ON MEASURES OF σ-NONCOMPACTESS IN F-SPACES

D. CAPONETTI*, A. TROMBETTA**, AND G. TROMBETTA***

ABSTRACT. Given an F-space (X,τ) and $\sigma \leq \tau$ a linear topology on X, this paper provides the definition of a general set function that allows us to define measures of σ-noncompactness in X. In particular, we construct measures of nonconvex σ-noncompactness that are monotonic, invariant under passage to the closed convex hull, and satisfy the Cantor intersection property. Additionally, we derive a fixed point theorem for maps that satisfy the classical Darbo condition with respect to a given measure of nonconvex σ-noncompactness.

1. Introduction

Strong and weak measures of noncompactness in locally convex spaces are valuable tools in functional analysis and operator theory. Specifically, the fixed point theorems that derive from them have many applications. It is recognized and well-known that the crucial problem to an axiomatic approach to measure of noncompactness in non-locally convex spaces is that the property of invariance under passage to the closed convex hull, expressed as $\varphi(M) = \varphi(\overline{\text{co}}M)$, that is necessary to obtain the Darbo's fixed point theorem. Strong measures of noncompactness in not necessarily locally convex spaces have been considered in the literature. In [9] measures of noncompactness have been introduced and investigated in general Hausdorff linear topological spaces, in connection with the notions of convexly total and strongly convexly total boundededness. Hadžić in [11] considers an approach in paranormed spaces in which the convex hull is controlled by a continuous mapping ϕ from $(0, +\infty)$ into itself, introducing the notion of ϕ-measure of noncompactness, as a mapping γ which verifies $\gamma(\overline{\text{co}}M) \leq \phi(\gamma(M))$. This paper primarily focuses on measures of weak noncompactness within the framework of non-locally convex spaces. Assuming (X,τ) to be an F-space, i.e. a complete metric linear space whose topology is hence

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induced by an F-norm, we are interested in topologies σ defined on X, weaker than the given topology τ and such that τ is σ-polar, that is τ has a base of σ-closed neighborhoods of the origin (see, for instance, [15, 21]).
In normed linear spaces, if σ denotes the weak topology, clearly the norm topology is σ-polar. In some constructions, one can consider τ the topology on X generated by the base at the origin θ consisting of all σ-closed neighborhoods of θ, so to obtain a σ-polar topology τ. Moreover, given an F-space (X, τ), in view of [14, Proposition 2.2 (iii)], whenever τ is locally bounded then there exists $\sigma < \tau$ such that τ is σ-polar. The following is an example of polar totopology in the classical Lebesgue spaces $L_p[0, 1]$.

Example 1. Let $X = (L_1[0, 1], \|\cdot\|_1)$ and τ the topology induced by $\|\cdot\|_1$. Assume σ to be the non-locally convex topology on X induced by the the F-norm

$$\|x\|_0 = \int_0^1 \frac{|x(t)|}{1 + |x(t)|} dt,$$

that is, σ is the topology of convergence in measure. Then τ is σ-polar. Indeed, given any $r > 0$ and (x_n) a sequence of functions in the closed ball $B_r(X) = \{x \in L_1[0, 1] : \|x\|_1 \leq r\}$ such that $\|x_n - x\|_0 \to 0$, then the sequence converges a.e. to x and applying Fatou's lemma we have that $x \in L_1(X)$ and $\|x\|_1 \leq r$. This shows that $\overline{B}_r^{\sigma}(X) = B_r(X)$, hence the assert. The same holds true if we consider the Lebesgue space $X = L^p[0, 1]$, either for $0 < p < 1$ or $1 < p < \infty$.

We point out that given $\sigma < \tau$, a condition stronger than σ-polarity is the requirement for τ to be σ-compatible. The topology τ is said to be σ-compatible if every τ-closed linear subspace of X is also σ-closed, and (see, [14, Corollary 5.4], [16, p.48]) if τ is σ-compatible then τ is σ-polar. As for compatible topologies, due to the Hanh-Banach extension theorem, the weak topology of a locally convex space is compatible with the original topology. But the same reasoning fails in non-locally convex F-spaces, since any F-space with the Hanh-Banach extension property is necessarily locally convex (see [14, Corallary 5.3]). But, despite of this, Kalton and Shapiro in [16] showed that a non-locally convex F-space can have a weaker compatible topology. Recall that the bounded weak topology is the strongest topology which agrees with the weak topology on bounded sets. Moreover, an F-space is said to be pseudo-reflexive if it has enough functionals to separate points, and every bounded subset is relatively compact in the weak topology of its closed linear span. Then from [16, Theorem 4.12] we have that the bounded weak topology of a locally bounded, pseudo-reflexive F-space is a weaker topology compatible with the original one.

Example 2. [16, Corollary 4.8] Consider, for $0 < p < 1 < q < \infty$, the space $\ell^q(p)$ of all complex sequences $z = \{z_n\}_{n=1}^{\infty}$ such that

$$\|z\|_{p,q} = \Big\{ \sum_{k=0}^{\infty} \Big\{ \sum_{2^k \le n \le 2^{k+1}} |z_n|^p \Big\}^{q/p} \Big\}^{1/q} < \infty.$$

Then $\| \cdot \|_{p,q}$ is a quasi-norm (in the sense of [15, p.6]), so $\ell^q(p)$ is a locally bounded F-space in the topology induced by $\| \cdot \|_{p,q}$. The space $\ell^q(p)$ is not locally convex and the bounded weak topology on the space $l^q(p)$ is strictly weaker than and compatible with the original topology [16, p.60].

