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Cite this article: Bagarello F. 2026 A pseudo-bosonic Klein–Gordon field with finite two-point function. *Proc. R. Soc. A* **482**: 20250994.
https://doi.org/10.1098/rspa.2025.0994

Received: 15 November 2025
Accepted: 22 January 2026

Subject Areas:
mathematical physics

Keywords:
Klein–Gordon equation, pseudo-bosons, non-Hermitian quantum field theory

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A pseudo-bosonic Klein–Gordon field with finite two-point function

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We introduce a class of pseudo-bosonic Klein–Gordon fields (KGFs) in $1+1$ dimensions, and we discuss some of their properties. This approach originates from non-Hermitian quantum mechanics and deformed canonical commutation relations (CCRs). We show that, within this class of fields, there exists a specific subclass with the quite interesting property of having finite equal space-time two-point function, contrarily to what happens for ordinary KGFs. This, in our opinion, is a relevant aspect of our proposal, which is a good motivation to undertake a deeper analysis of this (and related) quantum fields.

1. Introduction

Quantum field theory (QFT) is an extremely interesting area of research, linked to many crucial aspects of physics, from elementary particles to many body systems and statistical mechanics. There are thousands of papers and monographs dealing with QFT, in its various aspects. We only cite here [1–4], where more references can be found. What is particularly interesting for us in this article is the fact that, see [1], a quantum field is necessarily an operator valued distribution and, as such, taking its powers, or multiplying two such fields, is a dangerous operation: distributions cannot be multiplied, in general, even at a classical level. For instance, $\delta(x)^2$ is something for which the meaning is not clear. Quite often, in the literature, one finds claims stating simply that $\delta(x)^2$ does not exist. However, several attempts have been proposed in the past to define, in some rigorous way, this and other products of distributions. Results in this direction can be found in [5] or in some specific cases in [6–11]. In particular, in [10], we have proposed

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a regularization of a Klein–Gordon field (KGF) $\varphi(x,t)$ in the attempt to produce finite equal-time two-point function $\Delta_+(x,y) = \langle\Psi_0, \varphi(x,t)\varphi(y,t)\Psi_0\rangle$ in the limit $y \to x$. Here, Ψ_0 is the vacuum of the bosonic operators used in the usual expansion of the field $\varphi(x,t)$ [2,3]. However, the approach proposed in [10] does not work, in the sense that the limit of $\Delta_+(x,y)$ for $y \to x$, even after the regularization proposed therein, diverges. Hence, our conclusion was that, even if the approach proposed first in [9] works, in particular, to multiply $\delta(x)$ functions, making it possible to define a sort of square of $\delta(x)$, it is not particularly useful when applied to KGFs. In ordinary QFT, the way to approach this kind of problem is the so-called *renormalization theory*, which removes infinities from diverging results (appearing, e.g. when computing transition probabilities via Feynman graphs) to obtain finite result. What is really important, and surprising, is that these finite results are often in extremely good agreement with the experimental data. However, renormalization appears as an ad hoc procedure, and it would be important to have a theory of elementary particles, which does not need any such procedure since it produces results that are already finite. As we shall see, what we discuss in this article is indeed a first step in this direction. Hopefully not the last.

In the past few decades, people started to be interested in a sort of *extended* version of quantum mechanics, in which it is not required that the observables of the system under analysis, and the Hamiltonian in particular, are self-adjoint. Since the first paper in 1998, [12], this line of research became very famous and several researchers started to work on it, not only for its physical implications but also for the interesting mathematics arising when working with observables which are not necessarily self-adjoint. We refer to [13–18] and the references therein for some books and edited books on several aspects on non-Hermitian[1] operators. In particular, in the past years, we have analysed in much detail some deformed versions of the canonical (anti-)commutation relations, and of other, less known, algebraic rules, in this new context. Ladder operators of several kinds, where the creation operator is not the adjoint of the lowering operator, have been extensively studied and a recent review can be found in [18]. Among all these ladders, the pseudo-bosonic operators (those which we used to extend the canonical commutation relation (CCR)) will be adopted here to define a sort of deformed KGF (our pseudo-bosonic KGF (PBKGF)), and we show that interesting features appear. In particular, the two-point function of the field will return a finite result, under certain conditions. In view of what was outlined earlier, this result can be seen as a completely different way to *regularize* the original KGF, which of course suggests several related problems, i.e. the possibility of bypassing the renormalization in QFT, which is a very difficult task.

It is surely worth stressing that ours here is not the first attempt to use techniques and results arising for non-Hermitian Hamiltonians in a QFT setting. We refer to [19–21] for some results in this direction. However, up to today, this topic does not seem to us *particularly popular*. On the other hand, also in view of the results, which we are going to discuss in this article, we believe that a deeper work on this kind of QFT is worthwhile.

This article is organized as follows:

In §2, we give a short review on pseudo-bosons (PBs) since they will be used as the main tool in our approach.

In §3, we introduce PBKGFs in $1+1$ dimensions, we study their properties and we show that a two-point function of this field is indeed finite, even in the limit $y \to x$.

Section 4 contains our final considerations and plans for the future.

2. A short review on pseudo-bonons (PBs)

In this section, to keep the paper self-contained, we briefly review few facts on $\mathcal{D}$-PBs. We refer to [18,22] for more details.

