and also $exp^{z}(D_0) = 2$.

The center of D_0 is $Z(D_0) = Fe_{14}$, hence $[e_{12}, e_{22}][e_{23}, e_{33}][e_{33}, e_{34}] = e_{14}$ and $[D_0, D_0][D_0, D_0][D_0, D_0] \subseteq Fe_{14}$ imply that $[x_1, x_2][x_3, x_4][x_5, x_6]$ is a proper central polynomial of D_0. It follows that $Fe_{22} \oplus Fe_{33}$ is a maximal central admissible subalgebra and, so, $exp^{\delta}(D_0) = 2$. This proves 1).

In order to prove 2) we notice that $\mathrm{Id}(\{\begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}) = \langle x_1[x_2, x_3]\rangle_T$ and $\mathrm{Id}(\{\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}) = \langle [x_1, x_2]x_3\rangle_T$. Hence by Lewin theorem (see [18, Corollary 1.8.2]) we have that

$$\mathrm{Id}(D_0) = \langle x_1[x_2, x_3]\rangle_T \cdot \langle [x_1, x_2]x_3\rangle_T = \langle x_1[x_2, x_3][x_4, x_5]x_6\rangle_T.$$

To prove 3), consider a monomial

$$w = x_a x_{u_1} \cdots x_{u_{n-2}} x_b \in P_n.$$

Since $x_1[x_2, x_3][x_4, x_5]x_6 \in \mathrm{Id}(D_0)$, by the Poincaré-Birkhoff-Witt theorem (see [12]) w can be written modulo $\mathrm{Id}(D_0)$ as a linear combination of polynomials of the type

$$x_a x_{i_1} \cdots x_{i_{k-1}} [x_{j_{k+1}}, \ldots, x_{j_{n-1}}] x_b.$$

Let us write for simplicity $[x_{j_{k+1}}, \ldots, x_{j_{n-1}}] = [u]$.

We claim that the variables in $x_{i_1} \cdots x_{i_{k-1}}$ can be ordered. In fact if we want to exchange the variables x_{i_h} and $x_{i_{h+1}}$, we do the following

$$x_a x_{i_1} \cdots x_{i_h} x_{i_{h+1}} \cdots x_{i_{k-1}} [u] x_b = x_a x_{i_1} \cdots [x_{i_h}, x_{i_{h+1}}] \cdots x_{i_{k-1}} [u] x_b + x_a x_{i_1} \cdots x_{i_{h+1}} x_{i_h} \cdots x_{i_{k-1}} [u] x_b$$
$$\equiv x_a x_{i_1} \cdots x_{i_{h+1}} x_{i_h} \cdots x_{i_{k-1}} [u] x_b \mod. \mathrm{Id}(D_0).$$

Here we have used the fact that $x_1[x_2, x_3]x_7[x_4, x_5]x_6$ is a consequence of $x_1[x_2, x_3][x_4, x_5]x_6$.

Next we claim that the variables in $[u]$ can be ordered. In fact, if $[v]$ is any long Lie commutator, $m_1([v][x_1, x_2] - [x_1, x_2][v])m_2 \in \mathrm{Id}(D_0)$, for any m_1, m_2 monomials, implies that

$$m_1([v]x_1x_2 + x_2x_1[v])m_2 \equiv m_1([v]x_2x_1 + x_1x_2[v])m_2 \quad (\mathrm{mod.}\ \mathrm{Id}(D_0)).$$

Hence if we want to exchange two variables inside $[u]$, say x_1 and x_2, by opening part of the long Lie commutator we write $x_a x_{i_1} \cdots x_{i_{k-1}} [u] x_b$ as a sum with plus or minus sign of polynomials of the type

$$x_a m_1[[u'], x_1, x_2] m_2 x_b = x_a m_1([u']x_1x_2 - x_1[u']x_2 - x_2[u']x_1 + x_2x_1[u'])m_2 x_b$$
$$\equiv x_a m_1([u']x_2x_1 + x_1x_2[u'] - x_1[u']x_2 - x_2[u']x_1)m_2 x_b = x_a m_1[[u'], x_2, x_1] m_2 x_b \quad (\mathrm{mod.}\ \mathrm{Id}(D_0)).$$

If we now rebuild the long Lie commutator, the two variables have been exchanged. This proves the claim.

We have proved that any monomial in P_n can be written modulo $\mathrm{Id}(D_0)$ as a linear combination of polynomials of the type

$$x_a x_{i_1} \cdots x_{i_{k-1}} [x_{j_{k+1}}, \ldots, x_{j_{n-1}}] x_b, \quad i_1 < \cdots < i_{k-1},\ j_{k+1} > j_{k+2} < \cdots < j_{n-1},\ n-1-k \geq 0. \tag{3}$$

Thus the polynomials in (3) span P_n mod. $\mathrm{Id}(D_0)$. They are also linearly independent mod. $\mathrm{Id}(D_0)$.

In fact write for simplicity $\bar{i} = \{i_1, \ldots, i_{k-1}\}, \bar{j} = \{j_{k+1}, \ldots, j_{n-1}\}$, $x_{i_1} \cdots x_{i_{k-1}} = x_{\bar{i}}$ and $[x_{j_{k+1}}, \ldots, x_{j_{n-1}}] = [x_{\bar{j}}]$ and let f be a linear combination of the polynomials in (3), say,

$$f = \sum_{a,b} \sum_{\bar{i},\bar{j}} \alpha_{a,b,\bar{i},\bar{j}} x_a x_{\bar{i}} [x_{\bar{j}}] x_b.$$

Suppose $f \in \mathrm{Id}(D_0)$ and fix a sequence $(a, b, \bar{i}, \bar{j})$.

