VARIETIES OF SPECIAL JORDAN ALGEBRAS OF ALMOST POLYNOMIAL GROWTH

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Abstract. Let J be a special Jordan algebra and let $c_n(J)$ be its corresponding codimension sequence. The aim of this paper is to prove that in case J is finite dimensional, such a sequence is polynomially bounded if and only if the variety generated by J does not contain UJ_2, the special Jordan algebra of 2×2 upper triangular matrices. As an immediate consequence, we prove that UJ_2 is the only finite dimensional special Jordan algebra that generates a variety of almost polynomial growth.

1. Introduction

Let $\mathcal{J}(X)$ be the free Jordan algebra over a field F of characteristic zero on a countable set X and let J be a Jordan algebra over F. A polynomial of $\mathcal{J}(X)$ vanishing under every evaluation in J is called polynomial identity of J and we denote by $\mathrm{Id}(J)$ the T-ideal of polynomial identities satisfied by J. If P_n is the space of multilinear polynomials in the variables $x_1, x_2, \ldots, x_n$, we also denote by

$$c_n(J) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(J)}, \quad n = 1, 2, \ldots,$$

the codimension sequence of J. Such a sequence was introduced firstly in the setting of associative algebras by Regev in [23] and it gives a quantitative measure of the identities satisfied by a given algebra. In the same paper, the author showed that if A is an associative algebra satisfying a non-trivial polynomial identity (PI-algebra), then $c_n(A)$ is exponentially bounded.

Later on, Kemer in [16] and [17] proved several properties about codimension sequence of associative algebras. On one hand he showed that $c_n(A)$ is polynomially bounded or grows exponentially, on the other he gave a characterization of the varieties of associative algebras of polynomial growth of the codimensions proving that $c_n(A)$ is polynomially bounded if and only if G and UT_2 do not satisfy all the identities of A, where G is the infinite dimensional Grassmann algebra and UT_2 is the algebra of 2×2 upper triangular matrices. Then Giambruno and Zaicev in [10] and [11] showed that the exponential growth of $c_n(A)$ is always an integer called the PI-exponent (sometimes just the exponent) of the algebra A. The scale provided by the exponent has been exploited in the past years in order to classify some significant classes of algebras (see [12] for more details).

Similar results about codimension sequence and polynomial growth for associative algebras were given by several authors in various settings, such as superalgebras ([7]), algebras graded by a finite abelian group ([24]), algebras with involution ([6]), superinvolution ([5]) and pseudoinvolution ([19]).

Concerning non-associative algebras, in general the codimension sequence has overexponential growth, in fact if J is such an algebra, then

$$c_n(J) \leq \frac{1}{n}\binom{2n-2}{n-1} n!$$

where $\frac{1}{n}\binom{2n-2}{n-1}$ is the Catalan number. A remarkable case is represented by the family of (not necessarily associative) finite dimensional algebras, since it is well-known that in this setting the codimensions are bounded by d^n where d is the dimension of the algebra. Moreover, in [8] it was proved that if $\dim_F A = d$, then either $c_n(A)$ is polynomially bounded or $c_n(A) > \frac{1}{n^2} 2^{\frac{n}{3d^2}}$ for n large enough. Thus no intermediate growth is allowed between exponential and polynomial in the finite dimensional case.

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Here we deal with finite dimensional special Jordan algebras, i.e., Jordan algebras that have an associative enveloping algebra. If one considers a variety $\mathcal{V}$ of such algebras, the growth of $\mathcal{V}$ is defined as the growth of an algebra J generating $\mathcal{V}$, that means $\mathcal{V} = \mathrm{var}(J)$ and in this case we write $c_n(\mathcal{V}) = c_n(J)$. Thus we say that $\mathcal{V}$ has polynomial growth if $c_n(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n(\mathcal{V})$ grows exponentially but every proper subvariety grows polynomially.

The main goal of this paper is to characterize the varieties of finite dimensional special Jordan algebras by showing that $\mathcal{V}$ has polynomial growth if and only if it does not contain $UJ_2(F)$, the 2×2 upper triangular matrix algebra endowed with the Jordan product, $a \circ b = \frac{1}{2}(ab + ba)$. Moreover, as a corollary, we classify up to PI-equivalence the varieties of finite dimensional special Jordan algebras of almost polynomial growth. Recall that two algebras A and B are said to be PI-equivalent if and only if they have the same polynomial identities.

2. Preliminaries

Throughout this paper F will denote an algebraically closed field of characteristic zero unless explicitly written otherwise.

Let X be a countable set of indeterminates and let $\mathcal{J}(X)$ be the free Jordan algebra generated by the set X over F. We say that a polynomial $f(x_1, \dots, x_n) \in \mathcal{J}(X)$ is a polynomial identity for the Jordan algebra J if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in J$. In this case we write $f \equiv 0$. The identities of J form a T-ideal of $\mathcal{J}(X)$, i.e., an ideal closed under all endomorphisms of the free Jordan algebra. Let us denote by $\mathrm{Id}(J) = \{f \in \mathcal{J}(X) |\ f \equiv 0 \text{ on } J\}$ the T-ideal of polynomial identities of J. It is well-known (see for example [12, Theorem 1.3.7]) that, in characteristic 0, $\mathrm{Id}(J)$ is determined by the multilinear polynomials it contains. Recall that a multilinear polynomial is an element of the vector subspace

$$P_n = span_F\langle x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} |\ \sigma \in S_n\rangle,$$

where S_n is the symmetric group and $x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)}$ stands for a monomial with all possible brackets arrangements. Thus, the relatively free algebra $\frac{\mathcal{J}(X)}{\mathrm{Id}(J)}$ is determined by the sequence of vector subspaces

$$P_n(J) = \frac{P_n}{P_n \cap \mathrm{Id}(J)},\ n \geq 1.$$

In this way, we can attach to the Jordan algebra J a numerical sequence $c_n(J)$ called the codimension sequence, by defining

$$c_n(J) = \dim_F P_n(J).$$

We shall refer to the growth of the Jordan algebra J as the asymptotic behaviour of its codimension sequence.

