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A version of Hake's theorem for Kurzweil–Henstock integral in terms of variational measure

Abstract: We introduce the notion of variational measure with respect to a derivation basis in a topological measure space and consider a Kurzweil–Henstock-type integral related to this basis. We prove a version of Hake's theorem in terms of a variational measure.

Keywords: Topological measure space, derivation basis, Kurzweil–Henstock integral, variational measure, Hake property

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A classical Hake theorem in the theory of integration (see for example [10, Lemma 3.1, Chapter VIII]) states that, in contrast to the Lebesgue integral, the Perron integral on a compact interval is equivalent to the improper Perron integral. As the Perron integral on the real line is known to be equivalent to the Kurzweil–Henstock integral (see [9]), the same property is true for the latter integral. The general idea of computing the improper integral as a limit of the integral over increasing families $\{A_\alpha\}$ of sets can be realized in the multidimensional case in several different ways depending on the type of integral and on what family $\{A_\alpha\}$ is chosen to generalize the compact intervals of the one-dimensional construction. This gives rise to various types of the Hake property. A version of this property for certain Kurzweil–Henstock-type integrals in $\mathbb{R}^n$ was studied in [3, 5, 8].

A generalized Hake theorem in terms of the limit of an integral over increasing families of sets for a Kurzweil–Henstock-type integral on a topological space with respect to an abstract derivation basis was considered in [15]. Another version of the Hake theorem in terms of so-called variational measures generated by an indefinite integral was proved in [11, 12] for the Kurzweil–Henstock integral in $\mathbb{R}^n$ and in a metric space, respectively.

In this paper, we obtain a generalization of the latter results to our case of a Kurzweil–Henstock-type integral on a topological space and show that the conditions for the Hake property in terms of increasing families of sets as in [15] and in terms of variational measures as in [11, 12] are in fact equivalent.

The ambient set X in this paper is a Hausdorff topological space with an outer regular Borel measure μ on it. For any set $E \in X$ we use the notation int E, $\overline{E}$ and ∂E for the interior, the closure and the boundary of E, respectively. The notation $\mathrm{int}_L(E)$ will mean the interior of $E \subset L$ with respect to the topology in L induced by the topology of the space X.

We use the following version of the general definition of a derivation basis (see [9, 17, 18]): a *derivation basis* (or simply a *basis*) $\mathcal{B}$ in $(X, \mathcal{M}, \mu)$ is a filter base on the product space $\mathcal{I} \times X$, where $\mathcal{I}$ is a family of closed subsets of X having finite positive measure μ and called *generalized intervals* or $\mathcal{B}$-*intervals*. That is, $\mathcal{B}$ is a nonempty collection of subsets of $\mathcal{I} \times X$ so that each $\beta \in \mathcal{B}$ is a set of pairs (I, x), where $I \in \mathcal{I}$, $x \in X$ and $\mathcal{B}$ has the *filter base property*: $\emptyset \notin \mathcal{B}$ and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$. So, each basis is a directed set with the order given by "reversed" inclusion. We shall refer to the elements β of $\mathcal{B}$ as *basis sets*. Some particular examples of a derivation basis of topological spaces of various types can be found in [1, 9, 13, 14, 16]. In this paper, we shall suppose that all pairs (I, x) making up each $\beta \in \mathcal{B}$ are such that $x \in I$, although this is not the case in the general theory (see [6, 9]). We assume that $\mu(\partial I) = 0$ for any $\mathcal{B}$-interval I. We say that two $\mathcal{B}$-intervals I' and I'' are *non-overlapping* if $\mu(I' \cap I'') = 0$. We call a $\mathcal{B}$-*figure* a finite union of non-overlapping $\mathcal{B}$-intervals. We denote by $\mathrm{Sub}(L)$ the collection of all $\mathcal{B}$-subfigures of L. We suppose that the intersection of two overlapping $\mathcal{B}$-intervals is a $\mathcal{B}$-figure, so the intersection of two overlapping $\mathcal{B}$-figures is also a figure as well as the union of any two $\mathcal{B}$-figures. For a set $E \subset X$ and $\beta \in \mathcal{B}$ we write

$$\beta(E) := \{(I, x) \in \beta : I \subset E\} \quad \text{and} \quad \beta[E] := \{(I, x) \in \beta : x \in E\}.$$

We refer to $\beta(E)$ as basis sets *in* E. We call $\{\beta(E)\}_{\beta \in \mathcal{B}}$ the basis *in* E using the same notation $\mathcal{B}$ for it.

