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Abstract. We propose a new simple construction of hyperbolas, via a string passing through the foci, that shares properties of the classic "gardener's ellipse" construction and Perrault's construction of the tractrix as the locus of a dragged point, subject to frictional forces, at the end of a link of fixed length. We show that a frictional device such as this, with a single frictional element, traces the same locus regardless of the friction model, provided only that this is isotropic. This allows the introduction of a "purely geometrical" principle for tractional constructions more general than that of Huygens (1693).

1. STRING CONSTRUCTIONS OF HYPERBOLAS. It is well known that an ellipse is the locus of a point P constrained to move so that the sum $|FP| + |PC|$ of its distances to two fixed points F, C is a constant. The points are the foci of the ellipse, and the constant distance is the major axis. When this constraint is realized physically using a length of cord and two stakes, it provides a very simple way to lay out an elliptical path or flower bed; it is thus often called the "gardener's method" of constructing an ellipse.

We can group the hyperbolae with the ellipses as "bifocal conics." It is well known that the ellipses (on the one hand), and the hyperbolae (on the other), with a given pair of foci, are mutually orthogonal families of curves. (This is the basis of a useful system of curvilinear coordinates for the plane.)

If the signed difference $|FP| - |PC|$ of the distances, rather than the sum, is constrained, the resulting locus is one branch of a hyperbola, a straight line, or, in the limiting case, a ray. (The triangle inequality requires that we must have $||FP| - |PC|| \leq |FC|$.) Again, the points F and C are the foci of the hyperbola. This fact, too, gives rise to mechanical constructions. These require a little more ingenuity, because the constraint cannot be modeled by a "law of conservation of string" alone.

One elegant solution was discovered by the Persian mathematician Ibn Sahl at the end of the first millennium [17]. This is illustrated in Figure 1(a), taken from a woodcut in Frans van Schooten's 1646 treatise *De Organica Conicarum Sectionum*. The segment FP is not a taut string but part of a rigid rod FN, fixed at the end F; the other segment is a string from C that meets the rod at P and continues beside the rod to N. As the rod moves, the point P slides on it so that $|FP| - |PC|$ is the difference between the lengths of the rod and the string, and thus the locus of P is a hyperbola.[1] The requirement of a long rigid rod, however, makes this method inconvenient for landscaping (and other practical applications).

At the beginning of the 17th century, Kepler gave a closely related construction (Figure 1(b)) [9, Chapter IV, Section 4], in which two taut strings, constrained to pass through the point P and thence to the foci, are paid out at the same rate. Various ways to do this in practice suggest themselves: rough ropes sliding together through a gloved hand, cords held taut by a gardener's assistant, or cords passed around a common

[1] See also imaginary.org/film/mathlapse-constructions-by-pin-and-string-conics.

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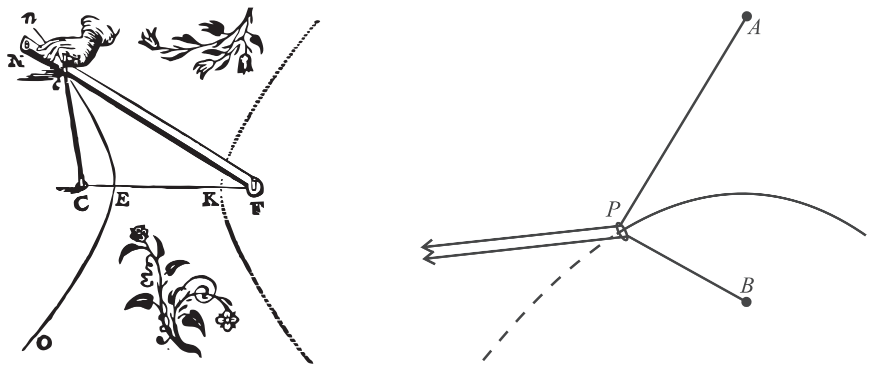

Figure 1. How to trace a hyperbola with strings. Left: by string and ruler [**19**, p. 67]. Right: by two strings to be equally extended.

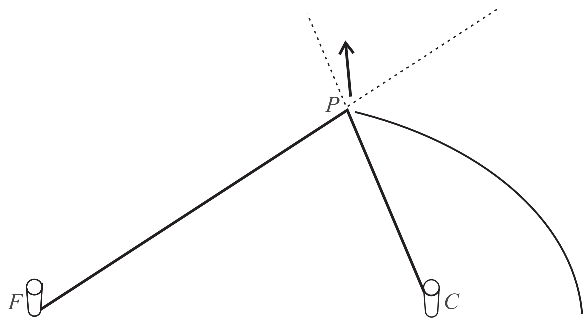

Figure 2. The gardener's ellipse: the force at P need not be directed precisely.

capstan. In many ways, this seems like a cruder version of Ibn Sahl's construction; we may speculate that Ibn Sahl was aware of both but chose to disseminate the more sophisticated one.