Given a non-empty set $S \subseteq \mathcal{J}(X)$, the class of all Jordan algebras J such that $f \equiv 0$ on J for all $f \in S$, is called variety $\mathcal{V} = \mathcal{V}(S)$ determined by S. Similarly, given a Jordan algebra J, the variety of Jordan algebras generated by J, $\mathrm{var}(J)$, is the set of all Jordan algebras satisfying the identities of J. Hence we say that $A \in \mathrm{var}(J)$ if and only if $\mathrm{Id}(J) \subseteq \mathrm{Id}(A)$. It is clear that there exists a one-to-one correspondence between T-ideals and varieties, thus given a variety $\mathcal{V}$, we can naturally define $\mathrm{Id}(\mathcal{V})$, $P_n(\mathcal{V})$ and $c_n(\mathcal{V})$. The growth of $\mathcal{V}$ will be the asymptotic behaviour of $c_n(\mathcal{V})$. Moreover, we say that $\mathcal{V}$ has almost polynomial growth if its codimension sequence is exponentially bounded and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$, $c_n(\mathcal{U})$ grows polynomially.

Now let J be a finite dimensional special Jordan algebra. Recall that a Jordan algebra is special if there exists an associative algebra A such that $J \subseteq A^{(+)}$, where $A^{(+)}$ is the Jordan algebra obtained by the same vector space structure of A endowed with a new multiplication, called the Jordan product, that is $a \circ b = \frac{1}{2}(ab + ba)$ for all $a, b \in A$. In this case A is called the associative enveloping algebra of J. Jordan algebras which do not share this property are called exceptional.

In case $c_n(J)$ is exponentially bounded, as in finite dimensional one, we can construct the bounded sequence $\sqrt[n]{c_n(J)}$, $n = 1, 2, \dots$, and ask if $\exp(J) = \lim\limits_{n \to +\infty} \sqrt[n]{c_n(J)}$ exists. By [9, Lemma 5],we get that a variety $\mathcal{V}$ has polynomial growth if and only if $\exp(\mathcal{V}) = 1$.

In [11] it was proved that for any associative algebra A the PI-exponent $\exp(A)$ exists and is an integer. In case of finite dimensional Lie algebras, in [25] it was shown that the PI-exponent also exists and it is an integer. The same conclusion was achieved in [13, 14] for special simple Jordan algebras. In particular, the authors proved that the exponent of such an algebra J equals the dimension of J over its center.

Recall that since F is algebraically closed, any finite dimensional special simple Jordan algebra is isomorphic to one of the following:

1. $M_n^{(+)}$, the $n \times n$ matrix algebra with respect to the Jordan product $\circ$.
2. $H(M_n, t)$, the Jordan algebra of $n \times n$ symmetric matrices under transpose involution, with multiplication $\circ$.
3. $H(M_{2n}, s)$, the Jordan algebra of $2n \times 2n$ symmetric matrices under symplectic involution, with multiplication $\circ$.
4. $B_n = F \oplus V$, the Jordan algebra defined by a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the n-dimensional vector space V. Here the multiplication is defined as follows: for all $\alpha + u, \beta + v \in B_n$, $(\alpha + u) \circ (\beta + v) = (\alpha\beta + \langle u, v \rangle) + (\alpha v + \beta u)$, $\alpha, \beta \in F$, $u, v \in V$.

If J is not simple, than it is well-known that it has a Wedderburn–Malcev type decomposition, i.e., there exist simple unitary subalgebras $A_1, \ldots, A_k$ of J such that

$$J = A_1 \oplus \ldots \oplus A_k + R,$$

where $R = \mathrm{Rad}(J)$ is the radical of J. Since we deal with Jordan algebras, we have that R is a strongly nilpotent ideal, thus there exists an integer $T \geq 1$ such that any product of elements of J containing at least T elements of R must be zero. One can find a proof of this decomposition for instance in [21].

We fix a basis $B = B_0 \cup B_1$ of J such that B_0 is a basis of R and B_1 is the union of bases of $A_1, \ldots, A_k$, respectively. In what follows any product of elements of B will be called a monomial of J. Next we define the height of a monomial as follows.

Let $M = M(a_1, \ldots, a_m, b_1, \ldots, b_n)$ be a non-zero monomial of J where $a_1, \ldots, a_m \in B_1$ and $b_1, \ldots, b_n \in B_0$. Then the height of M is

$$ht(M) = \dim_F(A_{i_1} + \ldots + A_{i_m})$$

where $a_{i_1} \in A_{i_1}, \ldots, a_{i_m} \in A_{i_m}$. Since J is a finite dimensional algebra, we can define the integer

$$d = \max \{ht(M) | 0 \neq M \in J\}.$$

In [9, Theorem 3] it was proved that $d = \exp(J)$.

