Mediterr. J. Math. (2026) 23:128
https://doi.org/10.1007/s00009-026-03124-0

Mediterranean Journal of Mathematics

Cauchy–Schwarz Inequalities for Maps in Noncommutative L^p-Spaces

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Abstract. In this paper, some generalized Cauchy–Schwarz inequalities for positive sesquilinear maps with values in noncommutative L^p-spaces for $p > 1$ are obtained. Bound estimates for their real and imaginary parts are also provided and, as an application, a generalization of the uncertainty relation in the context of noncommutative L^2-spaces is given. Next, a new norm on a noncommutative L^2-space which generalizes the classical numerical radius norm of bounded linear operators on a Hilbert space is proposed and a Cauchy–Schwarz inequality for positive sesquilinear maps with values in the space of bounded linear operators from a von-Neumann algebra into the noncommutative L^2-space equipped with this new norm is proved. These results are used to get representations of general positive linear maps with values into a noncommutative L^p-space and into certain operator spaces in several different situations. Some concrete examples are also given.

Mathematics Subject Classification. Primary 46K10, 47A07, 16D10.

Keywords. Cauchy–Schwarz inequalities, noncommutative L^p-spaces, positive maps, uncertainty relations, von Neumann algebras, numerical radius norm, quasi *-algebras.

1. Introduction

In this paper, we continue the generalization of the Cauchy–Schwarz inequality for various classes of positive maps and examine those cases that we have not studied in our previously published papers [1–3]. The basic motivation for this study relies in the representation theory of a variety of structures such as algebras, modules, etc. Indeed, the key role that the Cauchy–Schwarz inequality plays in the ordinary representation theory of either *-algebras or partial *-algebras, as basic ingredient for the Gelfand–Naimark–Segal representation constructed by means of either a positive linear functional or from a sesquilinear form with values in the complex field is well known. Replacing these latter with either positive linear or sesquilinear maps makes sense in view of the construction of more general representations.

These considerations seem to be one of the main reasons for which, since the famous Kadison–Schwarz inequality for positive maps between C^*-algebras [18], several variants of the Cauchy–Schwarz inequality for operator-valued maps have been studied; see Refs.[2,4,8,11,13–17,19,27]. In a recent paper [2], we have proved that some kind of Cauchy–Schwarz-like inequalities hold for positive maps taking values in certain ordered Banach bimodules over *-algebras, in particular, in noncommutative L^1-spaces. The natural question which arises is whether this result can be extended to positive maps taking values in the noncommutative L^p-spaces, with $p > 1$ and this is one of the topics of our present paper.

The paper is organized as follows. After having introduced in Sect. 2, the notation and some preliminary results, in Sect. 3, we obtain Cauchy–Schwarz-like inequalities for positive maps on a von Neumann algebra with values in the noncommutative L^p-spaces, with $p > 1$. The approach adopted in Ref. [2] is not applicable if $p > 1$. For this reason, we develop here some different techniques. The result is a generalized Cauchy–Schwarz inequality where a factor 2 appears in the right hand side (Proposition 3.1). However, as proved in Proposition 3.7, if Φ is a positive sesquilinear map from a vector space $\mathfrak{X}$ into the noncommutative L^p-space, with $p > 1$, the *proper* Cauchy–Schwarz inequality is satisfied for couples of elements $x, y \in \mathfrak{X}$ for which $\Phi(x, y)$ is a normal operator. Proposition 3.7 and Corollary 3.8 find their motivation in the Kadison–Schwarz inequality for normal elements (see Remark 3.9). Furthermore, in Proposition 3.3, we give norm estimates and bounds for the real and imaginary parts of a positive sesquilinear map with values in the noncommutative L^2-space. These bound estimates allow to obtain a generalization of *uncertainty relations* in this framework (Propositions 3.4 and 3.7). In Remark 3.5, a possible physical interpretation of this result is proposed. Next, motivated by a broad international research on the numerical radius norm and related operator inequalities (see Ref. [5]), in Sect. 4, we introduce a new norm on the noncommutative L^2-space, which generalizes the classical numerical radius norm of the space of bounded linear operators in Hilbert space (see subsection 4.2 and Remark 4.2), and in Lemma 4.1, we show that this is a well-defined norm. Then, in Corollary 4.3, we prove that every positive linear map with values in a noncommutative L^2-space satisfies the Cauchy–Schwarz inequality with respect to this new norm. Finally, we consider the space of bounded linear operators from a von Neumann algebra into a noncommutative L^2-space equipped with this new norm and we prove that every positive sesquilinear map with values in this operator space satisfies the Cauchy–Schwarz inequality in the operator norm. This result can be used to represent, in a Banach space, positive linear maps from a unital *-algebra into the space of bounded linear operators from a von Neumann algebra into the noncommutative L^2-space, with its standard norm (Remark 4.7). As explained in Remark 4.7, by applying the Cauchy–Schwarz inequalities obtained in this paper and repeating the procedure of generalized GNS-construction from Refs. [1–3], we obtain representations in (quasi) Banach spaces of the general positive maps that we considered. Positive maps play in general an important role in quantum physics, in quantum information theory (see Refs.

[7,9,20]) and also in operator theory and linear dynamic, see Refs. [12,24,25], and this was our main motivation for studying representations of these maps. Thanks to Corollary 4.3, we are able to represent in a Banach space (and not just in a quasi-Banach space) every positive linear map from an initial *-algebra into a noncommutative L^2-space, see Remark 4.7 for the details. This fact was the main reason and motivation for introducing this new generalized numerical radius norm on the noncommutative L^2-space. Finally, we notice that the Cauchy–Schwarz inequalities for general positive sesquilinear maps with values either in ordered Banach bimodules or in operator spaces play an important role for generating representations and, moreover, produce new related inequalities for certain classes of completely positive maps; see the discussion in Ref. [2, Section 4]. At the end of the paper, we also discuss a concrete example.

2. Notation and Preliminary Results

Throughout the paper, we will denote by $\mathfrak{B}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear maps from the normed space $\mathcal{X}$ into the normed space $\mathcal{Y}$. If $\mathcal{X} = \mathcal{Y}$, we will simply write $\mathfrak{B}(\mathcal{X}) = \mathfrak{B}(\mathcal{X}, \mathcal{X})$. Let $\mathcal{H}$ be a Hilbert space. If $T \in \mathfrak{B}(\mathcal{H})$, we will denote by T^* its adjoint, by $N(T)$ and $R(T)$ its null space and the range of T, respectively, and by $\|T\|$ its norm.

Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace ρ defined on $\mathfrak{M}^+$. For every $p \geq 1$, we denote by $L^p(\rho)$ the Banach space completion of the *-ideal of $\mathfrak{M}$ (see Refs. [21,22]):

$$\mathcal{J}_p = \{X \in \mathfrak{M} : \rho(|X|^p) < \infty\}$$

(where, as usual, $|X| = (X^*X)^{1/2}$) with respect to the norm

$$\|X\|_p = \rho(|X|^p)^{1/p}, \quad X \in \mathcal{J}_p.$$

We maintain the notation ρ for the natural extension of ρ to $L^p(\rho)$. If $p = \infty$, one defines $L^\infty(\rho) = \mathfrak{M}$ and $\|\cdot\|_\infty = \|\cdot\|$ (the operator norm in $\mathfrak{B}(\mathcal{H})$); then if ρ is finite, it is $L^\infty(\rho) \subset L^p(\rho)$ for every $p \geq 1$. For $1 \leq p < \infty$, we denote by $L^p(\rho)$ the noncommutative L^p-space associated with ρ on $\mathfrak{M}$ and consists of all (possibly unbounded) operators X affiliated to $\mathfrak{M}$ such that $\|X\|_p < \infty$. If $1 < p < \infty$, we write q for its conjugate exponent, $1/p + 1/q = 1$.

Throughout this paper, we will often use the noncommutative Hölder's inequality (see, e.g., Ref. [22, Section 3]).

Lemma 2.1. *[1, Lemma 4.8] Let $\mathfrak{M}$ be a von Neumann algebra which is a factor of type I or II, and ρ be a semifinite trace on $\mathfrak{M}$. Let $W \in \mathfrak{M}$ such that $W \geq 0$ and $W \in L^p(\rho)$. Then, there exists a sequence $\{P_n\}_n$ of finite projections in $\mathfrak{M}$ such that*

$$\lim_{n\to\infty} \|W(I - P_n)\|_p = 0.$$

Remark 2.2. By the proof of Ref. [1, Lemma 4.8], each $P_n = E_W(1/n, \infty)$, where E_W is the spectral measure corresponding to $W \in \mathfrak{M}$.

Lemma 2.3. *Let ρ be a faithful, semifinite normal trace on a von Neumann algebra $\mathfrak{M}$ on the Hilbert space $\mathcal{H}$. Let $p > 1$ and $q = \frac{p}{p-1}$. Then,*

- *if $A \in L^p(\rho)$ and $B \in L^q(\rho)$, with $A, B \geq 0$, then $\rho(AB) \geq 0$;*
- *if $A \in L^p(\rho)$ and $B \in L^q(\rho)$, with $A = A^*$ and $B = B^*$, then $\rho(AB) \in \mathbb{R}$.*

Proof. Let first $A_+ \in L^p(\rho)$ and $B_+ \in L^q(\rho) \cap L^\infty(\rho)$ with $A_+, B_+ \geq 0$. If $P \in L^\infty(\rho)$ is any finite projection commuting with B_+, then by Ref. [23, Lemma 3.1], it is not hard to check that

$$\rho\big(A_+B_+P\big) = \rho(PB_+^{1/2}PA_+PB_+^{1/2}P) \geq 0.$$

By Lemma 2.1, we can choose a sequence of finite spectral projections $\{P_n\}$ corresponding to B_+ such that

$$\|B_+ - B_+P_n\|_p \to 0, \qquad \text{as } n \to \infty.$$

Since $\rho(A_+B_+P_n) \geq 0$ for all n, letting $n \to \infty$ and applying Hölder's inequality yields

$$\rho(A_+B_+) \geq 0.$$

Since this holds, for all $B_+ \in L^\infty(\rho) \cap L^q(\rho)$, by passing to the limit, we deduce that it also holds for all $\widetilde{B}_+ \in L^q(\rho)$, with $\widetilde{B}_+ \geq 0$.