We now describe the organization of the paper. We assume (X, τ) to be an F-space and $\sigma \le \tau$ a linear topology on X weaker than, possibly equal to, τ. We assume τ to be σ-polar. In the preliminaries we give some basic definitions, we introduce the notion of measure of noncompactness which we adopt for the purposes of this study. Section 3 is devoted to introducing the general set function $\alpha_{(\mathcal{K},\mathcal{F})}$, depending on suitable pairs $(\mathcal{K}, \mathcal{F})$ of families of subsets of X, and to establishing its fundamental properties. In the fourth section, by restricting ourselves to pairs $(\mathcal{K}, \mathcal{F})$ such that sets of $\mathcal{K}$ are σ-totally bounded, together with their convex hull, and $\mathcal{F}$ is the family of all convex subsets of X, we obtain a measure of nonconvex σ-noncompactness. Then, under the Cauty's result [5] which proves the Schauder conjecture, that in every topological vector space every compact convex subset has the fixed point property, we derive the Darbo's fixed point theorem for nonconvex measures of σ-noncompactness in non-locally convex spaces. We provide concrete examples of nonconvex σ-measures of noncompactness. The first example arises from the full measure whose kernel consists of all σ-totally bounded sets whose convex hull is also σ-totally bounded. Additionally, we consider the measure whose kernel consists of all σ-convexly totally bounded sets whose convex hull is also σ-convexly totally bounded and the one whose kernel is the family of all σ-strongly convexly totally bounded subsets of X.

2. Notation and Preliminaries

In this section, we recall some basic definitions and results which will be needed in the sequel. We recall that an F-seminorm on a linear space X is a function $\| \cdot \| : X \to [0, +\infty[$ which satisfies:
(i) $\|x + y\| \le \|x\| + \|y\|$; (ii) $\lim_{n\to+\infty} \|\frac{1}{n}x\| = 0$; (iii) $\|\lambda x\| \le \|x\|$ $(|\lambda| \le 1)$.
If $\|x\| = 0$ implies $x = \theta$, then $\| \cdot \|$ is called an F-norm. Any linear topology on X may be defined by a collection of continuous F-seminorms, and a metrizable topology may be defined by one F-norm. All topologies we consider in the following are assumed to be Hausdorff.

Throughout we assume $(X, \| \cdot \|)$ to be an F-space (complete metric linear space), τ will be the topology induced by the F-norm and $\sigma \le \tau$ a linear topology on X weaker than, possibly equal to, τ. We denote by θ the zero element of X and by $\tau(\theta)$ and $\sigma(\theta)$, respectively, the families of all τ-open and σ-open neighborhoods of θ. For a set $M \subseteq X$ we denote by co$M$ its convex hull. Further, $\overline{M}^{\sigma}$ and $\overline{\text{co}}^{\sigma} M$ will stand for the σ-closure and the σ-closed convex hull of M, respectively. Set $2^X = \{M \subseteq X : M \neq \emptyset\}$. We use the symbol $\mathcal{W}_{\sigma\text{-}tb}$ to denote the family consisting of all σ-totally bounded subsets of 2^X and define $\mathcal{W}^{\text{co}}_{\sigma\text{-}tb} = \left\{M \in 2^X : \text{co}M \in \mathcal{W}_{\sigma\text{-}tb}\right\}$. Obviously, $\mathcal{W}_{\sigma\text{-}tb} = \mathcal{W}^{\text{co}}_{\sigma\text{-}tb}$ if σ is a locally convex topology. Let $B_r = \{x \in X : \|x\| \leq r\}$, for $r > 0$. In the following we always assume the topology τ to be σ-polar in the sense of [15], i.e. the topology τ has a fundamental system of σ-closed τ-neighborhoods of θ. Such a fundamental system can be chosen to be the subfamily $\{\overline{B}_r^{\sigma},\ r > 0\}$ of $\tau(\theta)$.

In the sequel we use the following axiomatic approach of the notion of a measure of noncompactness (see, [1, 2, 3, 9]).

Definition 1. A set function $\varphi : 2^X \to [0, +\infty]$ is said to be *a measure of σ-noncompactness* if it satisfies the following properties ($M, N \in 2^X$):

(i) The family $\ker \varphi = \{M \in 2^X : \varphi(M) = 0\}$ is nonempty and $\ker \varphi \subseteq \mathcal{W}^{\text{co}}_{\sigma\text{-}tb}$.
(ii) $M \subseteq N$ implies $\varphi(M) \leq \varphi(N)$.
(iii) $\varphi(M) = \varphi(\overline{\text{co}}^{\sigma} M)$.
(iv) If (M_n) is a decreasing sequence of nonempty σ-complete subsets of X with $\lim_{n\to\infty} \varphi(M_n) = 0$, then the intersection $\bigcap_{n\in\mathbb{N}} M_n$ is nonempty.

Notice that the measure φ will have the following additional properties:

(v) $\varphi(M) = \varphi(\overline{M}^{\sigma})$.
(vi) $\varphi(M) = \varphi(\overline{M}^{\tau})$ and $\varphi(M) = \varphi(\overline{\text{co}}^{\tau} M)$.

