A NOTE ON HOMOCLINIC SOLUTIONS OF (p, q)-LAPLACIAN DIFFERENCE EQUATIONS

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Abstract. We prove the existence of at least two positive homoclinic solutions for a discrete boundary value problem of equations driven by the (p, q)-Laplace operator. The properties of the nonlinearity ensure that the energy functional, corresponding to the problem, satisfies a mountain pass geometry and a Palais-Smale compactness condition.

1. Introduction

Let $\mathbb{N}$ be the set of positive integers. We study the discrete boundary value problem

$$\begin{gathered} -\Delta_p u(z-1) - \Delta_q u(z-1) + a_p(z)\phi_p(u(z)) + a_q(z)\phi_q(u(z)) = \lambda g(z, u(z)), \\ \text{for all } z \in \mathbb{N} \ (1 < q < p < +\infty,\ \lambda \in]0, +\infty[), \\ u(0) = 0, \quad u(z) \to 0 \quad \text{as } z \to +\infty, \end{gathered} \tag{1.1}$$

where

- $\phi_r : \mathbb{R} \to \mathbb{R}$ is given as $\phi_r(u) = |u|^{r-2}u$ with $u \in \mathbb{R}$,
- $\Delta u(z-1) = u(z) - u(z-1)$ is the forward difference operator $(z \in \mathbb{N})$,
- $\Delta_r u(z-1) := \Delta\phi_r(\Delta u(z-1)) = \phi_r(\Delta u(z)) - \phi_r(\Delta u(z-1))$ is the discrete r-Laplace operator $(z \in \mathbb{N})$,
- $a_p, a_q : \mathbb{N} \to \mathbb{R}$, $g : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is continuous.

Let $G : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be given as

$$G(z, x) = \int_0^x g(z, y)dy, \quad \text{for all } x \in \mathbb{R},\ z \in \mathbb{N}.$$

Here, we impose the following conditions:

(H1) $\lim_{x\to 0} \frac{|g(z,x)|}{|x|^{q-1}} = 0$ uniformly for all $z \in \mathbb{N}$;
(H2) $\sup_{|x|\le\xi} |G(\cdot, x)| \in l^1$ for all $\xi > 0$;
(H3) $\limsup_{|x|\to+\infty} \frac{G(z,x)}{|x|^p} \le 0$ uniformly for all $z \in \mathbb{N}$;
(H4) $G(\alpha, \beta) > 0$ for some $\alpha \in \mathbb{N}$, $\beta \in \mathbb{R}$;
(H5) $a_r(z) \in [a_{r_0}, +\infty[$ for all $z \in \mathbb{N}$, $a_{r_0} > 0$ and $a_r(z) \to +\infty$ as $z \to +\infty$, $r \in \{p, q\}$.

Remark 1.1. *By* (H1) *it follows that* $g(z, 0) = 0$ *for all* $z \in \mathbb{N}$.

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A simple function satisfying the above conditions is the following:

$$g_s(z,x) = f(z)|x|^{s-2}x \quad \text{for all } x \in \mathbb{R}, \tag{1.2}$$

where $1 < q < s < p < +\infty$, $f : \mathbb{Z} \to \mathbb{R}$ is any l^1 function, with $f(z) > 0$ for all $z \in \mathbb{N}$.

The solution of abstract differential problems, driven by a r-Laplace operator, was a largely investigated field over the last fifty years. In fact, in connection with the use of various nonlinearities g, this kind of operator is useful to model the dynamical behaviour of real phenomena in biological, chemical and physical applications (see the recent books of Diening-Harjulehto-Hästö-Růžička [5] and Motreanu-Motreanu-Papageorgiou [10]). There is an extensive literature on this topic, where it can be found a deeper discussion on the methods of the calculus of variations, critical points and Morse theories. On the other hand, the study of differential problems involving the sum of a p-Laplace operator and of a q-Laplace operator (with $q < p$) is more recent. Here, we recall some interesting contributions due to Marano-Mosconi-Papageorgiou [9], Mugnai-Papageorgiou [13] and Motreanu-Vetro-Vetro [11, 12]. Precisely, in [9, 13] the authors consider equations, while in [11, 12] the authors work with systems of equations.

