Chapter

Derivation and Integration on a Fractal Subset of the Real Line

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Abstract

Ordinary calculus is usually inapplicable to fractal sets. In this chapter, we introduce and describe the various approaches made so far to define the theory of derivation and integration on fractal sets. In particular, we study some Riemann-type integrals (the s-Riemann integral, the sHK integral, the s-first-return integral) defined on a closed fractal subset of the real line with finite and positive s-dimensional Hausdorff measure (s-set) with particular attention to the Fundamental Theorem of Calculus. Moreover, we pay attention to the relation between the s-Riemann integral, the sHK integral, and the Lebesgue integral with respect to the Hausdorff measure $\mathcal{H}^s$, respectively, and we give a characterization of the primitives of the sHK integral.

Keywords: Hausdorff measure, s-set, s-Riemann integral, sHK integral, s-derivative

1. Introduction

For many years, it was thought that the structure of fractal sets had so many irregularities to render too difficult the definition of standard-type methods and techniques of ordinary calculus in such type of sets. In fact, for example, the usual derivative of the classical Lebesgue-Cantor staircase function is zero almost everywhere, and the usual Riemann integral of a function defined on a fractal set is undefined (see Refs. [1–3]). Therefore analysis of fractals has been studied by different methods such as harmonic analysis, stochastic processes, and fractional processes. Very recently, a non-Newtonian calculus on fractal sets of the real line that starts by elementary non-Diophantine arithmetic operations of a Burgin type was formulated by M. Czachor (see Ref. [4]).

In this chapter, we present a method of standard calculus for fractal subsets of the real line, that was independently formulated by various authors (see Refs. [5–9]). Such formulation is aimed at those self-similar fractal subsets of the real line with finite and positive s-dimensional Hausdorff measure, briefly called s-sets. Moreover, it differs from the classical one, for the use of the Hausdorff measure instead of the natural distance. The idea of replacing the usual distance with the Hausdorff measure was used for the first time in the definition of s-derivative given in 1991 by De Guzmann, Martin, and Reyes (see Ref. [5]), in order to study the problem of existence and uniqueness of the solutions of ordinary differential equations in which the independent variable takes value in a fractal set. Later, in 1998, this concept was taken up by Jung and Su in Ref. [6] to define an integral of the Riemann type called s-integral.

Moreover, Parvate and Gangal in Ref. [7], independently by Jung and Su, introduced an integral of Riemann type on an s-set of the real line, called F^s-integral. Such integration processes were defined as the classical Riemann integral but with the Hausdorff measure and the mass function, respectively, taking over the role of the distance. Since the Hausdorff measure and the mass function are proportional (see Ref. [7], Section 4), it follows that Jiang and Su in Ref. [6] and Parvate and Gangal in Ref. [7] defined, independently, the same integral. In this chapter, we call it the s-Riemann integral.

Both authors proved a version of the Fundamental Theorem of Calculus. About this, we recall that, in the real line, such fundamental theorem states that: *if* $f : [a,b] \to \mathbb{R}$ *is differentiable on* $[a, b]$, *then the function* $f'(x)$ *is integrable (in some sense) on* $[a, b]$ *and* $\int_a^b f'(t)dt = f(b) - f(a)$.

Unfortunately, as it happens in the real line, the s-Riemann integral is not the best integral for the formulation of the Fundamental Theorem of Calculus. This was the reason that motivated Bongiorno and Corrao in Ref. [8] to define an Henstock-Kurzweil integration process on an s-set of the real line and to formulate in Ref. [9] the best version of the Fundamental Theorem of Calculus on such s-sets. We recall that, in the real line, it is the Henstock-Kurzweil integral that, solving the problem of primitives, provides the best version of the Fundamental Theorem of Calculus (see Ref. [10]). Later and independently, also Golmankhaneh and Baleanu extended, in Ref. [11], the s-Riemann integral by introducing an integral of the Henstock–Kurzweil type. However, no version of the Fundamental Theorem of Calculus is proved in Ref. [11].

More precisely, the integral introduced in Ref. [8] is based on the use of the Hausdorff measure instead of the notion of the classical distance, as it was already done by Jung and Su in Ref. [6]. On the other hand, the mass function, instead of the classical distance, is used in Ref. [11], as it was already done by Parvate and Gangal in Ref. [7]. Precisely, Golmankhaneh and Baleanu have revised the notion of mass function given in Ref. [7] by defining a special mass function through the use of a gauge function previously introduced in Ref. [12]. However, since the Hausdorff measure and the mass function are proportional, without loss of generality, we infer that the two integrals coincide and we call the resulting integral the *sHK* integral. Moreover, following the literature, we prefer the use of the Hausdorff measure instead of the mass function to define the *sHK* integral. Finally, through the characterization of the *sHK* primitives, a descriptive definition of the *sHK* integral is given.

All sections of this chapter are related to each other; moreover, in order not to burden the reader, some proofs are reported in Appendix A, while Appendix B contains a brief history of the Henstock-Kurzweil integral. Particular attention is paid to the formulation of the best version of the Fundamental Theorem of Calculus. Afterward, the last section of this chapter is devoted to the formulation of a new integration process. Precisely an integral of the first return-type, called s-first-return integral, is defined. The idea of the first-return technique comes from the Poincaré first-return map of differentiable dynamics, and it was already used in differentiation and in integration theory (see Refs. [13–15]).

2. The s-derivatives

Let $\mathcal{L}(\cdot)$ be the usual Lebesgue measure, and let E be a compact subset of the real line. Given s, with $0 < s \leq 1$, let

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \mathcal{L}(A_i)^s : E \subset \bigcup_{i=1}^{\infty} A_i, \ \mathcal{L}(A_i) \le \delta \right\} \tag{1}$$

be the *s-dimensional Hausdorff measure* of E.

Recall that $\mathcal{H}^s(\cdot)$ is a Borel regular measure and that the unique number s for which $\mathcal{H}^t(E) = 0$ if $t > s$ and $\mathcal{H}^t(E) = \infty$ if $t < s$ is called the Hausdorff dimension of E.

Definition 2.1. Let E be a compact subset of the real line and let $0 < s < 1$. E is called an s-set if it is measurable with respect to the s-dimensional Hausdorff measure $\mathcal{H}^s$ (briefly $\mathcal{H}^s$-measurable) and $0 < \mathcal{H}^s(E) < \infty$.

Therefore $\mathcal{H}^s$ is a Radon measure on each s-set (see Ref. [16]).

From now on we will denote by $E \subset [a, b]$ an s-set of the real line, by $a = minE$ and by $b = maxE$.

Definition 2.2. For $x, y \in E$ we set

$$d(x,y) = \begin{cases} \mathcal{H}^s([x,y] \cap E), & \text{if } \ x < y; \\ \mathcal{H}^s([y,x] \cap E), & \text{if } \ y < x. \end{cases} \tag{2}$$

Proposition 2.1. The function $(x,y) \rightsquigarrow d(x,y)$ from $E \times E \to \mathbb{R}^+ \cup \{0\}$ is a metric, and the space (E, d) is a complete metric space.