If $\bar{j} = \emptyset$, evaluating $x_a = e_{12}$, $x_b = e_{23}$, $x_{\bar{i}} = e_{22} \cdots e_{22}$, we get $\alpha_{a,b,\bar{i},\emptyset} e_{13} = 0$ and, so, $\alpha_{a,b,\bar{i},\emptyset} = 0$. Now let $\bar{j}$ be a largest set such that $\alpha_{a,b,\bar{i},\bar{j}} \neq 0$.

Make the substitution $x_a = e_{12}$, $x_b = e_{34}$, $x_{\bar{i}} = e_{22} \cdots e_{22}, [x_{\bar{j}}] = [e_{23}, e_{33}, \ldots, e_{33}]$. Then the value of f is $\alpha_{a,b,\bar{i},\bar{j}} e_{14} = 0$. Thus $\alpha_{a,b,\bar{i},\bar{j}} = 0$, a contradiction. This proves part 3) of the theorem.

Computing codimensions we have:

$$c_n(D_0) = n(n-1)c_{n-2}(UT_2) = n(n-1)(2^{n-3}(n-4) + 2) = 2^{n-3}n(n-1)(n-4) + 2n(n-1).$$

□

It is worth noticing that St_5, the standard polynomial of degree five, is also a proper central polynomial of D_0. In fact the only non-zero product of distinct standard basis elements is $e_{12}e_{22}e_{23}e_{33}e_{34} = e_{14}$. More is true: $\sum_{\sigma \in S_4} [x_5, x_{\sigma(1)}] x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$ and $\sum_{\sigma \in S_4} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} [x_{\sigma(4)}, x_5]$ are proper central polynomials of D_0.

Remark 2.2. *If $A, B \in \{D, D_0, E\}$ are distinct, then* $\mathrm{Id}(A) \not\subseteq \mathrm{Id}(B)$.

Proof. Recall that $\mathrm{Id}(E) = \langle [x_1, x_2, x_3] \rangle_T$ (see [18]). Therefore since D and D_0 do not satisfy $[x_1, x_2, x_3]$, we get that $\mathrm{Id}(E) \nsubseteq \mathrm{Id}(D)$ and $\mathrm{Id}(E) \nsubseteq \mathrm{Id}(D_0)$.

The other relations are obtained by observing that $x_1[x_2, x_3][x_4, x_5]x_6$ is an identity of D_0 but does not vanish in D or E. Moreover $St_4 \in \mathrm{Id}(D)$ but St_4 is not an identity of D_0 or E. □

Theorem 2.3. *Let A be a PI-algebra such that $exp^\delta(A) \geq 2$. Then* $\mathrm{Id}(A) \subseteq \mathrm{Id}(D)$ *or* $\mathrm{Id}(A) \subseteq \mathrm{Id}(D_0)$ *or* $\mathrm{Id}(A) \subseteq \mathrm{Id}(E)$.

Proof. Suppose first that A is a finite dimensional algebra. Then $A = \bar{A} + J$ with $\bar{A} = A_1 \oplus \cdots \oplus A_q$ and since F is algebraically closed, $A_i \cong M_{k_i}(F)$, $1 \leq i \leq q$.

Now, if $k_i \geq 2$, for some i, then by Remark 2.1 $\mathrm{Id}(A) \subseteq \mathrm{Id}(A_i) = \mathrm{Id}(M_{k_i}(F)) \subseteq \mathrm{Id}(M_2(F)) \subseteq \mathrm{Id}(D)$ and we are done. Therefore we may assume that $A_1 \cong \cdots \cong A_q \equiv F$.

Let C be a centrally admissible subalgebra of A of maximal dimension and assume, as we may, that $C = A_1 \oplus \cdots \oplus A_d$. Let $f = f(x_1, \ldots, x_t, y_1, \ldots, y_k)$ be a proper multilinear central polynomial corresponding to C. Then there are elements $a_i \in A_i$, $1 \leq i \leq d$, and $b_1, \ldots, b_k \in A$ such that $0 \neq \bar{f} = f(a_1, \ldots, a_d, b_1, \ldots, b_k) \in Z(A)$.

Let e_i be the unit element of A_i, $1 \leq i \leq q$. Notice that since the simple components A_i are isomorphic to F, we may evaluate any variable of f into the idempotents e_i or in J.

Since $\bar{f} \in Z(A)$, for any $i \neq k$ we have that $e_i \bar{f} e_k = e_i e_k \bar{f} = 0$. Also, since $exp^\delta(A) \geq 2$, A_1 and A_2 must participate in the evaluation $\bar{f}$.

Suppose first that f has a non zero monomial whose evaluation lies in $e_1 J e_1 + e_2 J e_2$. Let such evaluation be for instance in $e_1 J e_1$. Then since $exp^\delta(A) \geq 2$, e_2 must participate and, so, the value of the monomial is $e_1 w_1 e_2 w_2 e_1$, for some $w_1, w_2 \in J$.

let B be the subalgebra of A generated by the elements

$$e_1, e_2, e_1 w_1 e_2, e_2 w_2 e_1.$$

If I is the ideal of B generated by $e_2 w_2 e_1 w_1 e_2$, then $B/I \simeq D$. Hence $\mathrm{Id}(A) \subseteq \mathrm{Id}(B) \subseteq \mathrm{Id}(B/I) = \mathrm{Id}(D)$ and we are done in this case.

Hence we may assume that $e_1 \bar{f} e_1 = e_2 \bar{f} e_2 = 0$. Since $\bar{f}$ is a central value it follows that $e_1 \bar{f} = \bar{f} e_1 = e_2 \bar{f} = \bar{f} e_2 = 0$.

Next suppose that a monomial of f has an evaluation of the type $w_1 e_i w_2 e_k w_3 \neq 0$, where $\{e_i, e_k\} = \{e_1, e_2\}$ and $w_1, w_2, w_3 \in J$. Let for instance $w_1 e_1 w_2 e_2 w_3 \neq 0$.