The family $\ker \varphi$ is the kernel of the measure φ, and the measure φ is called full if $\ker \varphi = \mathcal{W}^{\text{co}}_{\sigma\text{-}tb}$. A measure of σ-noncompactness is called a measure of noncompactness if $\sigma = \tau$. If σ is a non-locally convex linear topology on X, thought $\mathcal{W}^{\text{co}}_{\sigma\text{-}tb} \neq \mathcal{W}_{\sigma\text{-}tb}$, axiom (iii) of Definition 1 implies that the assumptions $\ker \varphi \subseteq \mathcal{W}^{\text{co}}_{\sigma\text{-}tb}$ and $\ker \varphi = \mathcal{W}^{\text{co}}_{\sigma\text{-}tb}$ are equivalent, respectively, to $\ker \varphi \subseteq \mathcal{W}_{\sigma\text{-}tb}$ and $\ker \varphi = \mathcal{W}_{\sigma\text{-}tb}$. Note that when $(X, \|\cdot\|)$ is a Banach space and σ is the weak topology on X, axiom (iii) is equivalent to $\varphi(M) = \varphi(\overline{\text{co}}^{\|\cdot\|} M)$ for each $M \in 2^X$, since every nonempty convex subset of X is $\|\cdot\|$-closed if and only if it is σ-closed.

Classical measures of noncompactness in Banach spaces are the Hausdorff measure [10] and the weak measure introduced by De Blasi [8], which for any nonempty subset of X are defined, respectively, by

$$\gamma(M) = \inf\left\{\epsilon > 0 : M \subseteq F + B_\epsilon \text{ for some finite } F \subseteq X\right\},$$

$$\omega(M) = \inf\left\{\epsilon > 0 : M \subseteq K + B_\epsilon \text{ for some weakly compact } K \subseteq X\right\}.$$

3. A general set function

We introduce a general extended real set function $\alpha_{(\mathcal{K},\mathcal{F})}$, defined on the set of all nonempty subsets of X, depending on a pair of families $\mathcal{K}$ and $\mathcal{F}$ consisting of subsets of X. This section is devoted to study the properties of the set function $\alpha_{(\mathcal{K},\mathcal{F})}$ under suitable and natural conditions for the pair $(\mathcal{K},\mathcal{F})$. Set

$$\Gamma = \{(\mathcal{K},\mathcal{F}) : \ \mathcal{K} \subseteq \mathcal{W}_{\sigma\text{-}tb} \text{ and } \mathcal{F} \text{ is a family of subsets of } 2^X\}. \qquad (1)$$

Throughout we assume that any pair $(\mathcal{K},\mathcal{F})$ of Γ satisfies the following conditions:

$(a1)$ $\mathcal{K} \neq \emptyset$:
$(a2)$ $K \in \mathcal{K}$, implies $\overline{K}^{\sigma} \in \mathcal{K}$
$(b1)$ $S \in \mathcal{F}$ implies $\overline{S}^{\sigma} \in \mathcal{F}$;
$(b2)$ for all $\varepsilon > 0$ there is $S \in \mathcal{F}$ with $\theta \in S \subseteq \overline{B}_{\varepsilon}^{\sigma}$.

Observe that (b2) is satisfied whenever $\{\theta\}$ is an element of the family $\mathcal{F}$.

Definition 2. Given $(\mathcal{K},\mathcal{F}) \in \Gamma$, the set function $\alpha_{(\mathcal{K},\mathcal{F})} : 2^X \to [0,+\infty]$ is defined by

$$\alpha_{(\mathcal{K},\mathcal{F})}(M) = \inf\{\varepsilon > 0 : \text{there are } K \in \mathcal{K} \text{ and } S \in \mathcal{F} \text{ with } S \subseteq \overline{B}_{\varepsilon}^{\sigma} \\ \text{such that } M \subseteq K + S\},$$

where $\inf \emptyset = +\infty$.

Example 3. Let $(X, \|\cdot\|)$ be a Banach space, and $\mathcal{F} = \{B_r : \ r > 0\}$. Then the Hausdorff measure of noncompactness may be obtained as a special case of $\alpha_{(\mathcal{K},\mathcal{F})}$ by choosing $\mathcal{K} = \mathcal{W}_{\tau\text{-}tb}$ and the weak measure of noncompactness corresponds to the choice $\mathcal{K} = \mathcal{W}_{\sigma\text{-}tb}$, when σ is the weak topology on X.

Now we show that the set function $\alpha_{(\mathcal{K},\mathcal{F})}$ satisfies the axioms of Definition 1 unless, in the general case, the invariance under the passage to the closed convex hull. The study of this crucial property will be the object of the subsequent section. For our purposes, we need the following result.

Lemma 1. *A subset M of X is σ-totally bounded if and only if for every $r > 0$ there exists a σ-totally bounded subset H of X such that*

$$M \subseteq H + \overline{B}_r^{\sigma}.$$

Proof. The necessity is obvious. Vice versa, let U a σ-neighborhood of θ be given. Choose W a σ-open neighborhood of θ such that $W + W \subseteq U$. Since $\sigma \leq \tau$, W is a τ-neighborhood of θ. Using that $\{\overline{B}_r^{\sigma} : r > 0\}$ is a fundamental system of τ-neighborhoods of 0, we find $r > 0$ such that $\overline{B}_r^{\sigma} \subseteq W$. Then, corresponding to r, by the hypothesis, there is a σ-totally bounded subset H of X such that $M \subseteq H + \overline{B}_r^{\sigma}$. On the other hand, since the set H is σ-totally bounded, there is a finite subset F of X such that $H \subseteq F + W$. Then, $M \subseteq F + W + \overline{B}^{\sigma}_r \subseteq F + W + W \subseteq F + U$. Hence M is σ-totally bounded. □