We assume that the basis has the following properties:

- The basis $\mathcal{B}$ *ignores no point*, i.e., $\beta[\{x\}] \neq \emptyset$ for any point $x \in X$ and for any $\beta \in \mathcal{B}$.
- The basis $\mathcal{B}$ has a *local character* by which we mean that for any family of basis sets $\{\beta_\tau\}$, $\beta_\tau \in \mathcal{B}$, and for any pairwise disjoint sets E_τ there exists $\beta \in \mathcal{B}$ such that $\beta[\bigcup_\tau E_\tau] \subset \bigcup_\tau \beta_\tau[E_\tau]$.
- The basis $\mathcal{B}$ is a *Vitali basis* by which we mean that for any x and for any neighborhood $U(x)$ of x there exists $\beta_x \in \mathcal{B}$ such that $I \subset U(x)$ for each pair $(I, x) \in \beta_x$.

For a fixed basis set β, a β-*partition* is a finite collection π of β, where the distinct elements (I', x') and (I'', x'') in π have I' and I'' non-overlapping. Let $L \subset X$. If $\pi \subset \beta(L)$, then π is called a β-partition *in* L. If $\pi \subset \beta[L]$, then π is called a β-partition *on* L. If $\bigcup_{(I,x)\in\pi} I = L$, then π is called β-partition *of* L. For a set E and a β-partition π we set $\pi[E] := \{(I, x) \in \pi : (I, x) \in \beta[E]\}$.

We also assume that the basis $\mathcal{B}$ has the *partitioning property* by which we mean: (i) for each finite collection $I_0, I_1, \ldots, I_n$ of $\mathcal{B}$-intervals with $I_1, \ldots, I_n \subset I_0$ being non-overlapping there exists a finite number of $\mathcal{B}$-intervals $I_{n+1}, \ldots, I_m$ such that $I_0 = \bigcup_{s=1}^m I_s$, all I_s being pairwise non-overlapping $\mathcal{B}$-intervals; (ii) for each $\mathcal{B}$-interval I and for any $\beta \in \mathcal{B}$ there exists a β-partition of I.

We note that condition (ii) of the partitioning property in fact implies the existence of a β-partition for any $\mathcal{B}$-figure. The union of all $\mathcal{B}$-intervals involved in a β-partition π will be called the $\mathcal{B}$-*figure generated by* π.

A typical example of a basis satisfying our condition is the basis formed by usual intervals in $\mathbb{R}^n$. An interesting example of a basis in a metric space formed by closed balls, scalloped balls and their finite intersections was considered in [12].

The following lemma on an extension of a β-partition is a direct consequence of the partitioning property of the basis.

Lemma 1. *Let π_1 be a β-partition in a $\mathcal{B}$-figure L. Then there exists a β-partition π_2 in L such that $\pi = \pi_1 \cup \pi_2$ is a β-partition of L.*

We call the β-partition π_2 of the above lemma a β-*complementary to* π_1 in L. If F is a $\mathcal{B}$-figure generated by the partition π_1, then the $\mathcal{B}$-figure generated by the partition π_2 which is β-complementary to π_1 in L is called β-*complementary to* F *in* L and is denoted by $C_\beta F$.

The following lemma is proved in [15].

