An elliptic equation on n-dimensional manifolds

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We consider an elliptic equation driven by a p-Laplacian-like operator, on an n-dimensional Riemannian manifold. The growth condition on the right-hand side of the equation depends on the geometry of the manifold. We produce a nontrivial solution, by using a Palais-Smale compactness condition and a mountain pass geometry.

Keywords: Isocapacitary inequality, mountain pass geometry, Orlicz space, p-Laplacian-like operator; Sobolev space.

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1. Introduction

Let N be a connected, without boundary, n-dimensional Riemannian manifold. In this paper we study the following elliptic equation:

$$\begin{aligned}\int_N |\nabla x|^{p-2}\nabla x \nabla y d\mathcal{H}^n + \int_N \frac{|\nabla x|^{2p-2}\nabla x}{\sqrt{1+|\nabla x|^{2p}}}\nabla y d\mathcal{H}^n \\ = \int_N g(x) y\, d\mathcal{H}^n \quad \text{for all } y \in W^{1,p}(N). \qquad (1)\end{aligned}$$

In this equation, we incorporate the p-Laplacian-like operator, with $1 < p < +\infty$. We mention that $\mathcal{H}^n$ is the volume measure on N induced by the Riemannian metric on N (here, we suppose $\mathcal{H}^n(N) < +\infty$). Also, $g : \mathbb{R} \to \mathbb{R}$ is continuous and has certain other regularity and growth properties listed in the sequel. The basic hypotheses on g are as follows:

(g_1) $\dfrac{g(s)}{|s|^{p-1}} \to 0$ as $s \to 0$,
(g_2) $G(s) > 0$ for $s \neq 0$, with $G : \mathbb{R} \to \mathbb{R}$ given as $G(s) = \int_0^s g(t)dt$, $s \in \mathbb{R}$,
(g_3) there exists $s_0 > 0$ such that $sg(s) - 2pG(s) > 0$ if $|s| > s_0$.

Remark 1.1 If (g_1) and (g_2) hold, then for every ε there exists $s_0 > 0$ such that

$$G(s) \le \varepsilon |s|^p \quad \text{if } |s| < s_0. \qquad (2)$$

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Remark 1.2 If (g_2) and (g_3) hold, then for all $\kappa \in [p, 2p]$ there exists $s_0 > 0$ such that

$$G(s) \geq a_0 |s|^\kappa \quad \text{if } |s| \geq s_0, \text{ for some } a_0 > 0. \tag{3}$$

So, we can find $a_0, a_1 > 0$ such that

$$G(s) \geq a_0 |s|^p - a_1 \quad \text{for } s \in \mathbb{R}. \tag{4}$$

Also, we have

$$G(r) \leq G(s)(r/s)^\kappa \quad \text{if } s_0 \leq r \leq s \text{ or } s \leq r \leq -s_0. \tag{5}$$

So, for all $\sigma > 0$, we can find $a > 0$ such that

$$G(s) \leq a b^\kappa G(s/b) \quad \text{if } |s| \geq \sigma,\ 0 < b < 1. \tag{6}$$

Our work here continues the one by Barletta-Cianchi-Maz'ya [3]. In [3] the differential operator is the p-Laplacian. The authors prove the existence of a nontrivial solution, by using the isocapacitary inequality of N, which for all measurable set $E \subset N$, with $2\mathcal{H}^n(E) \leq \mathcal{H}^n(N)$, gives us

$$\eta_{N,p}(\mathcal{H}^n(E)) \leq C_{N,p}(E), \tag{7}$$

where $\eta_{N,p} : [0, \mathcal{H}^n(N)/2] \to [0, +\infty]$ is the isocapacitary function

$$\eta_{N,p}(s) = \inf \{C_{N,p}(E) : E \subset N,\ s \leq \mathcal{H}^n(E) \leq \mathcal{H}^n(N)/2\} \quad \text{for } s \in [0, \mathcal{H}^n(N)/2]$$

and $C_{N,p}(E)$ is the condenser capacity of E

$$C_{N,p}(E) = \inf \left\{ \int_N |\nabla u|^p d\mathcal{H}^n : u \in W^{1,p}(N), u \geq 1\ C_p\text{-quasi everywhere in } E \right\},$$

where C_p-quasi everywhere means that the condition holds outside a set of p-capacity zero. Recall that the p-capacity is defined by

$$C_p(E) = \inf \left\{ \int_N |\nabla u|^p d\mathcal{H}^n : u \in W_0^{1,p}(N), u \geq 1 \text{ in some neighbourhood of } E \right\}.$$

The isocapacitary inequality was defined by Maz'ya [9] to obtain a priori bounds for solutions to nonlinear elliptic equations in Sobolev spaces. For earlier related discussion and results, we refer to Cianchi-Maz'ya [6], Maz'ya [8], Milman [10], and the references therein. For other results on elliptic equations see Vétois [18] (existence of multiple solutions on compact manifold) and Sun [17] (nonexistence of positive solutions on noncompact manifold).

We mention that elliptic equations driven by the p-Laplacian-like operator attract considerable interest since this operator arises in problems of mathematical physics (for instance, it is useful to model the phenomenon of capillarity). So, one can found various existence and multiplicity results for such equations in the recent literature. We recall the papers of Chen-Luo [5], Papageorgiou-Rocha [12], Rodrigues [15], Vetro [21], Zhou [23] (Dirichlet problem), Afrouzi-Kirane-Shokooh [1], Shokooh [16] (Neumann problem).

Here, we produce a nontrivial solution of (1), by using a Palais-Smale compactness condition and a mountain pass geometry. For other types of differential operators see also Vetro [19, 20] (Neumann and Robin problems) and Vetro-Vetro [22] (Dirichlet problem).

2. Mathematical background

In our analysis of equation (1), we will use the Sobolev type space $\mathbb{W}^{1,p}(N)$, $1 \le p \le +\infty$, and the Orlicz space $L^\Phi(N)$ of a Young function $\Phi : [0,+\infty[\to [0,+\infty]$. Clearly $L^\Phi(N)$ reduces to $L^p(N)$ whenever $\Phi(s) = s^p$, $1 \le p < +\infty$. Also $L^\Phi(N) = L^\infty(N)$, where $\Phi(s) = \begin{cases} 0 & \text{if } 0 \le s \le 1, \\ +\infty & \text{if } s > 1. \end{cases}$

According to Barletta-Cianchi-Maz'ya [3], we start by the following definition.

Definition 2.1 We say that $\Phi : [0,+\infty[\to [0,+\infty]$ is a Young function if

$$\Phi(s) = \int_0^s \phi(t)dt \quad \text{for } s \ge 0,$$

for some non-decreasing, left-continuous function $\phi : [0,+\infty[\to [0,+\infty]$ such that neither $\phi \equiv 0$ nor $\phi \equiv +\infty$. Also, $\Phi^*(s) = \sup_{t\ge 0}\{ts - \Phi(t)\}$ is called the Young conjugate of Φ.

We have

$$\Phi^*(s) = \int_0^s \phi^{-1}(t)dt \quad \text{for } s \ge 0$$

(ϕ^{-1} is the (generalized) left-continuous inverse of ϕ).

Now, $\Phi_p : [0,+\infty[\to [0,+\infty]$ defined by $\Phi_p(s) = |s|^p p^{-1}$, $1 \le p < +\infty$, is an easy example of Young function.