Proposition 1. *Let $(\mathcal{K}, \mathcal{F}) \in \Gamma$. Then the set function $\alpha_{(\mathcal{K},\mathcal{F})}$ has the following properties ($M, N \in 2^X$):*

1. *$M \subseteq N$ implies $\alpha_{(\mathcal{K},\mathcal{F})}(M) \leq \alpha_{(\mathcal{K},\mathcal{F})}(N)$.*
2. *$\mathcal{K} \subseteq \ker \alpha_{(\mathcal{K},\mathcal{F})} \subseteq \mathcal{W}_{\sigma\text{-}tb}$.*
3. *$\alpha_{(\mathcal{K},\mathcal{F})}(M) = \alpha_{(\mathcal{K},\mathcal{F})}(\overline{M}^{\sigma})$.*

Proof. (1) is immediate.
(2) Let $M \in \mathcal{K}$, in view of (b2) for any $\varepsilon > 0$ there is $S \in \mathcal{F}$ with $\theta \in S \subseteq \overline{B}^{\sigma}_{\varepsilon}$, hence we can write $M \subseteq M + S$ which implies $M \in \ker \alpha_{(\mathcal{K},\mathcal{F})}$, and so $\mathcal{K} \subseteq \ker \alpha_{(\mathcal{K},\mathcal{F})}$ To prove the second inclusion, let $M \in \ker \alpha_{(\mathcal{K},\mathcal{F})}$, so that $\alpha_{(\mathcal{K},\mathcal{F})}(M) = 0$. Hence Lemma 1 implies $M \in \mathcal{W}_{\sigma\text{-}tb}$, as desired.
(3) Let $M \in 2^X$. By (1) $\alpha_{(\mathcal{K},\mathcal{F})}(M) \leq \alpha_{(\mathcal{K},\mathcal{F})}(\overline{M}^{\sigma})$. Therefore it is sufficient to prove that $\alpha_{(\mathcal{K},\mathcal{F})}(\overline{M}^{\sigma}) \leq \alpha_{(\mathcal{K},\mathcal{F})}(M)$. Let $r > \alpha_{(\mathcal{K},\mathcal{F})}(M)$ and choose $K \in \mathcal{K}$ and $S \in \mathcal{F}$ with $S \subset \overline{B}^{\sigma}_r$ such that $M \subseteq K + S$. Then, since $\overline{K}^{\sigma}$ is σ-compact and $\overline{S}^{\sigma}$ is σ-closed, $\overline{K}^{\sigma} + \overline{S}^{\sigma}$ is σ-closed. Moreover, by (a2) and (b1), $\overline{K}^{\sigma} \in \mathcal{K}$ and $\overline{S}^{\sigma} \in \mathcal{F}$. Then from $\overline{S}^{\sigma} \subseteq \overline{B}^{\sigma}_r$ and $\overline{M}^{\sigma} \subseteq \overline{K}^{\sigma} + \overline{S}^{\sigma}$, it follows $\alpha_{(\mathcal{K},\mathcal{F})}(\overline{M}^{\sigma}) \leq r$, which proves $\alpha_{(\mathcal{K},\mathcal{F})}(\overline{M}^{\sigma}) \leq \alpha_{(\mathcal{K},\mathcal{F})}(M)$. □

The following result says that if we take $\mathcal{K} = \mathcal{W}_{\sigma\text{-}tb}$ and $\mathcal{F} = \tau(\theta)$, then for each $M \in 2^X$ the value of the general set function $\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M)$ is the smallest possible one and, moreover, $\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}$ enjoys the maximum property.

Lemma 2. *The pair $(\mathcal{W}_{\sigma\text{-}tb}, \tau(\theta)) \in \Gamma$ and the following properties hold ($M, N \in 2^X$):*

1. *$\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M) \leq \alpha_{(\mathcal{K},\mathcal{F})}(M)$ for each $(\mathcal{K}, \mathcal{F}) \in \Gamma$,*
2. *$\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M \cup N) = \max\{\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M), \alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(N)\}$.*

Proof. Clearly the pair $(\mathcal{W}_{\sigma\text{-}tb}, \tau(\theta))$ belongs to Γ.
(1) Let $(\mathcal{K}, \mathcal{F}) \in \Gamma$, $M \in 2^X$ and $r > \alpha_{(\mathcal{K}_i,\mathcal{F})}(M)$ be given. Then there are $K \in \mathcal{K}$ and $S \in \mathcal{F}$ with $S \subseteq \overline{B}^{\sigma}_r$ such that $M \subseteq K + S$. By $K \in \mathcal{W}_{\sigma\text{-}tb}$, $\overline{B}^{\sigma}_r \in \tau(\theta)$ and $M \subseteq K + \overline{B}^{\sigma}_r$ we find $\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M) \leq r$, so that $\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M) \leq \alpha_{(\mathcal{K},\mathcal{F})}(M)$, hence the thesis.
(2) Let $M, N \in 2^X$. We prove $\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M \cup N) \leq \max\{\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M), \alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(N)\}$ as the reverse inequality follows from the property (1) of Proposition 1. Let $r > \max\{\alpha_{\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M), \alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(N)\}$ and $K_1, K_2 \in \mathcal{W}_{\sigma}$ such that $M \subseteq K_1 + \overline{B}^{\sigma}_r$ and $N \subseteq K_2 + \overline{B}^{\sigma}_r$. By $K_1 \cup K_2 \in \mathcal{W}_{\sigma\text{-}tb}$ and $M \cup N \subseteq (K_1 \cup K_2) + \overline{B}^{\sigma}_r$, we have $\alpha_{(\mathcal{W}_{\sigma},\tau(\theta))}(M \cup N) \leq r$, and therefore $\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M \cup N) \leq \max\{\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M), \alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(N)\}$. □

The following result is a generalization of the Cantor intersection theorem. The proof is classical.