Lemma 2. *For any $\beta \in \mathcal{B}$ in a $\mathcal{B}$-figure L and any open set $G \subset L$ there exists a basis set $\beta' \subset \beta$ in L such that $\beta'[G] \subset \beta'(G)$, i.e., $I \subset G$ for each $(I, x) \in \beta'[G]$.*

Definition 3 (see [9]). Let $\mathcal{B}$ be a basis having the partitioning property and let L be a $\mathcal{B}$-figure. A real-valued function f on L is said to be *Kurzweil–Henstock integrable with respect to the basis* $\mathcal{B}$ (or $H_{\mathcal{B}}$-*integrable*) on L with $H_{\mathcal{B}}$-*integral* A if for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ such that for any β-partition π of L we have

$$\Big|\sum_{(I,x)\in\pi} f(x)\mu(I) - A\Big| < \varepsilon. \tag{1}$$

We denote the integral value A by $(H_{\mathcal{B}}) \int_L f$.

We say that a function f is $H_{\mathcal{B}}$-integrable on a set $E \subset L$ if the function $f \cdot \chi_E$ is $H_{\mathcal{B}}$-integrable on L and $\int_E f := \int_L f \cdot \chi_E$.

We shall need the following result proved in [15].

Proposition 4. *A function which is equal to zero almost everywhere on a $\mathcal{B}$-figure L is $H_{\mathcal{B}}$-integrable on L with integral value zero.*

It follows from this proposition that if we change the value of a function f on a set of measure zero, then it does not influence the $H_{\mathcal{B}}$-integrability of f and the value of the integral.

We note that if f is $H_{\mathcal{B}}$-integrable on a $\mathcal{B}$-figure L, then it is $H_{\mathcal{B}}$-integrable also on any $\mathcal{B}$-figure $J \subset L$. It can be easily proved that the $\mathcal{B}$-interval function $\Phi : J \mapsto (H_{\mathcal{B}}) \int_J f$ is additive on the family of all $\mathcal{B}$-figures and we call it the *indefinite $H_{\mathcal{B}}$-integral* of f.

An essential part of the classical theory of the Kurzweil–Henstock-type integral is based on the so-called Kolmogorov–Henstock lemma (see [7], the name is justified by the fact that one version of this lemma was stated by Kolmogorov in [4]). This lemma can be extended also to the case of our basis in a topological space.

Lemma 5. *If a function f is $H_{\mathcal{B}}$-integrable on a $\mathcal{B}$-figure L, with Φ being its indefinite $H_{\mathcal{B}}$-integral, then for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ such that for any β-partition π in L we have*

$$\sum_{(I,x)\in\pi} |f(x)\mu(I) - \Phi(I)| < \varepsilon.$$

Proof. The proof follows the lines of the proof in the classical case of a usual interval basis on $\mathbb{R}^n$ (see [6, Theorem 3.2.1] and [9, Theorem 1.6.1]).

Take β for which (1) holds for any β-partition π of L with ε replaced by $\frac{\varepsilon}{4}$. By the additivity of the $H_{\mathcal{B}}$-integral we can rewrite this inequality in the form

$$\Big|\sum_{(I,x)\in\pi} (f(x)\mu(I) - \Phi(I))\Big| < \frac{\varepsilon}{4}. \tag{2}$$

Now take any subpartition $\pi_1 \subset \pi$. Let F_1 be a figure generated by π_1 and let F_2 be the figure β-complementary to F_1 in L. As f is $H_{\mathcal{B}}$-integrable on F_2, there exists β_1 in F_2 such that for any β_1-partition π_2 of F_2 we have

$$\Big|\sum_{(I,x)\in\pi_2} (f(x)\mu(I) - \Phi(I))\Big| < \frac{\varepsilon}{4}.$$

We can assume that $\beta_1 \subset \beta(F_2)$. Then $\pi_1 \cup \pi_2$ is a β-partition of L and (2) holds. So we have

$$\begin{aligned}\Big|\sum_{(I,x)\in\pi_1} &(f(x)\mu(I) - \Phi(I))\Big| \\ &= \Big|\sum_{(I,x)\in\pi_1\cup\pi_2} (f(x)\mu(I) - \Phi(I))\Big| + \Big|\sum_{(I,x)\in\pi_2} (f(x)\mu(I) - \Phi(I))\Big| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.\end{aligned} \tag{3}$$