Remark 2.2 Given a Young function $\Phi : [0,+\infty[\to [0,+\infty]$, we have

$$\Phi(s) \le s\phi(s) \le \Phi(2s) \quad \text{for } s > 0. \tag{8}$$

Given two Young functions $\Phi, \Psi : [0,+\infty[\to [0,+\infty]$, we say that Φ dominates Ψ globally (near infinity) if there is $k > 0$ such that

$$\Psi(s) \le \Phi(ks) \quad \text{for } s > 0 \ (\text{for } s \ge s_0 \ge 0). \tag{9}$$

Provided that Φ and Ψ dominate each other globally (near infinity), we say that Φ and Ψ are equivalent globally (near infinity). The notion of equivalent functions applies not necessarily to Young functions (as shown in the following example).

Example 2.3 Let $\omega :]0,+\infty[\to [0,+\infty[$ be such that $s \to \dfrac{\omega(s)}{s}$ is non-decreasing and consider the Young function defined as

$$\Phi(s) = \int_0^s t^{-1}\omega(t)dt \quad \text{for } s > 0.$$

From (8) and since $s^{-1}\omega(s)$ is non-decreasing, we have

$$\Phi(s) \leq \omega(s) \leq \Phi(2s) \quad \text{for } s > 0. \tag{10}$$

It follows that Φ and ω are globally equivalent.

Given two finite-valued Young functions $\Phi, \Psi : [0, +\infty[\to [0, +\infty[$, we say that Ψ increases essentially more slowly than Φ near infinity if

$$\frac{\Psi(\chi s)}{\Phi(s)} \to 0 \quad \text{as } s \to +\infty, \text{ for all positive } \chi.$$

We recall the Luxemburg norm related to a Young function Φ.

$$\|x\|_{L^\Phi(N)} = \inf\left\{\chi > 0 : \int_N \Phi\left(\frac{|x|}{\chi}\right) d\mathcal{H}^n \leq 1 \text{ with } x : N \to \mathbb{R} \text{ measurable}\right\}.$$

Also, we have

$$\begin{aligned}
L^\Phi(N) &= \{x : x \text{ is measurable and } \|x\|_{L^\Phi(N)} < +\infty\}, \\
\mathbb{W}^{1,p}(N) &= \{x : x \text{ is weakly differentiable on } N \text{ and } |\nabla x| \in L^p(N)\}, \\
W^{1,p}(N) &= \mathbb{W}^{1,p}(N) \cap L^p(N) \quad \text{(the standard Sobolev space)}, \\
\|x\|_{W^{1,p}(N)} &= \|\nabla x\|_{L^p(N)} + \|x\|_{L^p(N)} \text{ (here } \|\nabla x\|_{L^p(N)}\| = |\nabla x|\|_{L^p(N)}), \qquad (11) \\
W_0^{1,p}(N) &= \text{ the closure in } W^{1,p}(N) \text{ of the set of smooth} \\
&\qquad \text{compactly supported functions on } N, \\
W_\perp^{1,p}(N) &= \left\{x \in W^{1,p}(N) : x_m = \frac{1}{\mathcal{H}^n(N)}\int_N x d\mathcal{H}^n = 0\right\}, \\
\|x\| &= \|\nabla x\|_{L^p(N)} + |x_m|, \\
W^{1,p}(N) &= \mathbb{R} \oplus W_\perp^{1,p}(N).
\end{aligned}$$

For all $x \in L^\Phi(N)$ and $y \in L^{\Phi^*}(N)$, we have

$$\int_N |xy| d\mathcal{H}^n \leq 2\|x\|_{L^\Phi(N)}\|y\|_{L^{\Phi^*}(N)} \quad \text{(Hölder's inequality)}. \tag{12}$$

If Φ dominates Ψ globally, it follows that

$$\|x\|_{L^\Psi(N)} \leq k\|x\|_{L^\Phi(N)} \quad \text{for all } x \in L^\Phi(N),\ k > 0 \text{ as in (9)}. \tag{13}$$

Moreover, if Φ dominates Ψ near infinity, then (13) remains true with $k = k(\Phi, \Psi, \mathcal{H}^n(N))$ (see [4, 13, 14]).

Finally, we put $\operatorname{supp}(x) = \{z \in N : x(z) \neq 0\}$ and

$$\operatorname{med}(x) = \inf\big\{s \in \mathbb{R} : 2\mathcal{H}^n(\{z \in N : x(z) > s\}) \leq \mathcal{H}^n(N)\big\},$$

that is, $\operatorname{med}(x)$ is the median of a measurable function $x : N \to \mathbb{R}$.

Here, a key tool is the following theorem of Barletta-Cianchi-Maz'ya [3, Theorem 2.1].

THEOREM 2.4 *Let $\Phi : [0, +\infty[\to [0, +\infty[$ be such that $\Phi(s^{1/p})$, $1 \leq p < +\infty$, is a Young function (Φ is a Young function, too). We put*

$$\mu(\alpha) = \sup_{t \in]0,\alpha[} \frac{1}{\eta_{N,p}(t)\Phi^{-1}(1/t)^p}, \quad \alpha \in \big]0, \tfrac{\mathcal{H}^n(N)}{2}\big].$$

The following assertions are equivalent:

(i) There is $\alpha \in \big]0, \frac{\mathcal{H}^n(N)}{2}\big]$ such that

$$\mu(\alpha) < +\infty. \tag{14}$$

(ii) $\mu(\alpha) < +\infty$ for all $\alpha \in \big]0, \frac{\mathcal{H}^n(N)}{2}\big]$.

(iii) For all $x \in \mathbb{W}^{1,p}(N)$ with $\mathcal{H}^n(\operatorname{supp}(x)) \leq \alpha$, there is a_0 satisfying

$$\|x\|_{L^\Phi(N)} \leq a_0 \|\nabla x\|_{L^p(N)}. \tag{15}$$

(iv) For all $x \in \mathbb{W}^{1,p}(N)$ with $\mathcal{H}^n(\operatorname{supp}(x)) \leq \alpha$, there is a_0 satisfying

$$\int_N \Phi\Big(\frac{|x|}{a_0\|\nabla x\|_{L^p(N)}}\Big) d\mathcal{H}^n \leq 1. \tag{16}$$

(v) For all $x \in \mathbb{W}^{1,p}(N)$, there is a_0 satisfying

$$\|x - \operatorname{med}(x)\|_{L^\Phi(N)} \leq a_0 \|\nabla x\|_{L^p(N)}. \tag{17}$$

(vi) We have

$$\mathbb{W}^{1,p}(N) \hookrightarrow L^\Phi(N). \tag{18}$$

Also, in (15)-(17), *we can choose*

$$a_0 = k\mu(\alpha)^{1/p} \quad \textit{for some constant } k = k(p). \tag{19}$$

Indeed, the results in Barletta-Cianchi-Maz'ya [3] are proved for the following class of manifolds:

$$\begin{aligned}\mathcal{D}_p(\eta) = \{N : {}& \eta_{N,p}(s) \geq \eta(s) \text{ with } s \text{ near at zero},\\ & \eta :]0, +\infty[\to [0, +\infty[\text{ quasi-concave}\}.\end{aligned}$$

If $N \in \mathcal{D}_p(\eta)$, we have

$$\inf_{s \in]0, \mathcal{H}^n(N)/2[} \frac{\eta_{N,p}(s)}{s} > 0.$$

Remark 2.5 If $\lim_{s \to 0^+} \eta_{N,p}(s) > 0$, then (14) is true for each $\Phi : [0, +\infty[\to [0, +\infty]$, and so also for $L^\Phi(N) = L^\infty(N)$. Indeed inequality (15) becomes

$$\|x\|_{L^\infty(N)} \leq a_0 \|\nabla x\|_{L^p(N)} \tag{20}$$

for all $x \in \mathbb{W}^{1,p}(N)$ such that $\alpha \geq \mathcal{H}^n(\mathrm{supp}(x))$, where a_0 is as in (19).