Proposition 2.2. The topology of the metric space (E, d) coincides with the topology induced on E by the usual topology of $\mathbb{R}$.

Definition 2.3. Let $F : E \to \mathbb{R}$ and let $x_0 \in E$. The *s-derivatives* of F on the left and on the right at the point x_0 are defined, respectively, as follows:

$$F'^{-}_{s}(x_0) = \lim_{\substack{x \to x_0^- \\ x \in E}} \frac{F(x_0) - F(x)}{d(x,x_0)} \tag{3}$$

$$F'^{+}_{s}(x_0) = \lim_{\substack{x \to x_0^+ \\ x \in E}} \frac{F(x) - F(x_0)}{d(x,x_0)} \tag{4}$$

when these limits exist.

We say that the *s-derivative* of F at x_0 exists if $F'^{-}_{s}(x_0) = F'^{+}_{s}(x_0)$ or if the s-derivative of F on the left (resp. right) at x_0 exists, and for some, $\varepsilon > 0$ we have $d(x_0, x_0 + \varepsilon) = 0$ (resp. $d(x_0 - \varepsilon, x_0) = 0$). The s-derivative of F at x_0, when it exists, will be denoted by $F'_s(x_0)$.

It is trivial to observe that by the previous definition, it follows the linearity of the s-derivative. Moreover.

Remark 2.1. If F is s-derivable at the point x_0, then F is continuous at x_0 according to the topology induced on E by the usual topology of $\mathbb{R}$.

Hereafter, for each interval $A \subset [a, b]$, we set $\widetilde{A} = A \cap E$.

Example 2.1. Let $E \subset [0, 1]$ be the ternary Cantor set. E is an s-set for $s = \log_3 2$ and $\mathcal{H}^s(E) = 1$ (see Ref. [2], Theorem 1.14). Moreover $\mathcal{H}^s \widetilde{\left[\frac{2}{3^n}, \frac{1}{3^{n-1}}\right]} = \frac{1}{2^n} = \mathcal{H}^s \widetilde{\left[0, \frac{1}{3^n}\right]}$, and $\mathcal{H}^s \widetilde{\left[\frac{2}{3^n}, \frac{7}{3^{n+1}}\right]} = \frac{1}{4^n} = \mathcal{H}^s \widetilde{\left[\frac{8}{3^{n+1}}, \frac{1}{3^{n-1}}\right]}$.

The function

$$F(x) = \begin{cases} \dfrac{(-2)^n}{n}\mathcal{H}^s \widetilde{\left[\dfrac{2}{3^n}, x\right]}, & x \in \widetilde{\left[\dfrac{2}{3^n}, \dfrac{7}{3^{n+1}}\right]}; \\ \dfrac{(-2)^n}{n}\mathcal{H}^s \widetilde{\left[\dfrac{8}{3^{n+1}}, x\right]}, & x \in \widetilde{\left[\dfrac{8}{3^{n+1}}, \dfrac{1}{3^{n-1}}\right]}; \\ 0, & x = 0. \end{cases} \tag{5}$$

is s-derivable on E with

$$F'_s(x) = \begin{cases} \dfrac{(-2)^n}{n}, & x \in \widetilde{\left[\dfrac{2}{3^n}, \dfrac{1}{3^{n-1}}\right]}; \\ 0, & x = 0. \end{cases} \tag{6}$$

Infact, for $x_0 \in \widetilde{\left[\frac{2}{3^n}, \frac{1}{3^{n-1}}\right]}$ it is:

$$F'^{-}_s(x_0) = \lim_{\substack{x \to x_0^- \\ x \in E}} \frac{F(x_0) - F(x)}{\mathcal{H}^s\left(\widetilde{[x, x_0]}\right)} = \frac{(-2)^n}{n}, \tag{7}$$

$$F'^{+}_s(x_0) = \lim_{\substack{x \to x_0^+ \\ x \in E}} \frac{F(x) - F(x_0)}{\mathcal{H}^s\left(\widetilde{[x_0, x]}\right)} = \frac{(-2)^n}{n}. \tag{8}$$

So $F'_s(x_0) = (-2)^n/n$.

Moreover, for $x \in \widetilde{\left[\frac{2}{3^n}, \frac{7}{3^{n+1}}\right]}$, it is

$$\left|\frac{F(x) - F(0)}{\mathcal{H}^s\left(\widetilde{[0, x]}\right)}\right| = \frac{2^n}{n} \frac{\mathcal{H}^s\left(\widetilde{\left[\frac{2}{3^n}, x\right]}\right)}{\mathcal{H}^s\left(\widetilde{[0, x]}\right)} \leq \frac{2^n}{n}\frac{1}{4^n}2^n = \frac{1}{n} \tag{9}$$

and, for $x \in \widetilde{\left[\frac{8}{3^{n+1}}, \frac{1}{3^{n-1}}\right]}$, it is

$$\left|\frac{F(x) - F(0)}{\mathcal{H}^s\left(\widetilde{[0, x]}\right)}\right| = \frac{2^n}{n} \frac{\mathcal{H}^s\left(\widetilde{\left[\frac{8}{3^{n+1}}, x\right]}\right)}{\mathcal{H}^s\left(\widetilde{[0, x]}\right)} \leq \frac{2^n}{n}\frac{1}{4^n}2^n = \frac{1}{n} \tag{10}$$

Thus

$$F'_s(0) = \lim_{\substack{x \to 0 \\ x \in E}} \frac{F(x) - F(0)}{\mathcal{H}^s\left(\widetilde{[0, x]}\right)} = 0. \tag{11}$$

Definition 2.4. Let $F : \mathbb{R} \to \mathbb{R}$ be a function. A point $x \in \mathbb{R}$ is said to be a *point of change of* F if it is not a constant over any open interval (c, d) containing x. The set of all points of change of F is called the set of change of F, and it is denoted by $Sch(F)$.

Theorem 2.1 Let $F : [a,b] \to \mathbb{R}$ be a continuous function such that $Sch(F) \subset E$. Let us suppose that F is s-derivable at each point $x \in [a,b]$ and that $F(a) = F(b) = 0$. Then there exists a point $c \in E$ such that $F_s'(c) \geq 0$ and a point $d \in E$ such that $F_s'(d) \leq 0$.

In Ref. [7], it is possible to find the proof of the previous theorem, and an example that shows as the "fragmented nature" of the fractal set does not allow us to make an analog of Rolle's theorem. Furthermore, the law of mean and the Leibniz rule is also discussed in Ref. [7].

3. The s-Riemann integral

Definition 3.1. Let E be a closed s-set of the real line, $a = \min E$ and $b = \max E$. Let $A \subset [a,b]$ be an interval. We call *interval of* the set $\widetilde{A} = A \cap E$.