Difference equations give the discrete versions of continuous problems. This approach is useful in connection with numerical analysis to approximate numerically the solutions and investigate their stability. We refer to the books of Agarwal [1] and Kelly-Peterson [8]. Also, some interesting contributions are given by Cabada-Iannizzotto-Tersian [2], Iannizzotto-Tersian [6], Jiang-Zhou [7] (for discrete r-Laplace operator) and Nastasi-Vetro-Vetro [14] (for discrete (p, q)-Laplace operator).

The multiplicity of solutions and their sign are crucial arguments to investigate, in respect of the above problems. To this aim variational methods lead to significant results. Here, we work with the critical points of the energy functional, corresponding to problem (1.1), to get the existence of two positive homoclinic solutions for discrete (p, q)-Laplacian equations. Precisely, the hypotheses on the nonlinearity g ensure that the involved energy functional satisfies a mountain pass geometry and a Palais-Smale compactness condition (see Definition 2.3 and Theorem 2.4 below).

2. Mathematical background

Let us recall some basic definitions. Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). We use the Banach space

$$X_{r,h} = \left\{ u : \mathbb{N} \cup \{0\} \to \mathbb{R} \text{ such that } u(0) = 0 \text{ and } \sum_{z\in\mathbb{N}} h(z)|u(z)|^r < +\infty \right\}.$$

Remark 2.1. *From the above definition of $X_{r,h}$, it follows that $u(z) \to 0$ as $z \to +\infty$ for each $u \in X_{r,h}$.*

Also, we consider the norm

$$\|u\|_{r,h} := \left(\sum_{z\in\mathbb{N}} h(z)|u(z)|^r \right)^{1/r},$$

where $h : \mathbb{N} \to \mathbb{R}$, with $h(z) > 0$ for all $z \in \mathbb{N}$ and $h(z) \to +\infty$ as $z \to +\infty$, and $1 < r < +\infty$. Let l^r, l^∞ be the sets of all sequences $u : \mathbb{N} \cup \{0\} \to \mathbb{R}$ such that $\|u\|_r^r := \sum_{z\in\mathbb{N}\cup\{0\}} |u(z)|^r < +\infty$ and $\|u\|_\infty := \max_{z\in\mathbb{N}\cup\{0\}} |u(z)| < +\infty$, respectively. Thus, $(l^r, \|\cdot\|_r)$ is a reflexive Banach space, the embedding $l^r \hookrightarrow l^\infty$ is continuous ($\|u\|_\infty \leq \|u\|_r$ for all $u \in l^r$) and $l^q \subseteq l^p$ with $1 < q < p < +\infty$, see Iannizzotto-Tersian [6, Proposition 2], Cabada-Li-Tersian [3] and the references therein. Also, $(X_{r,h}, \|\cdot\|_{r,h})$ is a reflexive Banach space and the embedding $X_{r,h} \hookrightarrow l^r$ is compact, see [6, Proposition 3]. We point out that we can consider $\mathbb{N}$ instead of $\mathbb{Z}$, without affecting validity or applicability of the results in [3, 6].

Now, consider $\mathbb{X} = X_{p,a_p} \cap X_{q,a_q}$ and the norm $\|\cdot\|$ given as

$$\|u\| = \|u\|_{p,a_p} + \|u\|_{q,a_q},$$

where a_p and a_q (see (H5)) are the coefficients of ϕ_p and ϕ_q in (1.1), respectively.