We recall some consequences of Theorem 2.4.

COROLLARY 2.6 *Let $\Phi : [0, +\infty[\to [0, +\infty[$ be such that $\Phi(s^{1/p})$, $1 \leq p < +\infty$, is a Young function. If* (14) *holds for some $\alpha \in]0, \frac{\mathcal{H}^n(N)}{2}]$ (and so for all $\alpha \in]0, \frac{\mathcal{H}^n(N)}{2}]$), we have $W^{1,p}(N) = \mathbb{W}^{1,p}(N)$, up to equivalent norms. Also, there is $k = k(p)$ such that*

$$\|x\|_{L^\Phi(N)} \leq k\mu\left(\frac{\mathcal{H}^n(N)}{2}\right) \|\nabla x\|_{L^p(N)} \quad \textit{for all } x \in W^{1,p}_\perp(N) \tag{21}$$

and

$$\int_N \Phi\left(\frac{|x|}{k\mu\left(\frac{\mathcal{H}^n(N)}{2}\right)\|\nabla x\|_{L^p(N)}}\right) d\mathcal{H}^n \leq 1 \quad \textit{for all } x \in W^{1,p}_\perp(N). \tag{22}$$

Proof. Note that $\dfrac{\Phi(s^{1/p})}{s}$ is non-decreasing, and so $\Phi(s)$ dominates s^p near infinity. Thus $L^\Phi(N) \hookrightarrow L^p(N)$ and $\mathbb{W}^{1,p}(N) \hookrightarrow L^p(N)$ (by (18)). So $\mathbb{W}^{1,p}(N) \hookrightarrow W^{1,p}(N)$. This leads to $W^{1,p}(N) = \mathbb{W}^{1,p}(N)$ since $W^{1,p}(N) \hookrightarrow \mathbb{W}^{1,p}(N)$ holds trivially. Moreover (21) follows from (17), since $\|x - x_m\|_{L^\Phi(N)} \leq 2\|x - \mathrm{med}(x)\|_{L^\Phi(N)}$. Finally, we mention that (22) and (21) are equivalent (by the Luxemburg norm). ■

The following result concerns the compactness of Sobolev embeddings.

COROLLARY 2.7 *Let Φ and p as in Theorem 2.4. If* (14) *holds and Ψ is a Young function increasing essentially more slowly than Φ near infinity, then*

$$W^{1,p}(N) \hookrightarrow L^\Psi(N) \tag{23}$$

is compact.

Proof. Since Ψ increases essentially more slowly than Φ near infinity, appealing to a property of Orlicz-Sobolev embeddings (see Theorem 3.4 of Hajlasz-Liu [7]) we have the compactness of (23). Recall that (18) says us $W^{1,p}(N) \hookrightarrow L^\Phi(N)$. ■

Denote by $g : \mathbb{R} \to \mathbb{R}$ a continuous function and set $G(s) = \int_0^s g(t)dt$, $s \in \mathbb{R}$. Also consider $\widehat{g} : \mathbb{R} \to [0, +\infty[$ given by

$$\widehat{g}(s) = \max_{t \in [-|s|, |s|]} |g(t)| \quad \text{for } s \in \mathbb{R},$$

and $\widehat{G} : [0, +\infty[\to [0, +\infty[$ defined as

$$\widehat{G}(s) = \int_0^s \widehat{g}(t)dt \quad \text{for } s \in [0, +\infty[. \tag{24}$$

Clearly $\widehat{g}$ is even in $\mathbb{R}$ and non-decreasing in $[0, +\infty[$, so that $\widehat{G}$ is a Young function.

We mention the following basic facts.

LEMMA 2.8 *[3, Lemma 3.5] Let $f :]0, t_0[\to]0, +\infty[$ be a quasi-concave function, with $t_0 > 0$, and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying:*

(i) $sf^{-1}(1/s) \to 0$ as $s \to +\infty$;
(ii) $sf^{-1}(1/s)g(qs) \to 0$ as $s \to +\infty$, for all $q \in \mathbb{R}$.

For all $q \in \mathbb{R}$, we have

$$\lim_{s\to+\infty} sf^{-1}(1/s)\widehat{g}(qs) = \lim_{s\to+\infty} f^{-1}(1/s)G(qs) = \lim_{s\to+\infty} f^{-1}(1/s)\widehat{G}(|q|s) = 0. \tag{25}$$

Definition 2.9 Let (X, X^*) be a Banach topological pair. We say that $J : X \to \mathbb{R}$ has the Palais-Smale property if any sequence $\{x_i\}$ satisfying:

(i) $\{J(x_i)\}$ is bounded;
(ii) $\|J'(x_i)\|_{X^*} \to 0$ as $i \to +\infty$,

admits a convergent subsequence.

3. Existence of a nontrivial solution

To obtain our result, we need a mountain pass geometry. So, we recall an useful version of the mountain pass theorem (see Theorem 5.3 of Rabinowitz [11]). Let $B(0, \rho) = \{x \in W^{1,p}(N) : \|x\| < \rho\}$, $\rho > 0$. Fixed $\widehat{x} \in W^{1,p}(N)$ and $\rho > 0$, we set

$$[0, \rho\widehat{x}] = \{x \in W^{1,p}(N) : x = \chi\widehat{x} \quad \text{for some } \chi \in [0, \rho]\}.$$

We mention that by ∂Q_ρ we denote the boundary of $Q_\rho = [-\rho, \rho] \oplus [0, \rho\widehat{x}]$.

THEOREM 3.1 *Consider $W^{1,p}(N)$ and $W^{1,p}_\perp(N)$ given as above (recall $W^{1,p}(N) = \mathbb{R} \oplus W^{1,p}_\perp(N)$). Let $J : W^{1,p}(N) \to \mathbb{R}$ be a C^1-functional with the Palais-Smale property such that:*
(i) there are $r, \gamma > 0$ satisfying $J_{|\partial B(0,r)\cap W^{1,p}_\perp(N)} \geq \gamma$;
(ii) there are $\widehat{x} \in \partial B(0, 1) \cap W^{1,p}_\perp(N)$ and $\rho > r$ satisfying $J_{|\partial Q_\rho} \leq 0$.
Thus J admits a critical point $\widetilde{x}$ such that $J(\widetilde{x}) = c \geq \gamma$, with $c = \inf_{\beta\in\Theta} \sup_{x\in Q_\rho} J(\beta(x))$, where $\Theta = \{\beta \in C^0(Q_\rho, W^{1,p}(N)) : \beta_{|\partial Q_\rho} = id_{|\partial Q_\rho}\}$.