Now, for arbitrary selfadjoint $A \in L^p(\rho) \cap L^\infty(\rho)$ and $B \in L^q(\rho) \cap L^\infty(\rho)$, with $A = A^*$ and $B = B^*$, we can write

$$A = A_+ - A_-, \qquad B = B_+ - B_-,$$

with $A_\pm \in L^p(\rho) \cap L^\infty(\rho)$ and $B_\pm \in L^q(\rho) \cap L^\infty(\rho)$ and $A_\pm, B_\pm \geq 0$. Hence, by the above,

$$\rho(AB) = \rho(A_+B_+) - \rho(A_+B_-) - \rho(A_-B_+) + \rho(A_-B_-) \in \mathbb{R}.$$

Finally, for general selfadjoint $A \in L^p(\rho)$ and $B \in L^q(\rho)$, choose sequences of selfadjoint elements $\{A_n\} \in L^p(\rho) \cap L^\infty(\rho)$ and $\{B_n\} \in L^q(\rho) \cap L^\infty(\rho)$, such that

$$A_n \to A \quad \text{in } L^p(\rho), \qquad B_n \to B \quad \text{in } L^q(\rho), \qquad \text{as } n \to \infty.$$

Since $\rho(A_nB_n) \in \mathbb{R}$ for all n, letting $n \to \infty$ and applying Hölder's inequality gives $\rho(AB) \in \mathbb{R}$. □

A *quasi *-algebra* $(\mathfrak{A}, \mathfrak{A}_0)$ is a pair consisting of a vector space $\mathfrak{A}$ and a *-algebra $\mathfrak{A}_0$ contained in $\mathfrak{A}$ as a subspace and such that

- $\mathfrak{A}$ carries an involution $a \mapsto a^*$ extending the involution of $\mathfrak{A}_0$;
- $\mathfrak{A}$ is a bimodule over $\mathfrak{A}_0$ and the module multiplications extend the multiplication of $\mathfrak{A}_0$. In particular, the following associative laws hold:

$$(ca)d = c(ad); \quad a(cd) = (ac)d, \quad \forall\ a \in \mathfrak{A},\ c, d \in \mathfrak{A}_0;$$

- $(ac)^* = c^*a^*$, for every $a \in \mathfrak{A}$ and $c \in \mathfrak{A}_0$.

The *identity* or *unit element* of $(\mathfrak{A}, \mathfrak{A}_0)$, if any, is a necessarily unique element $\mathsf{e} \in \mathfrak{A}_0$, such that $a\mathsf{e} = a = \mathsf{e}a$, for all $a \in \mathfrak{A}$.

We will always suppose that

$$\begin{aligned} &ac = 0,\ \forall c \in \mathfrak{A}_0 \Rightarrow a = 0 \\ &ac = 0,\ \forall a \in \mathfrak{A} \Rightarrow c = 0. \end{aligned}$$

Clearly, both these conditions are automatically satisfied if $(\mathfrak{A}, \mathfrak{A}_0)$ has an identity e.

Definition 2.4. Let $\mathfrak{Y}$ be a Banach bimodule over the *-algebra $\mathfrak{Y}_0$. We say that $\mathfrak{Y}$ is an *ordered Banach bimodule* over $\mathfrak{Y}_0$ if

- (i) $\mathfrak{Y}$ is ordered as a vector space; that is, $\mathfrak{Y}$ contains a (positive) cone $\mathfrak{K}$, i.e., $\mathfrak{K} \subset \mathfrak{Y}$ is such that $\mathfrak{K}+\mathfrak{K} \subset \mathfrak{K}$, $\lambda\mathfrak{K} \subset \mathfrak{K}$ for $\lambda \geq 0$ and $\mathfrak{K} \cap (-\mathfrak{K}) = \{0\}$;
- (ii) $z^*\mathfrak{K}z \subset \mathfrak{K}, \quad \forall z \in \mathfrak{Y}_0$.

Example. $L^p(\rho)$ is an ordered Banach bimodule over $L^\infty(\rho)$.

Let $\mathfrak{X}$ be a vector space and $\mathfrak{Y}$ an ordered Banach module over $\mathfrak{Y}_0$ with positive cone $\mathfrak{K}$. Let Φ be a $\mathfrak{Y}$-valued positive sesquilinear map on $\mathfrak{X} \times \mathfrak{X}$

$$\Phi : (x_1, x_2) \in \mathfrak{X} \times \mathfrak{X} \to \Phi(x_1, x_2) \in \mathfrak{Y},$$

i.e., a map with the properties

- (i) $\Phi(x_1, x_1) \in \mathfrak{K}$,
- (ii) $\Phi(\alpha x_1 + \beta x_2, \gamma x_3) = \overline{\gamma}[\alpha\Phi(x_1, x_3) + \beta\Phi(x_2, x_3)]$,

for every $x_1, x_2, x_3 \in \mathfrak{X}$ and $\alpha, \beta, \gamma \in \mathbb{C}$.

Definition 2.5. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra and $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{Y}$ be a positive sesquilinear map. The map Φ is called

- *left-invariant* if

$$\Phi(ac, d) = \Phi(c, a^*d), \quad \forall\ a \in \mathfrak{A},\ c, d \in \mathfrak{A}_0. \tag{2.1}$$

If the quasi *-algebra is also normed, Φ is called

- *bounded* if there exists a constant $M > 0$ such that

$$\|\Phi(a, b)\|_{\mathfrak{Y}} \leq M\|a\|\|b\|, \quad \forall a, b \in \mathfrak{A}.$$

3. Generalized Cauchy–Schwarz Inequality for L^p-Valued Positive Maps

In this section, a generalized Cauchy–Schwarz inequality for general positive sesquilinear maps into $L^p(\rho)$ is proved. The inequality has been studied in Ref. [2] just in the case $p = 1$ (and not for $p > 1$). With this inequality at hand, we can represent such positive $L^p(\rho)$-valued maps in a quasi-Banach space (see Remark 4.7).

Since $L^p(\rho)$ carries the operator involution, a positive sesquilinear map $\Phi : \mathfrak{X} \times \mathfrak{X} \to L^p(\rho)$ is called hermitian if $\Phi(y, x) = \Phi(x, y)^*$ for every $x, y \in \mathfrak{X}$.

Proposition 3.1. *Let $\mathfrak{M}$ be a von Neumann algebra with a faithful normal semifinite trace ρ, let $1 < p < \infty$, and let $\mathfrak{X}$ be a complex vector space. If $\Phi : \mathfrak{X} \times \mathfrak{X} \longrightarrow L^p(\rho)$ is a positive sesquilinear map, then*

$$\|\Phi(x,y)\|_p \leq 2\|\Phi(x,x)\|_p^{1/2}\,\|\Phi(y,y)\|_p^{1/2}, \quad \forall x,y \in \mathfrak{X}.$$

Proof. Fix $x, y \in \mathfrak{X}$. If they are such that $\Phi(x,y) = 0$, then the thesis follows. Now, let $x, y \in \mathfrak{X}$ be such that $\Phi(x,y) \neq 0$ and choose a sequence $\{z_n\}_{n\in\mathbb{N}} \subset L^p(\rho) \cap L^\infty(\rho)$ such that

$$z_n \longrightarrow \Phi(x,y) \quad \text{in } L^p(\rho) \quad \text{ as } n \to \infty.$$

Let $z_n = u_n|z_n|$ be the polar decomposition of z_n for all $n \in \mathbb{N}$. Since $z_n \in L^p(\rho)$, we have $|z_n|^{p-1} \in L^q(\rho)$, with $q = \frac{p}{p-1}$ and by Hölder's inequality,

$$z_n|z_n|^{p-1} \in L^1(\rho), \quad \forall n \in \mathbb{N}.$$

Therefore,

$$\rho(|z_n|^p) = \rho\big(u_n^* z_n\,|z_n|^{p-1}\big) = \rho\big(z_n|z_n|^{p-1}u_n^*\big), \quad \forall n \in \mathbb{N}.$$

Hence,

$$\begin{aligned}
\big|\|\Phi(x,y)\|_p^p - &\rho\big(\Phi(x,y)\,|z_n|^{p-1}u_n^*\big)\big| \\
&\leq \big|\|\Phi(x,y)\|_p^p - \|z_n\|_p^p\big| + \big|\rho\big((\Phi(x,y) - z_n)\,|z_n|^{p-1}u_n^*\big)\big| \\
&\leq \big|\|\Phi(x,y)\|_p^p - \|z_n\|_p^p\big| + \|\Phi(x,y) - z_n\|_p\,\||z_n|^{p-1}u_n^*\|_q \\
&\leq \big|\|\Phi(x,y)\|_p^p - \|z_n\|_p^p\big| + \|\Phi(x,y) - z_n\|_p\,\||z_n|^{p-1}\|_q \\
&= \big|\|\Phi(x,y)\|_p^p - \|z_n\|_p^p\big| + \|\Phi(x,y) - z_n\|_p\,\|z_n\|_p^{p-1}, \ \forall n \in \mathbb{N}.
\end{aligned}$$

Since $z_n \to \Phi(x,y)$ in $L^p(\rho)$ as $n \to \infty$, it follows that $\|z_n\|_p^{p-1} \leq M$ for all $n \in \mathbb{N}$ and some $M > 0$, hence we conclude that

$$\rho\big(\Phi(x,y)\,|z_n|^{p-1}u_n^*\big) \longrightarrow \|\Phi(x,y)\|_p^p, \qquad \text{as } n \to \infty.$$

