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CONICS CONSTRUCTIONS BY PINS AND STRING: TANGENTIAL AND PHYSICAL PROPERTIES

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Abstract. We introduce conic sections' construction by pins and string, without rigid linear components. In addition to well-known methods (as gardener's ellipse), we analyze some recent constructions based on tangent properties and, as an original contribution, extend these constructions to all kinds of conics. In this setting, the physical analysis of the string's tension permits us to smoothly analyze these constructions since the first years of high school.

From a didactical perspective, we think that such ideas can be fruitfully adopted in laboratory activities to deepen the tangential properties of conics by a rich interaction between mathematics and physics. Furthermore, the simplicity of the required materials allows the introduction of these manipulative activities in distance learning.

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1. Introduction

In education, historical machines' adoption is a well-established laboratory experience for mediating mathematical meanings [6]. Specifically, many studies deepened the construction of geometric curves as traces realized by practical methods. As it is well known, there are many ways to trace conic sections, both by elementary tools (the so-called "gardener's ellipse") and by more complex machines (the various parabolo/ellipso/hyperbolo-graphs[1]). The utility of such different approaches in education is widely accepted [2].

The gardener's method of drawing an ellipse is one of the best known and elegant examples of geometric constructions beyond ruler and compass. The required tools are elementary: a string with its ends tied in two points of the plane or, as a little variation, a loop of string put around two pins. Call "focal" the vectors connecting any point of the curve with the two foci: the taut string naturally embodies the property of keeping constant the sum of the two focal-vectors lengths (such a sum is the length of the major axis of the ellipse). Besides this construction, string constructions of conics (not only ellipses) have been introduced by the Persian Ibn Sahl at the end of the 1st millennium, with the adoption of a rod for the constructions of hyperbolas and parabolas [11][2].

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[1]Cf. http://www.mmlab.unimore.it/site/home/laboratorio-visite-mostre/theatrum-machinarum/1.-sezioni-coniche/artCattm-sez-coniche.16002690.1.99.1.4.html

[2]See also https://imaginary.org/film/mathlapse-constructions-by-pin-and-string-conics.

Recently, Dawson et al. [4] proposed new constructions for gardener's hyperbolas, adding to the classical gardener's tools a point in friction with the base plane that is dragged by the string. The dragged point can be a heavy load, a magnet on a metal plate or merely a pen/crayon pressed on a paper base. In this work, we propose an elementary analysis of these constructions by focusing on strings' physical tensions. Furthermore, as a new variation, we extend these constructions to ellipses and parabolas. We think that teachers or trainers can fruitfully introduce the constructions proposed in this paper in didactical laboratory activities. Furthermore, the use of cheap and easy-to-get materials appears very useful to perform laboratory activities also for distance learning (during the lockdown, we tested it with Italian University students).

For some of these constructions, the peculiarity is that they trace conics in a way not directly linked to the length of the focal vectors but related to conical tangent properties. Although such properties have many physical applications (as conical mirrors and antennas), the link between these properties and the focal-distance definitions of conics is generally not sufficiently deepened (at least in the Italian high school system). Furthermore, the analysis of these constructions constitutes a profound interplay between mathematics and physics, an increasing topic in the educational community [14].

To conclude the introduction, we have to mention the historical role of constructions by a point dragged by a string, the so-called "tractional motion" [3]. At the basis of Leibniz's geometrical legitimation of infinitesimal analysis [1], constructions by traction constituted, at the end of the 17th century, a general method to trace transcendental curves by solving the inverse tangent problem (i.e. to find the curve whose tangent has to satisfy certain properties). Although almost forgotten, in recent years, a repurposing of these constructions arose with foundational [8], computational [9], and didactical [7] goals. From this perspective, we hope that this paper can provide another step in bringing in educational activities the various aspects of tractional motion.

2. Gardener's constructions

The most famous construction of a conic is the so-called *gardener's method* for the ellipse. As illustrated in Figure 1, it traces an ellipse thanks to the string's behavior, which is flexible but inextensible. Therefore, the string, its total length remaining constant, satisfies the ellipse's definition.