Here, we consider the norm $\|x\| = \|\nabla x\|_{L^p(N)} + |x_m|$ for all $x \in W^{1,p}(N)$ (by Corollary 2.5 of [3], this norm is equivalent to the standard one $\|\cdot\|_{W^{1,p}(N)}$, see (11)), and define $\mathcal{J} : W^{1,p}(N) \to \mathbb{R}$ by

$$\mathcal{J}(x) = \frac{1}{p}\int_N |\nabla x|^p d\mathcal{H}^n + \frac{1}{p}\int_N \left[\sqrt{1 + |\nabla x|^{2p}} - 1\right] d\mathcal{H}^n - \int_N G(x)d\mathcal{H}^n$$

for all $x \in W^{1,p}(N)$. It is well-known that the solutions of (1) are the critical points of $\mathcal{J}$.

We collect some useful results derived from [3].

PROPOSITION 3.2 *[3, Proposition 3.2] Let* $\Phi : [0,+\infty[\to [0,+\infty[$ *be such that* $\Phi(s^{\frac{1}{p}})$, $1 \le p < +\infty$, *is a Young function. Let* η *be a quasi-concave function with* $N \in \mathcal{D}_p(\eta)$, *and* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* (g_1), (g_2). *If* $\lim_{s\to 0^+} \eta(s) = 0$, *we suppose that*

$$\sup_{t\in]0,\alpha[} \frac{1}{\eta(t)\Phi^{-1}(\frac{1}{t})^p} < +\infty \text{ for some } \alpha \in \left]0, \frac{\mathcal{H}^n(N)}{2}\right[, \tag{26}$$

$$g \text{ satisfies } (g_3) \text{ and } \lim_{s\to+\infty} \eta^{-1}(s^{-p})sg(qs) = 0 \text{ for all } q \in \mathbb{R},$$
$$G(s) \le \Phi(\gamma|s|) \text{ if } |s| \ge s_1, \text{ for some } \gamma > 0 \text{ and } s_1 > 0. \tag{27}$$

Then

$$\lim_{x\in W_\perp^{1,p}(N), \|\nabla x\|_{L^p(N)}\to 0} \frac{\int_N G(x)d\mathcal{H}^n}{\|\nabla x\|^p_{L^p(N)}} = 0. \tag{28}$$

We can find $\delta > 0$ *so that*

$$\mathcal{J}_{|\partial B(0,r)\cap W_\perp^{1,p}(N)} > 0 \quad \text{if } \delta > r. \tag{29}$$

Proof. We show that (28) implies (29). By (28), we can find $\delta > 0$ such that

$$\int_N G(x)d\mathcal{H}^n < \frac{1}{p}\|\nabla x\|^p_{L^p(N)} \quad \text{for all } x \in W_\perp^{1,p}(N) \text{ satisfying } \|\nabla x\|^p_{L^p(N)} < \delta.$$

From

$$\begin{aligned}
\mathcal{J}(x) &= \frac{1}{p}\int_N |\nabla x|^p d\mathcal{H}^n + \frac{1}{p}\int_N \left[\sqrt{1+|\nabla x|^{2p}} - 1\right] d\mathcal{H}^n - \int_N G(x)d\mathcal{H}^n \\
&\ge \frac{1}{p}\|\nabla x\|^p_{L^p(N)} - \int_N G(x)d\mathcal{H}^n \\
&> \frac{1}{p}\|\nabla x\|^p_{L^p(N)} - \frac{1}{p}\|\nabla x\|^p_{L^p(N)} = 0
\end{aligned}$$

for all $x \in \partial B(0,r) \cap W_\perp^{1,p}(N)$ with $\delta > r$, we deduce (29).

Now, we have to prove (28). We distinguish the cases $\lim_{s\to 0^+} \eta(s) = 0$ and $\lim_{s\to 0^+} \eta(s) > 0$. In the first case, as G is continuous, and Φ is increasing and goes to infinity (at infinity), by (27) for all $s_0 > 0$, there is $\gamma_0 > 0$ such that

$$G(s) \le \Phi(\gamma_0|s|) \quad \text{if } |s| \ge s_0. \tag{30}$$

Now, given $\varepsilon > 0$, one can find s_0 so that (2) is true. For $x \in W_\perp^{1,p}(N)$, put $r = \|\nabla x\|^p_{L^p(N)}$. From (26), we have

$$\inf_{t\in]0,\mathcal{H}^n(N)/2[} \frac{\eta_{N,p}(t)}{t} > 0 \quad \text{for all } N \in \mathcal{D}_p(\eta).$$

By Corollary 2.5 of [3], one can find $a_0 = a_0(p, N)$ so that

$$\|x\|_{L^p(N)} \leq a_0 \|\nabla x\|_{L^p(N)} \quad \text{for all } x \in W^{1,p}_\perp(N). \tag{31}$$

Thus, by inequalities (2) and (31), we get

$$\frac{\int_{\{|x|\leq s_0\}} G(x) d\mathcal{H}^n}{r} \leq \varepsilon \frac{\int_N |x|^p d\mathcal{H}^n}{r} \leq \varepsilon a_0^p. \tag{32}$$

Put $\sigma = s_0$ in (6), and choose r so that $b = \gamma_0 k \mu(\mathcal{H}^n(N)/2) r^{1/p} < 1$ (k as in (21)). From (6), (30) and by the equivalence of (21) and (22), we have

$$\begin{aligned} &\frac{\int_{\{|x|>s_0\}} G(x) d\mathcal{H}^n}{r} \\ &\leq a b^\kappa \frac{\int_{\{|x|>s_0\}} G\big(\frac{x}{b}\big) d\mathcal{H}^n}{r} \\ &= a \gamma_0^\kappa k^\kappa \mu(\mathcal{H}^n(N)/2)^\kappa r^{(\kappa-p)/p} \int_{\{|x|>s_0\}} G\Big(\frac{x}{b}\Big) d\mathcal{H}^n \\ &\leq a \gamma_0^\kappa k^\kappa \mu(\mathcal{H}^n(N)/2)^\kappa r^{(\kappa-p)/p} \int_{\{|x|>s_0\}} \Phi\left(\frac{|x|}{k\mu(\mathcal{H}^n(N)/2) r^{1/p}}\right) d\mathcal{H}^n \\ &\leq a \gamma_0^\kappa k^\kappa \mu(\mathcal{H}^n(M)/2)^\kappa r^{(\kappa-p)/p}, \quad \text{where } \kappa \in]p, 2p]. \end{aligned} \tag{33}$$

Since ε is arbitrary, (32) and (33) imply (28).