Let $\mathcal{H}$ be a given Hilbert space with scalar product $\langle ., . \rangle$ and related norm $\|.\|$. Let a and b be two operators on $\mathcal{H}$, with domains $D(a) \subset \mathcal{H}$ and $D(b) \subset \mathcal{H}$, respectively, $a^\dagger$ and $b^\dagger$ their adjoint,

[1]In this article, Hermitian and self-adjoint will be used synonymously.

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and $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ such that $a^\sharp\mathcal{D}\subseteq\mathcal{D}$ and $b^\sharp\mathcal{D}\subseteq\mathcal{D}$, where with $x^\sharp$ we indicate x or $x^\dagger$. Of course, $\mathcal{D}\subseteq D(a^\sharp)$ and $\mathcal{D}\subseteq D(b^\sharp)$.

Definition 2.1. The operators (a,b) are $\mathcal{D}$-pseudo-bosonic if, for all $f\in\mathcal{D}$, we have

$$abf-baf=f. \tag{2.1}$$

When $b=a^\dagger$, this is simply the CCR for ordinary bosons. The interesting situation is $b\neq a^\dagger$. Now we consider the following:

Assumption $\mathcal{D}$-pb 1.—There exists a non-zero $\varphi_0\in\mathcal{D}$ such that $a\,\varphi_0=0$.
Assumption $\mathcal{D}$-pb 2.—There exists a non-zero $\Psi_0\in\mathcal{D}$ such that $b^\dagger\,\Psi_0=0$.

It is obvious that, since $\mathcal{D}$ is stable under the action of b and $a^\dagger$, in particular, $\varphi_0\in D^\infty(b):=\cap_{k\geq 0}D(b^k)$ and $\Psi_0\in D^\infty(a^\dagger)$, so that the vectors

$$\varphi_n:=\frac{1}{\sqrt{n!}}b^n\varphi_0 \quad\text{and}\quad \Psi_n:=\frac{1}{\sqrt{n!}}{a^\dagger}^n\Psi_0, \tag{2.2}$$

$n\geq 0$, can be defined and they all belong to $\mathcal{D}$, and, as such, to the domains of $a^\sharp$, $b^\sharp$ and $N^\sharp$, where $N=ba$. Let us put $\mathcal{F}_\Psi=\{\Psi_n,n\geq 0\}$ and $\mathcal{F}_\varphi=\{\varphi_n,n\geq 0\}$. It is simple to deduce the following lowering and raising relations:

$$\begin{cases} b\varphi_n=\sqrt{n+1}\varphi_{n+1}, & n\geq 0,\\ a\varphi_0=0,\, a\varphi_n=\sqrt{n}\varphi_{n-1}, & n\geq 1,\\ a^\dagger\Psi_n=\sqrt{n+1}\Psi_{n+1}, & n\geq 0,\\ b^\dagger\Psi_0=0,\, b^\dagger\Psi_n=\sqrt{n}\Psi_{n-1}, & n\geq 1, \end{cases} \tag{2.3}$$

as well as the eigenvalue equations, $N\varphi_n=n\varphi_n$ and $N^\dagger\Psi_n=n\Psi_n$, $n\geq 0$. In particular, as a consequence of these two last equations, if we choose the normalization of φ_0 and Ψ_0 in such a way that $\langle\varphi_0,\Psi_0\rangle=1$, we deduce that

$$\langle\varphi_n,\Psi_m\rangle=\delta_{n,m}, \tag{2.4}$$

for all $n,m\geq 0$. Hence, $\mathcal{F}_\Psi$ and $\mathcal{F}_\varphi$ are biorthogonal. It is easy to see that, if $b=a^\dagger$, then $\varphi_n=\Psi_n$, so that biorthogonality is replaced by a simpler orthonormality. Also, the relations in equation (2.3) collapse, and only one number operator exists, since in this case $N=N^\dagger$.

The analogy with ordinary bosons suggests considering the following:

Assumption $\mathcal{D}$-pb 3.—$\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$.

This is equivalent to requiring that $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$ as well [23]. However, several physical models show that $\mathcal{F}_\varphi$ is **not** necessarily a basis for $\mathcal{H}$, but it is still complete in $\mathcal{H}$. For this reason, we adopt the following weaker version of assumption $\mathcal{D}$-pb 3, [22]:

Assumption $\mathcal{D}$-pbw 3.—For some subspace $\mathcal{G}$ dense in $\mathcal{H}$, $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{G}$-quasi bases.

This means that, for all f and g in $\mathcal{G}$,

$$\langle f,g\rangle=\sum_{n\geq 0}\langle f,\varphi_n\rangle\langle\Psi_n,g\rangle=\sum_{n\geq 0}\langle f,\Psi_n\rangle\langle\varphi_n,g\rangle, \tag{2.5}$$

which can be seen as a weak form of the resolution of the identity, restricted to $\mathcal{G}$.