3. The variety generated by $UJ_2(F)$

The variety generated by $UJ_2 = UJ_2(F)$ was extensively studied in the past years. For instance, in [1, 3, 18] a basis for the graded identities, the corresponding cocharacter sequence and the Gelfand–Kirillov dimension were found. Moreover, in [2] it was proved that $\mathrm{var}(UJ_2)$, endowed with any grading, has the Specht property, i.e. the T-ideal of any subvariety if finitely generated.

In this section we prove that $\mathrm{var}(UJ_2)$ has almost polynomial growth, i.e. it grows exponentially but every proper subvariety grows polynomially, by following the lines of [22]. We will improve the original proof by dealing with the free Jordan algebra instead of the free special Jordan algebra as the author did.

From now to the end of the section, if we omit the brackets in a monomial, then we are assuming them left-normalized, hence

$$x_1 x_2 x_3 \cdots x_n = (\cdots((x_1 x_2) x_3) \cdots) x_n.$$

Moreover, recall that an associator between three elements a, b and c of a Jordan algebra J is defined as $(a, b, c) = (ab)c - a(bc)$. We can define by induction an associator between more than three elements of J by left-normalizing the brackets. Thus $(a, b, c, d, e) = ((a, b, c), d, e)$ and so on. Notice that this definition makes sense only if the number of elements inside the associator is odd.

We can summarize the results of [18] about ordinary polynomial identities in the following Theorem.

Theorem 3.1. *Let UJ_2 be the Jordan algebra of 2×2 upper triangular matrices over an infinite field F of characteristic different from 2 and 3. Then:*

1) *$Id(UJ_2) = \langle (x_1 x_2, x_3, x_4) - (x_1, x_3, x_4) x_2 - (x_2, x_3, x_4) x_1,\ (x_1, (x_2, x_3, x_4), x_5) \rangle_T$.*
2) *A basis of $\mathcal{J}(X)$ modulo $Id(UJ_2)$ is given by the following set:*
   $\{(x_k, x_{i_1}, x_{i_2}, \ldots, x_{i_r}) x_{j_1} \ldots x_{j_s} |\ k \leq i_1 \leq i_3 \leq \ldots \leq i_r,\ k \leq i_2,\ j_1 \leq \ldots \leq j_s,\ r + s + 1 = n,\ r \text{ even}\}$,
   for all $n > 1$.

3) $c_n(UJ_2) = (n-2)2^{n-2} + 1$, *hence* $exp(UJ_2) = 2$.

Since in our case char$F = 0$, then we can consider only multilinear polynomials and so we have to translate the basis of the relatively free algebra appearing in the previous theorem in these terms.

In order to prove the main result of this section, we need the following technical lemma.

Lemma 3.1. *The following identity holds in every Jordan algebra:*

$$(x_1, x_2(x_3x_4), x_5) \equiv (x_1, x_2, x_5)x_3x_4 + (x_1, x_3, x_5)x_4x_2 + (x_1, x_4, x_5)x_3x_2 - (x_1, x_2, x_5, x_3, x_4).$$

Proof. We start by recalling that for all Jordan algebras, the following identity holds

$$(x_1, x_2x_3, x_4) \equiv (x_1, x_2, x_4)x_3 + (x_1, x_3, x_4)x_2. \tag{3.1}$$

Let us use (3.1) to expand $(x_1, x_2(x_3x_4), x_5)$.

$$\begin{aligned} (x_1, x_2(x_3x_4), x_5) &\equiv (x_1, x_2, x_5)(x_3x_4) + (x_1, x_3x_4, x_5)x_2 \\ &\equiv (x_1, x_2, x_5)(x_3x_4) + (x_1, x_4, x_5)x_3x_2 + (x_1, x_3, x_5)x_4x_2. \end{aligned} \tag{3.2}$$

Moreover, it is clear that

$$(x_1, x_2, x_5, x_3, x_4) = (x_1, x_2, x_5)x_3x_4 - (x_1, x_2, x_5)(x_3x_4). \tag{3.3}$$

Thus, by putting (3.3) in the right-hand side of (3.2), we get

$$(x_1, x_2(x_3x_4), x_5) \equiv (x_1, x_2, x_5)x_3x_4 + (x_1, x_3, x_5)x_4x_2 + (x_1, x_4, x_5)x_3x_2 - (x_1, x_2, x_5, x_3, x_4),$$

as claimed. □

We can now prove the following theorem that states that var(UJ_2) has almost polynomial growth.

Theorem 3.2. *Let* $\mathcal{V} = var(UJ_2)$ *be the variety of Jordan algebras generated by* UJ_2*. Then for any proper subvariety* $\mathcal{U} \subset \mathcal{V}$, $c_n(\mathcal{U}) \approx qn^k$, *for some positive integer* k *and* $q > 0$.

Proof. Let f be a multilinear polynomial identity of $\mathcal{U}$ such that $f \notin \mathrm{Id}(UJ_2)$. Then, according to 2) of Theorem 3.1, we can write

$$f = \sum_{i_2, I, J} \alpha_{i_1, I, J}(x_k, x_{i_1}, x_{i_2}, \dots, x_{i_r})x_{j_1} \dots x_{j_s} \pmod{\mathrm{Id}(UJ_2)},$$

where $r + s + 1 = n$ and r even. Moreover, for any fixed r and s, $k < i_1 < i_3 < i_4 < \dots < i_r$, $k < i_2$, $j_1 < j_2 < \dots < j_s$, $I = \{i_1, i_2, \dots, i_r\}$ and $J = \{j_1, j_2, \dots, j_s\}$.