Theorem 1. *Let* $(\mathcal{K},\mathcal{F})\in\Gamma$. *If* (M_n) *is a decreasing sequence of nonempty* σ-*complete subsets of* X *and* $\lim_{n\to\infty}\alpha_{(\mathcal{K},\mathcal{F})}(M_n)=0$ *then* $\bigcap_{n=1}^{\infty}M_n$ *is nonempty.*

Proof. Let $(\mathcal{K},\mathcal{F})\in\Gamma$ and let (M_n) be a decreasing sequence of nonempty σ-complete subsets of X with $\alpha_{(\mathcal{K},\mathcal{F})}(M_n)\to 0$ $(n\to\infty)$ and choose $x_n\in M_n$ for $n=1,2,\ldots$. Then, using properties (1) and (2) of Proposition 1 and properties (1) and (2) of Lemma 2, we have

$$\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(\bigcup_{n=1}^{\infty}\{x_n\})=\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(\bigcup_{n=k}^{\infty}\{x_n\})\le\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(M_k)\le\alpha_{(\mathcal{K},\mathcal{F})}(M_k).$$

Since $\lim_{n\to\infty}\alpha_{(\mathcal{K},\mathcal{F})}(M_n)=0$, it follows $\alpha_{(\mathcal{W}_{\sigma\text{-}tb},\tau(\theta))}(\bigcup_{n=1}^{\infty}\{x_n\})=0$ and $\{x_n : n=1,2,\ldots\}$ is σ-totally bounded. Therefore the set $\{x_n : n=1,2,\ldots\}$ has a cluster point $x\in X$. Since all the sets are σ-complete, x belongs to the intersection $\bigcap_{n=1}^{\infty}M_n$, and we obtain the desired result. $\square$

Due to Proposition 1 and Theorem 1, we have that the general set function $\alpha_{(\mathcal{K},\mathcal{F})}$, for every pair $(\mathcal{K},\mathcal{F})\in\Gamma$, satisfies properties (ii), (iv) and (v) of a measure of σ-noncompactness. Invariance under the passage to the closed convex hull is not guaranteed, without additional conditions.

4. Measures of nonconvex σ-noncompactness

This section of the paper contains our main results. We consider a more restricted class of pairs from Γ, in order to obtain a measure of σ-noncompactness, according to Definition 1, and to prove a Darbo's fixed-point theorem. A general set function $\alpha_{(\mathcal{K},\mathcal{F})}$, with $(\mathcal{K},\mathcal{F})\in\Gamma$, to be a measure of σ-noncompactness, has to satisfies property (iii) of invariance under the passage to the closed convex hull. To this end in this section we consider the subfamily $\Gamma_{\mathcal{C}}$ of Γ, defined in (1), by setting

$$\Gamma_{\mathcal{C}}=\{(\mathcal{K},\mathcal{C}):\mathcal{K}\subseteq\mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}\text{ and }\mathcal{C}\text{ is the family of convex subsets of }X\},\quad(2)$$

still requiring conditions (a1) and (a2), being (b1) and (b2) satisfied by the family $\mathcal{C}$. We can write, explicitly,

$$\alpha_{(\mathcal{K},\mathcal{C})}(M)=\inf\{\varepsilon>0:\text{ there are }K\in\mathcal{K}\text{ and a convex subset }C\text{ of }\overline{B}^{\sigma}_{\varepsilon}\text{ such that }M\subseteq K+C\},$$

having in mind that K and $\mathrm{coK}\in\mathcal{W}_{\sigma\text{-tb}}$, if $K\in\mathcal{K}$.

Proposition 2. *Let* $(\mathcal{K},\mathcal{C})\in\Gamma_{\mathcal{C}}$, *then* $\alpha_{(\mathcal{K},\mathcal{C})}(M)=\alpha_{(\mathcal{K},\mathcal{C})}(\overline{co}^{\sigma}M)$.

Proof. From Proposition 1, we have $\alpha_{(\mathcal{K},\mathcal{C})}(M) \leq \alpha_{(\mathcal{K},\mathcal{C})}(\overline{\text{co}}^{\sigma} M)$ and $\alpha_{(\mathcal{K},\mathcal{C})}(\text{co}M) = \alpha_{(\mathcal{K},\mathcal{C})}(\overline{\text{co}}^{\sigma} M)$, respectively. Therefore it suffices to prove that $\alpha_{(\mathcal{K},\mathcal{C})}(\text{co}M) \leq \alpha_{(\mathcal{K},\mathcal{C})}(M)$. Let $r > \alpha_{(\mathcal{K},\mathcal{C})}(M)$. Choose $K \in \mathcal{K}$ and $C \in \mathcal{C}$ with $C \subseteq \overline{B}_r^{\sigma}$ such that $M \subseteq K + C$. By the hyphothes we have $\text{coK} \in \mathcal{K}$ and $coM \subseteq \text{coK} + \text{C}$, so that $\alpha_{(\mathcal{K},\mathcal{C})}(\text{coM}) \leq \text{r}$. Hence $\alpha_{(\mathcal{K},\mathcal{C})}(\text{coM}) \leq \alpha_{(\mathcal{K},\mathcal{C})}(\text{M})$. □

In view of the above proposition, for $(\mathcal{K},\mathcal{C}) \in \Gamma_{\mathcal{C}}$ the set function $\alpha_{(\mathcal{K},\mathcal{C})}$ is a measure of σ-noncompactness.