For all $u \in \mathbb{X}$, let $I_p, I_q, I_g : \mathbb{X} \to \mathbb{R}$ be the functionals given as

$$\begin{aligned}
I_p(u) &= \frac{1}{p}\left[\sum_{z\in\mathbb{N}} |\Delta u(z-1)|^p + a_p(z)|u(z)|^p\right],\\
I_q(u) &= \frac{1}{q}\left[\sum_{z\in\mathbb{N}} |\Delta u(z-1)|^q + a_q(z)|u(z)|^q\right],\\
I_g(u) &= \sum_{z\in\mathbb{N}} G(z, u(z)).
\end{aligned}$$

We have that $I_p, I_q, I_g \in C^1(\mathbb{X}, \mathbb{R})$ (see [6]). Next, for all $u, v \in \mathbb{X}$, the following equalities hold:

$$\begin{aligned}
\langle I_p'(u), v\rangle &= \sum_{z\in\mathbb{N}} [\phi_p(\Delta u(z-1))\Delta v(z-1) + a_p(z)\phi_p(u(z))]v(z),\\
\langle I_q'(u), v\rangle &= \sum_{z\in\mathbb{N}} [\phi_q(\Delta u(z-1))\Delta v(z-1) + a_q(z)\phi_q(u(z))]v(z),\\
\langle I_g'(u), v\rangle &= \sum_{z\in\mathbb{N}} g(z, u(z))v(z).
\end{aligned}$$

So, for $1 < r < +\infty$, we deduce that

$$\begin{aligned}
&\sum_{z\in\mathbb{N}} \phi_r(\Delta u(z-1))\Delta v(z-1)\\
&= \sum_{z\in\mathbb{N}} [\phi_r(\Delta u(z-1))v(z) - \phi_r(\Delta u(z-1))v(z-1)]\\
&= \sum_{z\in\mathbb{N}} \phi_r(\Delta u(z-1))v(z) - \sum_{z\in\mathbb{N}} \phi_r(\Delta u(z))v(z)\\
&= -\sum_{z\in\mathbb{N}} \Delta\phi_r(\Delta u(z-1))v(z).
\end{aligned}$$

It follows that

$$\begin{aligned}
\langle I_p'(u), v\rangle &= \sum_{z\in\mathbb{N}} [-\Delta\phi_p(\Delta u(z-1)) + a_p(z)\phi_p(u(z))]v(z),\\
\langle I_q'(u), v\rangle &= \sum_{z\in\mathbb{N}} [-\Delta\phi_q(\Delta u(z-1)) + a_q(z)\phi_q(u(z))]v(z),
\end{aligned}$$

for all $u, v \in \mathbb{X}$. So, using the previous functionals, we define $J_\lambda : \mathbb{X} \to \mathbb{R}$ by

$$J_\lambda(u) = I_p(u) + I_q(u) - \lambda I_g(u), \quad \text{for all } u \in \mathbb{X}.$$

Clearly $J_\lambda(0) = 0$ and, for all $u, v \in \mathbb{X}$, we get

$$\langle J'_\lambda(u), v\rangle = \sum_{z\in\mathbb{N}} [-\Delta\phi_p(\Delta u(z-1)) - \Delta\phi_q(\Delta u(z-1)) \\ + a_p(z)\phi_p(u(z)) + a_q(z)\phi_q(u(z)) - \lambda g(z, u(z))]v(z).$$

We conclude that $u \in \mathbb{X}$ is a solution of problem (1.1) iff u is a critical point of J_λ.

We recall the following notions.

Definition 2.2. By a positive homoclinic solution of problem (1.1) we mean a solution $u \in \mathbb{X}$ such that $u(z) > 0$ for all $z \in \mathbb{N}$.

Definition 2.3. Let X be a real Banach space and X^* its topological dual. Then, $J : X \to \mathbb{R}$ satisfies the Palais-Smale condition if any sequence $\{u_n\}$ such that

(i) $\{J(u_n)\}$ is bounded;
(ii) $\lim_{n\to+\infty} \|J'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

From Pucci-Serrin [16] we have the following mountain pass theorem.

Theorem 2.4. *Let X be a reflexive Banach space, $J \in C^1(X)$ be a functional satisfying the Palais-Smale condition and suppose that there are $\widetilde{u} \in X$ and positive real numbers s_1, s_2 with $s_1 < s_2 \leq \|\widetilde{u}\|$ such that*

$$\inf_{s_1\leq\|u\|\leq s_2} J(u) = c \geq \max\{J(0), J(\widetilde{u})\}.$$

Then the functional J has a critical point $\widehat{u} \in X$ with $J(\widehat{u}) \geq c$. Moreover, if $J(\widehat{u}) = c$ then $s_1 \leq \|\widehat{u}\| \leq s_2$.