In this case we let B be the subalgebra of A generated by $e_1, e_2, w_1 e_1, e_1 w_2 e_2, e_2 w_3$ and I the ideal of B generated by the elements $e_2 w_3 e_1, e_2 w_1 e_1, e_1 w_1 e_1, e_2 w_3 e_2, e_2 w_3 w_1 e_1$. Then $B/I \simeq D_0$ and $\mathrm{Id}(A) \subseteq \mathrm{Id}(D_0)$ is a desired conclusion.

Hence we may assume that $\bar{f} \in e_1 J + J e_1 + e_2 J + J e_2$. Write

$$\bar{f} = e_1 \bar{f}_1 + \bar{f}_2 e_1 + e_2 \bar{f}_3 + \bar{f}_4 e_2, \tag{4}$$

where $e_1 \bar{f}_1$ is the sum of the evaluations of the monomials starting with e_1, $\bar{f}_2 e_1$ is the sum of the evaluations of the monomials ending with e_1 and $e_2 \bar{f}_3 + \bar{f}_4 e_2$ is the sum of the evaluations of the remaining monomials starting or ending with e_2.

Since $e_1 \bar{f} = 0$, we get $e_1 \bar{f}_1 + e_1 \bar{f}_2 e_1 + e_1 \bar{f}_4 e_2 = 0$, and by the first part of the proof we may assume that $e_1 \bar{f}_2 e_1 = 0$. Hence

$$e_1 \bar{f}_1 + e_1 \bar{f}_4 e_2 = 0 \Longrightarrow e_1 \bar{f}_1 e_2 = -e_1 \bar{f}_4 e_2$$

and this says that $e_1 \bar{f}_1 = e_1 \bar{f}_1 e_2$.

Also, since $\bar{f} e_2 = 0$, we get $e_1 \bar{f}_1 e_2 + e_2 \bar{f}_3 e_2 + \bar{f}_4 e_2 = 0$ and since we may assume $e_2 \bar{f}_3 e_2 = 0$, we get

$$e_1 \bar{f}_1 e_2 + \bar{f}_4 e_2 = 0.$$

Putting together the last equality with $e_1 \bar{f}_1 = e_1 \bar{f}_1 e_2$, we get $e_1 \bar{f}_1 + \bar{f}_4 e_2 = 0$, and, so,

$$\bar{f} = \bar{f}_2 e_1 + e_2 \bar{f}_3.$$

Since $e_2 \bar{f} = 0$, we get $e_2 \bar{f}_2 e_1 + e_2 \bar{f}_3 = 0$. Hence $\bar{f} = \bar{f}_2 e_1 - e_2 \bar{f}_2 e_1$ and $\bar{f} e_1 = \bar{f} \neq 0$, a contradiction.

In case A does not have the same identities as a finite dimensional algebra, then we know that $\mathrm{Id}(A) = \mathrm{Id}(E(B))$, for some finite dimensional superalgebra B with non trivial grading. But then by [23, Theorem 2.3, pag. 64] $\mathrm{Id}(E(B)) \subseteq \mathrm{Id}(E)$ and we are done.

□

Regarding the Grassmann algebra in [29] Regev computed the central cocharacter and the proper central cocharacter of E. As a consequence it turns out that

$$c_n^\delta(E) = c_n^z(E) = \frac{1}{2} c_n(E) = 2^{n-2}. \tag{5}$$

Thus $exp(E) = exp^z(E) = exp^\delta(E) = 2$.

Now, if A is an algebra such that $\mathrm{Id}(A) \supsetneq \mathrm{Id}(D)$, by Remark 2.2 $\mathrm{Id}(A) \not\subseteq \mathrm{Id}(D_0)$ and $\mathrm{Id}(A) \not\subseteq \mathrm{Id}(E)$. Hence by the previous theorem $exp^\delta(A) \leq 1$ and D has almost polynomial δ-growth. The same remark applies to D_0 and E. Thus we have the following.

Corollary 2.1. *The algebras E, D and D_0 generate the only varieties of almost polynomial δ-growth.*

3. G-graded algebras and δ-growth

In this section we shall classify the varieties of (finite abelian) group-graded algebras of almost polynomial δ-growth.

Throughout this section G will be a finite abelian group and F an algebraically closed field of characteristic zero.

Recall that an F-algebra A is G-graded if it has a decomposition $A = \oplus_{g\in G} A^{(g)}$, into subspaces $A^{(g)}$ such that $A^{(g)}A^{(h)} \subseteq A^{(gh)}$, for all $g, h \in G$. Each subspace $A^{(g)}$ is a homogeneous component of A and its elements are said homogeneous of degree g.

We shall consider the standard $\mathbb{Z}_2$-grading on the Grassmann algebra $E = E^{(0)} \oplus E^{(1)}$, where $E^{(0)}$ ($E^{(1)}$ resp.) is the span of all monomials in the generators of even (odd resp.) length. We shall denote such superalgebra by $E^{\mathbb{Z}_2}$.

Let $F\langle X, G\rangle$ be the free G-graded algebra on a set X. We write $X = \oplus_{g\in G} X^{(g)}$, where the sets $X^{(g)} = \{x_1^{(g)}, x_2^{(g)}, \dots\}$ are disjoint and the elements of $X^{(g)}$ have homogeneous degree g. Recall that $F\langle X, G\rangle = \oplus_{g\in G}\mathcal{F}^{(g)}$ is the decomposition of $F\langle X, G\rangle$ into homogeneous subspaces where $\mathcal{F}^{(g)}$ is the subspace spanned by all monomials having homogeneous degree g.

Let A be a G-graded PI-algebra, i.e., A satisfies a non-trivial polynomial identity.