We consider the second case, that is, $\lim_{s\to 0^+} \eta(s) > 0$. For s_0 as in (2) and a_0 as in (20), if $\frac{s_0}{a_0} > r^{1/p}$ we have

$$\begin{aligned} \frac{\int_N G(x) d\mathcal{H}^n}{r} &= \frac{\int_{\{|x|\leq s_0\}} G(x) d\mathcal{H}^n}{r} \leq \varepsilon \frac{\int_N |x|^p d\mathcal{H}^n}{r} \\ &\leq \varepsilon \|x\|^p_{L^\infty(N)} \frac{\int_N d\mathcal{H}^n}{r} \leq \varepsilon a_0^p \mathcal{H}^n(N) \quad (\text{fixed } \varepsilon > 0). \end{aligned}$$

This implies that (28) holds true (again ε is arbitrary). ∎

PROPOSITION 3.3 *If $g : \mathbb{R} \to \mathbb{R}$ is continuous and (g_2), (g_3) hold, then we can find $\widehat{x} \in \partial B(0,1) \cap W^{1,p}_\perp(N)$ and $\rho > 0$ such that*

$$\mathcal{J}_{|\partial Q_\rho} \leq 0. \tag{34}$$

Proof. To simplify the notation, let us define $N_1 = \{z \in N : \widehat{x}(z) \geq 1\}$, $N_2 = \{z \in N : \widehat{x}(z) \leq -1\}$, $N_+ = \{z \in N : \widehat{x}(z) \geq 0\}$ and $N_- = \{z \in N : \widehat{x}(z) \leq 0\}$ with $\widehat{x} \in \partial B(0,1) \cap W^{1,p}_\perp(N)$ such that

$$\begin{aligned} &\mathcal{H}^n(N_1) > 0 \text{ and } \mathcal{H}^n(N_2) > 0, \\ \Rightarrow \quad &\mathcal{H}^n(N_+) \geq \mathcal{H}^n(N_1) > 0 \text{ and } \mathcal{H}^n(N_-) \geq \mathcal{H}^n(N_2) > 0. \end{aligned}$$

Given $s_0 > 0$ such that (5) holds, we put

$$\rho > \max\left\{\left(\frac{2s_0^\kappa}{p\mathcal{H}^n(N_1)G(s_0)}\right)^{\frac{1}{\kappa-p}}, \left(\frac{2s_0^\kappa}{p\mathcal{H}^n(N_2)G(-s_0)}\right)^{\frac{1}{\kappa-p}}, s_0\right\}, \quad \kappa \in]p, 2p].$$

We have

$$\int_N |\nabla(\mu\widehat{x})|^p d\mathcal{H}^n = \|\mu\widehat{x}\|_{1,p}^p = \mu^p \quad \text{for } \mu > 0 \text{ (as } \widehat{x} \in \partial B(0,1) \cap W_\perp^{1,p}(N)).$$

If $\mu \in]0, \rho]$, by (g_2) and (5), we get

$$\begin{aligned}
&\mathcal{J}(\rho + \mu\widehat{x}) \\
&= \frac{1}{p}\int_N |\nabla(\rho + \mu\widehat{x})|^p d\mathcal{H}^n + \frac{1}{p}\int_N \left[\sqrt{1 + |\nabla(\rho + \mu\widehat{x})|^{2p}} - 1\right] d\mathcal{H}^n - \int_N G(\rho + \mu\widehat{x}) d\mathcal{H}^n \\
&= \frac{1}{p}\int_N |\nabla(\mu\widehat{x})|^p d\mathcal{H}^n + \frac{1}{p}\int_N [\sqrt{1 + |\nabla(\mu\widehat{x})|^{2p}} - 1] d\mathcal{H}^n - \int_N G(\rho + \mu\widehat{x}) d\mathcal{H}^n \\
&\le \frac{2}{p}\mu^p - \int_N G(\rho + \mu\widehat{x}) d\mathcal{H}^n \\
&\le \frac{2}{p}\rho^p - \int_{N_+} G(\rho + \mu\widehat{x}) d\mathcal{H}^n \\
&\le \frac{2}{p}\rho^p - \frac{G(s_0)\rho^\kappa \mathcal{H}^n(N_+)}{s_0^\kappa} \\
&\le \rho^p\left(\frac{2}{p} - \frac{G(s_0)\mathcal{H}^n(N_+)}{s_0^\kappa}\rho^{\kappa-p}\right) \\
&\le \rho^p\left(\frac{2}{p} - \frac{G(s_0)\mathcal{H}^n(N_+)}{s_0^\kappa}\frac{2s_0^\kappa}{p\mathcal{H}^n(N_+)G(s_0)}\right) = 0.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\mathcal{J}(-\rho + \mu\widehat{x}) &\le \frac{2}{p}\rho^p - \frac{G(-s_0)\rho^\kappa \mathcal{H}^n(N_-)}{s_0^\kappa} \\
&\le \rho^p\left(\frac{2}{p} - \frac{G(-s_0)\mathcal{H}^n(N_-)}{s_0^\kappa}\frac{2s_0^\kappa}{p\mathcal{H}^n(N_-)G(-s_0)}\right) = 0.
\end{aligned}$$

Now, we consider $\mu \in [0, \rho]$. So, we get

$$\begin{aligned}
\mathcal{J}(\mu + \rho\widehat{x}) &\le \frac{2}{p}\rho^p - \int_{N_1} G(\mu + \rho\widehat{x}) d\mathcal{H}^n \\
&\le \frac{2}{p}\rho^p - \frac{G(s_0)\rho^\kappa \mathcal{H}^n(N_1)}{s_0^\kappa} \\
&\le \rho^p\left(\frac{2}{p} - \frac{G(s_0)\mathcal{H}^n(N_1)}{s_0^\kappa}\frac{2s_0^\kappa}{p\mathcal{H}^n(N_1)G(s_0)}\right) = 0.
\end{aligned}$$

Next, if $\mu \in [-\rho, 0]$, we have

$$\begin{aligned}\mathcal{J}(\mu + \rho\widehat{x}) &\le \frac{2}{p}\rho^p - \int_{N_2} G(\mu + \rho\widehat{x})d\mathcal{H}^n \\ &\le \frac{2}{p}\rho^p - \frac{G(-s_0)\rho^\kappa \mathcal{H}^n(N_2)}{s_0^\kappa} \\ &\le \rho^p\left(\frac{2}{p} - \frac{G(-s_0)\mathcal{H}^n(N_2)}{s_0^\kappa}\frac{2s_0^\kappa}{p\mathcal{H}^n(N_2)G(-s_0)}\right) = 0,\end{aligned}$$

Thus, if $\mu \in [-\rho, \rho]$, we get

$$\mathcal{J}(\mu) = -\int_N G(\mu)d\mathcal{H}^n \le 0.$$

So, (34) is an immediate consequence of the above inequalities. ■

LEMMA 3.4 *[2, Lemma 3.5] Let Φ, Ψ be Young functions with Ψ increasing essentially more slowly than Φ near infinity. If the function $g : \mathbb{R} \to \mathbb{R}$ is continuous such that $|g(s)| \le k(1 + (\Psi^*)^{-1}(\Psi(k|s|)))$ for $s \in \mathbb{R}$, we have:*

(i) if $x \in L^\Psi(N)$ and the sequence $\{x_i\}$ is bounded in $L^\Phi(N)$ with $x_i \to x$ in $L^\Psi(N)$, then

$$\int_N g(x_i)(x_i - x)d\mathcal{H}^n \to 0 \quad \textit{as } i \to +\infty;$$

(ii) if $x \in L^\Phi(N)$ and $\{x_i\}$ is a bounded sequence in $L^\Phi(N)$ such that $x_i \to x$ in $L^\Phi(N)$, then

$$\|g(x_i) - g(x)\|_{L^{\Phi^*}(N)} \to 0 \quad \textit{as } i \to +\infty.$$

The following lemma is useful to show that the Gâteaux derivative of $\int_N G(x)d\mathcal{H}^n$ is continuous.