Definition 3.2. *A partition of* E is any collection $P = \left\{ \left(\widetilde{A}_i, x_i \right) \right\}_{i=1}^{p}$ of pairwise disjoint intervals $\widetilde{A}_i$ and points $x_i \in \widetilde{A}_i$ such that $E = \cup_i \widetilde{A}_i$.

Definition 3.3. Let $f : E \to \mathbb{R}$ be a function. It is said that f is s-Riemann integrable on E, if there exists a number I such that, for each $\varepsilon > 0$ there is a $\delta > 0$ with

$$\left| \sum_{i=1}^{p} f(x_i)\, \mathcal{H}^s\left(\widetilde{A}_i\right) - I \right| < \varepsilon \tag{12}$$

for each partition $P = \left\{ \left(\widetilde{A}_i, x_i \right) \right\}_{i=1}^{p}$ of E with $\mathcal{H}^s\left(\widetilde{A}_i\right) < \delta,\ \ i = 1, 2, \dots, p$. The number I is called the s-Riemann integral of f on E and we set

$$I = (s)\int_E f(t)\, d\mathcal{H}^s(t). \tag{13}$$

The collection of all functions that are s-Riemann integrable on E will be denoted by $sR(E)$.

Theorem 3.1 If $f : E \to \mathbb{R}$ is continuous on E with respect to the induced topology, then $f \in sR(E)$.

Remark 3.1. The classical Riemann integral properties, such as linearity, additivity with respect to integrating domain and the mean-value theorem for integrals hold too for this new integral. See Refs. [6, 7] for more details.

Moreover, it is useful to remark that, as it happens in the real case, the Lebesgue integral of f with respect to the Hausdorff measure $\mathcal{H}^s$, here denoted by $(L)\int_E f(t)\, d\mathcal{H}^s(t)$, includes the s-Riemann integral.

Theorem 3.2 If $f \in sR(E)$ then f is Lebesgue integrable on E with respect to the Hausdorff measure $\mathcal{H}^s$, and

$$(L)\int_E f(t)\, d\mathcal{H}^s(t) = (s)\int_E f(t)\, d\mathcal{H}^s(t). \tag{14}$$

The proof can be found in Appendix A.

3.1 The Fundamental Theorem of Calculus

Definition 3.4. Let $F : E \to \mathbb{R}$ be a function. We say that F is $\mathcal{H}^s$-absolutely continuous on E if $\forall \varepsilon > 0 \exists \delta > 0$ such that

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| < \varepsilon \tag{15}$$

whenever $\sum_{k=1}^{n} \mathcal{H}^s\left[\widetilde{(a_k, b_k)}\right] < \delta$, with $a_k, b_k \in E,\ k = 1, \dots, n,$ and $a_1 < b_1 \le a_2 < b_2 \le \dots \le a_n < b_n$.

Jiang and Su in Ref. [6] announced, without proof, the following version of the Fundamental Theorem of Calculus:

Theorem 3.2 Let $f : E \to \mathbb{R}$ be a continuous function. If $F : E \to \mathbb{R}$ is $\mathcal{H}^s$-absolutely continuous on E with $F_s'(x) = f(x)$ $\mathcal{H}^s$- a.e. in E, then

$$(s)\int_E f(t)\, d\mathcal{H}^s(t) = F(b) - F(a). \tag{16}$$

On the other hand Parvate and Gangal in Ref. [7] proved the following version of the Fundamental Theorem of Calculus:

Theorem 3.3 If $F : \mathbb{R} \to \mathbb{R}$ is s-derivable on E, and if F_s' is continuous with $Sch(F) \subseteq E$, then

$$(s)\int_E F_s'(x)\, d\mathcal{H}^s(x) = F(b) - F(a). \tag{17}$$

Now, we give an example of a very simple function, that is not $\mathcal{H}^s$-absolutely continuous on a fractal set E and that satisfies the condition $Sch(F) \not\subseteq E$ for which the Fundamental Theorem of Calculus fails.

Example 3.1. Let $E \subset [0, 1]$ be the classical Cantor set. Let F be a real function on $[0, 1]$ defined by

$$F(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{3} \\ 3x - 1, & \frac{1}{3} < x < \frac{2}{3} \\ 1, & \frac{2}{3} \le x \le 1. \end{cases} \tag{18}$$

and let f be the restriction of F on the Cantor set E.

Since, the Hausdorff dimension of the Cantor set E is $s = \log_3 2$ and $\mathcal{H}^{\log_3 2}(E) = 1$ it is trivial to observe that:

- f is $\log_3 2$-derivable on E with $f'_{\log_3 2}(x) = 0, \forall x \in E,$
- $f'_{\log_3 2}$ is $\log_3 2$-Riemann integrable on E,

$$(\log_3 2)\int_E f'_{\log_3 2}(x)\, d\mathcal{H}^{\log_3 2}(x) = 0 \ne f(1) - f(0) = 1. \tag{19}$$

In the next section, we will prove a more general formulation of the Fundamental Theorem of Calculus on an s-set, that enclose Theorem 3.2 and Theorem 3.3.

4. The *sHK* integral

Definition 4.1. It is called *gauge* on E any positive real function δ defined on E.

Definition 4.2. It is called *partition* of E any collection $P = \left\{ \left(\widetilde{A}_i, x_i \right) \right\}_{i=1}^{p}$ of pairwise disjoint intervals $\widetilde{A}_i$ of E, and points $x_i \in \widetilde{A}_i, i = 1, \cdots, p,$ such that $E = \bigcup_i \widetilde{A}_i$.

Definition 4.3. Let $P = \left\{ \left(\widetilde{A}_i, x_i \right) \right\}_{i=1}^{p}$ be a partition of E. If δ is a gauge on E, then we say that P is a δ-fine *partition* of E whenever $\widetilde{A}_i \subseteq]x_i - \delta(x_i), x_i + \delta(x_i)[$, for $i = 1, 2, \dots, p$.

Lemma 4.1. If δ is a gauge on E, then there exists a δ-fine partition of E.

This lemma, known in the literature as the Cousin lemma, is fundamental for the definition of an Henstock-Kurzweil type integral because it addresses the existence of δ-fine partitions. For completeness, we report the proof of this lemma in Appendix A, even if it is possible to find it in Ref. [8].

Now, let $P = \left\{ \left(\widetilde{A}_i, x_i \right) \right\}_{i=1}^{p}$ be a partition of E, let $f : E \to \mathbb{R}$ be a function and let

$$\sigma(f, P) = \sum_{i=1}^{p} f(x_i)\, \mathcal{H}^s \left(\widetilde{A}_i \right), \tag{20}$$

be the s-Riemann sum of f with respect to P.

Definition 4.4. We say that f is *sHK* integrable on E if there exists $I \in \mathbb{R}$ such that, for all $\varepsilon > 0$, there is a gauge δ on E with:

$$|\sigma(f, P) - I| < \varepsilon, \tag{21}$$

for each δ-fine partition $P = \left\{ \left(\widetilde{A}_i, x_i \right) \right\}_{i=1}^{p}$ of E.