The families $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ can be used to introduce two densely defined operators S_φ and S_Ψ via their action, respectively, on $\mathcal{F}_\Psi$ and $\mathcal{F}_\varphi$:

$$S_\varphi\Psi_n=\varphi_n \quad\text{and}\quad S_\Psi\varphi_n=\Psi_n, \tag{2.6}$$

for all n, which also implies that $\Psi_n=(S_\Psi S_\varphi)\Psi_n$ and $\varphi_n=(S_\varphi S_\Psi)\varphi_n$, again for all n. Of course, these equalities can be extended to the linear spans of the φ_n's, $\mathcal{L}_\varphi$, and of the Ψ_n's, $\mathcal{L}_\Psi$. This means that, for instance, $S_\Psi S_\varphi f = f$ and $S_\varphi S_\Psi g = g$ for all $f \in \mathcal{L}_\Psi$ and $g \in \mathcal{L}_\varphi$. With a little abuse of language, we could say that S_φ is the inverse of S_ψ. Quite often one writes these operators in bra-ket form:

$$S_\varphi = \sum_n |\varphi_n\rangle\langle\varphi_n| \quad \text{and} \quad S_\Psi = \sum_n |\Psi_n\rangle\langle\Psi_n|, \tag{2.7}$$

where $(|f\rangle\langle f|)g = \langle f, g\rangle f$, for all $f, g \in \mathcal{H}$. These expressions may probably be only formal since the series are not necessarily convergent in the uniform topology, as it happens when the operators S_φ and S_Ψ are unbounded. However, this is not the case if $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are Riesz bases. In this case, we call our $\mathcal{D}$-PBs regular [18,22]. Another interesting aspect of the operators S_φ and S_ψ is that they give rise to the following intertwining relations between N and $N^\dagger$:

$$S_\Psi N g = N^\dagger S_\Psi g \quad \text{and} \quad N S_\varphi f = S_\varphi N^\dagger f, \tag{2.8}$$

with $f \in \mathcal{L}_\psi$ and $g \in \mathcal{L}_\varphi$, which are in agreement with the fact that N and $N^\dagger$ have the same eigenvalues (other than related eigenvectors equation (2.6)).

These results can be extended to more modes of PBs, as one does for bosons [24]. In this case, we have two families of operators, $\{a_j, b_j,\ j \in \mathcal{J}\}$, $\mathcal{J}$ being a certain set of indices (discrete or continuous), such that $a_j \neq b_k^\dagger$ for all $j, k \in \mathcal{J}$, and

$$[a_j, b_k]f = \delta_{j,k} f, \tag{2.9}$$

with $\forall f \in \mathcal{D}$. Here, $\mathcal{D}$ is, as mentioned earlier, a dense subspace of the Hilbert space where all these operators act and which is invariant under the action of all the $a_j^\sharp$ and $b_j^\sharp$. Equation (2.9) is the pseudo-bosonic rule if $\mathcal{J}$ is discrete. In case $\mathcal{J}$ is continuous, which is what is interesting for us in §3, equation (2.9) must be replaced with

$$[a(k), b(q)]f = \delta(k-q) f, \tag{2.10}$$

with $\forall f \in \mathcal{D}$. In this formula, δ is, of course, the Dirac (rather than the Kronecker) delta.

In a way similar to what we showed earlier, these are ladder operators which can be used to construct two biorthogonal families of eigenstates of $N_j = b_j a_j$ and $N_j^\dagger = a_j^\dagger b_j^\dagger$, $j \in \mathcal{J}$, or of $N(k) = b(k)a(k)$ and of the adjoint in the continuous case. All that has been stated for a single mode of PBs can be extended to many modes without particular problems.

In the next section, we shall be particularly interested in a specific version of PBs, those related to the Swanson Hamiltonian [25]. In [26], we showed how this Hamiltonian (together with others) can be rewritten in terms of PBs, and we applied and extended these results in several ways, including also a distributional approach to some quantum mechanical systems [18,26,27]. In particular, we showed that the manifestly non-Hermitian Hamiltonian

$$H_\theta = \frac{1}{2}(p^2 + x^2) - \frac{\mathrm{i}}{2}\tan(2\theta)(p^2 - x^2),$$

proposed in [28] and strictly connected to the Swanson Hamiltonian, can be rewritten in terms of pseudo-bosonic operators. Here, $\theta \in (-\pi/4, \pi/4) \setminus \{0\} =: I$. We see that $H_\theta^\dagger = H_{-\theta} \neq H_\theta$, for all $\theta \in I$. The operators x and p are the self-adjoint position and momentum, respectively, they satisfy $[x, p] = \mathrm{i}\mathbb{1}$. Introducing the bosonic annihilation and creation operators $c = \frac{1}{\sqrt{2}}(x + ip)$ and $c^\dagger = \frac{1}{\sqrt{2}}(x - ip)$, and putting further

$$\left.\begin{aligned} A_\theta &= \cos(\theta) c + \mathrm{i}\sin(\theta) c^\dagger \\ \text{and} \qquad B_\theta &= \cos(\theta) c^\dagger + \mathrm{i}\sin(\theta) c, \end{aligned}\right\} \tag{2.11}$$

we have rewritten H_θ as follows:

$$H_\theta = \omega_\theta \left(B_\theta A_\theta + \frac{1}{2}\mathbb{1} \right). \tag{2.12}$$

Here, $\omega_\theta = 1/\cos(2\theta)$, which is well defined since $\cos(2\theta) \neq 0$ for all $\theta \in I$. It is clear that $A_\theta^\dagger \neq B_\theta$ and that $[A_\theta, B_\theta] = \mathbb{1}$. We refer to [26] for a full analysis of this model, including the expression of the eigenstates of H_θ and $H_\theta^\dagger$. Here, we see how to use equation (2.11), extended to a continuous set of modes, in the construction of our PBKGF.