In the next lemma, we establish some properties of the functional J_λ to show that it satisfies the hypotheses of Theorem 2.4.

Lemma 2.5. *If* (H1), (H2), (H3), (H5) *hold, then J_λ is coercive and satisfies the Palais-Smale condition.*

Proof. We show that J_λ is coercive, that is

$$\lim_{\|u\|\to+\infty} J_\lambda(u) = +\infty. \tag{2.1}$$

Fix $\lambda > 0$ so that, for all $\varepsilon_p \in]0, a_{p_0}(\lambda p)^{-1}[$, by (H3) there is $\xi > 0$ such that

$$G(z, x) \leq \varepsilon_p|x|^p \quad \text{for all } |x| > \xi.$$

Also let $w \in l^1$ be such that

$$|G(z, x)| \leq w(z) \quad \text{for all } z \in \mathbb{N}, \quad |x| \leq \xi \quad \text{(by (H2))}.$$

So, for all $u \in \mathbb{X}$, we get

$$\begin{aligned} J_\lambda(u) &= I_p(u) + I_q(u) - \lambda I_g(u) \\ &\geq \frac{\|u\|^p_{p,a_p}}{p} + \frac{\|u\|^q_{q,a_q}}{q} - \lambda \sum_{|u(z)|\leq\xi} G(z, u(z)) - \lambda \sum_{|u(z)|>\xi} G(z, u(z)) \\ &\geq \frac{\|u\|^p_{p,a_p}}{p} + \frac{\|u\|^q_{q,a_q}}{q} - \lambda\|w\|_1 - \lambda\varepsilon_p\|u\|^p_p \end{aligned}$$

$$\geq \left(\frac{1}{p} - \lambda\frac{\varepsilon_p}{a_{p_0}}\right)\|u\|_{p,a_p}^p + \frac{\|u\|_{q,a_q}^q}{q} - \lambda\|w\|_1, \quad (\text{since } a_{p_0}\|u\|_p^p \leq \|u\|_{p,a_p}^p)$$

which tends to $+\infty$ as $\|u\| \to +\infty$. So J_λ is coercive.

Next, we prove that J_λ satisfies the Palais-Smale condition. To this aim, we need the following inequalities, that is, there exists $m_r > 0$ such that for all $t_1, t_2 \in \mathbb{R}$ we have

$$(\phi_r(t_1) - \phi_r(t_2))(t_1 - t_2) \geq \begin{cases} m_r|t_1 - t_2|^r & \text{if } r \geq 2, \\ \dfrac{m_r|t_1 - t_2|^2}{(|t_1| + |t_2|)^{2-r}} & \text{if } 1 < r < 2. \end{cases} \tag{2.2}$$

(see [15, Lemma A.0.5]).

Let $\{u_n\} \subseteq \mathbb{X}$ be such that $\{J_\lambda(u_n)\}$ is bounded in $\mathbb{R}$ and $\lim_{n\to+\infty}\|J_\lambda'(u_n)\|_{\mathbb{X}^*} = 0$. By (2.1) we deduce that $\{u_n\}$ is bounded and by [6, Proposition 3] (i.e., compactness of the embedding $\mathbb{X} \hookrightarrow l^q$), we may suppose without any loss of generality (passing to a subsequence if necessary) that

$$u_n \rightharpoonup u \text{ in } \mathbb{X} \quad \text{and} \quad u_n \to u \text{ in } l^q, \quad \text{for some } u \in \mathbb{X}.$$