The number I is called the *sHK* integral of f on E, and we write

$$I = (sHK) \int_E f \, d\mathcal{H}^s. \tag{22}$$

The collection of all functions that are *sHK* integrable on E will be denoted by $sHK(E)$.

Remark 4.1. Let us notice that the difference between the *sHK* integral (see Definition 4.4) and the s-Riemann integral (see Definition 3.3) is due to the fact that while in Definition 4.4 the gauge δ is a positive real function in Definition 3.3 it is a positive constant.

About this, remark that, if f is an *sHK* integrable function, but it is not s-Riemann integrable, then $\inf_{x \in E} \delta(x) = 0,$ for the gauge δ involved in the definition of the *sHK* integral. In fact, the condition $\inf_{x \in E} \delta(x) = \delta > 0$ would imply that the choice of points x_i inside the intervals $\widetilde{A}_i, i = 1, 2, \cdots, p,$ may be arbitrary; therefore f would be s-Riemann integrable.

Here we list some basic properties of the *sHK* integral:

a. the number I from Definition 4.4 is unique,

b. if $f, g \in sHK(E)$ therefore $f + g \in sHK(E)$ and

$$(sHK) \int_E (f + g)(t)\, d\mathcal{H}^s(t) = (sHK) \int_E f(t)\, d\mathcal{H}^s(t) + (sHK) \int_E g(t)\, d\mathcal{H}^s(t),$$

c. if $f \in sHK(E)$ and $k \in \mathbb{R}$, then $kf \in sHK(E)$ and

$$(sHK)\int_E kf(t)\, d\mathcal{H}^s(t) = k(sHK)\int_E f(t)\, d\mathcal{H}^s(t),$$

d. if $f, g \in sHK(E)$ with $f \leq g$, $\mathcal{H}^s$-almost everywhere on E, therefore

$$(sHK)\int_E f(t)\, d\mathcal{H}^s(t) \leq (sHK)\int_E g(t)\, d\mathcal{H}^s(t),$$

e. if $f \in sR(E)$, then $f \in sHK(E)$ and

$$(sHK)\int_E f(t)\, d\mathcal{H}^s(t) = (s)\int_E f(t)\, d\mathcal{H}^s(t),$$

f. if $f \in sHK(E)$ and $a = \min E < x < b = \max E$, then the function

$$F(x) = (sHK)\int_{\widetilde{[a,x]}} f\, d\mathcal{H}^s \tag{23}$$

is continuous and

$$(sHK)\int_E f\, d\mathcal{H}^s = (sHK)\int_{\widetilde{[a,x]}} f\, d\mathcal{H}^s + (sHK)\int_{\widetilde{[x,b]}} f\, d\mathcal{H}^s. \tag{24}$$

Just as in the case of the Henstock-Kurzweil integral (see Ref. [10]), there is a Cauchy criterion for a function to be *sHK* integrable on E. This is the content of the following theorem.

Theorem 4.1 A function $f : E \to \mathbb{R}$ is *sHK*-integrable on E if and only if for each $\varepsilon > 0$ there exists a gauge δ on E such that

$$|\sigma(f, P_1) - \sigma(f, P_2)| < \varepsilon, \tag{25}$$

for each pair P_1 and P_2 of δ-fine partitions of E.

The proof can be found in Ref. [8].

4.1 Relation with the Lebesgue integral

In this section, we prove that the *sHK* integral includes the Lebesgue integral with respect to $\mathcal{H}^s$. In order to do this, we recall the following Vitali-Carathéodory Theorem:

The Vitali-Carathéodory Theorem Let f be a real function defined on E. If f is Lebesgue integrable on E with respect to $\mathcal{H}^s$ and $\varepsilon > 0$, then there exist functions u and v on E such that $u \leq f \leq v$, u is upper semicontinuous and bounded above, v is lower semicontinuous and bounded below, and

$$(L)\int_E (v - u)\, d\mathcal{H}^s < \varepsilon. \tag{26}$$

Theorem 4.2 Let $f : E \to \mathbb{R}$ be a function. If f is Lebesgue integrable on E with respect to $\mathcal{H}^s$, then f is sHK-integrable on E and

$$(L)\int_E f\, d\mathcal{H}^s = (sHK)\int_E f\, d\mathcal{H}^s. \tag{27}$$

The proof can be found in Appendix A.

Now we give a simple example of an sHK integrable function which is not Lebesgue integrable with respect to the Hausdorff measure $\mathcal{H}^s$.

Example 4.1. Let $E \in [0,1]$ be the classical Cantor set, and let $f : E \to \mathbb{R}$ be the function defined as follows

$$f(x) = \begin{cases} \dfrac{(-1)^{n+1}2^n}{n}, & \text{for } x \in \widetilde{\left[\dfrac{2}{3^n}, \dfrac{1}{3^{n-1}}\right]} \quad n = 1, 2, 3, \dots \\ 0, & \text{for } x = 0. \end{cases} \tag{28}$$

We show that $f \in sHK(E)$ where $s = \log_3 2$.

In order to do that, fixed $\varepsilon > 0$, we can find a gauge δ on E such that

- if $x \in E$ and $x \neq 0$, f is constant on $\left(\widetilde{x - \delta(x), x + \delta(x)}\right)$;
- $\delta(0) < \frac{1}{3^{n+1}}$ with $\frac{1}{n} < \varepsilon$.

Choose $k \in \mathbb{N}$ with $k > n + 1$ and set $c = \frac{1}{3^k}$.

Let us consider $P = \left\{\left(\widetilde{A}_1, x_1\right), \left(\widetilde{A}_2, x_2\right) \dots, \left(\widetilde{A}_m, x_m\right)\right\}$ a δ-fine partition of E such that $\widetilde{A_1} = \widetilde{[0, c]}$ and $\widetilde{A}_i \subseteq \widetilde{\left[\frac{2}{3^p}, \frac{1}{3^{p-1}}\right]}$ $(i = 2, \dots, m)$ for some $p \in \mathbb{N}$. Our choice of δ implies that $x_1 = 0$ and $\bigcup_{i=2}^m \widetilde{A}_i = \bigcup_{i=1}^k \widetilde{\left[\frac{2}{3^i}, \frac{1}{3^{i-1}}\right]}$.

$$\begin{aligned} (sHK)\int_{\widetilde{\left[\frac{2}{3^k},1\right]}} f\, d\mathcal{H}^s &= \sum_{i=1}^k (sHK)\int_{\widetilde{\left[\frac{2}{3^i},\frac{1}{3^{i-1}}\right]}} f\, d\mathcal{H}^s = \\ &= \sum_{i=1}^k \frac{(-1)^{i+1}\, 2^i}{i}\, \mathcal{H}^s\left(\widetilde{\left[\frac{2}{3^i}, \frac{1}{3^{i-1}}\right]}\right) = \\ &= \sum_{i=1}^k \frac{(-1)^{i+1}\, 2^i}{i} \cdot \frac{1}{2^i} = \sum_{i=1}^k \frac{(-1)^{i+1}}{i}. \end{aligned} \tag{29}$$