For every $n \in \mathbb{N}$, write

$$|z_n|^{p-1}u_n^* = \xi_1^{(n)} - \xi_2^{(n)} + i\big(\xi_3^{(n)} - \xi_4^{(n)}\big),$$

where $\xi_j^{(n)} \in L^q(\rho) \cap L^\infty(\rho)$ are such that $\xi_j^{(n)} \geq 0$ for every $j \in \{1, \ldots, 4\}$, $\xi_1^{(n)}\xi_2^{(n)} = \xi_3^{(n)}\xi_4^{(n)} = 0$, $\xi_1^{(n)} - \xi_2^{(n)} = \Re(|z_n|^{p-1}u_n^*)$ and $\xi_3^{(n)} - \xi_4^{(n)} = \Im(|z_n|^{p-1}u_n^*)$ (the real and the imaginary part of $|z_n|^{p-1}u_n^*$, respectively). Then, for all $n \in \mathbb{N}$,

$$\|\xi_1^{(n)} - \xi_2^{(n)}\|_q = \|\xi_1^{(n)} + \xi_2^{(n)}\|_q \leq \|\,|z_n|^{p-1}u_n^*\|_q$$

and

$$\|\xi_3^{(n)} - \xi_4^{(n)}\|_q = \|\xi_3^{(n)} + \xi_4^{(n)}\|_q \leq \|\,|z_n|^{p-1}u_n^*\|_q.$$

Hence, we have

$$|\rho\big(\Phi(x,y)\,|z_n|^{p-1}u_n^*\big)| \leq \sum_{j=1}^{4} |\rho\big(\Phi(x,y)\,\xi_j^{(n)}\big)|, \quad \forall n \in \mathbb{N}.$$

Define the (scalar) sesquilinear forms

$$\varphi_j^{(n)}(a,b) := \rho\big(\Phi(a,b)\,\xi_j^{(n)}\big), \qquad a, b \in \mathfrak{X},\ j = 1, \ldots, 4.$$

Since Φ is positive and $\xi_j^{(n)} \geq 0$, each $\varphi_j^{(n)}$ is a positive sesquilinear form on $\mathfrak{X}$ by Lemma 2.3. Applying the scalar Cauchy–Schwarz inequality to $\varphi_j^{(n)}$ for each $j \in \{1, \ldots, 4\}$ gives

$$\begin{aligned}
\sum_{j=1}^{4} \left|\rho\big(\Phi(x,y)\,\xi_j^{(n)}\big)\right| &\leq \sum_{j=1}^{4} (\varphi_j^{(n)}(x,x))^{1/2}\,(\varphi_j^{(n)}(y,y))^{1/2} \\
&\leq \left(\sum_{j=1}^{4} (\varphi_j^{(n)}(x,x))^{1/2}\right)^{1/2} \left(\sum_{j=1}^{4} (\varphi_j^{(n)}(y,y))^{1/2}\right)^{1/2} \\
&= \left(\rho\left(\Phi(x,x)\,\sum_{j=1}^{4} \xi_j^{(n)}\right)\right)^{1/2} \left(\rho\left(\Phi(y,y)\,\sum_{j=1}^{4} \xi_j^{(n)}\right)\right)^{1/2} \\
&\leq \|\Phi(x,x)\|_p^{1/2}\,\|\Phi(y,y)\|_p^{1/2} \left\|\sum_{j=1}^{4} \xi_j^{(n)}\right\|_q \\
&\leq \|\Phi(x,x)\|_p^{1/2}\,\|\Phi(y,y)\|_p^{1/2} (\|\xi_1^{(n)} + \xi_2^{(n)}\|_q + \|\xi_3^{(n)} + \xi_3^{(n)}\|_q) \\
&\leq 2\|\Phi(x,x)\|_p^{1/2}\,\|\Phi(y,y)\|_p^{1/2} \|\,|z_n|^{p-1} u_n^*\|_q.
\end{aligned}$$

by using the Cauchy–Schwarz inequality in $\mathbb{R}^4$ and the Hölder's inequality. Since

$$\|\,|z_n|^{p-1} u_n^*\|_q \leq \|\,|z_n|^{p-1}\|_q \|u_n^*\|_\infty \leq \||z_n|^{p-1}\|_q = \|z_n\|_p^{p-1}, \quad \forall n \in \mathbb{N}$$

it is

$$\begin{aligned}
\left|\rho\big(\Phi(x,y)\,|z_n|^{p-1} u_n\big)\right| &\leq 2\,\|\Phi(x,x)\|_p^{1/2}\,\|\Phi(y,y)\|_p^{1/2}\,\|\,|z_n|^{p-1} u_n^*\|_q \\
&\leq 2\,\|\Phi(x,x)\|_p^{1/2}\,\|\Phi(y,y)\|_p^{1/2}\,\|z_n\|_p^{p-1}, \quad \forall n \in \mathbb{N}
\end{aligned} \tag{3.1}$$

and letting $n \to \infty$ yields

$$\|\Phi(x,y)\|_p^p \leq 2\,\|\Phi(x,x)\|_p^{1/2}\,\|\Phi(y,y)\|_p^{1/2}\,\|\Phi(x,y)\|_p^{p-1}.$$

Dividing both sides by $\|\Phi(x,y)\|_p^{p-1} \neq 0$ gives

$$\|\Phi(x,y)\|_p \leq 2\,\|\Phi(x,x)\|_p^{1/2}\,\|\Phi(y,y)\|_p^{1/2}.$$

□

It is worth noting that, when $p = 2$, the constant 2 in the inequality can be reduced to $\sqrt{2}$.

Remark 3.2. Let $p = 2$. If $A \in L^2(\rho) \cap L^\infty(\rho)$, then

$$\begin{aligned}
\rho(AA^*) &= \rho\big((\Re A - i\Im A)(\Re A + i\Im A)\big) \\
&= \rho\big((\Re A)^2\big) + \rho\big((\Im A)^2\big) + i\big(\rho(\Re A\,\Im A) - \rho(\Im A\,\Re A)\big).
\end{aligned}$$

Since $\Re A, \Im A \in L^2(\rho)$, we have

$$\rho(\Re A\,\Im A) = \rho(\Im A\,\Re A).$$

Hence, $\rho(AA^*) = \rho((\Re A)^2)+\rho((\Im A)^2)$. By applying this equality in the proof of Proposition 3.1, we deduce that for every $n \in \mathbb{N}$,

$$\begin{aligned}\||z_n|^{p-1}u_n^*\|_2 = \||z_n|u_n^*\|_2 &= (\|\xi_1^n - \xi_2^n\|_2^2 + \|\xi_3^n - \xi_4^n\|_2^2)^{1/2}\\ &= (\|\xi_1^n + \xi_2^n\|_2^2 + \|\xi_3^n + \xi_4^n\|_2^2)^{1/2}\end{aligned}$$

since $\xi_1^n\xi_2^n = \xi_3^n\xi_4^n = 0$ for every $n \in \mathbb{N}$. However, by the Cauchy–Schwarz inequality applied to the inner product in $\mathbb{R}^2$, we get that for every $n \in \mathbb{N}$,

$$\begin{aligned}\|\xi_1^n + \xi_2^n\|_2 + \|\xi_3^n + \xi_4^n\|_2 &\leq \sqrt{2}\left(\|\xi_1^n + \xi_2^n\|_2^2 + \|\xi_3^n + \xi_4^n\|_2^2\right)^{1/2}\\ &= \sqrt{2}\||z_n|u_n^*\|_2.\end{aligned}$$

3.1. Estimates for the Real and Imaginary Parts of Positive $L^2(\rho)$-Valued Sesquilinear Maps

Motivated by Ref. [13, Lemma 1], we now provide norm estimates for the real and imaginary parts of positive $L^2(\rho)$-valued sesquilinear maps.

Proposition 3.3. *Let $\Phi : \mathfrak{X}\times\mathfrak{X} \to L^2(\rho)$ be a positive sesquilinear map. Then, for all $x, y \in \mathfrak{X}$,*

$$\|\Re\Phi(x, y)\|_2^2 \leq \|\Phi(x, x)\|_2 \, \|\Phi(y, y)\|_2,$$

and

$$\|\Im\Phi(x, y)\|_2^2 \leq \|\Phi(x, x)\|_2 \, \|\Phi(y, y)\|_2,$$

where

$$\Re\Phi(x, y) = \frac{\Phi(x, y) + \Phi(x, y)^*}{2}, \qquad \Im\Phi(x, y) = \frac{\Phi(x, y) - \Phi(x, y)^*}{2i}.$$

Proof. Fix any $x, y \in \mathfrak{X}$. We only prove the first inequality concerning $\Re\Phi(x, y)$; the proof of the second one is analogous. By Lemma 2.3, it is

$$\begin{aligned}\Re\left(\rho\big((\Re\Phi(x, y))\ \Phi(x, y)\big)\right) &= \rho\big(\Re\Phi(x, y)\,\Re\Phi(x, y)\big) \qquad (3.2)\\ &= \|\Re\Phi(x, y)\|_2^2.\end{aligned}$$

Indeed,

$$\rho\big((\Re\Phi(x, y))\ \Phi(x, y)\big) = \rho\big((\Re\Phi(x, y))^2\big) + i\,\rho\big((\Re\Phi(x, y))\ (\Im\Phi(x, y))\big),$$

and by (3.2), both $\rho\left((\Re\Phi(x, y))^2\right)$ and $\rho\left((\Re\Phi(x, y))\ (\Im\Phi(x, y))\right)$ are real, so

$$\Re\left(\rho\big((\Re\Phi(x, y))\ \Phi(x, y)\big)\right) = \rho\left((\Re\Phi(x, y))^2\right).$$

Let $\{z_n\} \subset L^2(\rho) \cap L^\infty(\rho)$ be such that $z_n \to \Phi(x, y)$ in $L^2(\rho)$ as $n \to \infty$. Then,

$$\Re z_n = \frac{z_n + z_n^*}{2} \to \Re\Phi(x, y) = \frac{\Phi(x, y) + \Phi(x, y)^*}{2} \quad \text{in } L^2(\rho), \text{ as } n \to \infty.$$