LEMMA 3.5 *[3, Proposition 3.7] Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, and Φ be a Young function such that*

$$W^{1,p}(N) \hookrightarrow L^\Phi(N), \quad 1 < p < +\infty. \tag{35}$$

Suppose that either Φ is finite-valued and there is a Young function Ψ, increasing essentially more slowly than Φ near infinity, such that

$$|g(s)| \le k(1 + (\Psi^*)^{-1}(\Psi(k|s|))) \quad \textit{for } s \in \mathbb{R}, \textit{ some } k > 0, \tag{36}$$

or Φ is infinite for large values of its argument (and hence $L^\Phi(N) = L^\infty(N)$, up to equivalent norms). Let $x \in W^{1,p}(N)$, then

$$\sup_{y \in W^{1,p}(N)\setminus\{0\}} \frac{\left|\int_N (g(x_i) - g(x))yd\mathcal{H}^n\right|}{\|y\|_{1,p}} \to 0 \quad \textit{as } i \to +\infty,$$

for any sequence $\{x_i\} \subset W^{1,p}(N)$ with $x_i \to x$ in $W^{1,p}(N)$.

Now, we prove the following result.

Proposition 3.6 *$\mathcal{J}$ is a C^1-functional, provided that Φ, Ψ and g satisfy the hypotheses of Lemma 3.5.*

Proof. We note that $\int_N |\nabla x|^p d\mathcal{H}^n + \int_N [\sqrt{1+|\nabla x|^{2p}} - 1] d\mathcal{H}^n$ is a C^1-functional. So, we need to show that $\int_N G(x) d\mathcal{H}^n$ is a C^1-functional too.

Given $0 < \varepsilon < 1$, for $x, y \in W^{1,p}(N)$ we have

$$\frac{1}{\varepsilon}\left[\int_N G(x+\varepsilon y) d\mathcal{H}^n - \int_N G(x) d\mathcal{H}^n\right] = \int_N \frac{G(x+\varepsilon y) - G(x)}{\varepsilon} d\mathcal{H}^n.$$

Since g is continuous, we get

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[G(x(z)+\varepsilon y(z)) - G(x(z))] = g(x(z))y(z) \quad \text{for a.e. } z \in N.$$

Also, we can find $0 < \vartheta_z < 1$ satisfying

$$\begin{aligned}
&\frac{1}{\varepsilon}[G(x(z)+\varepsilon y(z)) - G(x(z))] = g(x(z)+\varepsilon\vartheta_z y(z))y(z) \quad \text{for a.e. } z \in N,\\
\Rightarrow\ &\frac{1}{\varepsilon}[G(x(z)+\varepsilon y(z)) - G(x(z))] \le k[1 + (\Psi^*)^{-1}(\Psi(k(|x(z)+\varepsilon\vartheta_z y(z)|)))]y(z)\\
&\le k[1 + (\Psi^*)^{-1}(\Psi(k(|x(z)| + |y(z)|)))]|y(z)| \quad \text{for a.e. } z \in N,\\
&\qquad\qquad\qquad\qquad \text{(if } \Phi \text{ is finite-valued and (36) is true).}
\end{aligned}$$

The right-hand side of the last inequality belongs to $L^1(N)$ as

$$\begin{aligned}
&\int_N (\Psi^*)^{-1}(\Psi(k(|x|+|y|)))|y| d\mathcal{H}^n\\
&\le 2\|(\Psi^*)^{-1}(\Psi(k(|x|+|y|)))\|_{L^{\Psi^*}(N)}\|y\|_{L^{\Psi}(N)} \text{ (by (12))}\\
&< +\infty \quad \text{(by (35) and as } \Psi \text{ grows essentially more slowly than } \Phi \text{ near infinity).}
\end{aligned}$$

The above facts and Lemma 3.5 imply that the Gâteaux derivative of $\int_N G(x) d\mathcal{H}^n$ is continuous. Indeed

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\left[\int_N G(x+\varepsilon y) d\mathcal{H}^n - \int_N G(x) d\mathcal{H}^n\right] = \int_N g(x) y d\mathcal{H}^n,$$

for all $x, y \in W^{1,p}(N)$, by the dominated convergence theorem. The continuity of $\int_N g(x) y d\mathcal{H}^n$ is a consequence of Lemma 3.5. ■

Proposition 3.7 (Palais-Smale condition) *Let $\Phi : [0,+\infty[\to [0,+\infty[$ be such that $\Phi(s^{\frac{1}{p}})$, $1 < p < +\infty$, is a Young function. Let η be a quasi-concave function with $N \in \mathcal{D}_p(\eta)$ such that (26) holds, and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (g_2), (g_3). If either Φ is finite-valued and Ψ is a Young function increasing essentially more slowly than Φ near infinity for which (36) holds, or Φ is infinite for large values of its argument, then $\mathcal{J}$ has the Palais-Smale property.*

Proof. In view of the definition of Palais-Smale property, we consider a sequence $\{x_i\} \subset W^{1,p}(N)$. So, we may assume that $\mathcal{J}(x_i) \to k$ as $i \to +\infty$, for some $k \in \mathbb{R}$ (by passing to a subsequence, if necessary). Fixed $\varepsilon > 0$, we can find a natural number i_0 satisfying

$$k + \varepsilon > \mathcal{J}(x_i) > k - \varepsilon \quad \text{for } i > i_0. \tag{37}$$

From $\|\mathcal{J}'(x_i)\|_{W^{1,p}(N)^*} \to 0$ as $i \to +\infty$, it follows that we can find $\{\varepsilon_i\}$, with $\varepsilon_i \to 0^+$, such that

$$\begin{aligned} -\varepsilon_i \|y\|_{1,p} \leq & \int_N |\nabla x_i|^{p-2} \nabla x_i \nabla y d\mathcal{H}^n + \int_N \frac{|\nabla x_i|^{2p-2} \nabla x_i \nabla y}{\sqrt{1 + |\nabla x_i|^{2p}}} d\mathcal{H}^n - \int_N g(x_i) y d\mathcal{H}^n \\ \leq & \varepsilon_i \|y\|_{1,p} \quad \text{for all } y \in W^{1,p}(N). \end{aligned} \tag{38}$$