Therefore

$$(sHK)\int_{\widetilde{\left[\frac{2}{3^k},1\right]}} f\, d\mathcal{H}^s \sim \log 2. \tag{30}$$

Then

$$|\sigma(f,P) - \log 2| \leq \left|f(0)\mathcal{H}^s\left(\widetilde{A_1}\right)\right| + \left|\sum_{i=2}^{m} f(x_i)\mathcal{H}^s(\widetilde{A_i}) - (sHK)\int_{\widetilde{\left[\frac{2}{3^k},1\right]}} f\, d\mathcal{H}^s\right|$$
$$+\left|(sHK)\int_{\widetilde{\left[\frac{2}{3^k},1\right]}} f\, d\mathcal{H}^s - \log 2\right| < \varepsilon. \quad (31)$$

In conclusion, we prove that f is not Lebesgue integrable on E with respect to $\mathcal{H}^s$, where $s = \log_3 2$. In fact, if f were Lebesgue integrable on E with respect to $\mathcal{H}^s$, $|f|$ would be Lebesgue integrable on E with respect to $\mathcal{H}^s$. But we have

$$(L)\int_E |f|\, d\mathcal{H}^s = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty, \quad (32)$$

hence f is not Lebesgue integrable on E with respect to $\mathcal{H}^s$.

4.2 The Fundamental Theorem of Calculus

Definition 4.5. An interval (α, β) is said to be contiguous to a set E, if:

- $\alpha \in E$ and $\beta \in E$
- $(\alpha, \beta) \cap E = \emptyset$

In Ref. [9] we proved the following version of the Fundamental Theorem of Calculus.

Theorem 4.3 Let E be a closed s-set and let $\{(a_j, b_j)\}_{j\in\mathbb{N}}$ be the contiguous intervals of E. If $F : E \to \mathbb{R}$ is s-derivable on E and if $\sum_{j=1}^{\infty} |F(b_j) - F(a_j)| < +\infty$, then $F'_s \in sHK(E)$ and

$$(sHK)\int_E F'_s d\mathcal{H}^s = F(b) - F(a) - \sum_{j=1}^{\infty} \left(F(b_j) - F(a_j)\right). \quad (33)$$

Remark 4.2. By the previous theorem, it is possible to extend the versions of the Fundamental Theorem of Calculus given by Jung and Su (i.e., Theorem 3.2) and that given by Parvate and Gangal (i.e., Theorem 3.3).

Extension of Theorem 3.2.

Let E be a closed s-set and let $F : E \to \mathbb{R}$ be a function $\mathcal{H}^s$-absolutely continuous on E such that F'_s exists $\mathcal{H}^s$-almost everywhere in E. Then

$$(sHK)\int_E F'_s(t)\, d\mathcal{H}^s(t) = F(b) - F(a). \quad (34)$$

The proof of the Extension of Theorem 3.2 can be found in Ref. [9].

Extension of Theorem 3.3 If $F : \mathbb{R} \to \mathbb{R}$ is continuous and s-derivable on E and if $Sch(F) \subseteq E$, then

$$(sHK)\int_E F_s'(t)\,d\mathcal{H}^s(t) = F(b) - F(a). \tag{35}$$

Proof:

Condition $Sch(F) \subseteq E$ implies that F is constant on each contiguous interval (a_j, b_j) of E, then $F(a_j) = F(b_j)$ for $j \in \mathbb{N}$. Therefore $\sum_{j=1}^{\infty} |F(b_j) - F(a_j)| < +\infty,$ and Theorem 4.3 can be applied.

Remark 4.3. If we assume, like in Theorem 3.3 that F_s' is continuous on E, then $F_s' \in sR(E)$ (Ref. [7], Theorem 39) and by (e) and by Extension of Theorem 3.3, we have that:

$$(s)\int_E F_s'(t)\,d\mathcal{H}^s(t) = F(b) - F(a). \tag{36}$$

Remark 4.4. The absolute convergence of the series $\sum_{j=1}^{\infty} \left(F(b_j) - F(a_j)\right)$ is a necessary condition

- for the sHK integrability of F_s'
- for the validity of some formulation of the Fundamental Theorem of Calculus.

See Ref. [9] for more details.

4.3 The primitives

Bongiorno and Corrao in Ref. [17] introduced an Henstock-Kurzweil type integral defined on a complete metric measure space X endowed with a Radon measure μ and with a family $\mathcal{F}$ of μ-cells that enclose the sHK integral (see Ref. [17], Example 2.4). For such an integral, they proved an extension of the usual descriptive characterizations of the Henstock-Kurzweil integral on the real line in terms of ACG^* functions (see Ref. [17] and Appendix B). Here we report such descriptive characterization in the particular case in which $X = [a, b]$ is endowed with the Euclidean distance of $\mathbb{R}, E \subset [a, b]$ is an s-set and the family $\mathcal{F}$ of μ-cells coincides with the family $\mathcal{F}$ of all closed subintervals of $\widetilde{[a, b]}$.

Definition 4.6. A finite collection $\left\{\widetilde{A}_1, \dots, \widetilde{A}_m\right\}$ of pairwise disjoint elements of $\widetilde{[a, b]}$ is called a *division* of $\widetilde{[a, b]}$ if $\bigcup_{i=1}^{m} \widetilde{A}_i = \widetilde{[a, b]}$.

Definition 4.7. Let E be an s-set and let δ be a gauge on $\widetilde{[a, b]}$. A collection $P = \left\{\left(\widetilde{A}_i, x_i\right)\right\}_{i=1}^{m}$ of finite ordered pairs of intervals and points is said to be

- a *partial partition* of $\widetilde{[a, b]}$ if $\left\{\widetilde{A}_1, \dots, \widetilde{A}_m\right\}$ is a subsystem of a division of $\widetilde{[a, b]}$ and $x_i \in \widetilde{A}_i$ for $i = 1, 2, \dots, m$;
- *E-anchored* if the points $x_1, \dots, x_m$ belong to E.

Definition 4.8. Let $\pi : \mathcal{F} \to \mathbb{R}$ be a function. We say that π is an *additive function of interval* if for each $\widetilde{A} \in \mathcal{F}$ an for each division $\left\{\widetilde{A}_1, \dots, \widetilde{A}_m\right\}$ of $\widetilde{A}$ we have

$$\pi\left(\widetilde{A}\right) = \sum_{i=1}^{m} \pi\left(\widetilde{A}_i\right). \tag{37}$$

Definition 4.9. Let E be an s-set. Let π be a fixed additive function of interval. We say that π is AC^{Δ} on E if for $\varepsilon > 0$ there exists a gauge δ on E and a positive constant η such that the condition $\sum_{i=1}^{m} \mathcal{H}^s\left(\widetilde{A}_i\right) < \eta$, implies $\sum_{i=1}^{m} |\pi\left(\widetilde{A}_i\right)| < \varepsilon$, for each δ-fine E-anchored partial partition $P = \left\{\left(\widetilde{A}_i, x_i\right)\right\}_{i=1}^{m}$ of $\widetilde{[a,b]}$.