Then we denote by $\mathrm{Id}^G(A)$ the T_G-ideal of $F\langle X, G\rangle$ consisting of the G-graded polynomial identities satisfied by A and by $\mathrm{Id}^{G,z}(A)$ the T_G-space of central G-graded polynomials of A. Here T_G-stands for invariant under all endomorphisms of the G-graded free algebra.

A basic property of T_G-ideals that we shall use here is the following [2, 30]. If A is a G-graded PI-algebra then there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra $B = B^{(0)} \oplus B^{(1)}$ where $B^{(0)} = \oplus_{g\in G} B^{(g,0)}$, $B^{(1)} = \oplus_{g\in G} B^{(g,1)}$ such that $\mathrm{Id}^G(A) = \mathrm{Id}^G(E(B))$ where $E(B) = E^{(0)} \otimes B^{(0)} \oplus E^{(1)} \otimes B^{(1)}$ is the Grassmann envelope of B. The G-grading on $E(B)$ is given by setting $E(B)^{(g)} = E^{(0)} \otimes B^{(g,0)} \oplus E^{(1)} \otimes B^{(g,1)}$.

Notice that if A is a finite dimensional G-graded algebra, then we can regard A as a $G \times \mathbb{Z}_2$-graded algebra with trivial $\mathbb{Z}_2$-grading and we get that $\mathrm{Id}^G(A) = \mathrm{Id}^G(E^{(0)} \otimes A) = \mathrm{Id}^G(E(A))$. Thus we may consider A instead of $E(A)$.

As in the ordinary case one can attach some numerical invariants to a T_G-ideal as follows. Let $P_n^G \subset F\langle X, G\rangle$ be the space of multilinear G-graded polynomials in the variables $x_i^{(g)}$, $1 \leq i \leq n$, $g \in G$. One naturally defines the three numerical sequences $c_n^G(A) = \dim \frac{P_n^G}{P_n^G \cap \mathrm{Id}^G(A)}$, $c_n^{G,z}(A) = \dim \frac{P_n^G}{P_n^G \cap \mathrm{Id}^{G,z}(A)}$ and $c_n^{G,\delta}(A) = \dim \frac{P_n^G \cap \mathrm{Id}^{G,z}(A)}{P_n^G \cap \mathrm{Id}^G(A)}$, $n = 1, 2, \dots$. They are called the sequence of G-codimensions, central G-codimensions and proper central G-codimensions of A, respectively.

Comparing them with the ordinary codimensions we have that $c_n(A) \leq c_n^G(A)$, $c_n^z(A) \leq c_n^{G,z}(A)$ and $c_n^\delta(A) \leq c_n^{G,\delta}(A)$.

In [4], [14], [1] it was shown that if A is any G-graded PI-algebra, then $exp^G(A) = \lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$ exists and is an integer called the G-exponent of A. Moreover in [25] the authors proved that also the two limits

$$exp^{G,z}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{G,z}(A)}, \quad exp^{G,\delta}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{G,\delta}(A)}$$

exist and are called the central G-exponent and the proper central G-exponent of A, respectively.

The computation of the exponents is as follows. As we mentioned above we may assume that $A = E(B)$ with B a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B. We can decompose $B = \bar{B} + J$ with $\bar{B}$ a semisimple $G \times \mathbb{Z}_2$-graded algebra and J the Jacobson radical which is $G \times \mathbb{Z}_2$-invariant. We can write $\bar{B} = B_1 \oplus \cdots \oplus B_q$, a direct sum of $G \times \mathbb{Z}_2$-simple algebras.

Recall that a semisimple $G \times \mathbb{Z}_2$-algebra $C = B_{i_1} \oplus \cdots \oplus B_{i_k}$, where the B_{i_h}'s are distinct, is admissible if $B_{i_1} J \cdots J B_{i_k} \neq 0$. Moreover C is centrally admissible, if there exists a proper central G-graded polynomial f of $E(B)$ and a non-zero evaluation involving elements of $E(B_{i_h})$, for all $h \in \{1, \dots, k\}$. It is shown that the maximal dimension of an admissible subalgebra is $exp^G(E(B))$ and the maximal dimension of a centrally admissible subalgebra is $exp^{G,\delta}(E(B))$.

It is important to recall that it was also shown that the three sequences $c_n^G(A)$, $c_n^{G,z}(A)$ and $c_n^{G,\delta}(A)$, $n = 1, 2, \dots$, either grow exponentially or are polynomially bounded. In [31] the author classified the varieties of G-graded algebras of almost polynomial growth. They are generated either by UT_2 with some G-grading, or by FC_p, the group algebra of a cyclic group of order p, where p is a prime number such that $p \mid |G|$, or by E, the Grassmann algebra with trivial grading or, in case $|G|$ is even, by $E^{\mathbb{Z}_2}$ (see also [24]). We recall that if $C_p = \langle g \rangle \subseteq G$ then the G-grading on FC_p is defined by setting $FC_p^{(g^i)} = Fg^i$, $i = 0, \dots, p-1$, and $FC_p^{(h)} = 0$ for all $h \notin C_p$.

In search of almost polynomial growth of the proper central codimensions we notice that FC_p, E and $E^{\mathbb{Z}_2}$ are possible candidates. Nevertheless we need to define more G-graded algebras.

Next we consider matrix algebras endowed with elementary G-gradings. Let A be an algebra and $\mathbf{g} = (g_1, \dots, g_k) \in G^k$ an arbitrary k-tuple of elements of G. Then $\mathbf{g}$ defines an elementary G-grading on $M_k(A)$ by setting $M_k(A)^{(g)} = \operatorname{span}\{e_{ij} \mid g_i^{-1} g_j = g\}$ for all $g \in G$. We shall denote by $M_k(A)_{\mathbf{g}}$ the algebra $M_k(A)$ with elementary G-grading induced by $\mathbf{g}$.