Now, (g_3) implies that there is $s_0 > 0$ with

$$sg(s) - 2pG(s) > 0 \quad \text{if } |s| > s_0. \tag{39}$$

By (37), we have

$$\begin{aligned} k + \varepsilon > & \mathcal{J}(x_i) \\ = & \frac{1}{p} \int_N |\nabla x_i|^p d\mathcal{H}^n + \frac{1}{p} \int_N \left[\sqrt{1 + |\nabla x_i|^{2p}} - 1 \right] d\mathcal{H}^n - \int_N G(x_i) d\mathcal{H}^n, \end{aligned} \tag{40}$$

for $i > i_0$. By (40) and the first inequality of (38) (with $y = x_i$), we obtain

$$\begin{aligned} & \frac{1}{2p} \|\nabla x_i\|^p_{L^p(N)} + \frac{1}{p} \int_N \left[\sqrt{1 + |\nabla x_i|^{2p}} - 1 - \frac{1}{2} \frac{|\nabla x_i|^{2p}}{\sqrt{1 + |\nabla x_i|^{2p}}} \right] d\mathcal{H}^n \\ & \qquad - \frac{1}{2p} \int_N [2pG(x_i) - g(x_i) x_i] d\mathcal{H}^n \leq (k + \varepsilon) + \frac{\varepsilon_i}{2p} \|x_i\|_{1,p} \end{aligned} \tag{41}$$

if $i > i_0$. Since

$$\int_N \left[\sqrt{1 + |\nabla x_i|^{2p}} - 1 - \frac{1}{2} \frac{|\nabla x_i|^{2p}}{\sqrt{1 + |\nabla x_i|^{2p}}} \right] d\mathcal{H}^n \geq 0,$$

from (39) and (41) we have

$$\begin{aligned} & \frac{1}{2p} \|\nabla x_i\|^p_{L^p(N)} \\ \leq & (k + \varepsilon) + \frac{\varepsilon_i}{2p} \|\nabla x_i\|_{L^p(N)} + \frac{\varepsilon_i}{2p} |(x_i)_m| + \frac{1}{2p} \int_{|x_i| \leq t_0} [2pG(x_i) - g(x_i) x_i] d\mathcal{H}^n \end{aligned}$$

for $i > i_0$. By the continuity of g, we can find a_0 satisfying

$$\frac{1}{2p} \|\nabla x_i\|^p_{L^p(N)} \leq a_0 + \frac{\varepsilon_i}{2p} \|\nabla x_i\|_{L^p(N)} + \frac{\varepsilon_i}{2p} |(x_i)_m| \quad \text{for } i > i_0. \tag{42}$$

We have to prove that $\{x_i\}$ is bounded in $W^{1,p}(N)$. On the contrary, we assume that $\{x_i\}$ is unbounded. So, if $\{\|\nabla x_i\|_{L^p(N)}\}$ is unbounded, we can find a natural number $i_1 \geq i_0$ such that

$$\|\nabla x_i\|^p_{L^p(N)} < |(x_i)_m| \quad \text{if } i > i_1 \text{ (by (42))}. \tag{43}$$

By Remark 1.2 we can find $a_1 > 0$ and $a_2 > 0$ such that (4) holds. So

$$\mathcal{J}(x_i) \leq \frac{2}{p}\|\nabla x_i\|^p - \int_N G(x_i)\mathcal{H}^n < \frac{2}{p}|(x_i)_m| + a_1 - a_2\|x_i\|^p_{L^p(N)} \quad \text{(by (43))}$$

whenever $i > i_1$. We conclude that $\mathcal{J}(x_i) \to -\infty$, as $i \to +\infty$, which leads to contradiction with $\mathcal{J}(x_i) \to k \in \mathbb{R}$. So, $\{\|\nabla x_i\|_{L^p(N)}\}$ is bounded and hence $\{|(x_i)_m|\}$ is unbounded. It follows that $\{\|x_i\|_{L^p(N)}\}$ is unbounded. Next, by Remark 1.2, we get

$$\mathcal{J}(x_i) \leq \frac{2}{p}\|\nabla x_i\|^p_{L^p(N)} + a_1 - a_2\|x_i\|^p_{L^p(N)} \leq a_3 - a_2\|x_i\|^p_{L^p(N)} \quad \text{for some } a_3 > 0,$$

and again we have the absurd $\mathcal{J}(x_i) \to -\infty$ as $i \to +\infty$.

We proved that $\{x_i\}$ is bounded in $W^{1,p}(N)$. By Theorem 2.4, we have $W^{1,p}(N) \hookrightarrow L^\Phi(N)$ (see (18)), and hence $\{x_i\}$ is bounded in $L^\Phi(N)$. By Corollary 2.7, we have that $W^{1,p}(N) \hookrightarrow L^\Psi(N)$ is compact. So, we can find a subsequence, say again $\{x_i\}$ (without any loss of generality), which is convergent to x in $L^\Psi(N)$ (and hence also in $W^{1,p}(N)$). We have

$$(x_i)_m \to x_m \quad \text{as } i \to +\infty. \tag{44}$$

If Φ is finite-valued and (36) is true then, by (38) with $y = x - x_i$, Lemma 3.4(i) and the boundedness of $\{x_i\}$ in $W^{1,p}(N)$, we have

$$\lim_{i\to+\infty}\left(\int_N |\nabla x_i|^{p-2}\nabla x_i \nabla(x - x_i)d\mathcal{H}^n + \int_N \frac{|\nabla x_i|^{2p-2}\nabla x_i\nabla(x-x_i)}{\sqrt{1+|\nabla x_i|^{2p}}}d\mathcal{H}^n\right) = 0.$$

Passing to a subsequence if necessary, we can assume

$$\limsup_{i\to+\infty}\int_N |\nabla x_i|^{p-2}\nabla x_i\nabla(x-x_i)d\mathcal{H}^n \leq 0 \tag{45}$$

or

$$\limsup_{i\to+\infty}\int_N \frac{|\nabla x_i|^{2p-2}\nabla x_i\nabla(x-x_i)}{\sqrt{1+|\nabla x_i|^{2p}}}d\mathcal{H}^n \leq 0. \tag{46}$$

If (45) holds, then the convexity of $\|\cdot\|$ gives us

$$\limsup_{i\to+\infty}\|\nabla x_i\| \leq \|\nabla x\|.$$

If (46) holds, then the convexity of the function $\sqrt{1+t^{2p}}-1$ ensures again that

$$\limsup_{i\to+\infty} \|\nabla x_i\| \le \|\nabla x\|.$$

The previous inequality implies that $\{\nabla x_i\}$ is strongly convergent to ∇x in $L^p(N)$ (since $L^p(N)$ is uniformly convex). So, by (44), we have that $x_i \to x$ in $W^{1,p}(N)$. ■

THEOREM 3.8 *Let η be a quasi-concave function with $N \in \mathcal{D}_p(\eta)$. If (g_1)-(g_3) hold true and either $\lim_{s\to 0^+} \eta(s) > 0$ or*

$$\lim_{s\to 0^+} \eta(s) = 0 \quad \textit{and} \quad \lim_{s\to+\infty} \eta^{-1}(s^{-p}) s g(qs) = 0 \textit{ for all } q \in \mathbb{R}, \tag{47}$$

then problem (1) *admits a nontrivial (that is, non-constant) solution.*

Proof. We mention that $\eta\ :]0,+\infty[\to [0,+\infty[$ is a (continuous) quasi-concave function, means that η is increasing and

$$\frac{\eta(s)}{s} \text{ is non-increasing.} \tag{48}$$

Now, if $\lim_{s\to 0^+} \eta(s) = 0$, we consider $\omega\ :]0,+\infty[\to [0,+\infty[$ defined by

$$\omega(s) = \frac{1}{\eta^{-1}(\frac{1}{s^p})} \quad \text{for } s > 0, \tag{49}$$

which is continuous, increasing and such that

$$s \to \frac{\omega(s)}{s^p} \text{ is non-decreasing (as } \tfrac{s^p}{\eta^{-1}(s^p)} \text{ is non-increasing and recall (48)).}$$

We have

$$\eta(t) = \frac{1}{[\omega^{-1}(\frac{1}{t})]^p} \quad \text{for } t > 0,$$
$$\Rightarrow \quad (14) \text{ holds (by (26), with } \omega \text{ instead of } \Phi).$$

By Remark 2.2 of [3], we get easily $W^{1,p}(N) \hookrightarrow L^\Phi(N)$, where Φ is the Young function defined as

$$\Phi(s) = \int_0^s t^{-1}\omega(t)dt \quad \text{for } s > 0, \tag{50}$$

which is globally equivalent to ω, by Example 2.3.