We say that π is ACG^{Δ} on $\widetilde{[a,b]}$ if there exists a countable sequence of s-sets $\{E_k\}_k$ such that $\bigcup_k E_k = \widetilde{[a,b]}$ and π is AC^{Δ} on E_k, for each $k \in \mathbb{N}$.

Theorem 4.4 Let $E \subset [a,b]$ be an s-set. A function $f : E \to \mathbb{R}$ is *sHK* integrable on E if and only if there exists an additive function of interval F that is ACG^{Δ} on $\widetilde{[a,b]}$ and s-derivable on $\widetilde{[a,b]}$ such that $F'_s(x) = f(x)$ $\mathcal{H}^s$-almost everywhere on $\widetilde{[a,b]}$.

Remark 4.5. An interested reader can find the proof of Theorem 4.4 in a more general version (i.e. for an Henstock-Kurzweil type integral defined on a complete metric measure space X endowed with a Radon measure μ) in Ref. [17].

5. The *s*-first-return integral

Darji and Evans in Ref. [15] have defined on the real line the first-return integral. The motivation that led the authors to define such a new integration process lies in the fact that the gauge function involved in the definition of the first-return integral is a constant like in the Riemann integral. Borrowing such an idea, in this section, we define a new integral on an s-set E, called the s-first-return integral. Such a new integral is different from the *sHK* integral because, in the definition of this new integral, the function $\delta : E \to \mathbb{R}^+$ is a positive constant, but the choice of points x_i is not arbitrary in $\widetilde{A}_i$, for $i = 1, 2, \cdots, p$.

Definition 5.1. We call *trajectory* on E any sequence Γ of distinct points of E dense in E. Given a trajectory Γ on E and given an interval, $\widetilde{A}$ we denote by $r\left(\Gamma, \widetilde{A}\right)$ the first element of Γ that belongs to $\widetilde{A}$.

Definition 5.2. Let $f : E \to \mathbb{R}$ and let Γ be a trajectory on E. We say that f is s-first-return integrable on E with respect to Γ if there exists a number $I \in \mathbb{R}$ such that, for all $\varepsilon > 0$, there is a constant $\delta > 0$ with:

$$\left| \sum_{i=1}^{p} f\left(r\left(\Gamma, \widetilde{A}_i\right)\right) \mathcal{H}^s\left(\widetilde{A}_i\right) - I \right| < \varepsilon \tag{38}$$

for each division $P = \left\{\widetilde{A}_i\right\}_{i=1}^{p}$ of E with $\mathcal{H}^s\left(\widetilde{A}_i\right) < \delta$.

The number I is called the s-first-return integral of f on E with respect to Γ, and we write

$$I = (sfr)_{\Gamma} \int_E f \, d\mathcal{H}^s. \tag{39}$$

The collection of all functions that are s-first-return integrable on E with respect to Γ will be denoted by $(sfr)_{\Gamma}(E)$.

Theorem 5.1 Let $f : E \to \mathbb{R}$ such that $f \in sR(E)$, therefore, there exists a trajectory Γ on E such that $f \in (sfr)_\Gamma(E)$ and

$$(sfr)_\Gamma \int_E f(t) d\mathcal{H}^s(t) = (s) \int_E f(t) d\mathcal{H}^s(t). \tag{40}$$

Theorem 5.2 There exists an s-derivative $f : E \to \mathbb{R}$ such that $f \notin (sfr)_\Gamma(E)$, for each trajectory Γ on E.

Remark 5.1. To prove Theorem 5.2 it is enough to consider the function $f(x) = F'_s(x)$ of the Example 2.1 and to show that $f \notin (sfr)_\Gamma(E)$, for a given trajectory Γ. This is equivalent to find, for each $M > 0$ and for each $\delta > 0$, a finite system of pairwise disjoint intervals $\widetilde{A}_i$, with $i = 1, 2, \dots, p$, such that $\mathcal{H}^s\left(\widetilde{A}_i\right) < \delta$, $\cup_i \widetilde{A}_i = E$ and

$$\sum_{i=1}^{p} f\left(r\left(\Gamma, \widetilde{A}_i\right)\right) \mathcal{H}^s\left(\widetilde{A}_i\right) > M. \tag{41}$$

See Ref. [18] for more details.

6. Conclusions

In this chapter we have developed a method of calculus on a closed fractal subset of the real line with finite and positive s-dimensional Hausdorff measure. Much of the development of such new calculus on s-sets is carried in analogy with the ordinary calculus with some differences due to the "fragmented nature" of fractal sets, like in the formulation of Rolle's theorem or in the formulation of the Fundamental Theorem of Calculus. Therefore, in order to give the best version of the Fundamental Theorem of Calculus (see Theorem 4.3), we have generalized the s-Riemann integral by defining the *sHK* integral and the s-first-return integral. Finally, by Theorem 5.2, we have noted that to obtain the best version of the Fundamental Theorem of Calculus, we need to consider an Henstock-Kurzweil type integral, i.e., an integral in which the gauge δ is not a constant.

Appendix A

Proof of Theorem 3.1.

By standard techniques, it follows that f is bounded and $\mathcal{H}^s$-measurable; then f is Lebesgue integrable with respect to the Hausdorff measure $\mathcal{H}^s$ (briefly, $\mathcal{H}^s$-Lebesgue integrable).

Given $\varepsilon > 0$, by Definition 3.3, there exists $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(x_i)\, \mathcal{H}^s\left(\widetilde{A}_i\right) - (s) \int_E f(t)\, d\mathcal{H}^s(t) \right| < \varepsilon \tag{42}$$

holds for each partition $P = \left\{\left(\widetilde{A}_i, x_i\right)\right\}_{i=1}^{p}$ of E with $\mathcal{H}^s\left(\widetilde{A}_i\right) < \delta, i = 1, 2, \cdots, p$.

Hence

$$\sum_{i=1}^{p}\left(\sup_{\widetilde{A}_i} f(t) - \inf_{\widetilde{A}_i} f(t)\right)\mathcal{H}^s\left(\widetilde{A}_i\right) < 2\varepsilon. \tag{43}$$

Now remark that $\sum_{i=1}^{p} f(x_i)\,\mathcal{H}^s\left(\widetilde{A}_i\right)$ is the $\mathcal{H}^s$-Lebesgue integral of the $\mathcal{H}^s$-simple function $\sum_{i=1}^{p} f(x_i)\mathbf{1}_{\widetilde{A}_i}(t)$, where $\mathbf{1}_{\widetilde{A}_i}(t)$ denotes the characteristic function of $\widetilde{A}_i$; i.e., $\mathbf{1}_{\widetilde{A}_i}(t) = 1$ for $t \in \widetilde{A}_i$, and $\mathbf{1}_{\widetilde{A}_i}(t) = 0$ for $t \notin \widetilde{A}_i$.