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3. The pseudo-bosonic Klein–Gordon field (KGF)

As it is well known, the (standard) KGF is the solution of the following second-order differential equation:

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+m^2\right)\varphi(x,t)=0, \tag{3.1}$$

which is quantized by assuming the equal-time CCRs

$$\left.\begin{aligned}&[\varphi(x,t),\varphi(x',t)]=0,\\&[\dot\varphi(x,t),\dot\varphi(x',t)]=0\\ \text{and}\quad&[\varphi(x,t),\dot\varphi(x',t)]=\mathrm{i}\delta(x-x').\end{aligned}\right\} \tag{3.2}$$

Following the notation and the main steps of [3], we expand the solution of the Klein–Gordon equation in plane waves,

$$\varphi(x,t)=\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{\sqrt{4\pi\omega_k}}[c(k)\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega_k t}+c^\dagger(k)\,\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega_k t}], \tag{3.3}$$

where $\omega_k=\sqrt{k^2+m^2}$ and the operators $c(k)$ and Hermitian conjugate $c^\dagger(k)$ are the coefficients of the expansion. They satisfy the CCRs

$$[c(k),c(q)]=[c^\dagger(k),c^\dagger(q)]=0\quad\text{and}\quad[c(k),c^\dagger(q)]=\delta(k-q)\mathbb{1}, \tag{3.4}$$

for all $k,q\in\mathbb{R}$. More explicitly, assuming equation (3.3), it is possible to show that equation (3.4) implies equation (3.2), and vice versa.

Remark 3.1.

(1) The expansion in equation (3.3) can be thought as a classical function (or distribution), rather than an operator, if we do not explicitly require equation (3.2) or equation (3.4), i.e. if we assume $c(k)$ and $c^\dagger(k)$ are k-dependent (commuting) functions. In this case, of course, $c^\dagger(k)$ should be replaced by $\overline{c(k)}$.

(2) In this section, we mostly work formally, i.e. not paying much attention to the fact that the fields we are considering, here and in the rest of the section, are indeed operator valued distributions. Our main effort here is to show that, moving from bosonic to pseudo-bosonic operators in an expansion like that in equation (3.3), some infinite physical quantity turns out to become finite. We say more on this aspect in §4.

Let us call e_0 the ground state of the theory [3]. This is defined by requiring that $c(k)e_0=0$, with $\forall k\in\mathbb{R}$.

Interesting quantities to compute are the expectation values in e_0 of the field $\varphi(x,t)$ and of the product of the field, $\varphi(x,t)\varphi(x',t')$. It is well known that problems arise when we try to compute, in particular, the mean value of the product $\varphi(x,t)\varphi(x,t)$ on e_0. In particular, while the one-point function of the field is well defined, and trivial, $\langle e_0,\varphi(x,t)e_0\rangle=0$, we also find that

$$\Delta_+(x,y;t,s)=\langle e_0,\varphi(x,t)\varphi(y,s)e_0\rangle=\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{4\pi\omega_k}\,\mathrm{e}^{\mathrm{i}k(x-y)-\mathrm{i}\omega_k(t-s)}, \tag{3.5}$$

which, in the limit $(y,s)\to(x,t)$, diverges logarithmically[2] [3,29]. This is just the first divergence of many others which one has to face with when working with QFT.

[2]Incidentally, we recall that in four dimensions the analogous divergence is quadratic.

Let us now consider the following extended version of equation (3.3):

$$\Phi_\theta(x,t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\sqrt{4\pi\omega_k}} [A_\theta(k)\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega_k t} + B_\theta(k)\,\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega_k t}], \tag{3.6}$$

where θ is a (real) parameter in I. For the moment, we assume that $A_\theta(k)$ and $B_\theta(k)$ are simply coefficients of the expansion of $\Phi_\theta(x,t)$ in plane waves, and that, in principle, $B_\theta(k)$ is not necessarily the complex conjugate of $A_\theta(k)$. In other words, both $A_\theta(k)$ and $B_\theta(k)$ are functions and $A_\theta(k) \neq \overline{B_\theta(k)}$, in principle.

This reminds us what happens for charged fields, which are not Hermitian, see [29] for instance. Hence, this suggests introducing the following Lagrangian density for our field:

$$\mathcal{L}_\theta = \dot{\Phi}_\theta^\dagger\,\dot{\Phi}_\theta - (\partial_x\Phi_\theta)^\dagger(\partial_x\Phi_\theta) - m^2\Phi_\theta^\dagger\Phi_\theta, \tag{3.7}$$

where, to simplify the notation, we have omitted everywhere the dependence on x and t. We stress once more that, at this stage, we are looking at the fields appearing in equation (3.7) as classical objects, and, for this reason, the † symbol should be understood as a simple complex conjugation. We prefer to keep this symbol here since it will be relevant soon, when quantizing the field. By replacing $\mathcal{L}_\theta$ in the Euler–Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}_\theta}{\partial\dot{\Phi}_\theta^\dagger}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}_\theta}{\partial(\partial_x\Phi_\theta)^\dagger}\right) - \frac{\partial\mathcal{L}_\theta}{\partial\Phi_\theta^\dagger} = 0,$$