Given two foci, one can draw a hyperbola as visible in the left of Figure 2. Besides "pins-and-string", a rod imposes the constant focal-vectors difference. To our knowledge, the following variation of Ibn Sahl's construction without the rod appeared only recently [4]. As visible in the right of Figure 2, attach P on a string passing through the foci F and C. By dragging the string by both extremities, both distances PF and PC decrease the same length, so the taut string imposes P to keep the focal-vectors' difference. By traction, P traces the branch of a hyperbola

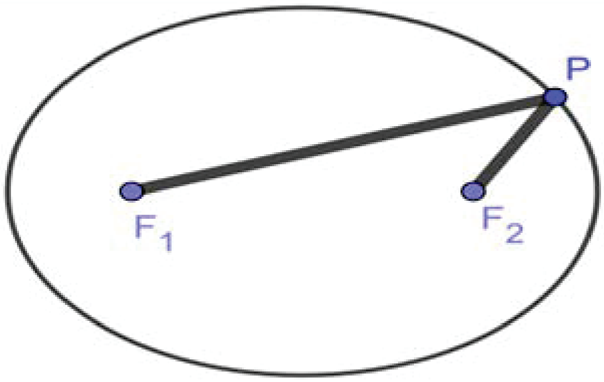

Figure 1: The gardener's ellipse. The constant length of the string ending in the foci F_1 and F_2 defines an ellipse.

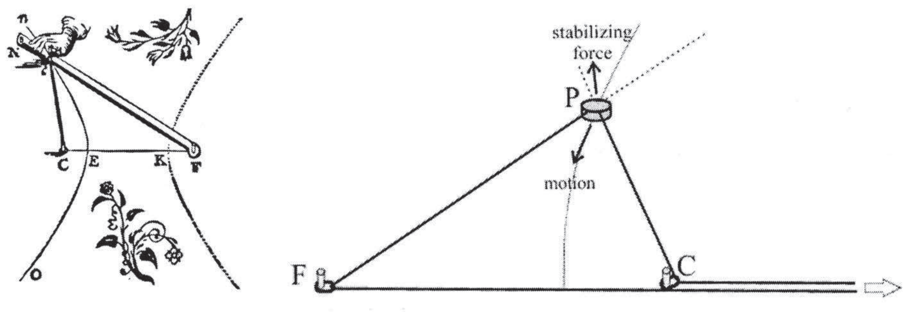

Fig. 2. Left: an early modern print of Ibn Sahl's hyperbola construction by string and ruler [12, p. 67]. Right: the new gardener's hyperbola without ruler [4, Fig. 6].

from a starting point to the vertex. For the similitude to the ellipse case, we call such a construction the "gardener's method for hyperbolas".

3. Introducing tractional constructions

Ellipse and hyperbola "gardener's" constructions rely on the distances from the foci, and the role of the taut string is to pose certain conditions on these distances. However, by a slight modification of the gardener's hyperbola, we get a new construction based on tangent properties [4]. In this case, it is unavoidable the role of the *traction*: given a material point P on a base surface *with non-neglectable friction*, we drag P by pulling a string in touch with P.

This kind of construction, called tractional motion, was foundationally relevant in early modern mathematics, cf. [3, 13]. The first appearance is attributed to the 1670s famous Perrault's construction of the tractrix. As visible in Figure 3, consider a pocket watch on a horizontal plane: if we slowly move the chain's end along a line, the pocket defines a tractrix. The curve is constructed because the taut chain remains tangent to the tractrix. Therefore, the tractrix is generated by the properties of its tangent (*inverse tangent problem*). Before the end of the 17th century, tractional constructions constituted the geometrical justification of Leibniz's calculus. However, these constructions used the dragged point tied on a string, while we generalize this idea by considering that the point-to-drag can slide along the string.