Now we split π into the two subpartitions

$$\pi^+ = \{(I,x) \in \pi : f(x)\mu(I) - \Phi(I) \geq 0\} \quad \text{and} \quad \pi^- = \{(I,x) \in \pi : f(x)\mu(I) - \Phi(I) < 0\}$$

and apply (3), taking $\pi_1 = \pi^+$ and $\pi_1 = \pi^-$. Then we get

$$\begin{aligned}\sum_{(I,x)\in\pi} &|(f(x)\mu(I) - \Phi(I))| \\ &= \sum_{(I,x)\in\pi^+} |(f(x)\mu(I) - \Phi(I))| + \sum_{(I,x)\in\pi^-} |(f(x)\mu(I) - \Phi(I))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

□

Let Φ be an additive set function on $\mathcal{I}$ and let E be an arbitrary subset of X. For fixed $\beta \in \mathcal{B}$, we set

$$\operatorname{Var}(E, \Phi, \beta) := \sup_{\pi\subset\beta[E]} \sum |\Phi(I)|.$$

We put also

$$V_\Phi(E) = V(E, \Phi, \mathcal{B}) := \inf_{\beta\in\mathcal{B}} \mathrm{Var}(E, \Phi, \beta).$$

The extended real-valued set function $V_\Phi(\cdot)$ is called a *variational measure* generated by Φ, with respect to the basis $\mathcal{B}$.

By following the proof given in [19] for the interval bases in $\mathbb{R}$, it is possible to show that $V_\Phi(\cdot)$ is an outer measure and a metric outer measure in the case of a metric space X (in the latter case the Vitali property of the basis is essential).

Lemma 6. *Let f be an $H_\mathcal{B}$-integrable function on L and let Φ be its indefinite $H_\mathcal{B}$-integral. Let $f(x) = 0$ on some set $E \subset L$. Then $V_\Phi(E) = 0$.*

Proof. For an arbitrary $\varepsilon > 0$, we choose β according to Lemma 5. Then for any partition $\pi \subset \beta[E]$ we get

$$\sum_{(I,x)\in\pi} |\Phi(I)| < \varepsilon.$$

Now from the definition of $\mathrm{Var}(E, \Phi, \beta)$ and V_Φ we get the assertion of the lemma. □

We remind that the variational measure generated by an additive set function Φ is absolutely continuous with respect to the measure μ if $V_\Phi(E) = 0$ for any set E with $\mu(E) = 0$.

Theorem 7. *Let f be an $H_\mathcal{B}$-integrable function on L and let Φ be its indefinite $H_\mathcal{B}$-integral. Then V_Φ is absolutely continuous with respect to μ.*

Proof. Let $E \subset L$ be any set of measure zero. By Proposition 4 we can assume that $f(x) = 0$ if $x \in E$. Now, the absolute continuity of V_Φ follows from the preceding lemma. □

Now we prove our versions of the Hake-type theorem.

Theorem 8. *Suppose that in a $\mathcal{B}$-interval L there exists a closed set E and an increasing sequence of $\mathcal{B}$-figures $\{F_k\}$ such that $L \setminus E = \bigcup_{k=1}^\infty \mathrm{int}_L F_k$, the function $f(x)$ equals 0 on E and is $H_\mathcal{B}$-integrable on any $\mathcal{B}$-figure $F \subset L \setminus E$, with $H_\mathcal{B}$-integral $\Phi(F) = \int_F f$. Then f is $H_\mathcal{B}$-integrable on L if and only if there exists an extension of the function Φ to* Sub(L) *such that $V_\Phi(E) = 0$. In this case $\int_L f = \Phi(L)$.*