For every $n \in \mathbb{N}$, write

$$\Re z_n = z_n^+ - z_n^-, \quad \text{where } z_n^\pm \geq 0 \qquad \text{and } z_n^+z_n^- = 0.$$

Then,

$$\left|\Re\left(\rho(\Phi(x, y)\,\Re\Phi(x, y))\right)\right| \leq \left|\rho\big(\Phi(x, y)\Re\Phi(x, y)\big)\right|,$$

and, on the other hand, for every $n \in \mathbb{N}$,

$$\begin{aligned}\left|\rho(\Phi(x,y)(z_n^+ - z_n^-))\right| &\leq \|\Phi(x,x)\|_2^{1/2}\|\Phi(y,y)\|_2^{1/2}\|z_n^+ + z_n^-\|_2 \\ &= \|\Phi(x,x)\|_2^{1/2}\|\Phi(y,y)\|_2^{1/2}\|z_n^+ - z_n^-\|_2\end{aligned}$$

as in (3.1). Letting $n \to \infty$ on both sides yields

$$\begin{aligned}\|\Re\left(\Phi(x,y)\right)\|_2^2 &\leq \left|\rho(\Phi(x,y)\,\Re\Phi(x,y))\right| \\ &\leq \|\Phi(x,x)\|_2^{1/2}\,\|\Phi(y,y)\|_2^{1/2}\,\|\Re\Phi(x,y)\|_2.\end{aligned}$$

This completes the proof. □

Let us consider the partial *-algebra $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ of all closable operators A such that the domain of A and its adjoint A^* are, respectively, $D(A) = \mathcal{D}$, $D(A^*) \supset \mathcal{D}$. The involution is defined by $A^\dagger := A^*_{\restriction\mathcal{D}}$.

Let $\Phi : \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) \times \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) \to L^2(\rho)$ be a positive sesquilinear map. Let $A, B \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. We say that A, B admit a Φ-*commutator*, if there exists $C \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that

$$\Phi(AX, B^\dagger Y) - \Phi(BX, A^\dagger Y) = \Phi(CX, Y), \quad \forall X, Y \in \mathcal{L}^\dagger(\mathcal{D})_b$$

where $\mathcal{L}^\dagger(\mathcal{D})_b$ denotes the *-algebra of bounded operators of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that $A\mathcal{D} \subset \mathcal{D}$, $A^\dagger\mathcal{D} \subset \mathcal{D}$. In particular, if $A = A^\dagger, B = B^\dagger$ are symmetric operators and a Φ-commutator C exists, then $C = iK$ with $K = K^\dagger$.

If $A \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a symmetric operator (i.e., $A = A^\dagger$), we define a map $(\Delta A)_\Phi : \mathbb{R} \to \mathbb{R}^+$ by

$$(\Delta A)_\Phi(\lambda) = \|\Phi(A - \lambda I, A - \lambda I)\|_2^{1/2}, \quad \lambda \in \mathbb{R}$$

which can be interpreted as a sort of *uncertainty* of A.

Proposition 3.3 allows us to get an uncertainty relation for certain pairs of symmetric operators.

Proposition 3.4. *Let $\Phi : \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) \times \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) \to L^2(\rho)$ be a positive sesquilinear map and $A, B, K \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ be symmetric operators satisfying*

(a) $\Phi(AX, Y) = \Phi(X, AY)$, $\Phi(BX, Y) = \Phi(X, BY)$, $\forall X, Y \in \mathcal{L}^\dagger(\mathcal{D})_b$.
(b) $\Phi(AX, BY) - \Phi(BX, AY) = \Phi(iKX, Y)$, $\quad \forall X, Y \in \mathcal{L}^\dagger(\mathcal{D})_b$.

Then, for every $\lambda, \mu \in \mathbb{R}$, the following uncertainty relation holds:

$$(\Delta A)_\Phi(\lambda)(\Delta B)_\Phi(\mu) \geq \frac{1}{2}\gamma_\Phi(K),$$

where $\gamma_\Phi(K) = \|\Phi(K, I)\|_2$

Proof. As one can prove by a direct easy computation, the condition (a) implies that if A, B satisfy condition (b), then the same holds true for $A - \lambda I$ and $B - \mu I$, for every $\lambda, \mu \in \mathbb{R}$, with the same operator K on the right hand side. Then, from (b) A and B, admit a Φ-commutator iK and one obviously has

$$\Phi(A - \lambda I, B - \mu I) - \Phi(B - \mu I, A - \lambda I) = \Phi(iK, I). \tag{3.3}$$

By Proposition 3.3, we obtain

$$\|\Re\Phi(iK, I)\|_2 \leq 2\|\Phi(A - \lambda I, A - \lambda I)\|_2^{1/2}\|\Phi(B - \mu I, B - \mu I)\|_2^{1/2}$$

and

$$\|\Im\Phi(iK, I)\|_2 \leq 2\|\Phi(A - \lambda I, A - \lambda I)\|_2^{1/2}\|\Phi(B - \mu I, B - \mu I)\|_2^{1/2}.$$

Taking into account (3.3), we can prove that $\Phi(K, I)$ is selfadjoint. Then,

$$\|\Phi(K, I)\|_2 \leq 2\|\Phi(A - \lambda I, A - \lambda I)\|_2^{1/2}\|\Phi(B - \lambda I, B - \lambda I)\|_2^{1/2}.$$

In other terms,

$$\|\Phi(K, I)\|_2 \leq 2(\Delta A)_\Phi(\lambda)(\Delta B)_\Phi(\mu).$$

□

Remark 3.5. The interest of Proposition 3.4 relies on its close analogy with the uncertainty relations that in quantum physics are linked to the commutation relations. The term $\gamma_\phi(K)$ can be interpreted as sort of *mean value* of K with respect to Φ, while a term like $(\Delta A)_\Phi$ can be understood as the *variance* of A with respect to Φ.

A statement analogous to Proposition 3.4 can be obtained in a more abstract setting. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit e and $\Phi : \mathfrak{A}\times\mathfrak{A} \to L^2(\rho)$ a positive sesquilinear map satisfying the left-invariance condition (2.1). As before, if $a, b \in \mathfrak{A}$, we say th.at a, b admit a Φ-*commutator*, if there exists $c \in \mathfrak{A}$ such that

$$\Phi(ax, b^*y) - \Phi(bx, a^*y) = \Phi(cx, y), \quad \forall x, y \in \mathfrak{A}_0.$$

For $a \in \mathfrak{A}$ and $\lambda \in \mathbb{R}$, we put

$$\Delta_\Phi(a)(\lambda) = \|\Phi(a - \lambda\mathsf{e}, a - \lambda\mathsf{e})\|_2.$$

Proposition 3.6. *Let $\Phi : \mathfrak{A} \times \mathfrak{A} \to L^2(\rho)$ a left-invariant positive sesquilinear map, $a, b \in \mathfrak{A}$ be symmetric and assume that they admit a Φ-commutator $c \in \mathfrak{A}$. Then, for every $\lambda, \mu \in \mathbb{R}$,*

$$(\Delta a)_\Phi(\lambda)(\Delta b)_\Phi(\mu) \geq \frac{1}{2}\gamma_\Phi(c),$$

where $\gamma_\Phi(c) = \|\Phi(c, \mathsf{e})\|_2$.

3.2. Cauchy–Schwarz Inequalities for Normal Elements in $L^p(\rho)$

Let ρ be a faithful, normal, semifinite trace on a von Neumann algebra $\mathfrak{M}$. We have the following proposition.

Proposition 3.7. *Let $p > 1$ and let $\Phi : \mathfrak{X} \times \mathfrak{X} \to L^p(\rho)$ be a positive sesquilinear map. If $x, y \in \mathfrak{X}$ are such that $\Phi(x, y) \in L^p(\rho) \cap L^\infty(\rho)$ and $\Phi(x, y)$ is normal, then*

$$\|\Phi(x, y)\|_p \ \leq \ \|\Phi(x, x)\|_p^{1/2}\, \|\Phi(y, y)\|_p^{1/2}.$$

Proof. If $x, y \in \mathfrak{X}$ are such that $\Phi(x, y) \in L^p(\rho) \cap L^\infty(\rho)$ and $\Phi(x, y) = 0$, the inequality holds true, so assume $\Phi(x, y) \neq 0$. Let $V_{x,y}$ denote the partial isometry in the polar decomposition of $\Phi(x, y)$

$$\Phi(x, y) = V_{x,y}\, |\Phi(x, y)|.$$

Let

$$\Pi = \sum_{j=1}^{N} c_j P_j$$

be a simple operator-valued function in $L^q(\rho) \cap L^\infty(\rho)$, where P_j's are pairwise orthogonal projections in $L^q(\rho) \cap L^\infty(\rho)$ and c_j's are scalars. Then,

$$\begin{aligned} |\rho(\Phi(x,y)\Pi)| &\le \sum_{j=1}^{N} |c_j|\,|\rho(\Phi(x,y)P_j)| \qquad (3.4)\\ &\le \sum_{j=1}^{N} |c_j|\, \left(\rho\left(\Phi(x,x)P_j\right)\right)^{1/2} \left(\rho\left(\Phi(y,y)P_j\right)\right)^{1/2}\\ &\le \left(\sum_{j=1}^{N} |c_j| \rho\left(\Phi(x,x)P_j\right)\right)^{1/2} \left(\sum_{j=1}^{N} |c_j| \rho\left(\Phi(y,y)P_j\right)\right)^{1/2}\\ &= \left(\rho\left(\Phi(x,x)\left(\sum_{j=1}^{N} |c_j| P_j\right)\right)\right)^{1/2} \left(\rho\left(\Phi(y,y)\left(\sum_{j=1}^{N} |c_j| P_j\right)\right)\right)^{1/2}\\ &= \left(\rho\left(\Phi(x,x)\left(|\Pi|\right)\right)\right)^{1/2} \left(\rho\left(\Phi(y,y)\left(|\Pi|\right)\right)\right)^{1/2} \end{aligned}$$

where in the second inequality, we have applied the Cauchy–Schwarz inequality to the sesquilinear forms $\varphi_j : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$, $j \in \{1, \ldots, N\}$, given by

$$\varphi_j(x,y) = \rho(\Phi(x,y)P_j), \quad \forall x, y \in \mathfrak{X}$$

which are positive by the positivity of Φ and by Lemma 2.3, whereas in the last inequality we applied the Cauchy–Schwarz inequality to the inner product in $\mathbb{R}^N$ and in the last equality the fact that $|\Pi| = \sum_{j=1}^{N} |c_j| P_j$ since $P_i P_j = 0$ whenever $i \neq j$. By the Hölder's inequality, we get

$$\begin{aligned} \left(\rho\left(\Phi(x,x)\left(\Pi\right)\right)\right)^{1/2} &\left(\rho\left(\Phi(y,y)\left(\Pi\right)\right)\right)^{1/2} \qquad (3.5)\\ &\le \|\Phi(x,x)\|_p^{1/2}\, \|\Phi(y,y)\|_p^{1/2}\, \|\Pi\|_q. \end{aligned}$$