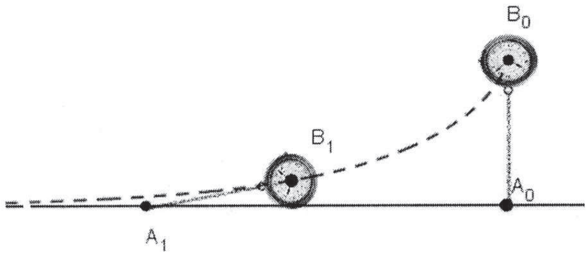

Fig. 3. Perrault's watch (construction of the tractrix). The motion of A to the left drags the clock B making the chain AB tangent to the curve traced by B.

In the following sections, we introduce methods to draw conics by a dragged point. These methods rest on well-known tangent properties [5]. As visible in Figure 4, the tangent to a conic has to bisects a certain angle related to the foci position. Such tangent properties are crucial to construct burning glasses and parabolic antennas.

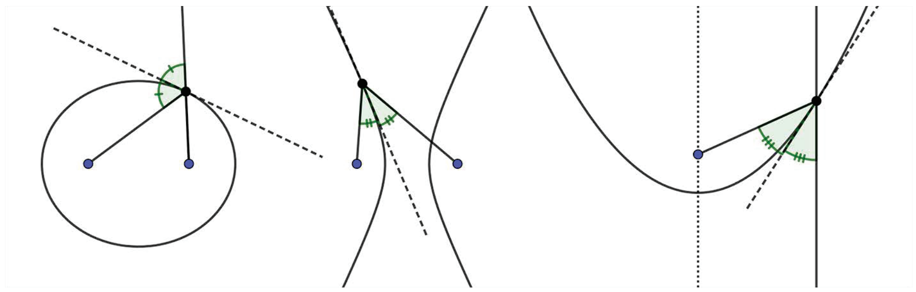

Fig. 4. Tangent properties of conics. The tangent line (represented by dashed lines) has to bisect the angles defined by the lines connecting the point on the curve and the foci. In the third case, one focus is a point at infinity; thus, we have to consider a line parallel to the parabola axis.

4. Between maths and physics: tensions and bisectors

Differently from the Lagrangian approach of Dawson et al [4], this manuscript proposes an elementary analysis of tractional constructions (suitable for high schools). Physically, we introduce the main property for string tension to pose a tangent condition about bisectors. Assume a heavy point P moving on a horizontal plane with non-neglectable friction. On the plane, let P be subject to two tensions $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$ of equal modulus and pointing respectively to the points P_1 and P_2, as in Figure 5. These tensions are generated by a string over which P can slide. The resulting force $\overrightarrow{T}$ acting on P, being the sum of two tensions of the same magnitude, bisects the angle formed by $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$. As explained below, if we drag P slowly, it has to move along the bisector of the angle P_1PP_2.

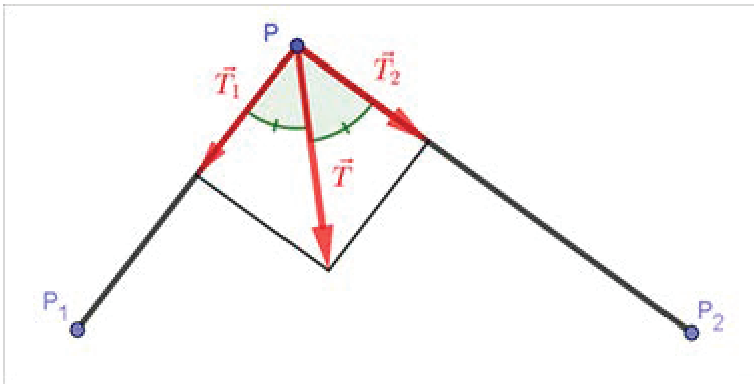

Fig. 5. A heavy point P subject to two tensions of the same magnitude, if dragged slowly enough, moves along the bisector of the angle defined by the tensions' directions.

First of all, assume neglectable sliding friction between the string and the body P. To clarify the ideas, consider P as a cylinder (not a dimensionless point) with the string around its lateral surface. If the string sliding friction is neglectable and the cylinder P is not rotating initially, P won't rotate when pushed by the string. Thus the motion of P has to be a pure translation. Therefore, being null the angular momentum, the tensions T_1 and T_2 have to be equal in magnitude.