Definition 3. Whenever $(\mathcal{K},\mathcal{C}) \in \Gamma_{\mathcal{C}}$, the set function $\alpha_{(\mathcal{K},\mathcal{C})}$ is called *a measure of nonconvex σ-noncompactness.*

Definition 4. Let $(\mathcal{K},\mathcal{C}) \in \Gamma_{\mathcal{C}}$ and let C be a nonempty subset of X. A σ-continuous function $f : C \to C$ is said to be an $\alpha_{(\mathcal{K},\mathcal{F})}$*-contraction* if there exists $0 \leq k < 1$ such that

$$\alpha_{(\mathcal{K},\mathcal{F})}(f(M)) \leq k \ \alpha_{(\mathcal{K},\mathcal{F})}(M)$$

for every subset M of C.

Now we are in a position to obtain a Darbo's fixed point theorem in our setting. The final step of the proof is established under the property that every convex σ-complete subset of $\ker \varphi$ has the fixed point property. Therefore, we have to refer to one of the oldest problems in fixed point theory, which is the *Schauder's conjecture* (Problem 54 in The Scottish Book, see [17]): every nonempty compact convex set in a topological vector space has the fixed point property. Let us recall that Schauder [22] proved the conjecture in normed vector spaces and Tychonoff [23] extended the result in locally convex spaces. Since then, numerous attempts have been made to address the problem in non-locally convex spaces. In particular, the approach by Idzik, discussed later in this section, led the author to prove the conjecture in non-locally convex linear spaces for compact convex sets that are convexly totally bounded. In 2001, Cauty in [4] proposed a solution to the problem, but later a gap was discovered in the proof (see, for instance [17, p.131], [13, p.91]). Thereafter, Cauty [5, 6, 7] filling the gap provided the affirmative solution of the Schauder conjecture. Therefore we state the following general result.

Theorem 2. *Let $(\mathcal{K},\mathcal{C}) \in \Gamma_{\mathcal{C}}$, C a nonempty σ-complete convex subset of X and $f : C \to C$ an $\alpha_{(\mathcal{K},\mathcal{F})}$-contraction. If $\alpha_{(\mathcal{K},\mathcal{C})}(C) \in [0, +\infty[$, then f has a fixed point.*

Proof. We define inductively a sequence of sets $C_0 = C$ and $C_{n+1} = \overline{\text{co}}^{\sigma} f(C_n)$. We have

$$\alpha_{(\mathcal{K},\mathcal{C})}(C_n) \leq k^n \ \alpha_{(\mathcal{K},\mathcal{C})}(C) \to 0 \ \ (n \to \infty).$$

By the Cantor intersection property, the σ-complete convex set $C_\infty = \cap_{n\in\mathbb{N}} C_n$ is nonempty, moreover σ-compact, since $\alpha_{(\mathcal{K},\mathcal{C})}(C_\infty) = 0$. Further $f(C_\infty) \subseteq C_\infty$, hence by Cauty's fixed point theorem the statement follows. $\square$

In the sequel, we want to consider concrete examples of measures of nonconvex σ-noncompactness, so that Darbo's theorem will be satisfied. Recall that a nonvoid subset M of X is said to be σ-convexly totally bounded (briefly, σ-ctb) if for every σ-neighborhood U of 0 there are $x_1, \ldots, x_n \in X$ and convex subsets $C_1, \ldots, C_n$ of U such that $M \subseteq \cup_{s=1}^{n}(x_s + C_s)$. This notion was introduced by Idzik [12], before Cauty's result, to establish fixed-point theorems in non-locally convex spaces. In particular, [12, Theorem 4.3] asserts that if a σ-compact, convex set C is σ-ctb, then C has the fixed-point property. In the same paper ([12, Problem 4.7]) Idzik posed the problem whether every σ-compact convex set is σ-ctb, and a positive answer would have proved Shauder's conjecture. As it is well known it is not so, examples of σ-compact convex set which are not σ-ctb have been provided in the literature (see for example, [9, 24, 26]). Inspired by Idzik's work, the concept of σ-strongly convexly totally bounded sets have been introduced and further explored in several papers, including [9, 20, 25, 26].

Definition 5. A nonempty subset M of X is said to be *σ-strongly convexly totally bounded* (briefly, σ-sctb) if for every σ-neighborhood U of θ there exist a convex subset C of U and a finite subset F of X such that $M \subseteq F + C$.

Observe that the convex hull of any σ-sctb set is still σ-sctb, while the convex hull of a σ-ctb is not necessarily σ-ctb. Denote by $\mathcal{W}_{\sigma\text{-}ctb}$ and $\mathcal{W}_{\sigma\text{-}sctb}$ the family, respectively, consisting of all σ-ctb and σ-sctb subsets of 2^X. Summarizing, we have $\mathcal{W}_{\sigma\text{-}sctb} = \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}sctb}$ and $\mathcal{W}_{\sigma\text{-}sctb} \subsetneq \mathcal{W}_{\sigma\text{-}ctb} \subsetneq \mathcal{W}_{\sigma\text{-}tb}$ and $\mathcal{W}_{\sigma\text{-}sctb} \subseteq \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb} \subsetneq \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}$. As far as we know, the question (see for instance, [9]) regarding whether there exist convex σ-ctb sets that are not σ-sctb remains still open. If this were be not the case, then $\mathcal{W}_{\sigma\text{-}sctb}$ and $\mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$ would coincide.