and its adjoint, we recover

$$(\partial_t^2 - \partial_x^2 + m^2)\Phi_\theta(x,t) = (\partial_t^2 - \partial_x^2 + m^2)\Phi_\theta^\dagger(x,t) = 0, \tag{3.8}$$

as expected. From equation (3.7), we also compute the conjugate momenta of $\Phi_\theta(x,t)$ and $\Phi_\theta^\dagger(x,t)$:

$$\Pi_\theta(x,t) = \frac{\partial\mathcal{L}_\theta}{\partial\dot{\Phi}_\theta^\dagger} = \dot{\Phi}_\theta(x,t) \quad \text{and} \quad \Pi_\theta^\dagger(x,t) = \frac{\partial\mathcal{L}_\theta}{\partial\dot{\Phi}_\theta} = \dot{\Phi}_\theta^\dagger(x,t). \tag{3.9}$$

Then the Hamiltonian density is

$$\mathfrak{H}_\theta = \Pi_\theta\dot{\Phi}_\theta^\dagger + \Pi_\theta^\dagger\dot{\Phi}_\theta - \mathcal{L}_\theta = \Pi_\theta^\dagger\Pi_\theta + (\partial_x\Phi_\theta)^\dagger(\partial_x\Phi_\theta) + m^2\Phi_\theta^\dagger\Phi_\theta. \tag{3.10}$$

To quantize the system, we now assume that $A_\theta(k)$ and $B_\theta(k)$ is a continuous family of pseudo-bosonic operators obeying the following commutation relations, which extend those in equation (3.4):

$$[A_\theta(k), A_\theta(q)] = [B_\theta(k), B_\theta(q)] = 0 \quad \text{and} \quad [A_\theta(k), B_\theta(q)] = \delta(k-q)\mathbb{1}, \tag{3.11}$$

for all $k, q \in \mathbb{R}$. For concreteness' sake (and because our choice will produce interesting results), we consider here a specific class of operators $A_\theta(k)$ and $B_\theta(k)$. We know, see [18], that many other choices are also possible, but we do not consider these alternatives here. To be concrete, we assume that $A_\theta(k)$ and $B_\theta(k)$ can be written in terms of $c(k)$ and $c^\dagger(k)$ satisfying equation (3.4) as follows:

$$\left.\begin{aligned} & A_\theta(k) = \cos(\theta)c(k) + \mathrm{i}\sin(\theta)c^\dagger(k) \\ \text{and} \quad & B_\theta(k) = \cos(\theta)c^\dagger(k) + \mathrm{i}\sin(\theta)c(k), \end{aligned}\right\} \tag{3.12}$$

with $\theta \in I$ and $k \in \mathbb{R}$, which is a multi-mode version of equation (2.11). It is clear that $\Phi_0(x,t) = \varphi(x,t)$ since, when $\theta = 0$, $A_0(k) = c(k)$ and $B_0(k) = c^\dagger(k)$.

The first obvious remark is that, since for all $\theta \in I \setminus \{0\}$ $B_\theta(k)^\dagger \neq A_\theta(k)$, $(\Phi_\theta(x,t))^\dagger \neq \Phi_\theta(x,t)$, while $(\varphi(x,t))^\dagger = \varphi(x,t)$.

Equations (3.11) are now deduced if we assume the following equal-time canonical commutation rules between the fields and their conjugate momenta:

$$[\Phi_\theta(x,t),\Phi_\theta(y,t)]=[\Pi_\theta(x,t),\Pi_\theta(y,t)]=0 \quad\text{and}\quad [\Phi_\theta(x,t),\Pi_\theta(y,t)]=\mathrm{i}\delta(x-y), \tag{3.13}$$

with similar rules for $\Phi_\theta^\dagger(x,t)$ and $\Pi_\theta^\dagger(x,t)$:

$$[\Phi_\theta^\dagger(x,t),\Phi_\theta^\dagger(y,t)]=[\Pi_\theta^\dagger(x,t),\Pi_\theta^\dagger(y,t)]=0 \quad\text{and}\quad [\Phi_\theta^\dagger(x,t),\Pi_\theta^\dagger(y,t)]=\mathrm{i}\delta(x-y). \tag{3.14}$$

If we now insert equation (3.6) and

$$\Pi_\theta(x,t)=\dot{\Phi}_\theta(x,t)-\mathrm{i}\int_{-\infty}^{\infty}\sqrt{\frac{\omega_k}{4\pi}}[A_\theta(k)\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega_k t}-B_\theta(k)\,\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega_k t}]\,\mathrm{d}k, \tag{3.15}$$

together with their adjoints, into equation (3.10) for $\mathfrak{H}_\theta$, and then integrate over x, we find the following expression for the Hamiltonian operator:

$$H_\theta=\int_{\mathbb{R}}\mathfrak{H}_\theta(x,t)\,\mathrm{d}x=\int_{\mathbb{R}}\mathrm{d}k\,\omega_k(B_\theta^\dagger(k)B_\theta(k)+A_\theta^\dagger(k)A_\theta(k)), \tag{3.16}$$

which is (formally) self-adjoint, and which reduces to the usual form of the Hamiltonian of the KGF if $\theta=0$. In fact, there is more than this: in terms of the $c(k)$ and $c^\dagger(k)$'s, it turns out that

$$H_\theta=\int_{\mathbb{R}}\mathrm{d}k\omega_k(c(k)c^\dagger(k)+c^\dagger(k)c(k)), \tag{3.17}$$

which is exactly the Hamiltonian of the KGF. Hence, in particular, H_θ does not depend on θ, in contrast to $\mathfrak{H}_\theta$. Notice that H_θ also does not depend explicitly on time, as it is clear from its integral expression in terms of pseudo-bosonic or of bosonic operators.