To underline the non-uniqueness of the modelization, we can provide different models to justify the motion of P:

1. We can assume P moving with many micro-displacements (before and after the impulse, its speed is null).

   For any micro-displacement, we can consider applying a constant force in an interval Δt. In this interval, being null the initial velocity, the instantaneous velocity is oriented as the driving force $\overrightarrow{T_1} + \overrightarrow{T_2}$. The frictional force $\overrightarrow{F}_{fr}$ has the same direction of the velocity; without initial velocity, $\overrightarrow{F}_{fr}$ does not modify the direction of the resulting force. Being the two tensions $\overrightarrow{T_1}$ and

$\overrightarrow{T_2}$ equal in magnitude, their sum bisects them and gives the direction of P's displacement.

2. We can analyze the forces involving a continuous motion (while P moves slowly).

   According to the trajectory of P, $\overrightarrow{T} = \overrightarrow{T_1} + \overrightarrow{T_2}$ can be split into tangential and centripetal components ($\overrightarrow{T}_\parallel$ and $\overrightarrow{T}_\perp$), as visible in Figure 6. Besides $\overrightarrow{T}$, also the dynamical friction force $\overrightarrow{F}_{fr}$ acts in P, with direction opposite to the motion. Assuming the Coulomb model, $F_{fr} = \mu mg$ (the dynamic friction coefficient μ depends on the mass and the contact surfaces; in our case F_{fr} is constant).

   Assuming P moving at a constant velocity, the tangential component of the resultant force has to be null, hence $T_\parallel = F_{fr}$. Considering the centripetal force $T_\perp = mv^2/r$ (r is the instantaneous radius of curvature) and assuming the curvature radius of P's trajectory not to become less than r_{min} on a given branch of trajectory, $T_\perp/T_\parallel = v^2/(\mu rg) \leq v^2/(\mu r_{min} g)$.

   The angular difference between the tension bisectrix and $\overrightarrow{T}$ is $\tan^{(-1)} T_\perp/T_\parallel$. Hence, the direction of $\overrightarrow{T}$ well approximates the tensions bisectrix if $T_\perp/T_\parallel$ is neglectable. To grant $T_\perp/T_\parallel < \varepsilon$, it suffices to keep $v < \sqrt{\varepsilon \mu r_{min} g}$

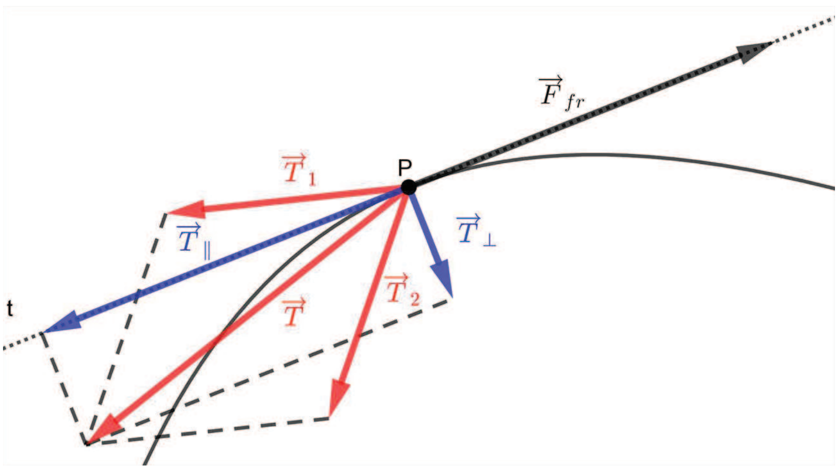

Fig. 6. Forces acting on P and decomposition of the driving force (tangential and normal component).

5. Hyperbolas by traction

In this paper, we would like to provide some activities based on pins-and-string constructions. After proposing the analysis of the gardener's hyperbola (Section 2), the instructor can pose the following problem.

Problem 1. In the gardener's hyperbola, point P is attached to the string. What happens if we let P free to move along the string while subject to the plane's friction?