So, we have

$$\lim_{n\to+\infty}\langle J_\lambda'(u), u_n - u\rangle = 0.$$

Now, we use (2.2) to get

$$\begin{aligned} &\langle I_p'(u_n) + I_q'(u_n) - I_p'(u) - I_q'(u), u_n - u\rangle - \sum_{z\in\mathbb{N}}[\phi_p(\Delta u_n(z-1)) - \phi_p(\Delta u(z-1)) \\ &\quad + \phi_q(\Delta u_n(z-1)) - \phi_q(\Delta u(z-1))](\Delta u_n(z-1) - \Delta u(z-1)) \\ &= \sum_{z\in\mathbb{N}} a_p(z)[\phi_p(u_n(z)) - \phi_p(u(z))](u_n(z) - u(z)) \\ &\quad + \sum_{z\in\mathbb{N}} a_q(z)[\phi_q(u_n(z)) - \phi_q(u(z))](u_n(z) - u(z)) \\ &\geq m_p\|u_n - u\|_{p,a_p}^p + m_q\|u_n - u\|_{q,a_q}^q \quad (\text{by (2.2) if } 2 \leq q < p). \end{aligned} \tag{2.3}$$

The other cases ($p \geq 2$ and $1 < q < 2$; $1 < q < p < 2$) can be derived in a similar way, so to avoid repetitions we omit the details. On the other hand, see [6], we obtain

$$\begin{aligned} &\lim_{n\to+\infty}\sum_{z\in\mathbb{N}}[\phi_r(\Delta u_n(z-1)) - \phi_r(\Delta u(z-1))](\Delta u_n(z-1) - \Delta u(z-1)) = 0, \\ &\lim_{n\to+\infty}\langle I_g'(u_n) - I_g'(u), u_n - u\rangle = 0, \\ &\lim_{n\to+\infty}\langle I_p'(u_n) + I_q'(u_n) - I_p'(u) - I_q'(u), u_n - u\rangle = 0. \end{aligned}$$

So, passing to the limit as $n \to +\infty$ in (2.3), we deduce easily that

$$u_n \to u \text{ in } X_{p,a_p} \quad \text{and} \quad u_n \to u \text{ in } X_{q,a_q}$$

It follows that $u_n \to u$ in $\mathbb{X} = X_{p,a_p} \cap X_{q,a_q}$. This concludes the proof. $\square$

3. Two non-zero homoclinic solutions

In this section we prove the existence of two non-zero solutions for problem (1.1). Let $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{R}$ be such that $G(\alpha,\beta) > 0$ (see (H4)). We put

$$\lambda_* := \frac{(2+a_p(\alpha))p^{-1}|\beta|^p + (2+a_q(\alpha))q^{-1}|\beta|^q}{G(\alpha,\beta)}.$$

We establish the following theorem.

Theorem 3.1. *If* (H1) - (H5) *hold, then there exists* $\lambda_* \geq 0$ *such that for all* $\lambda > \lambda_*$ *problem* (1.1) *has two non-zero solutions* $\widetilde{u}, \widehat{u} \in \mathbb{X}$.

Proof. We already pointed out that $J_\lambda(0) = 0$. Now, we show that J_λ has zero as strict local minimizer, for each $\lambda > 0$. So, for all $\varepsilon_q \in]0, a_{q_0}(\lambda q)^{-1}[$, by (H1) there exists $\sigma > 0$ such that

$$|G(z,x)| \leq \varepsilon_q |x|^q \quad \text{for all } z \in \mathbb{N},\ |x| \leq \sigma.$$

From the embeddings $\mathbb{X} \hookrightarrow l^q \hookrightarrow l^\infty$, we can find $s > 0$ such that $\|u\|_\infty < \sigma$ for all $u \in \mathcal{B}_s(0)$. So, we get

$$\begin{aligned} J_\lambda(u) &\geq \frac{\|u\|_{p,a_p}^p}{p} + \frac{\|u\|_{q,a_q}^q}{q} - \lambda\varepsilon_q \|u\|_q^q \\ &\geq \frac{\|u\|_{p,a_p}^p}{p} + \left(\frac{1}{p} - \frac{\lambda\varepsilon_q}{a_{q_0}}\right)\|u\|_{q,a_q}^q > 0 \quad \text{for all } u \in \mathcal{B}_s(0) \setminus \{0\}. \end{aligned}$$