Now if $\mathbf{g}'$ is obtained from $\mathbf{g}$ by multiplying all the elements by a fixed element of G, we obtain $M_k(A)_{\mathbf{g}} = M_k(A)_{\mathbf{g}'}$. Hence throughout we shall consider gradings on $M_k(A)_{\mathbf{g}}$ with $g_1 = 1$.

Also, if B is a graded subalgebra of $M_k(A)$ the induced grading on B will also be called elementary.

Remark 3.1. *Consider $M_2(F)_{(1,g)}$, $g \neq 1$. By* [10, Lemma 2] *and* [6, Theorem 4.5] *a basis of the graded identities of $M_2(F)_{(1,g)}$ is given by the polynomials*

$$[x_1^{(1)}, x_2^{(1)}], \quad x_1^{(g)} x_2^{(g)} x_3^{(g)} - x_3^{(g)} x_2^{(g)} x_1^{(g)}, \quad x_1^{(h)}, \quad h \neq 1, g,$$

in case $g^2 = 1$ and by the polynomials

$$[x_1^{(1)}, x_2^{(1)}], \quad x_1^{(g)} x_2^{(g)}, \quad x_1^{(g^{-1})} x_2^{(g^{-1})},$$

$$x_1^{(g)} x_1^{(g^{-1})} x_2^{(g)} - x_2^{(g)} x_1^{(g^{-1})} x_1^{(g)}, \quad x_1^{(g^{-1})} x_1^{(g)} x_2^{(g^{-1})} - x_2^{(g^{-1})} x_1^{(g)} x_1^{(g^{-1})}, \quad x_1^{(h)}, \quad h \neq 1, g, g^{-1},$$

in case $g^2 \neq 1$.

Remark 3.2. *Let $M_{1,1}(E) = \begin{pmatrix} E^{(0)} & E^{(1)} \\ E^{(1)} & E^{(0)} \end{pmatrix} \subseteq M_2(E)$ be endowed with the elementary G-grading induced by $(1, g)$, $g \neq 1$.*

A basis of the G-graded identities of $M_{1,1}(E)$ is given by the polynomials

$$[x_1^{(1)}, x_2^{(1)}], \quad x_1^{(g)} x_2^{(g)} x_3^{(g)} + x_3^{(g)} x_2^{(g)} x_1^{(g)}, \quad x_1^{(h)}, \quad h \neq 1, g,$$

in case $g^2 = 1$ and by the polynomials

$$[x_1^{(1)}, x_2^{(1)}], \quad x_1^{(g)} x_2^{(g)}, \quad x_1^{(g^{-1})} x_2^{(g^{-1})},$$

$$x_1^{(g)} x_1^{(g^{-1})} x_2^{(g)} + x_2^{(g)} x_1^{(g^{-1})} x_1^{(g)}, \quad x_1^{(g^{-1})} x_1^{(g)} x_2^{(g^{-1})} + x_2^{(g^{-1})} x_1^{(g)} x_1^{(g^{-1})}, \quad x_1^{(h)}, \quad h \neq 1, g, g^{-1},$$

in case $g^2 \neq 1$.

Proof. If $g^2 = 1$, the algebra $M_{1,1}(E)$ has an induced structure of superalgebra and the result was proved in [10, Theorem 2].

Now assume that $g^2 \neq 1$.

Let W_n be the space of multilinear polynomials of degree n in the first n variables of homogeneous degree $1, g, g^{-1}$. Hence every monomial of W_n contains only one variable among $x_i^{(1)}, x_i^{(g)}, x_i^{(g^{-1})}$, for every $1 \leq i \leq n$.

By generalizing the map defined in [23, page 17] we let $\tilde{}: W_n \to W_n$ be the linear isomorphism defined on monomials as follows. If $\omega = w_1 z_{\sigma(1)} w_2 \cdots w_r z_{\sigma(r)} w_{r+1} \in W_n$ where the z_i's have homogeneous degree g or g^{-1} and the w_i's are eventually empty words in variables of homogeneous degree 1, then $\tilde{\omega} = (\operatorname{sgn} \sigma)\omega$.

Now let $f(x_1, \dots, x_r, z_1, \dots, z_{n-r}) \in W_n$ and evaluate f on homogeneous elements of $M_{1,1}(E)$. Then if $a_1, \dots, a_r \in M_{1,1}(E)^{(1)}$, $b_1, \dots, b_{n-r} \in M_{1,1}(E)^{(g)} \cup M_{1,1}(E)^{(g^{-1})}$, $g_1, \dots, g_r \in E^{(0)}$, $h_1, \dots, h_{n-r} \in E^{(1)}$, we have

$$f(a_1 \otimes g_1, \dots, a_r \otimes g_r, b_1 \otimes h_1, \dots, b_{n-r} \otimes h_{n-r}) = \tilde{f}(a_1 \dots, a_r, b_1, \dots, b_{n-r}) \otimes g_1 \cdots g_r h_1 \cdots h_{n-r}.$$

It follows that f is a G-identity of $M_{1,1}(E)_{(1,g)}$ if and only if $\tilde{f}$ is a G-identity of $M_2(F)_{(1,g)}$. $\square$

When the algebra $M_{1,1}(E)$ has trivial grading, by a result of Kemer $M_{1,1}(E)$ and $E \otimes E$ have the same (ordinary) identities (see [5, Theorem 20.3.20]). But then by a result of Popov [26] the T-ideal of such algebra is generated by the polynomials $[[[x_1, x_2], [x_3, x_4]], x_5]$ and $[[x_1, x_2]^2, x_2]$. Notice that by Theorem 2.1, $\operatorname{Id}(M_{1,1}(E) \subsetneq \operatorname{Id}(D)$.