In this sum, let us consider the associators with maximal length, say r_{max}, and among them, the ones with x_2 in the third position of the associator. Write

$$f = \sum_{I,J} \alpha_{I,J}(x_k, x_{i_1}, x_2, \dots, x_{i_{r_{max}}})x_{j_1} \dots x_{j_{s_{min}}} + \sum_{i_2, I, J} \alpha_{i_2, I, J}(x_k, x_{i_1}, x_{i_2}, \dots, x_{i_r})x_{j_1} \dots x_{j_s} \pmod{\mathrm{Id}(UJ_2)}, \tag{3.4}$$

where $r \leq r_{max}$.

Substitute now the associator (x_2, x_{n+1}, x_{n+2}) into (3.4) instead of the variable x_2. Notice that we get as a consequence, a polynomial f' in which the maximal length of the associators is $r_{max} + 2$. Moreover, due to the identities of UJ_2, one can easily verify that with the previous substitution, the first sum of f corresponds in f' to a linear combination of polynomials of the type $(x_2, x_{n+1}, x_{n+2}, x_{i_1}, x_1, x_{i_3}, \dots, x_{i_{r_{max}}})x_{j_1} \dots x_{j_{s_{min}}}$. Here also recall that $(a, b, c) = -(c, b, a)$.

Concerning the second sum of f, it turns out that we have two possibilities. If x_2 lies in the third position of an associator of length $r \leq r_{max}$, then we get in f' an associator of length $r + 2$. If x_2 stands either inside an associator in the second position or in a position starting from the fourth or in a tail, then this summand becomes zero since $(x_1, (x_2, x_3, x_4), x_5)$, $(x_1, x_2, x_3)(x_4, x_5, x_6)$ and $(x_1, x_2, x_3, x_4, (x_5, x_6, x_7))$ are identities of UJ_2. Thus,

$$\begin{aligned} f' = &\sum_{I,J} \alpha_{I,J}(x_2, x_{n+1}, x_{n+2}, x_{i_1}, x_k, x_{i_3} \dots, x_{i_{r_{max}}})x_{j_1} \dots x_{j_{s_{min}}} \\ &+ \sum_{i_1, I, J} \alpha_{i_2, I, J}(x_2, x_{n+1}, x_{n+2}, x_{i_1}, x_k, x_{i_3} \dots, x_{i_r})x_{j_1} \dots x_{j_s} \pmod{\mathrm{Id}(UJ_2)}. \end{aligned}$$

Let us now consider in the first sum of f' a variable appearing at least in one of the associators of length $r_{max}+2$ but not in all of them, say x_i. By using the identities of UJ_2, let us move x_i to the second-last position of the associators and substitute it with $x_i(x_{n+4}x_{n+5})$, in order to obtain a new consequence f''. By Lemma 3.1, it is clear that in f'' we get a sum of elements with associator of maximal length $r_{max}+4$ and the number of such elements is strictly less then the number of elements of f' with associator of maximal length $r_{max}+2$. By induction, we finally get a consequence $\bar{f}$ which has exactly one associator of maximal length,

$$\bar{f} = (x_2, x_{n+1}, x_{n+2}, x_{i_1} \dots, x_{i_t})x_{j_1} \dots x_{j_m} + \sum_{K,L} \beta_{K,L}(x_2, x_{n+1}, x_{n+2}, x_{k_1} \dots, x_{k_r})x_{l_1} \dots x_{l_s} \pmod{\mathrm{Id}(UJ_2)},$$

where $r < t$. Thus, since $\bar{f} \in \mathrm{Id}(\mathcal{U})$, it is trivial that

$$(x_2, x_{n+1}, x_{n+2}, x_{i_1} \dots, x_{i_t})x_{j_1} \dots x_{j_m} = -\sum_{K,L} \beta_{K,L}(x_2, x_{n+1}, x_{n+2}, x_{k_1} \dots, x_{k_r})x_{l_1} \dots x_{l_s} \pmod{\mathrm{Id}(\mathcal{U})}.$$

This implies that any polynomial of $\mathcal{J}(X)$ which is a linear combination of summands in which appear associators of length greater or equal to t, can be written modulo $\mathrm{Id}(\mathcal{U})$ as linear combination of summands with associators of shorter length. Notice that if such x_i does not exist, then in f' there is already just one associator of length $r_{max}+2$, thus the previous argument applies.
By taking into account the previous remark, one gets the following upper bound for the codimension sequence of $\mathcal{U}$:

$$c_n(\mathcal{U}) \le 1 + \sum_{k=2}^{t-1}(k-1)\binom{n}{k} + \sum_{k=n-m+t+1}^{n}(k-1)\binom{n}{k} \approx qn^h, \ q > 0,$$

where $h = \max\{t-1, m-t-1\}$. Thus $\mathcal{U}$ has polynomial growth and we are done. $\square$

4. Polynomial growth and special Jordan varieties

In this section we shall prove the main theorem of this paper. In fact, we show that the growth of a variety $\mathcal{V}$ is polynomially bounded if and only if $UJ_2 \notin \mathcal{V}$. As a consequence, we will get that UJ_2 is the only finite dimensional special Jordan algebra generating a variety of almost polynomial growth.