We show that zero is not a global minimizer of J_λ for all $\lambda > \lambda_*$. We put $\widehat{u} = \beta e_\alpha$ with

$$e_\alpha(z) = \delta_{\alpha z} \text{ for all } z \in \mathbb{N}, \quad \delta_{\alpha z} = \begin{cases} 1 & \text{if } \alpha = z, \\ 0 & \text{otherwise.} \end{cases}$$

So, we have

$$J_\lambda(\widehat{u}) = (2+a_p(\alpha))p^{-1}|\beta|^p + (2+a_q(\alpha))q^{-1}|\beta|^q - \lambda G(\alpha,\beta) < 0, \quad \text{for all } \lambda > \lambda_*.$$

Next, we choose a negative real number γ with $J_\lambda(\widehat{u}) < \gamma$ and set $\Gamma := \{u \in \mathbb{X} : J_\lambda(u) < \gamma\} \neq \emptyset$ (Γ is bounded since J_λ is coercive). We show that J_λ is bounded from below on Γ. Assume that there is a sequence $\{u_n\} \subseteq \Gamma$ such that $J_\lambda(u_n) \to -\infty$ as $n \to +\infty$. Since $\{u_n\}$ is bounded and by [6, Propositions 2 and 3] (i.e., continuity of the embedding $l^q \hookrightarrow l^\infty$ and compactness of the embedding $\mathbb{X} \hookrightarrow l^q$), we may suppose without any loss of generality (passing to a subsequence if necessary) that

$$u_n \rightharpoonup u \text{ in } \mathbb{X} \quad \text{and} \quad u_n \to u \text{ in } l^q.$$

Recall that $J_\lambda = I_p + I_q - \lambda I_g$, where $I_p + I_q$ is continuous and I_g is weakly lower semicontinuous (see [6]). So, we have easily

$$\liminf_{n\to+\infty} J_\lambda(u_n) \geq J_\lambda(u),$$

which contradicts the assumption: "$J_\lambda(u_n) \to -\infty$ as $n \to +\infty$". Now, consider a sequence $\{u_n\} \subseteq \Gamma$ such that

$$J_\lambda(u_n) \to \inf_{u\in\Gamma} J_\lambda(u) = \inf_{u\in\mathbb{X}} J_\lambda(u) := \varrho \quad \text{as } n \to +\infty.$$

By passing to a subsequence if necessary, we conclude easily that

$$u_n \rightharpoonup \widetilde{u} \text{ in } \mathbb{X} \quad \text{and} \quad u_n \to \widetilde{u} \text{ in } l^q \quad \text{for some } \widetilde{u} \in \mathbb{X}.$$

So $J_\lambda(\widetilde{u}) = \varrho < 0$ and hence $\widetilde{u} \neq 0$. Clearly, $\widetilde{u}$ is a critical point of J_λ. To get a second critical point of J_λ, we note that there are positive real numbers s_1, s_2 with $s_1 < s_2 \leq \|\widetilde{u}\|$ such that

$$\inf_{s_1 \leq \|u\| \leq s_2} J_\lambda(u) = c \geq 0 \quad \text{(since zero is a local minimizer).}$$

So, by an application of Theorem 2.4 (see also Lemma 2.5), we deduce that the functional J_λ has a critical point $\widehat{u} \in \mathbb{X}$ with $J_\lambda(\widehat{u}) \geq c$ (recall $J_\lambda(\widetilde{u}) = \varrho < 0$). Clearly $\widehat{u} \neq \widetilde{u}$ and $\widehat{u} \neq 0$. So, $\widehat{u}$ and $\widetilde{u}$ are the two non-zero solutions of problem (1.1).

□

Example 3.2. *As we pointed out in the Introduction, the function g_s in* (1.2) *satisfies conditions (H1) - (H4). So, by an application of Theorem 3.1, the problem*

$$\begin{gathered}-\Delta_p u(z-1) - \Delta_q u(z-1) + a_p(z)\phi_p(u(z)) + a_q(z)\phi_q(u(z)) = \lambda g_s(z, u(z)),\\ \text{for all } z \in \mathbb{N},\ 1 < q < s < p < +\infty,\ \lambda \in]0, +\infty[,\\ u(0) = 0, \quad u(z) \to 0 \quad \text{as } z \to +\infty,\end{gathered}$$

has two non-zero solutions for all $\lambda > \lambda_*$, provided that a_p and a_q satisfy condition (H5). Here, we have

$$\lambda_* := \frac{(2 + a_p(\alpha))p^{-1}|\beta|^p + (2 + a_q(\alpha))q^{-1}|\beta|^q}{\beta^s}, \quad \beta \in \mathbb{R} \text{ such that (H4) holds.}$$