If $\Phi(x,y)$ is normal, then

$$|\Phi(x,y)|^{p-1} V_{x,y}^* = \int_{\sigma(\Phi(x,y))} \operatorname{sgn}(z) |z|^{p-1}\, dE(z)$$

where E is the spectral measure of $\Phi(x,y)$ and $\operatorname{sgn} : \sigma(\Phi(x,y)) \to \mathbb{C}$ is given by

$$\operatorname{sgn}(z) = \begin{cases} \frac{z}{|z|}, & \text{if } z \neq 0\\ 0, & \text{if } z = 0. \end{cases}$$

By Lemma 2.1, for each $m \in \mathbb{N}$, there exists a spectral projection P_m of $|\Phi(x,y)|$ such that

$$\||\Phi(x,y)|(I - P_m)\|_p \le \frac{1}{m}.$$

Now, by Remark 2.2, we have $P_m = F\left(\frac{1}{m}, \infty\right)$, where F is the spectral measure corresponding to $|\Phi(x,y)|$. Since $t^p \chi_{[0,\frac{1}{m}]}(t) = (t\chi_{[0,\frac{1}{m}]}(t))^p$ for all $t \in \sigma(|\Phi(x,y)|)$, by spectral functional calculus, we have

$$|\Phi(x,y)|^p (I - P_m)^p = (|\Phi(x,y)|(I - P_m))^p$$

and since $(I - P_m)^p = I - P_m = (I - P_m)^q$, we obtain:

$$\||\Phi(x,y)|^{p-1}(I - P_m)\|_q = \||\Phi(x,y)|(I - P_m)\|_p^{p-1} < m^{-(p-1)}. \tag{3.6}$$

Let $\{\Pi_n\}_n$ be a sequence of simple functions on $\sigma(\Phi(x,y))$ converging in supremum norm to the function

$$z \longrightarrow \operatorname{sgn}(z)|z|^{p-1},$$

then $\left\{\int_{\sigma(\Phi(x,y))} \Pi_n dE\right\}_n$ is a sequence of operator-valued functions converging to $|\Phi(x,y)|^{p-1}V_{x,y}^*$ in the operator norm. Let now for each $n \in \mathbb{N}$ consider $\{A_1, \ldots, A_N\}$, a set of pairwise disjoint Borel subsets of $\sigma(\Phi(x,y))$ with $\sigma(\Phi(x,y)) = \sum_{j=1}^N A_j$ such that

$$\int_{\sigma(\Phi(x,y))} \Pi_n dE = \sum_{j=1}^N c_j E(A_j)$$

for some scalars $c_1, \ldots, c_n$ (note that not only $\{A_1, \ldots, A_N\}$ but also N and the scalars $\{c_1, \ldots, c_N\}$ depend on n). Now, P_m mutually commutes with $E(A_j)$ for each j since P_m is a spectral projection corresponding to $|\Phi(x,y)|$ and E is the spectral measure corresponding to $\Phi(x,y)$ which is normal and hence commutes with $|\Phi(x,y)|$. Therefore, since the product of two mutually commuting orthogonal projections is an orthogonal projection too (see. e.g., Ref. [6, Theorem 2.8.4]), we get that, for every $n \in \mathbb{N}$, the operator $\left(\int_{\sigma(\Phi(x,y))} \Pi_n dE\right) P_m$ is also a simple operator-valued function in $\mathfrak{M}$ because

$$\left(\int_{\sigma(\Phi(x,y))} \Pi_n dE\right) P_m = \sum_{j=1}^N c_j E(A_j) P_m$$

and $E(A_j)P_m$ is orthogonal projection for every $j \in \{1, \ldots, N\}$; moreover, for all $i \neq j$, with $i, j \in \{1, \ldots, N\}$,

$$E(A_j)P_m E(A_i)P_m = P_m E(A_j)E(A_i)P_m = 0.$$

Furthermore, P_m commutes with $V_{x,y}^*$. Indeed, since $\Phi(x,y)$ is normal, then $V_{x,y}^*$ commutes with $|\Phi(x,y)|$ and since P_m is a spectral projection corresponding to $|\Phi(x,y)|$, we must have

$$P_m V_{x,y}^* = V_{x,y}^* P_m.$$

Now observe that since P_m is a finite projection (i.e. $\rho(P_m) < \infty$) and hence $P_m \in L^q(\rho)$,

$$\left\| \left(\int_{\sigma(\Phi(x,y))} \Pi_n dE \right) P_m - |\Phi(x,y)|^{p-1} V_{x,y}^* P_m \right\|_q$$
$$\leq \left\| \int_{\sigma(\Phi(x,y))} \Pi_n dE - |\Phi(x,y)|^{p-1} V_{x,y}^* \right\|_\infty \left\| P_m \right\|_q \to 0,$$

Therefore, by (3.6),

$$\begin{aligned} \| \, |\Phi(x,y)|^{p-1} V_{x,y}^* (P_m - I) \|_q &= \| \, |\Phi(x,y)|^{p-1} (P_m - I) V_{x,y}^* \|_q \\ &\leq \| \, |\Phi(x,y)|^{p-1} (P_m - I) \|_q \| V_{x,y}^* \|_\infty \\ &\leq \| \, |\Phi(x,y)|^{p-1} (P_m - I) \|_q < m^{-(p-1)}. \end{aligned}$$

Since $\left(\int_{\sigma(\Phi(x,y))} \Pi_n dE \right) P_m$ is a simple operator-valued function in $L^q(\rho) \cap L^\infty(\rho)$ for each n and $E(A_j) P_m \in L^q(\rho)$ for all $j \in \{1, \ldots, N\}$, from (3.4) and (3.5), we obtain

$$\left| \rho \left(\Phi(x,y) \left(\int_{\sigma(\Phi(x,y))} \Pi_n dE \right) P_m \right) \right| \tag{3.7}$$
$$\leq \|\Phi(x,x)\|_p^{1/2} \, \|\Phi(y,y)\|_p^{1/2} \left\| \left(\int_{\sigma(\Phi(x,y))} \Pi_n dE \right) P_m \right\|_q .$$

Letting $n \to \infty$ on both sides of (3.7) yields, for every $m \in \mathbb{N}$,

$$\begin{aligned} &\left| \rho(\Phi(x,y) \, |\Phi(x,y)|^{p-1} V_{x,y}^* P_m) \right| = \left| \rho(\Phi(x,y) \, |\Phi(x,y)|^{p-1} P_m V_{x,y}^*) \right| \\ &\leq \|\Phi(x,x)\|_p^{1/2} \, \|\Phi(y,y)\|_p^{1/2} \, \| |\Phi(x,y)|^{p-1} P_m V_{x,y}^* \|_q . \end{aligned}$$

Hence, letting $m \to \infty$, and using that

$$|\Phi(x,y)|^{p-1} P_m V_{x,y}^* \to |\Phi(x,y)|^{p-1} V_{x,y}^*, \text{ in } L^q(\rho) \text{ as } m \to \infty,$$

we obtain

$$\begin{aligned} |\rho\left(|\Phi(x,y)|^p\right)| &= \left| \rho\left(\Phi(x,y) |\Phi(x,y)|^p V_{x,y}^*\right) \right| \\ &\leq \|\Phi(x,x)\|_p^{1/2} \, \|\Phi(y,y)\|_p^{1/2} \, \| |\Phi(x,y)|^{p-1} V_{x,y}^* \|_q . \end{aligned}$$

Since $\| |\Phi(x,y)|^{p-1} V_{x,y}^* \|_q \leq \| |\Phi(x,y)|^{p-1} \|_q = \| |\Phi(x,y)| \|_q^{p-1}$, dividing both sides by $\|\Phi(x,y)\|_p^{p-1}$ gives

$$\|\Phi(x,y)\|_p \leq \|\Phi(x,x)\|_p^{1/2} \, \|\Phi(y,y)\|_p^{1/2} .$$

This completes the proof. □

Motivated by Ref. [26, Theorem 1.3.1 (ii)], we give the following

Corollary 3.8. *Let $p > 1$ and $\mathfrak{A}$ be a $*$-algebra and $\omega : \mathfrak{A} \to L^p(\rho)$ be a positive linear map. If $x, y \in \mathfrak{A}$ are such that $\omega(y^* x) \in L^p(\rho) \cap L^\infty(\rho)$ and $\omega(y^* x)$ is normal, then*

$$\|\omega(y^* x)\|_p \leq \|\omega(x^* x)\|_p^{1/2} \, \|\omega(y^* y)\|_p^{1/2} .$$

Proof. Let $x, y \in \mathfrak{A}$ and define $\Phi_\omega(x,y) = \omega(y^*x)$. If $\omega(y^*x) \in L^p(\rho) \cap L^\infty(\rho)$ is normal, the thesis follows by applying Proposition 3.7 to the positive sesquilinear map Φ_ω. $\square$

Remark 3.9. Note that if $\mathfrak{A}$ is a C^*-algebra with unit e and $\|\omega\| \leq 1$, then for all $a \in \mathfrak{A}$ such that $\omega(a) \in L^p(\rho) \cap L^\infty(\rho)$ and $\omega(a)$ is normal,

$$\|\omega(a)\|_p^2 \leq \|\omega(a^*a)\|_p \|\omega(\mathsf{e})\|_p \leq \|\omega(a^*a)\|_p,$$

hence, we obtain a certain link to Kadison–Schwarz inequality for normal elements (see, e.g., Ref. [26, Theorem 1.3.1 (ii)]). In fact, Corollary 3.8 can be viewed as an *opposite or symmetric version* of Ref. [26, Theorem 1.3.1 (ii)] in the setting of noncommutative L^p-spaces, since there it is assumed that a is a normal element, whereas we assume that $\omega(a)$ is normal in Corollary 3.8.