Proof. The necessity follows from Lemma 6. To prove the sufficiency, suppose that the required extension of Φ with $V_\Phi(E) = 0$ exists. For an arbitrary $\varepsilon > 0$ we choose, according to the definition of a variational measure, β_0 such that for any $\beta_0[E]$-partition π_0 in L we have

$$\sum_{(I,x)\in\pi_0} |\Phi(I)| < \frac{\varepsilon}{2}. \tag{4}$$

Consider the increasing sequence of $\mathcal{B}$-figures $\{F_k\}$ given by the assumption and put $T_k = \mathrm{int}_L(F_k) \setminus \mathrm{int}_L(F_{k-1})$ (where $F_0 = \emptyset$). Then $\bigcup_k^\infty T_k = L \setminus E$ and $T_r \cap T_s = \emptyset$ for every $r \neq s$.

By the assumption, f is $H_\mathcal{B}$-integrable on each F_k and so, by Lemma 5, there exists a basis set β_k in F_k such that

$$\Big|\sum_{(I,x)\in\pi_k} f(x)\mu(I) - \Phi(S_k)\Big| \le \sum_{(I,x)\in\pi_k} |f(x)\mu(I) - \Phi(I)| < \frac{\varepsilon}{2^{k+1}} \tag{5}$$

for any β_k-partition π_k in F_k, where S_k is the $\mathcal{B}$-figure generated by the partition π_k. Using Lemma 2, for each k we define $\beta_k' \subset \beta_k$ such that $I \subset \mathrm{int}_L F_k$ for each $(I, x) \in \beta_k'$ with $x \in \mathrm{int}\, F_k$. Now by the local character of $\mathcal{B}$ we determine β in L such that $\beta[T_k] \subset \beta_k'[T_k]$ for every k and $\beta[E] \subset \beta_0[E]$.

Take any β-partition π of L. We can represent π as a union of disjoint subpartitions $\pi[E]$ and $\pi[T_k]$ for a finite number of k. Applying (4) to $\pi_0 = \pi[E]$ and (5) to each $\pi[T_k]$, we finally get

$$\begin{aligned}\Big|\sum_{(I,x)\in\pi} f(x)\mu(I) - \Phi(L)\Big| &\le \sum_{(I,x)\in\pi[E]} |\Phi(I)| + \sum_k \Big|\sum_{(I,x)\in\pi[T_k]} (f(x)\mu(I) - \Phi(I))\Big| \\ &\le \frac{\varepsilon}{2} + \sum_{k=1}^\infty \frac{\varepsilon}{2^{k+1}} = \varepsilon,\end{aligned}$$

which proves that f is $H_\mathcal{B}$-integrable and $\Phi(L)$ is the value of the $H_\mathcal{B}$-integral of f over L. □

Using this theorem, we obtain its generalization stated as follows.

Theorem 9. *Suppose that in a $\mathcal{B}$-interval L there exists an increasing sequence of $\mathcal{B}$-figures $\{F_k\}$ such that a function f is $H_{\mathcal{B}}$-integrable on each F_k, on the set $E = L \setminus (\bigcup_{k=1}^{\infty} \operatorname{int}_L F_k)$ and on any $\mathcal{B}$-figure $F \subset L$, $F \cap E = \emptyset$, with $H_{\mathcal{B}}$-integral $\Phi(F) = \int_F f$. Then f is $H_{\mathcal{B}}$-integrable on L if and only if there exists an extension of the function Φ to* Sub(L) *such that $V_\Phi(E) = 0$. In this case $\int_L f = \Phi(L) + \int_E f$.*

Proof. Set $G = \bigcup_{k=1}^{\infty} \operatorname{int}_L F_k$ and apply Theorem 8 to the function $f\chi_G$. We obtain that this function is $H_{\mathcal{B}}$-integrable with integral value $\Phi(L)$ if and only if there exists an additive $\mathcal{B}$-interval function Φ for which $V_\Phi(E) = 0$ and $\Phi(L) = \int_L f\chi_G$. Since $f = f\chi_G + f\chi_E$, the $H_{\mathcal{B}}$-integrability of f on L is equivalent, under the condition of the theorem, to the $H_{\mathcal{B}}$-integrability of $f\chi_G$, and moreover $\int_L f = \Phi(L) + \int_E f$. □

A particular case of the above result is the case where $E = \partial L$. In this case, by our assumption $\mu(\partial L) = 0$ and by Theorem 7, we get the following corollary.