We start with the following simple remark.

Remark 4.1. $UJ_2 \in var(H(M_2, t))$.

Proof. Notice that $H(M_2, t) \cong B_2$, where $B_2 = F \oplus V$ is the Jordan algebra of a symmetric nondegerate bilinear form with $\dim_F V = 2$. Moreover, it is also clear that $UJ_2 \cong J_2$, where $J_2 = F \oplus V'$ is the Jordan algebra of a symmetric degenerate bilinear form of rank 1 with $\dim_F V' = 2$.
If we denote by $J_n = F \oplus V'$ the Jordan algebra of a symmetric degenerate bilinear form of rank $n-1$ and $\dim_F V' = n$, then in [20] it was proved that for all $n \ge 2$, B_{n-1} has the same polynomial identities of J_n, thus in particular, $\mathrm{Id}(B_2) = \mathrm{Id}(J_3)$. Since J_2 is a subalgebra of J_3, we have $\mathrm{Id}(J_2) \supseteq \mathrm{Id}(J_3) = \mathrm{Id}(B_2)$, hence $J_2 \cong UJ_2 \in \mathrm{var}(B_2) = \mathrm{var}(H(M_2, t))$ and we are done. $\square$

Let us now investigate the structure of a finite dimensional special Jordan algebra J in case $UJ_2 \notin \mathrm{var}(J)$. To this end, recall that we fixed $\{1, a, b\}$ as basis of UJ_2, where $1 = e_{11} + e_{22}$, $a = e_{11} - e_{22}$ and $b = e_{12}$ with $a^2 = 1$ and $ab = b^2 = 0$.

Lemma 4.1. *Let $\mathcal{V}$ be a variety of special Jordan algebras generated by a finite dimensional algebra J, $\mathcal{V} = var(J)$. Moreover, let*

$$J = A_1 \oplus \dots \oplus A_m + R,$$

where $R = Rad(J)$, be the Wedderburn–Malcev decomposition of J. If $UJ_2 \notin var(J)$, then $A_i \cong F$ for all $1 \le i \le m$.

Proof. By contradiction, let us suppose that there exists A_i not isomorphic to F. Thus, according to the classification of the finite dimensional special simple Jordan algebras, either $A_i \cong M_n^{(+)}$ or $A_i \cong H(M_{2n}, s)$ or $A_i \cong H(M_n, t)$ or $A_i \cong B_n$.

For each possibility, we shall prove that UJ_2 lies in the variety generated by A_i. Since $\mathrm{var}(A_i) \subseteq \mathrm{var}(J)$, we will reach the contradiction.

- Suppose that $A_i \cong M_n^{(+)}$, $n > 1$, then denoting by D the subalgebra generated by e_{11}, e_{12} and e_{22}, one gets trivially that $D \cong UJ_2$, then $UJ_2 \in \mathrm{var}(A_i)$, a contradiction.
- If $A_i \cong H(M_{2n}, s)$, $n > 1$, by setting $D = \langle e_{11}+e_{n+1,n+1}, e_{22}+e_{n+2,n+2}, e_{12}+e_{n+2,n+1}\rangle$ we have that $\varphi : UJ_2 \to D$ such that $\varphi(e_{11}) = e_{11}+e_{n+1,n+1}$, $\varphi(e_{22}) = e_{22}+e_{n+2,n+2}$ and $\varphi(e_{12}) = e_{12}+e_{n+2,n+1}$ is an isomorphism of Jordan algebras. As in the previous case, this implies a contradiction. Notice that if $n = 1$, then $H(M_2, s) \cong F$ and we have nothing to prove.
- Let $A_i \cong H(M_n, t)$, $n \geq 1$. It is clear that inside A_i we have a copy of $H(M_2, t)$, so by Remark 4.1 we get $UJ_2 \in \mathrm{var}(H(M_2, t)) \subseteq \mathrm{var}(A_i)$, a contradiction.
- Finally, let us suppose $A_i \cong B_n$, $n > 1$. By Remark 4.1, $UJ_2 \in \mathrm{var}(B_2)$ and, since B_2 is a subalgebra of B_n, for all $n > 1$, then $\mathrm{var}(B_2) \subseteq \mathrm{var}(B_n)$. Hence $UJ_2 \in \mathrm{var}(B_n)$, a contradiction.

Thus for all $1 \leq i \leq m$ we must have $A_i \cong F$ and we are done. □

In what follows we investigate the relation between each simple component of J and its radical.

Lemma 4.2. *Let $\mathcal{V}$ be a variety of special Jordan algebras generated by a finite dimensional algebra J, $\mathcal{V} = var(J)$, and let*

$$J = A_1 \oplus \ldots \oplus A_m + R,$$

where $R = Rad(J)$, be its Wedderburn–Malcev decomposition. If $UJ_2 \notin \mathcal{V}$, then $(A_iR)A_k = 0$ for all $1 \leq i, k \leq m$ and $i \neq k$.