4. Generalized Numerical Radius Norm on $L^2(\rho)$ and Related Cauchy–Schwarz Inequality

In this section, we introduce a new norm on $L^2(\rho)$ as a generalization of the numerical radius norm on $\mathfrak{B}(\mathcal{H})$. This norm is such that every $L^2(\rho)$-valued positive sesquilinear map satisfies the Cauchy–Schwarz inequality in this new norm. Hence, this allows to represent such maps in a Banach space and not just in a quasi-Banach space, see Remark 4.7.

Let $\mathfrak{M}$ be a factor of type either I or II on a Hilbert space $\mathcal{H}$, and let ρ be a faithful semifinite trace on $\mathfrak{M}$. Let $|||\cdot|||_2 : L^2(\rho) \to \mathbb{R}^+$ be given by

$$|||F|||_2 = \sup_{\substack{W \in L^\infty(\rho) \cap L^2(\rho), \\ W \geq 0 \\ \|W\|_2 \leq 1,\ \|W\|_\infty \leq 1}} \|WFW\|_1.$$

Lemma 4.1. *The map $|||\cdot|||_2$ is a norm on $L^2(\rho)$.*

Proof. Let us first show that $|||\cdot|||_2$ is well defined. Let $W \in L^\infty(\rho)$, $W \geq 0$. If $F \in L^2(\rho)$, then $FW \in L^2(\rho)$. If in addition $W \in L^2(\rho)$ and both $\|W\|_2 \leq 1$ and $\|W\|_\infty \leq 1$, then by the noncommutative Hölder inequality, $WFW \in L^1(\rho)$ and

$$\|WFW\|_1 \leq \|W\|_2 \, \|FW\|_2 \leq \|W\|_2 \, \|F\|_2 \|W\|_\infty \leq \|F\|_2.$$

Thus, $|||\cdot|||_2$ is well defined.

Homogeneity and the triangle inequality follow from the fact that $\|\cdot\|_1$ is a norm.

Now assume $|||F|||_2 = 0$, then $WFW = 0$ for every finite operator $W \in L^\infty(\rho)$ with $W > 0$. Let $D \in L^\infty(\rho) \cap L^2(\rho)$ and write it as

$$D = D_1 - D_2 + i(D_3 - D_4), \quad D_j \geq 0, \forall j \in \{1, \ldots, 4\}.$$

Then, $\|D_j\|_2 \leq \|D\|_2$ for each $j \in \{1, \ldots, 4\}$.

By Lemma 2.1, for each j, there exists a sequence of finite projections $\{P_n^{(j)}\}_n \subseteq \mathfrak{M}$ such that

$$\lim_{n\to\infty} \|D_j(I - P_n^{(j)})\|_2 = 0, \quad \forall j \in \{1, \ldots, 4\}.$$

Moreover, by the construction as in the proof of Ref. [1, Lemma 9.8], each $P_n^{(j)} = E_{D_j}\left(\frac{1}{n}, \infty\right)$ for all $n \in \mathbb{N}$ and $j \in \{1, \ldots, 4\}$ where E_{D_j} is the spectral measure corresponding to D_j, hence $P_n^{(j)}$ commutes with ${D_j}^{1/2}$ for all $n \in \mathbb{N}$ and $j \in \{1, \ldots, 4\}$.

Since for every $n \in \mathbb{N}$ and every $j \in \{1, \ldots, 4\}$, $P_n^{(j)} {D_j}^{1/2} P_n^{(j)}$ is a finite positive operator, we get

$$\begin{aligned} \rho(FD_jP_n^{(j)}) &= \rho(F{D_j}^{1/2}{D_j}^{1/2}P_n^{(j)}P_n^{(j)}) = \rho(F({D_j}^{1/2}P_n^{(j)})^2) \\ &= \rho({D_j}^{1/2}P_n^{(j)}F{D_j}^{1/2}P_n^{(j)}) \\ &= \rho(P_n^{(j)}{D_j}^{1/2}P_n^{(j)}FP_n^{(j)}{D_j}^{1/2}P_n^{(j)}), \ \forall n \in \mathbb{N}, \forall j \in \{1, \ldots, 4\} \end{aligned}$$

since ${D_j}^{1/2}P_n^{(j)} = {D_j}^{1/2}P_n^{(j)}P_n^{(j)} = P_n^{(j)}{D_j}^{1/2}P_n^{(j)}$, for all $n \in \mathbb{N}$ and $j \in \{1, \ldots, 4\}$. Since $WFW = 0$ for every positive finite operator W, we get

$$\rho(FD_jP_n^{(j)}) = 0, \quad \forall n \in \mathbb{N}, \forall j \in \{1, \ldots, 4\}.$$

On the other hand, since

$$|\rho(FD_j(I - P_n^{(j)}))| \le \|FD_j(I - P_n^{(j)})\|_1 \le \|F\|_2 \, \|D_j(I - P_n^{(j)})\|_2 \to 0,$$

as $n \to \infty$, we deduce that $\rho(FD_j) = 0$ for every $j \in \{1, \ldots, 4\}$. Thus $\rho(FD) = 0$ for every $D \in L^\infty(\rho) \cap L^2(\rho)$, and therefore $\rho(FF^*) = 0$. Thus $F = 0$ since both $\|F\|_2 = 0$ and $\|\cdot\|_2$ is a norm. □

Remark 4.2. If $A \in M_n(\mathbb{C})$, the space of the $n \times n$ matrices with complex entries, then for every $x \in \mathbb{C}^n$, we have

$$|\langle Ax, x\rangle| = \mathrm{tr}(|X^*AX|),$$

where $X \in M_n(\mathbb{C})$ is the operator having the vector $x \in \mathbb{C}^n$ in its first column and the zero vector of $\mathbb{C}^n$ in all the other ones. In this way, the norm $|||\cdot|||_2$ can be considered as a generalization of the numerical radius norm.

Corollary 4.3. *Let $\mathfrak{X}$ be a vector space, $\Phi : \mathfrak{X} \times \mathfrak{X} \to (L^2(\rho), |||\cdot|||_2)$ be a positive sesquilinear map. Then, for all $x, y \in \mathfrak{X}$,*

$$|||\Phi(x, y)|||_2 \le |||\Phi(x, x)|||_2^{1/2} \, |||\Phi(y, y)|||_2^{1/2}.$$

Proof. Let $W \in L^\infty(\rho) \cap L^2(\rho)$ be such that $W \ge 0$, $\|W\|_2, \|W\|_\infty \le 1$, and define

$$\Phi_W(x, y) := W\Phi(x, y)W, \quad \forall x, y \in \mathfrak{X}.$$

Then, Φ_W is a $L^1(\rho)$-valued positive sesquilinear map. By Ref. [2, Proposition 3.1, case 7], we have

$$\begin{aligned} \|\Phi_W(x, y)\|_1 &\le \|\Phi_W(x, x)\|_1^{1/2} \, \|\Phi_W(y, y)\|_1^{1/2} \\ &\le |||\Phi(x, x)|||_2^{1/2} \, |||\Phi(y, y)|||_2^{1/2}, \quad \forall x, y \in \mathfrak{X}. \end{aligned}$$

Taking the supremum over all such W yields the claim. □

Remark 4.4. As we will see later in Remark 4.7, by using Corollary 4.3, we will be able to represent in a quasi-Banach space, whose quasi-norm is induced by an $\mathfrak{Y}$-valued quasi-inner product (see Ref. [2]), every positive linear map from a unital *-algebra into $L^2(\rho)$.

Let $\mathfrak{Y} = \mathcal{L}^2(\rho)$ be the completion of $(L^2(\rho), |||\cdot|||_2)$ and $\mathfrak{M}$ be a factor of type I or II. Now we consider the space $\mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$ and denote the operator norm in $\mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$ by $\|\cdot\|_{o.n.}$.

Definition 4.5. A sesquilinear map $\Phi : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$ will be called *positive* if $\Phi(x, x)(T)$ is a positive element of $L^2(\rho)$ for all $x \in \mathfrak{X}$ whenever $T \in \mathfrak{M}$ and $T \geq 0$.

Note that if we let $\mathfrak{K}$ be the subset of $\mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$ consisting of all those operators that map positive elements of $\mathfrak{M}$ into positive elements of $L^2(\rho)$, then it can be checked that $\mathfrak{K}$ is a cone in $\mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$. Further, $\mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$ is in fact an ordered Banach bimodule over $\mathfrak{M}$ where the multiplication is defined as follows: for all $a \in \mathfrak{M}$ and $T \in \mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$,

$$(a \cdot T)(x) := T(ax) \text{ and } (T \cdot a)(x) := T(xa), \quad x \in \mathfrak{M}.$$

The proof of Ref. [2, Proposition 3.1] motivates the one of the next theorem.