Next we consider all possible G-gradings on the algebras D and D_0 and we make the following.

Definition 3.1.

1) *$D_{(1,g,h)}$ is the algebra D with elementary G-grading induced by $(1, g, h)$, with $g, h \in G$.*
2) *$D_{0,(1,g,h,h')}$ is the algebra D_0 with elementary G-grading induced by $(1, g, h, h')$, with $g, h, h' \in G$.*

Lemma 3.1. *Let $M_2(F)$ be endowed with an elementary $G\times\mathbb{Z}_2$-grading. The possible $G\times\mathbb{Z}_2$-gradings are given by the pairs*

$$((1,0),(g,i)),\quad g\in G, i\in\{0,1\}$$

and the corresponding Grassmann envelope $E^{(0)}\otimes M_2(F)^{(0)}\oplus E^{(1)}\otimes M_2(F)^{(1)}$ satisfies the same G-graded identities as $M_2(F)_{(1,g)}$ or $M_{1,1}(E)_{(1,g)}$ according as $i=0$ or $i=1$. It follows that the algebra $D_{(1,g,1)}$ satisfies all the graded identities of the Grassmann envolope of $M_2(F)$ with elementary $G\times\mathbb{Z}_2$-grading induced by $((1,0),(g,i)), i\in\{0,1\}$.

Proof. The proof of the first part is immediate. For the last part notice that if $g=1$ the result follows from the above comment and Remark 2.1.

Now let $g\neq 1$. Notice that $D_{(1,g,1)}$ satisfies the graded identities

$$[x_1^{(1)},x_2^{(1)}],\quad x_1^{(g)}x_2^{(g)}x_3^{(g)},\quad x_1^{(h)},\quad h\neq 1,g,$$

in case $g^2=1$ and the graded identities

$$[x_1^{(1)},x_2^{(1)}],\quad x_1^{(g^{-1})}x_2^{(g)},\quad x_1^{(g)}x_2^{(g)},\quad x_1^{(g^{-1})}x_2^{(g^{-1})},\quad x_1^{(h)},\quad h\neq 1,g,g^{-1},$$

in case $g^2\neq 1$. Now the result follows by Remarks 3.1 and 3.2. $\square$

Now we can prove the main result of this section.

In order to make the statement of the next theorem we introduce the notation $FC_{p,a}$ for the group algebra of the cyclic subgroup $C_{p,a}=\langle a\rangle\subseteq G$, with a of order p, and E^b for the Grassmann algebra with the canonical $\mathbb{Z}_2$-grading induced by the element $b\in G$ of order 2.

Theorem 3.1. *Let A be a G-graded PI-algebra. If $exp^{G,\delta}(A)\geq 2$, then $\mathrm{Id}^G(A)\subseteq\mathrm{Id}^G(FC_{p,a})$ or $\mathrm{Id}^G(A)\subseteq\mathrm{Id}^G(E)$ or $\mathrm{Id}^G(A)\subseteq\mathrm{Id}^G(D_{(1,g,h)})$ or $\mathrm{Id}^G(A)\subseteq\mathrm{Id}^G(D_{0,(1,g',h',l)})$, or $\mathrm{Id}^G(A)\subseteq\mathrm{Id}^G(E^b)$, in case $|G|$ is even, for some $a,b,g,h,g',h',l\in G$ and a prime p such that $p\mid |G|$.*

Proof. Assume, as we may, that $A=E(B)=E^{(0)}\otimes B^{(0)}\oplus E^{(1)}\otimes B^{(1)}$ is the Grassmann envelope of a $G\times\mathbb{Z}_2$-graded algebra $B=B^{(0)}\oplus B^{(1)}$. Write $B=\bar{B}+J$ with $\bar{B}=B_1\oplus\cdots\oplus B_q$, a direct sum of $G\times\mathbb{Z}_2$-simple algebras.

According to [7], for $1\leq i\leq q$, $B_i=M_{k_i}(F^{\alpha_i}H_i)$, where H_i is a subgroup of $G\times\mathbb{Z}_2$ and α_i is a 2-cocycle. The grading on B_i is as follows. There is a k_i-tuple $\mathbf{m}_i=(m_1=1,m_2,\ldots,m_{k_i})\in(G\times\mathbb{Z}_2)^{k_i}$ such that $B_i^{(m)}=span_F\{b_h\otimes e_{jl}\mid m=m_j^{-1}hm_l\}$, $m\in G\times\mathbb{Z}_2$. Here $b_h\in F^{\alpha}H$ is a representative of $h\in H$.

Suppose first that there is some $B_i=M_{k_i}(F^{\alpha_i}H_i)$ with $k_i\geq 2$.

Then B_i contains the graded subalgebra $M_2(F)$ with elementary $G\times\mathbb{Z}_2$ grading induced by $\mathbf{m}_i$. Hence

$$E(B)\supseteq E(B_i)\supseteq E(M_2(F)_{\mathbf{m}_i})$$

and, by Lemma 3.1, we get that $\mathrm{Id}^G(E(B))\subseteq\mathrm{Id}^G(D_{(1,g,1)})$, for some $g\in G$, a desired conclusion.

Therefore we may assume that for $1\leq i\leq q$, $B_i=F^{\alpha_i}H_i$, with H_i a subgroup of $G\times\mathbb{Z}_2$ and α_i a 2-cocycle.

Suppose first that $(a,0)\in H_i$ for some $a\in G$ of order a prime p. Then B_i contains the graded subalgebra $FC_{p,(a,0)}$ and we may assume that the cocycle α_i is trivial on $C_{p,(a,0)}$. Thus

$$E(B)\supseteq E(B_i)\supseteq E(FC_{p,(a,0)})=E^{(0)}\otimes FC_{p,(a,0)},$$

and since $E^{(0)}\otimes FC_{p,(a,0)}$ has the same G-graded identities as $FC_{p,a}$ we are done in this case.