Assume that a force $\overrightarrow{F}_{pull}$ is applied to pull strings. Such a force acts on P with two forces $\overrightarrow{T}_F$ and $\overrightarrow{T}_C$ pointing respectively to F and C (consider the pegs as in the right of Figure 2). For the magnitudes, considering a neglectable string mass, $F_{pull} = T_F = T_C$. For Section 4, by dragging P slowly enough, its direction has to bisect FPC. But the direction of P is tangent to its trajectory: for the tangent property of Figure 4 (center image), even in this case, P moves along the hyperbola of foci F and C.

Remark. As we get the same curve with P fixed or not on the string, the string sliding friction does not change the trajectory in this particular case. To provide an example in which sliding friction is relevant, we can imagine another variation. As illustrated in Figure 7, consider the string passing through C with an extremity attached in F (in the image, the pulling force is parallel to FC, but that is not necessary). If the heavy point P is constrained to touch the string (with neglectable sliding friction), the tensions $\overrightarrow{T}_F$ and $\overrightarrow{T}_C$ (pointing respectively to F and C) act on P with equal magnitude. Hence, also this time, the trajectory is a hyperbola of foci F and C. On the other side, if we consider non-neglectable sliding friction between P and the string, we should add a force (directed perpendicularly to $\overrightarrow{T} = \overrightarrow{T}_F + \overrightarrow{T}_C$) that would make P no longer move on a hyperbola.

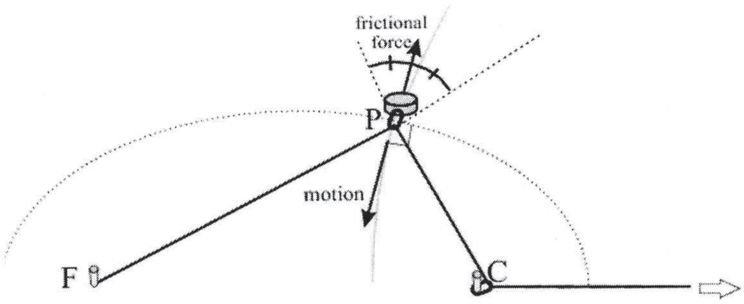

Fig. 7: Another hyperbola construction [4, Fig. 7]

6. Other conics by traction

Above, the proposed activities involved the analysis of some constructions. In this section, we imagine the instructor asking for the invention of some construction methods by modifying the previous sections' ideas. That could enrich students' approach to related topics and their operative reasoning.

Specifically, we generalize tractional constructions to ellipses and parabolas. For the hyperbola, the direction in which we pull the string does not modify the resulting trajectory (tensions point to the foci). On the contrary, the pulling direction becomes relevant for the following constructions. Even though elementary, such string constructions look new to us.

PROBLEM 2. Given two pins F_1 and F_2 on the plane, a string attached to F_1, in which direction do we have to pull the string to make the dragged point P move along an ellipse of foci F_1, F_2?

As illustrated in Figure 8, after fastening an extremity of the string in F_1 and passing across the heavy point P, we can pull the other extremity A of the string with a direction opposite to PF_2. By Section 4, neglecting sliding friction, we have that the direction of P has to bisect the angle F_1PA. As illustrated in the left of Figure 4, this is the tangent property of the ellipse of foci F_1 and F_2.