Theorem 4.6. *Let $\Phi : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$ be a positive sesquilinear map. Then, for all $x, y \in \mathfrak{X}$,*

$$\|\Phi(x, y)\|_{o.n.} \leq \|\Phi(x, x)\|_{o.n.}^{1/2} \, \|\Phi(y, y)\|_{o.n.}^{1/2}.$$

Proof. Let $W \in L^2(\rho) \cap L^\infty(\rho)$ with $W \geq 0$, $\|W\|_2 \leq 1$, $\|W\|_\infty \leq 1$ and Π and Θ be two simple operator-valued functions in $\mathfrak{M}$

$$\Pi = \sum_{i=1}^{N} c_i P_i, \qquad \Theta = \sum_{j=1}^{M} d_j Q_j,$$

where c_i, d_j are scalars for every $i \in \{1, \dots, N\}$, $j \in \{1, \dots, M\}$ and $\{P_i\}$ and $\{Q_j\}$ are families of mutually orthogonal projections and $\sum_{i=1}^N P_i = \sum_{j=1}^M Q_j = I$. Then, if $x, y \in \mathfrak{X}$

$$|\rho(\Pi\, W \Phi(x, y)(\Theta) W)| = \left| \sum_{i,j} c_i d_j \, \rho(P_i W \Phi(x, y)(Q_j) W) \right|.$$

Thus,

$$\begin{aligned}
&\left| \rho\left(\sum_{i=1}^{N} c_i P_i W \Phi(x, y) \left(\sum_{j=1}^{M} d_j Q_j \right) W \right) \right| \\
&\leq \sum_{i,j} |c_i|\, |d_j|\, |\rho(P_i W \Phi(x, y)\, (Q_j)\, W)| \\
&= \sum_{i,j} |c_i|\, |d_j|\, \big|\rho\big(P_i^2 W \Phi(x, y)\, (Q_j)\, W\big)\big| \\
&= \sum_{i,j} |c_i|\, |d_j|\, |\rho(P_i W \Phi(x, y)\, (Q_j)\, W P_i)|\,.
\end{aligned}$$

For each pair (i, j), define the sesquilinear form on $\mathfrak{X}$

$$\psi_{ij}(x, y) = \rho(P_i W \Phi(x, y)(Q_j) W P_i)\,, \quad x, y \in \mathfrak{X}.$$

Since Φ is positive, each ψ_{ij} is positive. Applying the Cauchy–Schwarz inequality to every ψ_{ij} and using the fact that $\sum_i P_i = \sum_j Q_j = I$, we obtain

$$\begin{aligned}
&\sum_{i,j} |c_i|\,|d_j|\,|\rho(P_i W \Phi(x,y)\,(Q_j)\,W P_i)|\\
&\leq \sum_{i,j} |c_i|\,|d_j|\,(\rho(P_i W\Phi(x,x)(Q_j)WP_i))^{1/2}\\
&\qquad\qquad \cdot (\rho(P_i W\Phi(y,y)(Q_j)WP_i))^{1/2}\\
&\leq \left(\sum_{i,j} |c_i|\,|d_j|\rho(P_i W\Phi(x,x)(Q_j)WP_i)\right)^{1/2}\\
&\qquad\qquad \cdot \left(\sum_{i,j} |c_i|\,|d_j|\rho(P_i W\Phi(y,y)(Q_j)WP_i)\right)^{1/2}\\
&= \left(\sum_{i,j} |c_i|\,|d_j|\rho(P_i W\Phi(x,x)(Q_j)W)\right)^{1/2}\\
&\qquad\qquad \cdot \left(\sum_{i,j} |c_i|\,|d_j|\rho(P_i W\Phi(y,y)(Q_j)W)\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{i=1}^{N}\sum_{j=1}^{M} \|\Pi\|_\infty\,\|\Theta\|_\infty \rho(P_i W\Phi(x,x)(Q_j)W)\right)^{1/2}\\
&\qquad\qquad \cdot \left(\sum_{i=1}^{N}\sum_{j=1}^{M} \|\Pi\|_\infty\,\|\Theta\|_\infty \rho(P_i W\Phi(y,y)(Q_j)W)\right)^{1/2}\\
&= \|\Pi\|_\infty\,\|\Theta\|_\infty\,(\rho(W\Phi(x,x)(I)W))^{1/2}\,(\rho(W\Phi(y,y)(I)W))^{1/2}\\
&\leq \|\Pi\|_\infty\,\|\Theta\|_\infty |||\Phi(x,x)(I)|||_2^{1/2}\,|||\Phi(y,y)(I)|||_2^{1/2}\\
&\leq \|\Pi\|_\infty\,\|\Theta\|_\infty \|\Phi(x,x)\|_{o.n.}^{1/2}\,\|\Phi(y,y)\|_{o.n.}^{1/2}. \qquad (4.1)
\end{aligned}$$

Since this holds for all simple functions Π, Θ in $\mathfrak{M}$, given an unitary operator $U \in \mathfrak{M}$, there exists a sequence $\{\Pi_n\}_n$ of simple operator-valued functions in $\mathfrak{M}$ such that $\Pi_n \to U$ as $n \to \infty$ in operator norm. Thus, for every $n \in \mathbb{N}$, by Ref. [10, Proposition 3.4.5, Corollary 3.4.6], we get, for every $x, y \in \mathfrak{X}$,

$$\begin{aligned}
&|\rho\left((U-\Pi_n)W\Phi(x,y)(\Theta)W\right)| \leq \rho(|(U-\Pi_n)W\Phi(x,y)(\Theta)W|)\\
&\leq \|U-\Pi_n\|_\infty \|W\Phi(x,y)(\Theta)W\|_1 \to 0, \qquad \text{as } n\to\infty.
\end{aligned}$$

On the other hand, by (4.1), we have

$$|\rho\left(\Pi_n W\Phi(x,y)(\Theta)W\right)| \leq \|\Pi_n\|_\infty \|\Theta\|_\infty \|\Phi(x,x)\|_{o.n.}^{1/2}\,\|\Phi(y,y)\|_{o.n.}^{1/2}$$

for every $x, y \in \mathfrak{X}$ and as $n \to \infty$ on both sides, we obtain

$$|\rho\,(UW\Phi(x,y)(\Theta)W)| \le \|\Theta\|_\infty \|\Phi(x,x)\|_{o.n.}^{1/2}\, \|\Phi(y,y)\|_{o.n.}^{1/2}, \quad \forall x, y \in \mathfrak{X}.$$

If now V is a convex combination of unitary operators in $\mathfrak{M}$, then we also easily get

$$|\rho\,(VW\Phi(x,y)(\Theta)W)| \le \|\Theta\|_\infty \|\Phi(x,x)\|_{o.n.}^{1/2}\, \|\Phi(y,y)\|_{o.n.}^{1/2}, \tag{4.2}$$

for all $x, y \in \mathfrak{X}$.

Let $m \in \mathbb{N}$ and $A_m \in L^1(\rho) \cap L^\infty(\rho)$ be such that

$$\|A_m - W\,\Phi(x,y)(\Theta)W\|_1 < \frac{1}{2^m}$$

and let Z_m be the partial isometry from the polar decomposition of A_m, choose a sequence $\{V_n^{(m)}\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ of convex combinations of unitaries in $\mathfrak{M}$ such that

$$V_n^{(m)} \to Z_m^* \quad \text{in the operator norm as } n \to \infty.$$

Then,

$$\left|\rho\left((V_n^{(m)} - Z_m^*)\,W\Phi(x,y)(\Theta)W\right)\right| \le \|V_n^{(m)} - Z_m^*\|_\infty\, \|W\Phi(x,y)(\Theta)W\|_1 \to 0,$$

as $n \to \infty$, for every $x, y \in \mathfrak{X}$. By (4.2), for every $n \in \mathbb{N}$, it is

$$\left|\rho\left(V_n^{(m)}W\Phi(x,y)(\Theta)W\right)\right| \le \|\Theta\|_\infty\, \|\Phi(x,x)\|_{o.n.}^{1/2}\, \|\Phi(y,y)\|_{o.n.}^{1/2},$$

for every $x, y \in \mathfrak{X}$. Letting $n \to \infty$, we obtain that

$$|\rho\,(Z_m^* W\Phi(x,y)(\Theta)W)| \le \|\Theta\|_\infty\, \|\Phi(x,x)\|_{o.n.}^{1/2}\, \|\Phi(y,y)\|_{o.n.}^{1/2},$$

for every $x, y \in \mathfrak{X}$. Now, observe that for every $x, y \in \mathfrak{X}$,

$$\begin{aligned}
&|\rho(|A_m|) - |\rho\,(Z_m^* W\Phi(x,y)(\Theta)W)|\,| \\
&= |\rho(Z_m^* A_m) - |\rho\,(Z_m^* W\Phi(x,y)(\Theta)W)|\,| \\
&\le |\rho(Z_m^*(A_m - W\Phi(x,y)(\Theta)W))| \\
&\le \|Z_m^*\|_\infty \|A_m - W\Phi(x,y)(\Theta)W\|_1;
\end{aligned}$$

then by Ref. [10, Proposition 3.4.5, Corollary 3.4.6],

$$\begin{aligned}
&|\,\|W\Phi(x,y)(\Theta)W\|_1 - |\rho\,(Z_m^* W\Phi(x,y)(\Theta)W)|\,| \\
&= |\,\|W\Phi(x,y)(\Theta)W\|_1 - \|A_m\|_1 + \|A_m\|_1 - |\rho\,(Z_m^* W\Phi(x,y)(\Theta)W)|\,| \\
&\le |\,\|W\Phi(x,y)(\Theta)W\|_1 - \|A_m\|_1\,| + |\rho(|A_m|) - |\rho\,(Z_m^* W\Phi(x,y)(\Theta)W)|\,| \\
&\le \frac{1}{2^m} + \|Z_m^*\|_\infty \|A_m - W\Phi(x,y)(\Theta)W\|_1 < \frac{1}{2^{m-1}}, \quad \forall x, y \in \mathfrak{X}.
\end{aligned}$$

Hence, for every $x, y \in \mathfrak{X}$,

$$\|W\Phi(x,y)(\Theta)W\|_1 \le \|\Theta\|_\infty \|\Phi(x,x)\|_{o.n.}^{1/2}\, \|\Phi(y,y)\|_{o.n.}^{1/2} + \frac{1}{2^{m-1}}.$$