Proof. By the previous Lemma, we have that $A_i \cong F$ for all $1 \leq i \leq m$. Suppose by contradiction that there exist $i \neq k$ such that $(A_iR)A_k \neq 0$, hence $(a_ir)a_k \neq 0$ for some $a_i \in A_i$, $a_k \in A_k$ and $r \in R$.
First notice that if e_i and e_k are the unit elemets of A_i and A_k, respectively, then $((e_ia_i)r)(e_ka_k) \neq 0$. Since $A_i \cong A_k \cong F$, then now one can consider a_i and a_k as scalars, thus $((e_ia_i)r)(e_ka_k) = a_ia_k(e_ir)e_k \neq 0$. Hence, without loss of generality, we may assume that $(a_ir)a_k \neq 0$ with $a_i^2 = a_i$ and $a_k^2 = a_k$.
Let $t \geq 1$ be the largest integer such that $(a_iR)a_k \subseteq R^t$ and let $D = J/R^{t+1}$. It is clear that $D \in \mathrm{var}(J)$. If one sets C as the subalgebra of D generated by $a_i + a_k + R^{t+1}$, $a_i - a_k + R^{t+1}$ and $(a_ir)a_k + R^{t+1}$, then we claim that $\varphi : C \to UJ_2$ such that $\varphi(a_i + a_k + R^{t+1}) = e_{11} + e_{22}$, $\varphi(a_i - a_k + R^{t+1}) = e_{11} - e_{22}$ and $\varphi((a_ir)a_k + R^{t+1}) = e_{12}$ is an isomorphism of algebras. To this end, remark that such elements are linearly independent since a_i and a_k are orthogonal elements of the semisimple part of J and $(a_ir)a_k \notin R^{t+1}$ by construction. Moreover, let us verify that C and UJ_2 have the same multiplication table according to the action of φ.
It is clear that $(a_i + a_k + R^{t+1})^2 = a_i^2 + a_k^2 + R^{t+1} = a_i + a_k + R^{t+1}$. Moreover, it is also trivial that $(a_i - a_k + R^{t+1})^2 = a_i^2 + a_k^2 + R^{t+1} = a_i + a_k + R^{t+1}$ and that $(a_i + a_k + R^{t+1})(a_i - a_k + R^{t+1}) = a_i^2 - a_k^2 + R^{t+1} = a_i - a_k + R^{t+1}$.
Let us now compute the product among $a_i + a_k + R^{t+1}$ and $(a_ir)a_k + R^{t+1}$ by using the speciality of the Jordan algebra J. From now on, we denote by xy the Jordan product among two elements of J and by $x \cdot y$ the associative product in the associative enveloping algebra of J. It is clear that $xy = \frac{1}{2}(x \cdot y + y \cdot x)$. Recall that if a_i and a_k are orthogonal elements in J, then by [15, Chapter 2, Section 2], a_i and a_k will be orthogonal also in the associative enveloping algebra of J. Thus for instance,

$$(a_ir)a_k = \left[\frac{1}{2}(a_i \cdot r + r \cdot a_i)\right] a_k = \frac{1}{4}\left(a_i \cdot r \cdot a_k + a_k \cdot r \cdot a_i\right) = (a_kr)a_i.$$

Notice that

$$\begin{aligned} a_i[(a_ir)a_k] &= a_i\left[\frac{1}{2}(a_i \cdot r + r \cdot a_i)a_k\right] = a_i\left[\frac{1}{4}(a_i \cdot r \cdot a_k + a_k \cdot r \cdot a_i)\right] = \frac{1}{8}(a_i^2 \cdot r \cdot a_k + a_k \cdot r \cdot a_i^2) \\ &= \frac{1}{2}(a_i^2r)a_k = \frac{1}{2}(a_ir)a_k. \end{aligned} \tag{4.1}$$

With similar arguments one can also prove that

$$a_k[(a_ir)a_k] = \frac{1}{2}(a_k^2r)a_i = \frac{1}{2}(a_kr)a_i. \tag{4.2}$$

Now, by using equations (4.1), (4.2), we get that

$$
\begin{aligned}
(a_i + a_k + R^{t+1})((a_i r)a_k + R^{t+1}) &= a_i[(a_i r)a_k] + a_k[(a_i r)a_k] + R^{t+1} = \frac{1}{2}[(a_i^2 r)a_k + (a_k^2 r)a_i] + R^{t+1} \\
&= \frac{1}{2}[(a_i r)a_k + (a_k r)a_i] + R^{t+1} = \frac{1}{2}[(a_i r)a_k + (a_i r)a_k] + R^{t+1} \\
&= (a_i r)a_k + R^{t+1}.
\end{aligned}
$$

Moreover, we get that

$$(a_i - a_k + R^{t+1})((a_i r)a_k + R^{t+1}) = \frac{1}{2}[(a_i r)a_k - (a_k r)a_i] + R^{t+1} = R^{t+1}.$$

Finally, since $((a_i r)a_k)^2$ is an element of R^{t+1}, it is clear that $((a_i r)a_k)^2 + R^{t+1} = R^{t+1}$. Thus φ is an isomorphism of Jordan algebras.
Hence, $UJ_2 \in \mathrm{var}(C) \subseteq \mathrm{var}(D) \subseteq \mathrm{var}(J)$, a contradiction. This implies that $(A_i R)A_k = 0$ for all $1 \le i, k \le m$ and $i \neq k$ as claimed. $\square$

Varieties as in Lemma 4.2 have also the following simple property.