The following Lemma is the analogous of Lemma 1 for σ-ctb and σ-sctb sets, in both cases it can be proven by making slightly changes to the proof of Lemma 1. For clarity, we demonstrate the first case.

Lemma 3. *Let $M \in 2^X$.*
(a) The set M is σ-ctb if and only if for every $r > 0$ there are a convex subset C of $\overline{B}_r^{\sigma}$ and a σ-ctb subset H of X such that

$$M \subseteq H + C.$$

(b) The set M is σ-sctb iff for every $r > 0$ there are a convex subset C of $\overline{B}_r^{\sigma}$ and a σ-sctb subset H of X such that

$$M \subseteq H + C.$$

Proof. (a) The necessity is obvious. To prove the sufficient part, following the proof of Lemma 1, given U a σ-neighborhood of θ, we choose a σ-open neighborhood W of θ such that $W + W \subseteq U$, and $r > 0$ such that $\overline{B}_r^{\sigma} \subseteq W$. Then, corresponding to r, by the hypothesis, there is a σ-ctb subset H of X such that $M \subseteq H + C$ with C convex subset of $\overline{B}_r^{\sigma}$. Now, since the set H is σ-ctb, there are $x_1, \dots, x_n \in X$ and convex subsets $C_1, \dots, C_n$ of W such that $H \subseteq \cup_{s=1}^n (x_s + C_s)$. We obtain $M \subseteq \cup_{s=1}^n (x_s + C_s + C)$, with each $C_s + C$ convex subsets of U. Hence the thesis. □

Next, we present examples of measures of nonconvex σ-noncompactness, which are obtained by considering $(\mathcal{K}, \mathcal{C}) \in \Gamma_C$ where the family $\mathcal{K}$ is chosen to be one of the following: $\mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}$, $\mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$ or $\mathcal{W}_{\sigma\text{-}sctb}$. Correspodingly, we denote the measure $\alpha_{(\mathcal{K},\mathcal{C})}$, respectively, by $\overline{\omega}_{tb}$, $\overline{\omega}_{ctb}$ and $\overline{\omega}_{sctb}$. Hence, using this notation, we can write:

$$\overline{\omega}_{tb}(M) = \inf\{\varepsilon \in]0, +\infty] : \text{ there are } H \in \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}$$
$$\text{and a convex subset } C \subseteq \overline{B}_{\varepsilon}^{\sigma} \text{ such that } M \subseteq H + C\}.$$
$$\overline{\omega}_{ctb}(M) = \inf\{\varepsilon \in]0, +\infty] : \text{ there are } H \in \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$$
$$\text{and a convex subset } C \subseteq \overline{B}_{\varepsilon}^{\sigma} \text{ such that } M \subseteq H + C\}.$$

and

$$\overline{\omega}_{sctb}(M) = \inf\{\varepsilon \in]0, +\infty] : \text{ there are } H \in \mathcal{W}_{\sigma\text{-}sctb}$$
$$\text{and a convex subset } C \subseteq \overline{B}_{\varepsilon}^{\sigma} \text{ such that } M \subseteq H + C\}.$$

For any locally convex topology σ, we have $\overline{\omega}_{tb} = \overline{\omega}_{ctb} = \overline{\omega}_{sctb}$.

In view of Proposition 1, Lemma 1 and Lemma 3, $\ker \overline{\omega}_{tb} = \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}$, $\ker \overline{\omega}_{ctb} = \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$ and $\ker \overline{\omega}_{sctb} = \mathcal{W}_{\sigma\text{-}sctb}$. So, in particular, the measure $\overline{\omega}_{tb}$ is full. The measure of nonconvex σ-noncompatness $\overline{\omega}_{sctb}$ is the weak counterpart of the measure $\overline{\gamma}_s$ introduced in [9], when $\overline{\gamma}_s$ is considered in an F-space, and if $\sigma = \tau$ the two measures coincide. Moreover notice that Darbo's theorem for the measures $\overline{\omega}_{ctb}$, $\overline{\omega}_{sctb}$ can be deduced from Idzik's fixed point theorem.
From [9, Proposition 2.2] we have the following result.

Proposition 3. *If K_1 and K_2 are subsets of X belonging to $\mathcal{W}_{\sigma\text{-}sctb}$, then the same is true for $K_1 + K_2$, $K_1 \cup K_2$ and λK_1 ($\lambda \in \mathbb{R}$).*

We want to prove that the same conclusion holds for the classes $\mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$ and $\mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}$. To this end recall the result of [19, Theorem 1.7], analogous to Proposition 3, for ctb sets.

Proposition 4. *If K_1 and K_2 are subsets of X belonging to $\mathcal{W}_{\sigma\text{-}ctb}$, then the same is true for $K_1 + K_2$, $K_1 \cup K_2$ and λK_1 ($\lambda \in \mathbb{R}$).*

Then we can prove the following.