Suppose now that $(a,1)\in H_i$ with $a\in G$ of order a prime p. If $p\neq 2$, considering the element $(a^2,0)$ we get into the previous case. Hence we may assume that $p=2$ and, so, G is of even order.

So we have:

$$E(B)\supseteq E(FC_{2,(a,1)})=E^{(0)}\otimes F(1,0)\oplus E^{(1)}\otimes F(a,1)$$

which is isomorphic to E^a with canonical $\mathbb{Z}_2$-grading.

The only cases left are $H_i=\{1\}\times\mathbb{Z}_2$ and $H_i=\{e\}$ trivial.

In the second case $B_i=F$ with trivial $G\times\mathbb{Z}_2$-grading, and in the first case

$$E(FH_i)=E^{(0)}\otimes F(1,0)\oplus E^{(1)}\otimes F(1,1)$$

which is isomorphic to E with trivial grading and, so, $\mathrm{Id}^G(E(B))\subseteq\mathrm{Id}^G(E)$. Therefore we may assume that for $1\leq i\leq q$, $B_i=F$ with trivial $G\times\mathbb{Z}_2$-grading.

Assume, as we may, that $C=B_1\oplus\cdots\oplus B_r$ is a centrally admissible $G\times\mathbb{Z}_2$-graded subalgebra of B of maximal dimension and let f be a proper central polynomial of $E(B)$ corresponding to C. Thus $exp^{G,\delta}(E(B))=\dim C=r$.

Let e_i be the unit element of B_i, $1\leq i\leq q$. Since $exp^{G,\delta}(E(B))\geq 2$, $E^{(0)}\otimes B_1$ and $E^{(0)}\otimes B_2$ must participate in a non-zero evaluation $\bar{f}$ of f.

Suppose first that in a non-zero evaluation of a monomial of f two non adjacent graded variables x_i, x_k are evaluated in $a \otimes e_1$ and a third variable between x_i and x_k is evaluated in $b \otimes e_2$.

Then we get a non-zero product of the form $(1 \otimes e_1)w_1(1 \otimes e_2)w_2(1 \otimes e_1)$, for some $w_1, w_2 \in E(J)$ and we may clearly assume that the elements w_1, w_2 are homogeneous, say of degree g and h, respectively.

Set $C_1 = 1 \otimes e_1$, $C_2 = 1 \otimes e_2$ and let A' be the homogeneous subalgebra of $E(B)$ generated by the elements $C_1,\ C_2,\ C_1w_1C_2,\ C_2w_2C_1$. If we let I be its homogeneous ideal generated by the element $C_2w_2C_1w_1C_2$, we obtain that A'/I is isomorphic to $D_{(1,g,gh)}$. Hence $\mathrm{Id}^G(E(B)) \subseteq \mathrm{Id}^G(A') \subseteq \mathrm{Id}^G(A'/I) = \mathrm{Id}^G(D_{(1,g,gh)})$ and we are done. It is clear that the same conclusion holds if we exchange the roles of C_1 and C_2.

In particular we may assume that $C_1\bar{f}C_1 = C_2\bar{f}C_2 = 0$ and since $\bar{f}$ is central, $C_1\bar{f} = \bar{f}C_1 = C_2\bar{f} = \bar{f}C_2 = 0$.

Now suppose without loss of generality that in a non-zero evaluation of a monomial of f two non adjacent graded variables x_i, x_k are evaluated in C_1 and C_2, respectively, and also these variables are not the leftmost or the rightmost variable of the monomial. Then the evaluation of such monomial contains an element of the form $w_1C_iw_2C_jw_3 \neq 0$ with $\{i,j\} = 1, 2$, for some $w_1, w_2, w_3 \in E(J)$.

We may clearly assume that the elements w_1, w_2, w_3 are homogeneous, say of degree g, h, h', respectively and that $i = 1, j = 2$. Let A' be the homogeneous subalgebra of $E(B)$ generated by $C_1, C_2, w_1C_1, C_1w_2C_2, C_2w_3$. Let I be the homogeneous ideal of B generated by the elements

$$C_2w_3C_1,\ C_2w_1C_1,\ C_1w_1C_1,\ C_2w_3C_2,\ C_2w_3w_1C_1.$$

Then we obtain that A'/I is isomorphic to the algebra $D_{0,(1,g,gh,ghh')}$ and $\mathrm{Id}^G(E(B)) \subseteq \mathrm{Id}^G(A'/I) \subseteq \mathrm{Id}^G(D_{0,(1,g,gh,ghh')})$, a desired conclusion. Clearly the same conclusion holds if we exchange the roles of C_1 and C_2.

In particular we may assume that any monomial of f with a non-zero evaluation in $\bar{f}$ must have either the leftmost variable evaluated in C_1 (C_2 resp.) and the rightmost not in C_1 (C_2 resp.) or the rightmost evaluated in C_1 (C_2 resp.) and the leftmost not in C_1 (C_2 resp.).

In particular this says that we may assume that $\bar{f}$ can be written $\bar{f} = C_1\bar{f}_1 + \bar{f}_2C_1 + C_2\bar{f}_3 + \bar{f}_4C_2$, as in (4). The proof is now completed as in Theorem 2.3. $\square$

Our final aim is to extend the result of Corollary 2.1 to the setting of G-graded algebras. To this end we make the following.

Definition 3.2. *A variety of G-graded algebras $\mathcal{V}$ has almost polynomial δ-growth if $exp^{G,\delta}(\mathcal{V}) \geq 2$ and for any proper subvariety $\mathcal{W} \subsetneq \mathcal{V}$ we have that $exp^{G,\delta}(\mathcal{W}) \leq 1$.*

We will show that the algebras $FC_{p,a}$, E, $D_{(1,g,h)}$, $D_{0,(1,g',h',l)}$, E^b of Theorem 3.1, generate varieties of G-graded algebras of almost polynomial δ-growth.