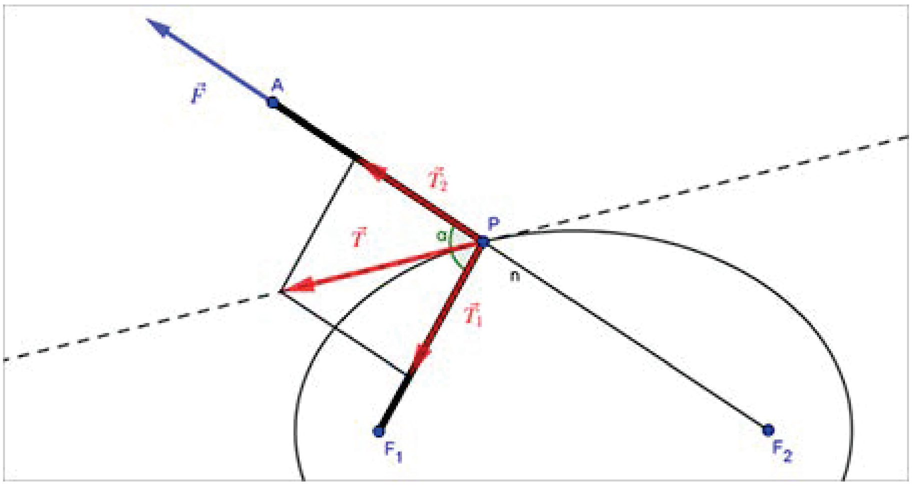

Fig. 8: A tractional construction of an ellipse of foci F_1 and F_2.

PROBLEM 3. Given a pin F_1 on the plane with a string attached and a line r passing through the pin, in which direction do we have to pull the string to make the dragged point P move along a parabola of focus F_1 and axis r?

A parabola is an ellipse with one focus F_2 at infinity. Call A the end of the string, that corresponds to keep constant the direction PA. Therefore, for a parabola of focus F_1 and axis r (as illustrated in Figure 9), after fastening an extremity of the string in F_1, we can pull the string downward with a direction parallel to r. For the usual reasoning, we have that the direction of P has to bisect the angle F_1PA. As illustrated in the right of Figure 4, this is the tangent property of the parabola of focus F_1 and axis r (perpendicular to the directrix).

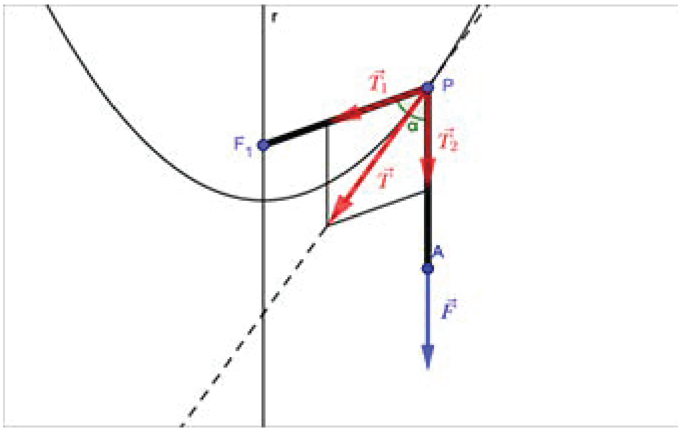

Fig. 9: A tractional construction of a parabola of focus F_1 and axis r.

7. Conclusions

This paper introduces tractional constructions of conics with some ideas for laboratory activities (some of these constructions are very recent, and others are original). For all of them, we provided elementary analyses, allowing their introduction since high school. These constructions can be implemented with simple tools (strings, pins, crayons, or pens) and appear didactically interesting for the following reasons.

Preliminary tests (with two Italian University students in mathematics and physics) showed the emergence of intense curiosity to construct conics by their tangent properties. Indeed, being habituated to define conics by focal distances, these methods implied a change of perspective for very familiar objects.

In today's digital era, the materiality of the proposed constructions can be considered an interesting plus. Indeed, concrete manipulations can help comprehend the underlying theoretical concepts. Such tracing methods allow us to feel the string tension and the resistances while dragging; others explicitly require visuomotor coordination (as when keeping the alignment for ellipses and parabolas). According to the embodiment studies in math education [10], we think that all these mind-body dynamics can produce profound and significant learning.

In these years, a special emphasis of the educational community involves the integration of mathematics and physics. As well expressed by Tzanakis [14], mathematics and physics have always been closely interwoven: mathematics is the language of physics, and physics constitutes a natural benchmark for mathematical theories. Our constructions constitute a crossing between the two disciplines, providing an exciting activity to mathematically model a simple (but not trivial) physical setting. Physical contents can be treated at different levels (forces/tensions in high school, Lagrangian mechanics at tertiary education) and can naturally introduce many interesting variations (e.g., considering what happens if the string is elastic or studying the friction between the dragged point and the string considering the traced curves).

To conclude, a future aim is to deepen these constructions' potentials in experimental researches with different audiences, from high-school students to teacher training.

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