So

$$(L)\int_E \inf_{\widetilde{A}_i} f(t)\cdot \mathbf{1}_{\widetilde{A}_i}(t)\, d\mathcal{H}^s(t) \le (L)\int_{\widetilde{A}_i} f(t)\, d\mathcal{H}^s(t) \le (L)\int_E \sup_{\widetilde{A}_i} f(t)\cdot \mathbf{1}_{\widetilde{A}_i}(t)\, d\mathcal{H}^s(t) \tag{44}$$

and

$$\begin{aligned}(L)\int_E \sum_{i=1}^{p} \inf_{\widetilde{A}_i} f(t)\cdot \mathbf{1}_{\widetilde{A}_i}(t)\, d\mathcal{H}^s(t) - \varepsilon \le (s)\int_E f(t)d\mathcal{H}^s(t) \le \\ \le (L)\int_E \sum_{i=1}^{p} \sup_{\widetilde{A}_i} f(t)\cdot \mathbf{1}_{\widetilde{A}_i}(t)\, d\mathcal{H}^s(t) + \varepsilon.\end{aligned} \tag{45}$$

Thus we have

$$\begin{aligned}\left|(L)\int_E f(t)\, d\mathcal{H}^s(t) - (s)\int_E f(t)\, d\mathcal{H}^s(t)\right| \le \\ \le (L)\int_E \sum_{i=1}^{p}\left(\sup_{\widetilde{A}_i} f(t) - \inf_{\widetilde{A}_i} f(t)\right)\mathbf{1}_{\widetilde{A}_i}(t)\, d\mathcal{H}^s(t) + 2\varepsilon < 4\varepsilon.\end{aligned} \tag{46}$$

By the arbitrariness of ε we end the proof.

Proof of Lemma 4.1.

Let c be the midpoint of $[a, b]$ and let us observe that if P_1 and P_2 are δ-fine partitions of $\widetilde{[a,c]}$ and $\widetilde{[c,b]}$, respectively, then $P = P_1 \cup P_2$ is a δ-fine partition of E. Using this observation, we proceed by contradiction.

Let us suppose that E does not have a δ-fine partition, then at least one of the intervals $\widetilde{[a,c]}$ or $\widetilde{[c,b]}$ does not have a δ-fine partition, let us say $\widetilde{[a,c]}$. Therefore $\widetilde{[a,c]}$ is not empty. Let us relabel the interval $[a, c]$ with $[a_1, b_1]$ and let us repeat indefinitely this bisection method. So, we obtain a sequence of nested intervals: $[a,b] \supset [a_1,b_1] \supset \ldots \supset [a_n,b_n] \supset \ldots$ such that $\widetilde{[a_n,b_n]}$ is not empty. Since the length of the interval $[a_n,b_n]$ is $(b-a)/2^n$, therefore, for the Nested Intervals Property, there is a unique number $\xi \in [a,b]$ such that:

$$\bigcap_{n=0}^{\infty}[a_n,b_n] = \{\xi\}. \tag{47}$$

Let $\xi_n \in \widetilde{[a_n,b_n]}$. Therefore $|\xi_n - \xi| < |b_n - a_n| = (b-a)/2^n$. So $\lim_{n\to\infty}\xi_n = \xi$. Now since E is a closed set, $\xi \in E$.

Since $\delta(\xi) > 0$, we can find $k \in \mathbb{N}$ such that $\widetilde{[a_k, b_k]} \subseteq]\xi - \delta(\xi), \xi + \delta(\xi)[$. Therefore $\left\{([a_k, b_k]\ , \xi)\right\}$ is a δ-fine partition of $\widetilde{[a_k, b_k]}$, contrarily to our assumption.

Proof of Theorem 4.2.

By Vitali-Carathéodory Theorem, given $\varepsilon > 0$ there exist functions u and v on E that are upper and lower semicontinuos respectively such that $-\infty \leq u \leq f \leq v \leq +\infty$ and $(L)\int_E (v - u)\, d\mathcal{H}^s < \varepsilon$. Define on E a gauge δ so that

$$u(t) \leq f(x) + \varepsilon \text{ and } v(t) \geq f(x) - \varepsilon, \tag{48}$$

for each $t \in E$ with $|x - t| < \delta(x)$.

Let $P = \left\{\left(\widetilde{A}_1, x_1\right), \left(\widetilde{A}_2, x_2\right), \dots \left(\widetilde{A}_p, x_p\right)\right\}$ be a δ-fine partition of E. Then, for each $i \in \{1, 2, \dots, p\}$, we have

$$(L)\int_{\widetilde{A}_i} u\ d\mathcal{H}^s \leq (L)\int_{\widetilde{A}_i} f\ d\mathcal{H}^s \leq (L)\int_{\widetilde{A}_i} v\ d\mathcal{H}^s. \tag{49}$$

Moreover, by $u(t) \leq f(x_i) + \varepsilon$ for each $t \in \widetilde{A}_i$, it follows

$$(L)\int_{\widetilde{A}_i} (u - \varepsilon)\ d\mathcal{H}^s \leq (L)\int_{\widetilde{A}_i} f(x_i)\ d\mathcal{H}^s \tag{50}$$

and therefore

$$(L)\int_{\widetilde{A}_i} u\ d\mathcal{H}^s - \varepsilon \mathcal{H}^s\left(\widetilde{A}_i\right) \leq f(x_i)\mathcal{H}^s\left(\widetilde{A}_i\right). \tag{51}$$

Similarly, by $v(t) \geq f(x_i) + \varepsilon$ for each $t \in \widetilde{A}_i$, it follows

$$f(x_i)\ \mathcal{H}^s\left(\widetilde{A}_i\right) \leq (L)\int_{\widetilde{A}_i} v\ d\mathcal{H}^s + \varepsilon\, \mathcal{H}^s\left(\widetilde{A}_i\right). \tag{52}$$

So, for $i = 1, 2, \dots, p$, we have

$$(L)\int_{\widetilde{A}_i} u\ d\mathcal{H}^s - \varepsilon \mathcal{H}^s\left(\widetilde{A}_i\right) \leq f(x_i)\mathcal{H}^s\left(\widetilde{A}_i\right) \leq (L)\int_{\widetilde{A}_i} v\ d\mathcal{H}^s + \varepsilon\, \mathcal{H}^s\left(\widetilde{A}_i\right). \tag{53}$$

Hence,

$$(L)\int_E u\ d\mathcal{H}^s - \varepsilon \leq \sigma(f, P) \leq (L)\int_E v\ d\mathcal{H}^s + \varepsilon, \tag{54}$$

and, by (Eq. (49)),

$$(L)\int_E u\ d\mathcal{H}^s \leq (L)\int_E f\ d\mathcal{H}^s \leq (L)\int_E v\, d\mathcal{H}^s. \tag{55}$$

Thus

$$\left|\sigma(f,P) - (L)\int_E f\, d\mathcal{H}^s\right| \leq (L)\int_E (v-u)\; d\mathcal{H}^s + 2\varepsilon < 3\varepsilon, \tag{56}$$

and the theorem is proved.