4. Two positive homoclinic solutions

In this section we prove the existence of two positive solutions for problem (1.1). To do this, we introduce some auxiliary facts (see also [4, 14]).

Let $y^+ = \max\{0, y\}$ and denote by $g_+ : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ the function given as $g_+(z, y) = g(z, y^+)$ for all $z \in \mathbb{N}$, all $y \in \mathbb{R}$.

Now, let $I_{g^+} : \mathbb{X} \to \mathbb{R}$ be the functional given as

$$I_{g^+}(u) = \sum_{z \in \mathbb{N}} G^+(z, u(z)), \quad \text{for all } u \in \mathbb{X},$$

where $G^+ : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$G^+(z, x) = \int_0^x g_+(z, y)dy, \quad \text{for all } x \in \mathbb{R},\ z \in \mathbb{N}.$$

Clearly, $I_{g^+} \in C^1(\mathbb{X}, \mathbb{R})$. So, $J_\lambda^+ : \mathbb{X} \to \mathbb{R}$ given as

$$J_\lambda^+(u) = I_p(u) + I_q(u) - \lambda I_{g^+}(u), \quad \text{for all } u \in \mathbb{X},$$

has as critical points the solutions of the problem

$$\begin{aligned} &- \Delta_p u(z-1) - \Delta_q u(z-1) + a_p(z)\phi_p(u(z)) + a_q(z)\phi_q(u(z))\\ &= \lambda g_+(z, u(z)), \quad \text{for all } z \in \mathbb{N}, \\ &\qquad u(0) = 0, \quad u(z) \to 0 \quad \text{as } z \to +\infty. \end{aligned} \tag{4.1}$$

Remark 4.1. *If g satisfies* $(H1)$*, then g_+ satisfies* $(H1)$*. If G satisfies* $(H2)$ *and* $(H3)$*, then G^+ satisfies* $(H2)$ *and* $(H3)$*. In such a case, Lemma 2.5 holds true for the functional J_λ^+.*

Next, let $h : \mathbb{N} \to [0, +\infty[$ and $r \in]1, +\infty[$. We note that

$$\begin{cases} \Delta\phi_r(\Delta u(z-1)) - h(z)\phi_r(u(z)) \leq 0, \\ u(z) \leq 0, \end{cases}$$
$$\Rightarrow \quad \Delta u(z) \begin{cases} \leq 0 & \text{if } \Delta u(z-1) \leq 0; \\ < 0 & \text{if } \Delta u(z-1) < 0. \end{cases} \tag{4.2}$$

By $u(z) \leq 0$, we deduce that $\phi_r(u(z)) \leq 0$ and so $\Delta\phi_r(\Delta u(z-1)) \leq 0$. It follows that $\phi_r(\Delta u(z)) \leq \phi_r(\Delta u(z-1))$ and so (4.2) is established.

Let

$$C_+ := \{u \in \mathbb{X} : u(z) > 0 \text{ for all } z \in \mathbb{N}\},$$

so that u (solution of problem (1.1)) is positive if $u \in C_+$. On this basis we develop a key-result.

Lemma 4.2. *Assume that condition* (H5) *holds and fix $u \in \mathbb{X}$ such that, for each $z \in \mathbb{N}$, one of the following conditions holds true:*

(i) $u(z) > 0$;
(ii) $\Delta\phi_p(\Delta u(z-1)) - a_p(z)\phi_p(u(z)) \leq 0$;
(iii) $\Delta\phi_q(\Delta u(z-1)) - a_q(z)\phi_q(u(z)) \leq 0$.

Then either $u \in C_+$ or $u \equiv 0$.