Proposition 4.1. *Let $\mathcal{V} = var(J)$ a variety of special Jordan algebras as in Lemma 4.2. If $(A_i R)A_k = 0$ for some $1 \le i, k, \le m$, $i \neq k$, then*

1. $(a_i r_1)(a_k r_2) = -(a_i r_2)(a_k r_1)$,
2. $a_i(r_1(a_k r_2)) = -a_k(r_1(a_i r_2))$,

for all $a_i \in A_i$, $a_k \in A_k$ and $r_1, r_2 \in R$.

Proof. Statement 1. can be easily proved by noticing that

$$(a_i r_1)(a_k r_2) + (a_i r_2)(a_k r_1) = [(a_i r_1)a_k]r_2 + [(a_i r_2)a_k]r_1 + [a_i(r_1 r_2)]a_k.$$

The proof of the latter equality is a straightforward computation by using the multiplication in the associative enveloping algebra of J, so we omit it.
Moreover, if one considers the identity (3.1) and makes the substitution $x_1 = r_1$, $x_2 = a_k$, $x_3 = r_2$ and $x_4 = a_i$, then recalling that $(A_i R)A_k = 0$, we get 2. $\square$

We are now in a position to prove the main theorem.

Theorem 4.1. *Let $\mathcal{V}$ be a variety of special Jordan algebras generated by a finite dimensional algebra J, $\mathcal{V} = var(J)$. Then $\mathcal{V}$ has polynomial growth if and only if $UJ_2 \notin \mathcal{V}$.*

Proof. Suppose first that $c_n(\mathcal{V}) \approx qn^k$, for some positive integer k and $q > 0$. Since in [18] it was proved that $\mathrm{var}(UJ_2)$ has exponential growth, it is clear that $UJ_2 \notin \mathcal{V}$.
Conversely, let us assume that $UJ_2 \notin \mathcal{V}$. If $J = A_1 \oplus \ldots \oplus A_m + R$ is the Wedderburn–Malcev decomposition of J, then by Lemma 4.1, $A_i \cong F$ for all $1 \le i \le m$. Furthermore, by Lemma 4.2, $(A_i R)A_k = 0$ for all $1 \le i, k \le m$ and $i \neq k$.
Let $d = \exp(J) = \max\{ht(M) | 0 \neq M \in J\}$. In [9, Lemma 5] it was proved that in case of finite dimensional algebra with strongly nilpotent radical, there exist constants C, k such that $c_n(J) \le Cn^k d^n$, for all $n \ge 1$. Thus by contradiction let us assume $d \ge 2$.
Since $A_i \cong F$ for all $1 \le i \le m$, by [9, Lemma 6] there exist $t \ge 0$ and a monomial $M(x_1, \ldots, x_{d+t+l})$ such that

$$M = M(a_1, \ldots, a_{d+l}, b_1, \ldots, b_t) \neq 0,$$

for some $a_1, \ldots, a_{d+l} \in B_1$, $b_1, \ldots, b_t \in B_0$, $l \ge 0$, where B_1 is the union of the bases of $A_1, \ldots A_m$, B_0 is a basis of R and $a_1, \ldots, a_d$ belong to distinct simple components $A_{i_1}, \ldots, A_{i_d}$, respectively. As in the proof of Lemma 4.2, we can assume $a_i^2 = a_i$, for all $1 \le i \le d + l$.
Let us consider the polynomial

$$f = (x_1 x_2, x_3, x_4) - x_1(x_2, x_3, x_4) - x_2(x_1, x_3, x_4)$$

and let us suppose that f is an identity of J. Thus, for all $a_i \in A_i$, $a_k \in A_k$ and $r_1, r_2 \in R$, if one sets $x_1 = a_i$, $x_2 = r_1$, $x_3 = a_k$ and $x_4 = r_2$ in f, then

$$(a_i r_1)(a_k r_2) = a_i(r_1(a_k r_2)) - a_i(r_2(a_k r_1)).$$

The previous one plus statements 1. and 2. of Proposition 4.1 allow us to write $M = a_{i_0}M' = a_{j_0}M''$, for some $i_0 \neq j_0$ and M', M'' non-zero monomials (here recall that $d \geq 2$).
Let $t \geq 0$ be the largest integer such that $M \in R^t$ and let $D = J/R^{t+1}$. If $j \in J$, we shall denote by $\bar{j}$ the image of j in D. Now let I be the ideal of D generated by $\overline{M' - M''}$ and $\overline{2M - M'}$. Remark that by construction, $\overline{M}$, $\overline{M'}$ and $\overline{M''}$ are not zero in D.
If we set C as the subalgebra of D/I generated by $\overline{a_{i_0} + a_{j_0}} + I$, $\overline{a_{i_0} - a_{j_0}} + I$ and $\overline{M'} + I$, then it is easily proved that $\varphi : C \to UJ_2$ such that $\varphi(\overline{a_{i_0} + a_{j_0}} + I) = e_{11} + e_{22}$, $\varphi(\overline{a_{i_0} - a_{j_0}} + I) = e_{11} - e_{22}$ and $\varphi(\overline{M'} + I) = e_{12}$ is an isomorphism of Jordan algebras. Indeed, a straightforward computation, as the one in Lemma 4.2, shows that the generators of C are linearly independent and they have the same multiplication table as the ones of UJ_2.
Hence $UJ_2 \in \mathrm{var}(C) \subseteq \mathrm{var}(D) \subseteq \mathrm{var}(J)$, a contradiction. Notice that if f is not an identity of J and if there exist $a_i \in A_i$, $a_k \in A_k$ and $r_1, r_2 \in R$, such that

$$b = (a_i r_1)(a_k r_2) - a_i(r_1(a_k r_2)) + a_i(r_2(a_k r_1)) \neq 0,$$

then by considering $J' = J/Q$, where Q is the ideal of J generated by b, the previous arguments apply. Thus $d = \exp(J) = 1$ and, by Lemma 5 of [9], J has polynomial growth. □

As a consequence we have the following corollaries.