Appendix B

The problem of primitives is the problem of recovering a function from its derivative (i.e. the problem of whether every derivative is integrable).

In the real line the Riemann integral is inadequate to solve it, in fact the function

$$F(x) = \begin{cases} x^2 \sin\left(1/x^2\right), & x \in (0,1]; \\ 0, & x = 0; \end{cases} \tag{57}$$

is differentiable everywhere on $[0, 1]$, but its derivative

$$F'(x) = \begin{cases} 2x \sin\left(1/x^2\right) - \dfrac{2}{x}\cos\left(1/x^2\right), & x \in (0,1]; \\ 0, & x = 0; \end{cases} \tag{58}$$

it is not Riemann integrable since it is unbounded. Moreover, a more detailed exam reveals that F' is neither Lebesgue integrable, since F is not absolutely continuous on $[0, 1]$. Therefore the Lebesgue integral does not solve the problem of primitives. So it was natural to find an integration process for which the following theorem holds:

The Fundamental Theorem of Calculus *If* $F : [a,b] \to \mathbb{R}$ *is differentiable on* $[a, b]$, *then the function* $F'(x)$ *is integrable on* $[a, b]$ *and* $\int_a^b F'(t)dt = F(b) - F(a)$.

The first solution to this problem was given in 1912 by Denjoy, shortly followed by Perron. Both definitions are constructive. While Denjoy developed a new method of integration, called totalization, that includes the Lebesgue integral and that gets the value of the integral of a function through a transfinite process of Lebesgue integrations and limit operations, Perron used an approach that does not require the theory of measure, based on families of major and minor functions previously introduced by de la Vallée Poussin. Later Hake, Alexandroff, and Looman independently proved that the Denjoy integral and the Perron integral are equivalent (see Ref. [18]). Hence from now on, we will refer to the integral of Denjoy–Perron.

Subsequently, denoted by $\omega(F, (c,d))$ the oscillation of F on a given interval $[c, d]$ (i.e. $\omega(F, (c,d)) = \sup_{(\alpha,\beta)\subset(c,d)}|F(\beta) - F(\alpha)|$), Luzin introduced the following notions of functions AC^* and ACG^*.

Definition 6.1. A function $F : [a,b] \to \mathbb{R}$ is said to be AC^* on a set $E \subset [a,b]$ if, given $\varepsilon > 0$, there is a constant $\eta > 0$ such that

$$\sum_{i=1}^{p} \omega(F, (a_i, b_i)) < \varepsilon \tag{59}$$

for each finite system $\{(a_i, b_i)\}$ of nonoverlapping intervals such that $(a_i, b_i) \cap E \neq \emptyset$.

Definition 6.2. A function $F : [a, b] \to \mathbb{R}$ is said to be ACG^* on $[a, b]$ if it is continuous and there is a decomposition $[a, b] = \cup_i E_i$ with E_i closed sets, such that F is AC^* on E_i, for each i.

Then Luzin proved the following characterization of the Denjoy–Perron integrable functions: *A function $f : [a, b] \to \mathbb{R}$ is Denjoy–Perron integrable on $[a, b]$ if and only if there exists a function $F : [a, b] \to \mathbb{R}$ that is ACG^* on $[a, b]$ and differentiable almost everywhere on $[a, b]$ such that $F'(x) = f(x)$ almost everywhere on $[a, b]$.*

Note that the "if" part of this characterization is often called "the descriptive" definition of the Denjoy–Perron integral.

Moreover, since the constructive definition of the Denjoy-Perron integral was not as immediate as that of the Riemann integral, only a few mathematicians were interested in working with it, so the descriptive definition became the most known definition of the Denjoy–Perron integral.

So, in spite of the general conviction that no modification of Riemann's method could possibly give such powerful results as that of the Lebesgue integral, Kurzweil in Ref. [20] and Henstock in Ref. [21] introduced independently, a generalized version of the Riemann integral that is known as the Henstock-Kurzweil integral. Such an integral solves the problem of primitives and encloses the Lebesgue integral, in the sense that if a function f is Lebesgue integrable, therefore, it is also integrable in the sense of Henstock-Kurzweil and the two integrals coincide.

The method of Henstock-Kurzweil is based on the notions of *gauge* and *partition*. Precisely, given a subinterval $[a, b]$ of the real line $\mathbb{R}$, a gauge on $[a, b]$ is, by definition, any positive function δ defined on it. A partition of $[a, b]$ is, by definition, any collection $P = \{(A_i, x_i)\}_{i=1}^{p}$ of pairwise disjoint intervals A_i and points $x_i \in A_i$ such that $[a, b] = \cup_i A_i$. Moreover, P is said to be *δ-fine* whenever $A_i \subset (x_i - \delta_i, x_i + \delta_i), i = 1, 2, \dots, p$.

Given a gage δ on $[a, b]$, the existence of δ-fine partitions of $[a, b]$ is ensured by the following lemma.

Cousin's lemma. *For any gage δ on $[a, b]$, there exists a δ-fine partition of $[a, b]$.*

Given a function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{(A_i, x_i)\}_{i=1}^{p}$ of $[a, b]$ the Riemann sum of f with respect to P is defined as follows

$$\sum_P f = \sum_{i=1}^{p} f(x_i)\,\mathcal{L}(A_i), \tag{60}$$

Definition 6.3. Let $f : [a, b] \to \mathbb{R}$. We say that f is Henstock–Kurzweil integrable on $[a, b]$ if there exists a number $I \in \mathbb{R}$ satisfying the following condition: to each positive ε there is a gage δ on $[a, b]$ such that

$$\left| \sum_P f - I \right| < \varepsilon, \tag{61}$$

for each δ-fine partition P of $[a, b]$.

The number I is called the Henstock–Kurzweil integral of f on $[a, b]$, and it is denoted by $(HK)\int_a^b f(x)\,dx$.

The proofs of the basic properties of the Henstock–Kurzweil integral, like the linearity and the Cauchy criterion, can be found in Ref. [10].

Moreover, the integral of Henstock-Kurzweil, even if its definition is completely different from that of Denjoy-Perron, it is equivalent to the Denjoy-Perron integral. This result was shown by several mathematicians, see Refs. [22, 23] for instance, by proving the following characterization of the Henstock-Kurzweil primitives: *The family of all Henstock-Kurzweil primitives coincides with the class of all* ACG^**-functions.*

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