Proof. Let $u \in \mathbb{X} \setminus \{0\}$ and $\mathcal{A} = \{z \in \mathbb{N} : u(z) \leq 0\}$. Obviously $u \in C_+$, whenever $\mathcal{A} = \emptyset$. Next, we assume that $\mathcal{A} \neq \emptyset$ and we lead to contradiction. We distinguish the cases whether $\min \mathcal{A} = 1$ or $\min \mathcal{A} = j \in \mathbb{N} \setminus \{1\}$. Indeed, in the first case, we have $u(1) \leq 0$ which implies $\Delta u(0) \leq 0$. From (4.2) we deduce that $\Delta u(1) \leq 0$ and hence $u(2) \leq u(1) \leq 0$. By iterating this procedure, we get

$$u(z+1) \leq u(z) \leq 0,$$

for all $z \in \mathbb{N}$. Since $u(z) \downarrow 0$ as $z \to +\infty$, we have $0 \leq u(z) \leq 0$ which leads to contradiction (i.e., $u \equiv 0$). In the second case (that is $\min \mathcal{A} = j \in \mathbb{N} \setminus \{1\}$), we have $\Delta u(j-1) = u(j) - u(j-1) < 0$ (by $u(j-1) > 0$). So, using (4.2), we deduce that

$$\Delta u(j) < 0 \;\Rightarrow\; u(j+1) < u(j) \leq 0.$$

Continuing this process, we conclude that

$$u(z+1) < u(z) < u(j) \leq 0,$$

for all $z > j$, which leads to a contradiction (i.e., $\lim_{z\to+\infty} u(z) < 0$). So, $\mathcal{A}$ is empty and hence $u \in C_+$. $\square$

Let $\alpha \in \mathbb{N}$ and $\beta > 0$ be such that $G^+(\alpha, \beta) > 0$. We put

$$\lambda_*^+ := \frac{(2 + a_p(\alpha))p^{-1}\beta^p + (2 + a_q(\alpha))q^{-1}\beta^q}{G^+(\alpha, \beta)}.$$

Summarizing, we can prove the following multiplicity result for problem (1.1).

Theorem 4.3. *If* (H1) - (H5) *hold with $\beta > 0$ in* (H4)*, then there exists $\lambda_*^+ > 0$ such that for all $\lambda > \lambda_*^+$ problem* (1.1) *has two positive solutions $\widetilde{u}, \widehat{u} \in C_+$.*

Proof. By Remark 4.1, Lemma 2.5 holds true for the functional J^+_λ, that is, J^+_λ is coercive and satisfies the Palais-Smale condition. So, Theorem 3.1 gives us two non-zero solutions $\widetilde{u}, \widehat{u} \in \mathbb{X}$ of problem (4.1), for all $\lambda > \lambda^+_*$.

We show that $\widetilde{u}, \widehat{u} \in \mathbb{X}$ are positive. So if we have $\widetilde{u}(z) \leq 0$ for some $z \in \mathbb{N}$, then

$$\begin{aligned} &-\Delta_p \widetilde{u}(z-1) - \Delta_q \widetilde{u}(z-1) + a_p(z)\phi_p(\widetilde{u}(z)) + a_q(z)\phi_q(\widetilde{u}(z)) \\ &= \lambda g(z, \widetilde{u}^+(z)) = \lambda g(z, 0) = 0 \text{ (by Remark 1.1)}, \quad \lambda > \lambda^+_*. \end{aligned}$$

Therefore either (ii) or (iii) of Lemma 4.2 holds for all $z \in \mathbb{N}$ such that $\widetilde{u}(z) \leq 0$. So, we obtain $\widetilde{u} \in C_+$, that is, $\widetilde{u}$ is positive. By reasoning in a similar fashion, one can deduce that $\widehat{u} \in C_+$. So, the two non-zero solutions of problem (4.1) are positive. Finally, since

$$g_+(z, u(z)) = g(z, u(z)) \text{ for all } z \in \mathbb{N},\ u \in C_+,$$

we point out that each positive solution of problem (4.1) is also a positive solution of problem (1.1). This concludes the proof. □

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