Corollary 10. *Suppose that in a $\mathcal{B}$-interval L there exists an increasing sequence of $\mathcal{B}$-figures $\{F_k\}$ such that a function f is $H_{\mathcal{B}}$-integrable on each F_k, on the set* $\operatorname{int} L = \bigcup_{k=1}^{\infty} \operatorname{int} F_k$ *and on any $\mathcal{B}$-figure $F \subset \operatorname{int} L$, with $H_{\mathcal{B}}$-integral $\Phi(F) = \int_F f$. Then f is $H_{\mathcal{B}}$-integrable on L if and only if there exists an extension of the function Φ to* Sub(L) *such that the variational measure generated by Φ is absolutely continuous with respect to μ. In this case $\int_L f = \Phi(L)$.*

In [15], the Hake property theorem was formulated using the so-called β-bordering (here we prefer the term β-halo as in [2]).

Definition 11. Given a $\mathcal{B}$-figure L, a closed set $E \subset L$ and a basis set β in L, we say that a $\mathcal{B}$-figure $O_E = \bigcup_{j=1}^{k} I_j$ is a *β-halo of E* if $E \subset \operatorname{int}_L O_E$ and O_E is generated by the $\beta[E]$-partition $\{(I_j, x_j)\}_{j=1}^{k}$.

It is easy to check, by using Lemma 2, that for any $\beta \in \mathcal{B}$ in the $\mathcal{B}$-figure L and a closed set $E \subset L$ there exists a β-halo O_E.

Theorem 12 ([15, Theorem 1]). *Under the conditions of Theorem 9, the function f is $H_{\mathcal{B}}$-integrable on L with integral value $A + \int_E f$ if and only if for any $\varepsilon > 0$ there exists a $\beta \in \mathcal{B}$ such that for any β-halo O_E the function f is $H_{\mathcal{B}}$-integrable on a β-complementary $\mathcal{B}$-figure $C_\beta(O_E)$ and the following inequality holds:*

$$\Big| \int_{C_\beta(O_E)} f - A \Big| < \varepsilon.$$

As a result we get the equivalence of two forms of the Hake-type theorem and we can summarize our results in the following statement.

Theorem 13. *Suppose that in a $\mathcal{B}$-interval L there exist a closed set G and an increasing sequence of $\mathcal{B}$-figures $\{F_k\}$ such that $L \setminus E = \bigcup \operatorname{int}_L F_k$, the function is $H_{\mathcal{B}}$-integrable on E with integral value $A + \int_E f$ and is $H_{\mathcal{B}}$-integrable on any $\mathcal{B}$-figure $F \subset L \setminus E$, with $H_{\mathcal{B}}$-integral $\Phi(F) = \int_F f$. Then the following assertions are equivalent:*

(i) *There exists an extension of the function Φ to* Sub(L) *such that $V_\Phi(E) = 0$.*

(ii) *For any $\varepsilon > 0$ there exists a $\beta \in \mathcal{B}$ such that for any β-halo O_E the function f is $H_{\mathcal{B}}$-integrable on a β-complementary $\mathcal{B}$-figure $C_\beta(O_E)$ and the following inequality holds:*

$$|\Phi(C_\beta(O_E)) - A| < \varepsilon.$$

(iii) *The function f is $H_{\mathcal{B}}$-integrable on L with integral value $\int_L f = \Phi(L) + \int_E f = A + \int_E f$.*

Note that the equivalence of conditions (i) and (ii) can be established directly. So the results of [12] could be obtained from the results of [15] as a particular case. By the same argument, [11, Theorem 6.1] can be deduced from [8, Theorem 1].

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