Let $var^G(A)$ be the variety of G-graded algebras generated by an algebra A. Then denote

$$\mathcal{V}_{(1,g,h)} = var^G(D_{(1,g,h)}),\ \mathcal{V}_{(0,1,g',h',l)} = var^G(D_{(0,(1,g',h',l))}),\ \mathcal{V}_{(a,p)} = var^G(FC_{p,a}),\ \mathcal{V}_b = var^G(E^b),\ \mathcal{V}_1 = var^G(E).$$

Let us say that two varieties $\mathcal{V}$ and $\mathcal{W}$ are not comparable if $\mathcal{V} \nsubseteq \mathcal{W}$ and $\mathcal{W} \nsubseteq \mathcal{V}$. Then we have the following.

Lemma 3.2. *For any $a, b, g, h, g', h', l \in G$ the varieties $\mathcal{V}_{(1,g,h)}$, $\mathcal{V}_{(0,1,g',h',l)}$, $\mathcal{V}_{(a,p)}$, $\mathcal{V}_b$, $\mathcal{V}_1$ are not comparable.*

Proof. We start by comparing the graded identities of $D_{(1,g,h)}$ and $D_{(1,g',h')}$ when $(1, g, h) \neq (1, g', h')$. Since $e_{12}e_{23} \neq 0$ we get that $x_1^{(g)}x_2^{(g^{-1}h)}$ is an identity of $D_{(1,g',h')}$ but is not an identity of $D_{(1,g,h)}$. Similarly $x_1^{(g')}x_2^{(g'^{-1}h')}$ is an identity of $D_{(1,g,h)}$ but is not an identity of $D_{(1,g',h')}$. Thus $\mathcal{V}_{(1,g,h)}$ and $\mathcal{V}_{(1,g',h')}$ are not comparable.

A similar remark applies to the algebras of the type $D_{(0,(1,g',h',l))}$ by considering the product $e_{12}e_{23}e_{34} \neq 0$. It is also clear that the varieties of the type $\mathcal{V}_{(a,p)}$ are not comparable among themselves and the same can be concluded for varieties of the type $\mathcal{V}_b$.

Comparing varieties of different types, notice that if we regard the ordinary identities as G-graded identities, then by Remark 2.2 we get that $\mathcal{V}_{(1,g,h)}$, $\mathcal{V}_{(0,1,g',h',l)}$ and $\mathcal{V}_1$ are non-comparable varieties.

Hence we only need to compare the variety $\mathcal{V}_{(a,p)}$ and the variety $\mathcal{V}_b$ with the other types. This is achieved by noticing that $FC_{p,a}$ is commutative, E is not commutative, $E_b^{(b)}$ is anticommutative and the components of $D_{(1,g,h)}$ and $D_{0,(1,g',h',l)}$ of homogeneous degree different from one are nilpotent of index at most two. $\square$

Now we can deduce the following.

Corollary 3.1. *A variety of G-graded algebras has almost polynomial δ-growth if and only if it is generated by one of the following algebras:*

1) *$FC_{p,a}$, if p is a prime such that $p \mid |G|$, for some $a \in G$,*
2) *$D_{(1,g,h)}$, for some $g, h \in G$,*

3) $D_{0,(1,g,h,h')}$, *for some* $g, h, h' \in G$,
4) E *with trivial grading,*
5) E^b, *if* $|G|$ *is even, for some* $b \in G$.

Proof. We start by proving that the varieties $\mathcal{V}_{(1,g,h)}$, $\mathcal{V}_{(0,1,g',h',l)}$, $\mathcal{V}_{(a,p)}$, $\mathcal{V}_b$ and $\mathcal{V}_1$ have exponential δ-growth.

In fact, it is clear that since $FC_{p,a}$ is G-simple and commutative, $exp^{G,\delta}(FC_{p,a}) = \dim FC_{p,a} = p$. Moreover by (5), $exp^{G,\delta}(E) = 2$ and if we regard the proper central polynomials constructed above for D and D_0 as G-graded polynomials, then we get that also $exp^{G,\delta}(D_{(1,g,h)}) = exp^{G,\delta}(D_{(0,(1,g',h',l)}) = 2$. Concerning $E^b = E(FC_{2,(b,1)})$, we notice that for $h_1^{(1)}, h_1^{(2)} \in E^{(1)}$, distinct, we have

$$[h_1^{(1)} \otimes (b,1), h_2^{(1)} \otimes (b,1)] = 2h_1^{(1)}h_2^{(1)} \otimes (b,1)(b,1) \in E^{(0)} \otimes F(1,0) = Z(E(FC_{2,(b,1)})).$$

Hence $[x_1^{(b)}, x_2^{(b)}]$ is a proper central polynomial of E^b and, so, $exp^{G,\delta}(E^b) = 2$.

Let $\mathcal{V}$ be one of the above varieties and let $\mathcal{W} \subsetneq \mathcal{V}$ be a proper subvariety. If $exp^{G,\delta}(\mathcal{W}) \geq 2$, then by Theorem 3.1 we get that $\mathcal{W}$ must contain one of the varieties $\mathcal{V}_{(1,g,h)}$, $\mathcal{V}_{(0,1,g',h',l)}$, $\mathcal{V}_{(a,p)}$, $\mathcal{V}_b$, $\mathcal{V}_1$, for some $a, b, g, h, g', h', l \in G$. Let $\mathcal{U}$ be such variety. But then $\mathcal{V} \supsetneq \mathcal{U}$, contrary to Lemma 3.2. Therefore $\mathcal{V}$ has almost polynomial δ-growth. The proof is completed by the result of Theorem 3.1. □

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