Summarizing what we have found so far is not particularly different from what is known for a standard (complex) KGF. Another similar result is the following: we can recover $A_\theta(k)$ and $B_\theta(k)$ from $\Phi_\theta(x,t)$ and $\Pi_\theta(x,t)$ as follows:

$$A_\theta(k)=\frac{1}{4\pi\,\omega_k}\int_{\mathbb{R}}\mathrm{d}x\,\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega_k t}(\omega_k\Phi_\theta(x,t)+\mathrm{i}\Pi_\theta(x,t)) \tag{3.18}$$

and

$$B_\theta(k)=\frac{1}{4\pi\,\omega_k}\int_{\mathbb{R}}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega_k t}(\omega_k\Phi_\theta(x,t)-\mathrm{i}\Pi_\theta(x,t)), \tag{3.19}$$

the latter being clearly different from $A_\theta^\dagger(k)$. These are essentially the same inverse formulas holding true for the KGF and can be deduced with similar computations.

Note that, because of equation (3.12), we have

$$B_\theta^\dagger(k)=A_{-\theta}(k) \quad\text{and}\quad A_\theta^\dagger(k)=B_{-\theta}(k), \tag{3.20}$$

and, equivalently

$$\Phi_\theta^\dagger(x,t)=\Phi_{-\theta}(x,t) \quad\text{and}\quad \Pi_\theta^\dagger(x,t)=\Pi_{-\theta}(x,t). \tag{3.21}$$

For this reason, in particular, we could rewrite H_θ in equation (3.15) replacing $B_\theta^\dagger(k)$ with $A_{-\theta}(k)$ and $A_\theta^\dagger(k)$ with $B_{-\theta}(k)$.

Remark 3.2. Other commutation rules can be computed using equations (3.6), (3.12) and (3.15). In particular, we obtain

$$[A_\theta(k),A_{-\theta}(q)]=-[B_\theta(k),B_{-\theta}(q)]=-\mathrm{i}\sin 2\theta\,\delta(k-k)\mathbb{1}, \tag{3.22}$$

$$[\Phi_\theta(x,t),\Phi_\theta^\dagger(y,t)]=-\frac{\sin 2\theta}{2\pi}\int\frac{\mathrm{d}k}{\omega_k}\,\mathrm{e}^{\mathrm{i}k(x-y)}\sin(2\omega_k t)\mathbb{1} \tag{3.23}$$

and

$$[\Pi_\theta(x,t),\Pi_\theta^\dagger(y,t)]=\frac{\sin 2\theta}{2\pi}\int\mathrm{d}k\,\omega_k\,\mathrm{e}^{\mathrm{i}k(x-y)}\sin(2\omega_k t)\mathbb{1}, \tag{3.24}$$

which all return *standard* results (i.e. those we know for ordinary KGFs) if $\theta=0$. In case $\sin 2\theta\neq 0$, we still observe that $[\Phi_\theta(x,0),\Phi_\theta^\dagger(y,0)]=[\Phi_\theta(x,0),\Phi_\theta^\dagger(y,0)]=0$, while these commutators are different from zero for $t>0$, in general.

(a) A finite two-point function

For what we have to do here, it is now convenient to rewrite $\Phi_\theta(x,t)$ in equation (3.6) in terms of the bosonic operators in equation (3.4). We obtain

$$\Phi_\theta(x,t)=\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{\sqrt{4\pi\omega_k}}[c(k)\alpha_\theta(k;x,t)+c^\dagger(k)\beta_\theta(k;x,t)],\tag{3.25}$$

where

$$\left.\begin{aligned}\alpha_\theta(k;x,t)&=\cos\theta\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega_k t}+\mathrm{i}\sin\theta\,\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega_k t}\\ \text{and}\qquad\beta_\theta(k;x,t)&=\cos\theta\,\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega_k t}+\mathrm{i}\sin\theta\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega_k t}.\end{aligned}\right\}\tag{3.26}$$

By using this expression for $\Phi_\theta(x,t)$, it is clear that

$$F_\theta^{(1)}(x,t)=\langle e_0,\Phi_\theta(x,t)e_0\rangle=0,\tag{3.27}$$

for all x and t. Similarly, we have

$$G_\theta^{(1)}(x,t)=\langle e_0,\Pi_\theta(x,t)e_0\rangle=0,\tag{3.28}$$

for all x and t. Here, we have used the following expression for $\Pi_\theta(x,t)$:

$$\Pi_\theta(x,t)=-\mathrm{i}\int_{-\infty}^{\infty}\sqrt{\frac{\omega_k}{4\pi}}\,\mathrm{d}k[c(k)\alpha_{-\theta}(k;x,t)-c^\dagger(k)\beta_{-\theta}(k;x,t)],\tag{3.29}$$

using the same definitions given in equation (3.26).