Let $\widehat{G}(s) = \int_0^s \widehat{g}(t)dt$, for $s \in [0,+\infty[$, as in (24), and assume that (47) holds. By

(25) and (49), with $f(t) = \eta(t)^{\frac{1}{p}}$, we have

$$\eta^{-1}(s^{-p})\widehat{G}(qs) = \frac{\widehat{G}(qs)}{\omega(s)} \to 0 \quad \text{as } s \to +\infty, \text{ for every } q > 0, \tag{51}$$

$$\Rightarrow \quad \frac{\widehat{G}(qs)}{\Phi(s)} \to 0 \quad \text{as } s \to +\infty, \text{ for every } q > 0, \text{ (by (10) and (51))}. \tag{52}$$

So, by (52), we conclude that $\widehat{G}$ increases essentially more slowly than Φ near infinity. Next, since

$$\widehat{g}(s) \leq 2(\widehat{G}^*)^{-1}(\widehat{G}(2s)) \quad \text{for } s > 0 \text{ (by (8))},$$
$$\Rightarrow \quad |g(s)| \leq 2(\widehat{G}^*)^{-1}(\widehat{G}(2|s|)) \quad \text{for } s \in \mathbb{R},$$

it follows that g satisfies the hypotheses of Lemma 3.4 (set $\Psi = \widehat{G}$ and Φ as in (50)). We conclude that all the hypotheses of Propositions 3.2, 3.3, 3.6 and 3.7 hold. The same hypotheses hold also in the case $\lim_{s\to 0^+} \eta(s) > 0$, provided that Φ is a Young function that equals infinity for large values of its argument.

So, by the mountain pass theorem (Theorem 3.1), the C^1-functional $\mathcal{J}$ admits a critical point $\widetilde{x}$, which is a solution to (1), with $\mathcal{J}(\widetilde{x}) > 0$. To conclude that $\widetilde{x}$ is non-constant (that is, nontrivial), we observe that if $\widetilde{x} = c$ for a certain $c > 0$, then $\widehat{J}(\widetilde{x}) = -\int_N G(c)d\mathcal{H}^n \leq 0$. This leads to contradiction with the inequality $\mathcal{J}(\widetilde{x}) > 0$ and hence $\widetilde{x}$ must be nontrivial. ■

We conclude with the following example.

Example 3.9 Let $\Omega \subset N$ be an open domain (as special case, $\Omega \subset \mathbb{R}^n$). We consider the problem:

$$-\mathrm{div}\left(\left(1 + \frac{|\nabla x|^p}{\sqrt{1 + |\nabla x|^{2p}}}\right)|\nabla x|^{p-2}\nabla x\right) = g(x) \quad \text{in } \Omega, \tag{53}$$

with Neumann boundary condition

$$\frac{\partial x}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where n is the unit inner normal vector field on $\partial\Omega$. Clearly, a function $x \in W^{1,p}(\Omega)$ satisfying (1) is a weak solution of (53). Recall that in order for (1) to have a solution is necessary that $g(x_0) = 0$ for some $x_0 \in \mathbb{R}$. The necessity of this condition can be proved on choosing y constant in (1).

References

[1] Afrouzi GA, Kirane M, Shokooh S. Infinitely many weak solutions for $p(x)$-Laplacian-like problems with Neumann condition. Complex Var Elliptic Equ. 2018;63:23–36.

[2] Barletta G, Cianchi A. Dirichlet problems for fully anisotropic elliptic equations. Proc Roy Soc Edinburgh. Sect A. 2017;147:25–60.

[3] Barletta G, Cianchi A, Maz'ya V. Quasilinear elliptic equations on noncompact Riemannian manifolds. J Funct Anal. 2017;273:3426–3462.
[4] Bennett C, Sharpley R. Interpolation of Operators. Boston: Academic Press; 1988.
[5] Chen ZC, Luo T. The eigenvalue problem for p-Laplacian-Like equations. Int J Math Math Sci. 2003;9:575–586.
[6] Cianchi A, Maz'ya V. Neumann problems and isocapacitary inequalities. J Math Pures Appl. 2008;89:71–105.
[7] Hajlasz P, Liu Z. A compact embedding of a Sobolev space is equivalent to an embedding into a better space. Proc Amer Math Soc. 2010;138:3257–3266.
[8] Maz'ya V. Lectures on isoperimetric and isocapacitary inequalities in the theory of Sobolev spaces. Contemp. Math. 2003;338:307–340.
[9] Maz'ya V. On p-conductivity and theorems on embedding certain functional spaces into a C-space. Dokl Akad Nauk SSSR. 1961;140:299–302 (in Russian).
[10] Milman E. On the role of convexity in functional and isoperimetric inequalities. Proc Lond Math Soc. 2009;99:32–66.
[11] Rabinowitz PH. Minimax methods in critical point theory with applications to differential equations. CBMS Reg Conf Ser in Math 65. Providence: Amer Math Soc; 1986.
[12] Papageorgiou NS, Rocha EM. On nonlinear parametric problems for p-Laplacian-like operators. Rev R Acad Cienc Exactas Fis Nat Ser A Math RACSAM. 2009;103:177–200.
[13] Rao MM, Ren ZD. Theory of Orlicz Spaces. New York: Marcel Dekker; 1991.
[14] Rao MM, Ren ZD. Applications of Orlicz Spaces. New York: Marcel Dekker; 2002.
[15] Rodrigues MM. Multiplicity of solutions on a nonlinear eigenvalue problem for $p(x)$-Laplacian-like operators. Mediterr J Math. 2012;9:211–223.
[16] Shokooh S. Existence and multiplicity results for elliptic equations involving the $p(x)$-Laplacian-like. Annals of the University of Craiova - Mathematics and Computer Science Series. 2017;44:249–258.
[17] Sun Y. On nonexistence of positive solutions of quasi-linear inequality on Riemannian manifolds. Proc Amer Math Soc. 2015;143:2969–2984.
[18] Vétois J. Multiple solutions for nonlinear elliptic equations on compact Riemannian manifolds. Internat J Math. 2007;18:1071–1111.
[19] Vetro C. Pairs of nontrivial smooth solutions for nonlinear Neumann problems. Appl Math Lett. 2020;103(106171):1–7.
[20] Vetro C. Semilinear Robin problems driven by the Laplacian plus an indefinite potential. Complex Var Elliptic Equ. 2019; https://doi.org/10.1080/17476933.2019.1597066
[21] Vetro C. Weak solutions to Dirichlet boundary value problem driven by $p(x)$-Laplacian-like operator. Electron J Qual Theory Differ Equ. 2017;2017(98):1–10.
[22] Vetro C, Vetro F. On problems driven by the $(p(\cdot), q(\cdot))$-Laplace operator. Mediterr J Math. 2020;17(24):1–11.
[23] Zhou Q-M. On the superlinear problem involving $p(x)$-Laplacian-like operators without AR-condition. Nonlinear Anal. 2015;21:161–169.