Corollary 4.1. *The algebra UJ_2 is the only finite dimensional special Jordan algebra generating a variety of almost polynomial growth.*

Corollary 4.2. *There is no variety of special Jordan algebras generated by a finite dimensional algebra of intermediate growth between polynomial and exponential.*

References

[1] L. Centrone, F. Martino, *A note on cocharacter sequence of Jordan upper triangular matrix algebra*, Comm. Algebra **45**, n.ro 4 (2017), 1687–1695.
[2] L. Centrone, F. Martino, M. da Silva Souza, *Specht property for some varieties of Jordan algebras of almost polynomial growth*, J. Algebra **521** (2019), 137 – 165.
[3] A. Cirrito, F. Martino, *Ordinary and graded cocharacter of the Jordan algebra of 2×2 upper triangular matrices*, Linear Algebra Appl. **541** (2014), 246–259.
[4] V. Drensky, *Polynomial identities for the Jordan algebra of a symmetric bilinear form*, J. Algebra **108** (1987), 66–87.
[5] A. Giambruno, A. Ioppolo, D. La Mattina, *Varieties of algebras with superinvolution of almost polynomial growth*, Algebr. Represent. Theory **19** (2016), no. 3, 599–611.
[6] A. Giambruno, S. Mishchenko, *On star-varieties with almost polynomial growth*, Algebra Colloq. **8** (2001), no. 1, 33–42.
[7] A. Giambruno, S. Mishchenko, M. Zaicev, *Polynomial identities on superalgebras and almost polynomial growth*, Comm. Algebra **29**(9) (2001), 3738–3800.
[8] A. Giambruno, S. Mishchenko, M. Zaicev, *Algebras with intermediate growth of the codimensions*, Adv. in Appl. Math. **37** (2006), 360–377.
[9] A. Giambruno, I. Shestakov, M. Zaicev, *Finite-dimensional non-associative algebras and codimension growth*, Adv. in Appl. Math. **47** (2011), 125–139.
[10] A. Giambruno, M. Zaicev, *On codimension growth of finitely generated associative algebras*, Adv. Math. **140** (1998), 145–155.
[11] A. Giambruno, M. Zaicev, *Exponential codimension growth of PI-algebras: An exact estimate*, Adv. Math **142** (1999), 221–243.
[12] A. Giambruno, M. Zaicev, *Polynomial identities and asymptotic methods*, AMS, Math. Surv. Monogr. **122** (2005).
[13] A. Giambruno, M. Zaicev, *Multialternating Jordan polynomials and codimension growth of matrix algebras*, Linear Algebra Appl. **422** (2007), 372–379.
[14] A. Giambruno, M. Zaicev, *Codimension growth of special simple Jordan algebras*, Trans. Amer. Math. Soc. **362** (2010), 3107–3123.
[15] N. Jacobson, *Structure and representations of Jordan algebras*, AMS Colloquium Publications, Vol. XXXIX American Mathematical Society, Providence, R.I. (1968).
[16] A. R. Kemer, *T-ideals with power growth of the codimensions are Specht*, Sibirsk. Mat. Zh. **19** (1978), 54-69 (in Russian); English translation: *Sib. Math. J.* **19** (1978), 37–48.
[17] A. R. Kemer, *Varieties of finite rank*, Proc. 15-th All the Union Algebraic Conf., Krasnoyarsk, Vol. 2, p. 73, (1979), (in Russian).
[18] P. Koshlukov, F. Martino, *Polynomial identities for the Jorda algebra of upper triangular matrices of order two*, J. Pure Appl. Algebra **216** (11) (2012), 2524–2532.

[19] A. Ioppolo, F. Martino, *Varieties of algebras with pseudoinvolution and polynomial growth*, Linear Multilinear Algebra **66** (11) (2017), 2286 – 2304.
[20] F. Martino, *Polynomial identities for the Jordan algebra of a degenerate symmetric bilinear form*, Linear Algebra Appl. **439** (2013), 4080–4089.
[21] A. J. Penico *The Wedderburn Principal Theorem for Jordan Algebras*, Trans. Amer. Math. Soc. **70**(3) (1951), 404–420.
[22] A. Popov, *Variety of Jordan algebras* $var(UT_2(F)^{(+)})$ *has almost polynomial growth*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **2012**, no. 5, 49–52 (in Russian); English translation: Moscow Univ. Math. Bull. **67** (2012), no. 5–6, 224–227.
[23] A. Regev, *Existence of identities in* $A \otimes B$, Israel J. Math. **11** (1972), 131–152.
[24] A. Valenti, *Group graded algebras and almost polynomial growth*, J. Algebra **334** (2011), 247–254.
[25] M. Zaicev, *Integrality of exponents of growth of identities of finite-dimensional Lie algebras*, Izv. Ross. Akad. Nauk Ser. Mat. **66** (2002), 23–48 (in Russian); English translation: Izv. Math. **66** (2002), 463–487.

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