Let us now consider the two-point function $F_\theta^{(2)}(x,t;y,s)=\langle e_0,\Phi_\theta(x,t)\Phi_\theta(y,s)e_0\rangle$. According to what we have shown in equation (3.5), see also [3,29], what is *dangerous* is the limit of this function when $(y,s)\to(x,t)$. For this reason, we only consider here $F_\theta^{(2)}(x,0;y,0)$ and then discuss what happens when $y\to x$. We find a different behaviour for $x=0$ and for $x\neq 0$.

By using equation (3.25), together with $c(k)e_0=0$, it is easy to find that

$$F_\theta^{(2)}(x,0;y,0)=\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{4\pi\omega_k}\overline{\beta_{-\theta}(k;x,0)}\beta_\theta(k;y,0),\tag{3.30}$$

which, when $y\to x$, produces

$$F_\theta^{(2)}(x,0;x,0)=\cos(2\theta)\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{4\pi\omega_k}+\mathrm{i}\sin(2\theta)\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{4\pi\omega_k}\,\mathrm{e}^{2\mathrm{i}kx}.\tag{3.31}$$

Our first remark is the following: if $x=0$, we obtain $F_\theta^{(2)}(0,0;0,0)=\mathrm{e}^{2\mathrm{i}\theta}\int_{-\infty}^{\infty}\mathrm{d}k/4\pi\omega_k$, and we are back to what we have seen in equation (3.5) and after, for the standard KGF: $F_\theta^{(2)}(0,0;0,0)=\infty$. However, if $x\neq 0$, the situation is quite different. In this case, and if we further fix[3] $\theta=\pi/4$, we deduce that

$$F_{\frac{\pi}{4}}^{(2)}(x,0;x,0)=\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{4\pi\omega_k}\,\mathrm{e}^{2\mathrm{i}kx}=\frac{\mathrm{i}}{2\pi}\int_0^{\infty}\frac{\mathrm{d}k}{\omega_k}\cos(2kx),\tag{3.32}$$

with easy computations. This integral can be written in terms of the Bessel function $K_0(x)$ as follows:

$$F_{\pi/4}^{(2)}(x,0;x,0)=\frac{\mathrm{i}}{2\pi}K_0(2m|x|).\tag{3.33}$$

Note that $K_0(2m|x|)$ diverges when $x\to 0$, in agreement with what we have found before for $F_\theta^{(2)}(0,0;0,0)$, but, if x is not zero, we obtain a finite result. This is different from what we have found in [10], and from what happens for ordinary KGFs, where $F_\theta^{(2)}(x,0;x,0)$ also diverges for $x\neq 0$. In other words, with the particular choice of $\theta=\pi/4$ (but also taking, e.g. $\theta=3\pi/4$), the two-point function $F_\theta^{(2)}(x,0;x,0)$ turns out to be finite for all $x\neq 0$.

[3] This is not the only useful choice of θ, as we shall see. Other choices are also possible.

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Similarly, we can consider

$$G_\theta^{(2)}(x,t;y,s)=\langle e_0,\Pi_\theta(x,t)\Pi_\theta(y,s)e_0\rangle, \tag{3.34}$$

and check if $G_\theta^{(2)}(x,0;x,0)$ is also finite for all $x\neq 0$. However, this is more complicated. In fact, by analogy with equation (3.32), we find that

$$G_{\frac{\pi}{4}}^{(2)}(x,0;x,0)=-\frac{\mathrm{i}}{2\pi}\int_0^\infty \omega_k\cos(2kx)\,\mathrm{d}k, \tag{3.35}$$

which, if $x=0$, diverges quadratically in k. If $x\neq 0$, the situation is slightly more involved. Despite of its apparent divergence, it is still possible to understand that $G_{\pi/4}^{(2)}(x,0;x,0)$ is a *manageable quantity*. For that, it is convenient to rewrite it as follows:

$$G_{\frac{\pi}{4}}^{(2)}(x,0;x,0)=-\frac{\mathrm{i}}{4\pi}\int_{\mathbb{R}}\omega_k\,\mathrm{e}^{2\mathrm{i}kx}\,\mathrm{d}k, \tag{3.36}$$

and then to consider its convolution with a (generic) function $f(x)\in\mathcal{S}(\mathbb{R})$, the set of test functions [30]:

$$G^{(2)}[f](y)=\int_{\mathbb{R}}f(y-x)G_{\pi/4}^{(2)}(x,0;x,0)\,\mathrm{d}x=-\frac{\mathrm{i}}{4\pi}\int_{\mathbb{R}}\omega_k\left(\int_{\mathbb{R}}f(y-x)\,\mathrm{e}^{2\mathrm{i}kx}\,\mathrm{d}x\right)\mathrm{d}k, \tag{3.37}$$

with a change of integration. Hence, we obtain

$$G^{(2)}[f](y)=-\frac{\mathrm{i}}{2\sqrt{2\pi}}\int_{\mathbb{R}}\omega_k\hat{f}(2k)\,\mathrm{e}^{2\mathrm{i}ky}\,\mathrm{d}k, \tag{3.38}$$