Theorem 3. *If K_1, K_2 are subsets of X both belonging either to $\mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}$ or $\mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$, then (i) $K_1 + K_2$, (ii) $K_1 \cup K_2$, (iii) λK_1 ($\lambda \in \mathbb{R}$) belong, respectively, to $\mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}$ and $\mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$.*

Proof. Let K_1 and K_2 be subsets of X.
(i) It follows from the following inclusions

$$\mathrm{co}(\mathrm{K}_1 + \mathrm{K}_2) \subseteq \mathrm{co}(\mathrm{coK}_1 + \mathrm{coK}_2) = \mathrm{coK}_1 + \mathrm{coK}_2.$$

Indeed, if $K_1, K_2 \in \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}$, then $\mathrm{coK}_1 + \mathrm{coK}_2$ is σ-tb, hence $\mathrm{co}(\mathrm{K}_1 + \mathrm{K}_2)$ is σ-tb. If $K_1, K_2 \in \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$, from Proposition 4 we have that $\mathrm{coK}_1 + \mathrm{coK}_2$ is σ-ctb, hence $\mathrm{co}(\mathrm{K}_1 + \mathrm{K}_2)$ is σ-ctb.
(ii) Let $K_1, K_2 \in \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}$. We have

$$\mathrm{co}(\mathrm{K}_1 \cup \mathrm{K}_2) \subseteq \mathrm{co}(\overline{\mathrm{co}}^{\sigma}\mathrm{K}_1 \cup \overline{\mathrm{co}}^{\sigma}\mathrm{K}_2),$$

where the set $\mathrm{co}(\overline{\mathrm{co}}^{\sigma}K_1 \cup \overline{\mathrm{co}}^{\sigma}K_2)$ is σ-tb (see, for instance, [18, p.73]). Hence, we obtain that the set $\mathrm{co}(K_1 \cup K_2) \in \mathcal{W}_{\sigma\text{-}tb}$, so that $K_1 \cup K_2 \in \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}tb}$.
Now let $K_1, K_2 \in \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$, and observe that the following inclusion holds

$$\mathrm{coK}_1 \cup \mathrm{coK}_2 \subseteq [0,1]\mathrm{coK}_1 + [0,1]\mathrm{coK}_2. \tag{3}$$

If A (here A is thought to be coK, for a set $K \in \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$) is convex and σ-ctb set, then $[0,1]A$ is clearly convex, but it is also σ-ctb. Indeed, given U any σ-neighborhood of the origin θ in X we can choose a σ-neighborhood V of θ such that $V + V \subset U$, a finite number of points $x_1, \dots, x_n$ in X and convex subsets $C_1, \dots, C_n$ of V, such that $A \subseteq \cup_{s=1}^{n}(x_s + C_s)$, and so

$$[0,1]A \subseteq \cup_{s=1}^{n}\left([0,1]x_s + [0,1]C_s\right)$$

Now $K_s = [0,1]C_s$ is a convex set, containing θ, still included in V, and considering that each $[0,1]x_s$ è σ-sctb we can find a finite set F_s and a convex subset D_s of V, such that

$$[0,1]A \subseteq \cup_{s=1}^{n}((F_s + D_s) + K_s).$$

Therefore, we have

$$[0,1]A \subseteq \cup_{i=1}^{n}(F + (D_s + K_s))$$

where F is the finite set union of all $F_s's$. Since $\mathrm{co}(\mathrm{D_s} \cup \{\theta\}) \subset \mathrm{V}$ we may assume $\theta \in D_s$, then $D_s + K_s$ is a convex subset of U containing D_s and K_s, so it follows that the set $[0,1]A$ is σ-ctb. Hence, using also that the sum of two σ-ctb sets is σ-ctb, we infer that

$$[0,1]\mathrm{coK}_1 + [0,1]\mathrm{coK}_2$$

is convex and σ-ctb. Then, since $\mathrm{co}(\mathrm{K}_1 \cup \mathrm{K}_2) \subseteq \mathrm{co}(\mathrm{coK}_1 \cup \mathrm{coK}_2)$, from (3) we find that $\mathrm{co}(\mathrm{K}_1 \cup \mathrm{K}_2)$ is σ-ctb, which means $K_1 \cup K_2 \in \mathcal{W}^{\mathrm{co}}_{\sigma\text{-}ctb}$.
(iii) The last assertion follows from $\mathrm{co}(\lambda K_1) = \lambda \mathrm{coK}_1$. □

From previous results, it follows that each of the measures of nonconvex σ-noncompactness, namely $\overline{\omega}_{tb}$, $\overline{\omega}_{ctb}$ and $\overline{\omega}_{sctb}$, satisfies the following properties.

Corollary 1. *If* $\overline{\omega}$ *is one of the measures* $\overline{\omega}_{tb}$, $\overline{\omega}_{ctb}$ *or* $\overline{\omega}_{sctb}$, *then* $\overline{\omega}$ *satisfies* $(M, N \in 2^X, \lambda \in \mathbb{R})$:

(i) $\overline{\omega}(M+N) \leq \overline{\omega}(M) + \overline{\omega}(N)$;
(ii) $\overline{\omega}(M \cup N) = \max\{\overline{\omega}(M), \overline{\omega}(N)\}$;
(iii) $\overline{\omega}(\lambda M) \leq |\lambda|\ \overline{\omega}(M)$.

In conclusion, by dealing with measures of weak noncompactness in non-locally convex spaces, we have presented a unifying approach for the study of measures of noncompactness, both in the strong and weak sense, within the setting of locally convex as well as non-locally convex spaces, for a systematic treatment of different topological contexts. We illustrated the applicability of the proposed approach providing concrete examples.

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* Dipartimento di Matematica e Informatica
Università di Palermo
I-90123 Palermo
ITALY
Email address: diana.caponetti@unipa.it

** Dipartimento di Matematica e Informatica
Università della Calabria
Arcavacata di Rende
Cosenza
ITALY
Email address: alessandro.trombetta@unical.it

*** Dipartimento di Matematica e Informatica
Università della Calabria
Arcavacata di Rende
Cosenza
ITALY
Email address: giulio.trombetta@unical.it