Since this holds for all $m \in \mathbb{N}$, we deduce that for every $x, y \in \mathfrak{X}$,

$$\|W\Phi(x,y)(\Theta)W\|_1 \le \|\Theta\|_\infty \|\Phi(x,x)\|_{o.n.}^{1/2}\, \|\Phi(y,y)\|_{o.n.}^{1/2}. \tag{4.3}$$

Next, given a unitary $U \in \mathfrak{M}$, choose a sequence $\{\Theta_n\}_n$ of simple operator-valued functions in $\mathfrak{M}$ such that $\Theta_n \to U$ as $n \to \infty$ in the operator norm. Then, for every $x, y \in \mathfrak{X}$

$$\begin{aligned}\| W\Phi(x,y)(\Theta_n - U)W\|_1 &\leq |||\Phi(x,y)(\Theta_n - U)|||_2 \\ &\leq \|\Theta_n - U\|_\infty \,\|\Phi(x,y)\|_{o.n.} \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

It follows that, for every $x, y \in \mathfrak{X}$,

$$\|W\Phi(x,y)(\Theta_n)W\|_1 \longrightarrow \|W\Phi(x,y)(U)W\|_1, \qquad \text{as } n \to \infty.$$

However, by (4.3), for every $x, y \in \mathfrak{X}$,

$$\|W\Phi(x,y)(\Theta_n)W\|_1 \leq \|\Theta_n\|_\infty \,\|\Phi(x,x)\|_{o.n.}^{1/2} \,\|\Phi(y,y)\|_{o.n.}^{1/2} \quad \text{for all } n,$$

hence, letting $n \to \infty$ gives

$$\|W\Phi(x,y)(U)W\|_1 \leq \|\Phi(x,x)\|_{o.n.}^{1/2} \,\|\Phi(y,y)\|_{o.n.}^{1/2}, \quad \forall x, y \in \mathfrak{X}.$$

Once again, if V is a convex combination of unitaries in $\mathfrak{M}$, the same argument shows that

$$\|W\Phi(x,y)(V)W\|_1 \leq \|\Phi(x,x)\|_{o.n.}^{1/2} \,\|\Phi(y,y)\|_{o.n.}^{1/2}, \quad \forall x, y \in \mathfrak{X}. \tag{4.4}$$

Now let $T \in \mathfrak{M}$ with $\|T\|_\infty \leq 1$ and let $\{V_n\}_n \subset \mathfrak{M}$ be a sequence of convex combinations of unitary operators in $\mathfrak{M}$ such that $V_n \to T$ as $n \to \infty$ in the operator norm. Then, for every $x, y \in \mathfrak{X}$,

$$\begin{aligned}&|\|W\Phi(x,y)(V_n)W\|_1 - \|W\Phi(x,y)(T)W\|_1| \\ &\leq \|W\Phi(x,y)(V_n - T)W\|_1 \leq \|V_n - T\|_\infty \,\|\Phi(x,y)\|_{o.n.} \to 0, \text{ as } n \to \infty.\end{aligned}$$

Hence,

$$\|W\Phi(x,y)(V_n)W\|_1 \to \|W\Phi(x,y)(T)W\|_1, \quad \text{as } n \to \infty, \quad \forall x, y \in \mathfrak{X}.$$

Since (4.4) holds for every V_n, then we can conclude that it is

$$\|W\Phi(x,y)(T)W\|_1 \leq \|\Phi(x,x)\|_{o.n.}^{1/2} \,\|\Phi(y,y)\|_{o.n.}^{1/2}, \quad \forall x, y \in \mathfrak{X}.$$

Since $W \in \mathfrak{M}$ with $W \geq 0$ and both $\|W\|_\infty \leq 1$ and $\|W\|_2 \leq 1$ has been chosen arbitrarily, taking the supremum over all such W yields

$$|||\Phi(x,y)(T)|||_2 \leq \|\Phi(x,x)\|_{o.n.}^{1/2} \,\|\Phi(y,y)\|_{o.n.}^{1/2}, \quad \forall x, y \in \mathfrak{X}.$$

Finally, taking the supremum over all $T \in \mathfrak{M}$ with $\|T\|_\infty \leq 1$, we conclude that

$$\|\Phi(x,y)\|_{o.n.} \leq \|\Phi(x,x)\|_{o.n.}^{1/2} \,\|\Phi(y,y)\|_{o.n.}^{1/2}, \quad \forall x, y \in \mathfrak{X}.$$

This completes the proof. □

Remark 4.7. Let $\mathfrak{A}$ be a unital *-algebra and $\mathfrak{Y}$ an ordered Banach bimodule over the *-algebra $\mathfrak{Y}_0$. If $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{Y}$ is a left-invariant positive sesquilinear map such that

$$\|\Phi(x_1, x_2)\|_{\mathfrak{Y}} \leq 2\|\Phi(x_1, x_1)\|_{\mathfrak{Y}}^{1/2} \|\Phi(x_2, x_2)\|_{\mathfrak{Y}}^{1/2}, \quad \forall x_1, x_2 \in \mathfrak{A}$$

then, by Ref. [3, Theorem 3.2], Φ can be represented in a quasi-Banach space whose quasi-norm is induced by an $\mathfrak{Y}$-valued quasi-inner product (see Ref. [2]), whereas if in fact

$$\|\Phi(x_1, x_2)\|_{\mathfrak{Y}} \leq \|\Phi(x_1, x_1)\|_{\mathfrak{Y}}^{1/2} \|\Phi(x_2, x_2)\|_{\mathfrak{Y}}^{1/2}, \quad \forall x_1, x_2 \in \mathfrak{A}$$

then Φ can be actually represented in a Banach space and not just in a quasi-Banach one (see Ref. [2, Proposition 3.9]). These representations induce further representations of positive linear maps from $\mathfrak{A}$ into $\mathfrak{Y}$, see Ref. [3, Corollary 3.10] and Ref. [2, Corollary 3.12] for the details. This fact was the main motivation and purpose of Cauchy–Schwarz inequalities obtained in this paper. However, since $|||\cdot|||_2 \leq \|\cdot\|_2$, then $\mathfrak{B}(\mathfrak{M}, (L^2(\rho), \|\cdot\|_2)) \subseteq \mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$, hence by obtaining a representation of general positive linear maps from $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$ we can get in this way representations of positive linear maps from $\mathfrak{A}$ in $\mathfrak{B}(\mathfrak{M}, (L^2(\rho), \|\cdot\|_2))$.

Example. Let ρ be a finite trace. Let $W \in L^\infty(\rho)$, with $W \geq 0$. Let $k \in C([0, \|W\|] \times [0, \|W\|])$ such that $k \geq 0$. Then, for each $x \in [0, \|W\|]$, the function $\eta_x : [0, \|W\|] \to \mathbb{C}$ defined by $\eta_x(t) = k(x, t)$ is a continuous, positive function on $[0, \|W\|]$. Therefore, by the functional calculus, $\eta_x(W)$ defines a positive operator in $L^\infty(\rho)$. Let us define, for every $x \in [0, \|W\|]$ and $X, Y \in L^2(\rho)$,

$$\varphi(X, Y)(x) = \rho(X\eta_x(W)Y^*).$$

Then, $\varphi(X, Y) \in C([0, \|W\|])$ for all $X, Y \in L^2(\rho)$. Moreover, $\varphi : L^2(\rho) \times L^2(\rho) \to C([0, \|W\|])$ is a bounded, left-invariant, positive sesquilinear map (see Ref. [3]).

Let $T \in L^4(\rho)$ and consider the map

$$\widetilde{\varphi} : L^2(\rho) \times L^2(\rho) \to L^2(\rho)$$

given by $\widetilde{\varphi}(X, Y) = T(\varphi(X, Y)(W))T^*$, for all $X, Y \in L^2(\rho)$, by functional calculus it is also a bounded, left-invariant positive sesquilinear map. Now, if $A, B \in L^\infty(\rho)$ are such that $A = A^*$ and $B = B^*$, let $K := i(AB - BA)$. Then, $K = K^*$. Moreover, by some calculations, it can be checked that the conditions of Proposition 3.6 are satisfied in this case.

Let us now suppose that $T \in L^4(\rho)$ and let

$$\Phi : L^2(\rho) \times L^2(\rho) \to \mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$$

be given by

$$\Phi(X, Y)(S) = T\rho\left(X\eta_x(W)S\eta_x(W)Y^*\right)(W)T^*$$

for all $S \in \mathfrak{M}$ and $X, Y \in L^2(\rho)$. For each $X, Y \in L^2(\rho)$ and $S \in \mathfrak{M}$, $\Phi(X, Y)(S) \in L^2(\rho) \cap L^\infty(\rho)$ and

$$|||\Phi(X, Y)(S)|||_2 \leq \|T\|_4^2 \|k\|_\infty^2 \,\|X\|_2 \,\|Y\|_2 \|S\|_\infty,$$

so Φ is a bounded, positive, left-invariant sesquilinear map from $L^2(\rho) \times L^2(\rho)$ into $\mathfrak{B}(\mathfrak{M}, \mathcal{L}^2(\rho))$ and such that $\Phi(X, Y)(\mathfrak{M}) \subset L^2(\rho)$ for all $X, Y \in L^2(\rho)$.

Acknowledgements

GB acknowledges that this work has been carried out within the activities of Gruppo UMI Teoria dell'Approssimazione e Applicazioni and of GNAMPA of the INdAM. SI is supported by the Ministry of Science, Technological Development and Innovations, Republic of Serbia, grant no. 451-03-66/2024-03/200029.

Author contributions All the authors contributed to the research and writing of this paper.

Funding Open access funding provided by Università degli Studi di Palermo within the CRUI-CARE Agreement.

Data Availability Statement No datasets were generated or analyzed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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Received: March 17, 2026.
Revised: March 17, 2026.
Accepted: May 3, 2026.