where $\hat{f}(2k)$ is the Fourier transform of $f(x)$ computed in $2k$. It is clear that $G^{(2)}[f](y)$ is well defined for all possible y and for all $f(x)\in\mathcal{S}(\mathbb{R})$ since $\hat{f}\in\mathcal{S}(\mathbb{R})$ as well. This means that $\hat{f}(2k)$ goes to zero very fast, while ω_k diverges linearly in $|k|$: their product is integrable. Furthermore, it is also possible to (try to) recover $G_{\pi/4}^{(2)}(x,0;x,0)$ out of $G^{(2)}[f](y)$. In fact, this could be formally done by simply fixing $f(x)=\delta(x)$, but this is not in agreement with our wish to have $f(x)\in\mathcal{S}(\mathbb{R})$. However, see [9–11,30], we could consider a δ-sequence of functions in $\mathcal{D}(\mathbb{R})$ and therefore of $\mathcal{S}(\mathbb{R})$. Some useful properties, and the construction of these sequences, are listed in the following theorem, see [9] for instance:

Theorem 3.1. *Let $\phi\in\mathcal{D}(\mathbf{R})$ be a given function with supp $\phi\subseteq[-1,1]$ and $\int\phi(x)\,dx=1$. We call $\delta-$sequence the sequence $\delta_n, n\in\mathbf{N}$, defined by $\delta_n(x)\equiv n\phi(nx)$.*

Then, $\forall\, T\in\mathcal{D}'(\mathbf{R})$, the set of distributions, the convolution $T_n\equiv T*\delta_n$ is a $C^\infty-$function, for any fixed $n\in\mathbf{N}$. This sequence converges to T in the topology of $\mathcal{D}'$, when $n\to\infty$.

Moreover, if $T(x)$ is a continuous function with compact support, then T_n converges uniformly to $T(x)$.

In other words, even if we cannot take $f(x)=\delta(x)$ in equation (3.37), we could still replace $f(x)$ with a sequence of functions $\delta_n(x)$ in $\mathcal{S}(\mathbb{R})$ converging to $\delta(x)$ as in this theorem, and then look for the limit of $G^{(2)}[\delta_n](y)$ when $n\to\infty$. This distributional analysis of $G_\theta^{(2)}$ is work in progress.

As a final remark, we observe that for $\theta=\pi/4$ our psuedo-bosonic operators can be written as follows:

$$A_{\pi/4}(k)=\frac{1+\mathrm{i}}{2}(x(k)+p(k))\quad\text{and}\quad B_{\pi/4}(k)=\frac{1+\mathrm{i}}{2}(x(k)-p(k)), \tag{3.39}$$

where $x(k)=\frac{1}{\sqrt{2}}(c(k)+c^\dagger(k))$ and $p(k)=1/\sqrt{2}\,\mathrm{i}(c(k)-c^\dagger(k))$ can be seen as a sort of *multi-mode* version of the standard position and momentum operators. Equation (3.39) clarifies once more that $A_{\pi/4}(k)$ is not the adjoint of $B_{\pi/4}(k)$. Also, a straight computation confirms that they obey the pseudo-bosonic rule $[A_{\pi/4}(k),B_{\pi/4}(q)]=\delta(k-q)\mathbb{1}$, as they should.

4. Conclusion

This article is a first *incursion* of PBs in QFT and, in our opinion, the results we have deduced are rather promising and suggest to carry on the analysis in many ways. In particular, we should consider other possible divergences arising in other relevant n-point functions and see if they are cancelled, or controlled, replacing bosons with PBs. The same should be done when considering higher spatial dimensions. Also, the KGF we have discussed here is a free field. Interacting fields make the system much more complicated, and more and more diverging Feynman graphs arise. Hence, it is natural to check if PBs can also be useful for these more complicated systems.

We should also stress that the specific form of pseudo-bosonic ladders we have considered here, see equation (3.12), are those arising from the Swanson model. But many other classes of PBs also exist [18]. An analysis of these alternative operators in this context would be interesting as, in particular, the relevance of, e.g. pseudo-fermions, [18], in connection with the Dirac field or other fermionic quantum fields. This analysis could possibly be relevant in connection with what is discussed in previous studies [19–21].

We end this article with a final remark concerning the apparent (lack of) mathematical rigour in some parts of this article. This aspect was not so relevant for us here, since, as we have already pointed out, we were more interested in understanding if PBs can somehow contribute to a very old problem of QFT, that of diverging quantities. Our results here suggest that this is the case. Hence, it makes sense to try to put our results on more rigorous bases, and this is part of our future plans.

Data accessibility. This article has no additional data.

Declaration of AI use. I have not used AI-assisted technologies in creating this article.

Conflict of interest declaration. I declare I have no competing interests.

Funding. The author acknowledges partial financial support from Palermo University and from G.N.F.M. of the INdAM. This work has also been partially supported by the PRIN grant *Transport phenomena in low dimensional structures: models, simulations and theoretical aspects*—project code 2022TMW2PY—CUP B53D23009500006, and partially by project ICON-Q, Partenariato Esteso NQSTI—PE00000023, Spoke 2.

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