Physically, the ellipse and hyperbola constructions above are from the field of statics: while the apparatus must be moved to generate the whole locus, the constraint is motion-independent, and individual points on the curve may be verified without motion. The compass and straightedge are both devices of this type; so are linkages such as the pantograph and Peaucellier's inverter. While string constructions of this type require a force to keep the string taut, the precise magnitude and direction of the force are not crucial. In the "gardener's ellipse" construction (Figure 2), the gardener must apply a force at P directed within the angle vertically opposite $\angle FPC$ in order to keep the string taut; and if this force has a component tangent to the ellipse, it will serve to move the point P along the string. The nature of the force is not important: it could be applied by the gardener's dog, tied to a ring at P, trying to chase the neighbor's cat. If an acrobat walks along a slack line, then the force of gravity will make his path an ellipse.

Similarly, in Figure 1(b) the "loose ends" of the cords need not have any particular direction. Provided that the applied force is within the angle vertically opposite $\angle APB$, the strings will stay taut and P will move along the hyperbola. Again, the source of this force is irrelevant: it could be provided by gardener, greyhound, or gravity.

To these we can contrast purely dynamic constructions, such as the tracing of an ellipse or hyperbola by an orbiting body, or the (differently positioned) elliptical locus of planar simple harmonic motion. These constructions involve the interaction of a central force and inertia. Usually, such constructions are only predictable enough to be considered as mechanical constructions when friction (and other dissipative forces) are small enough to ignore.

2. FRICTIONAL MOTION. If friction predominates and we can ignore inertia, then we are in the realm of *pure frictional motion*. The best known mechanical construction of this type is Perrault's construction of the tractrix (ca. 1670) as the locus of a pocket watch on a table top attached by a chain of fixed length to a point that is dragged along a line that does not pass through the watch (Figure 3). A watch on a flexible chain can only trace the part of the tractrix lying on one side of the cusp; if the chain is replaced by a rigid link, the entire tractrix may be traced, first pushing and then pulling.

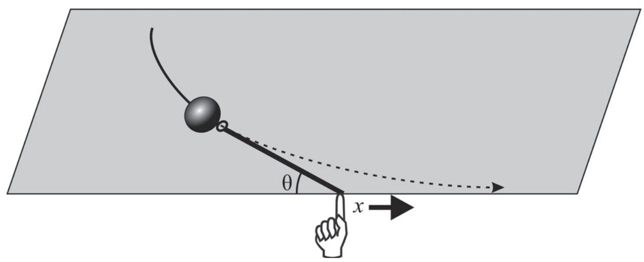

Figure 3. Perrault's tractrix.

A long-handled spoon demonstrates this well, but pulling or pushing the rigid handle requires care to exert no torque. This is made easier if the spoon is replaced by a "hatchet" with a sharp curved blade in line with the handle, allowing the free end of the device to move in that line much more easily than sideways. A wheel, mounted in line with the handle, has the same effect; Ferréol [**4**] uses this to construct (generalized) tractrices mechanically using an old-fashioned "high-wheeler" bicycle! The front wheel (which both drives and steers) follows the directrix; the small rear wheel traces the tractrix. The possibility of constructing a curve given the direction of a tracing wheel is discussed historically in [**21**], foundationally in [**13**], and computationally in [**14**].

The same spoon, hatchet, or bicycle may be used to approximately integrate the area inside a simple closed curve, an idea apparently pioneered by H. Prytz around 1875 [**16**]. The end of the spoon handle is dragged around a closed curve γ, being careful to exert no torque; when the handle returns to its original position, the bowl will be displaced from its original spot by an amount approximately proportional to the area inside γ. The mathematics of this has been described in many places, for instance in [**6**] and [**18**]. It may be noted that neither these calculations, nor Perrault's, take into account a specific model of friction. This is important, as physicists distinguish several very different types of friction.

- If frictional force is independent of speed, then we have *Coulomb friction*. This is a good approximation to the behavior of a body sliding on a dry, rough surface.
- If frictional force is proportional to speed, then we have *Stokes friction*. This occurs with lubricated surfaces, and with a body (e.g., a wheel in mud) that has to act on viscous matter along its track. There is a strong and useful analogy between (on the one hand) resistors, inductors, and capacitors and (on the other) viscous friction, masses, and springs.

- An object in a turbulent fluid exhibits friction proportional to the square or other (generally convex) function of its speed.

To see what difference the type of friction makes (or doesn't), we will need to go back to fundamentals. While Newton's formulation of mechanics in terms of force is the one taught explicitly in elementary textbooks, Lagrange gave an alternative foundation in 1788 [10] that will be better for our purposes. To get the flavor of Lagrangian mechanics, let's begin with a very familiar example than most readers will have worked out at some point.

Suppose that you want to find how fast a diver hits the water after a ten-meter dive. The purely Newtonian solution is to find the acceleration due to gravitational force, integrate to get velocity as a function of time, and again to get position. Solving the second equation for time and plugging into the first yields the answer. However, the lazy physics student (all good scientists are lazy when they can be) soon learns a shortcut. First, compute how much potential energy becomes kinetic energy during the dive; then figure out what velocity that corresponds to. Lagrangian mechanics formalizes this intuition: while mathematically equivalent to Newtonian mechanics, it considers *energy*, rather than force, as fundamental.

Consider the nondissipative system described by parameters $\{q_1, \ldots, q_n\}$. Let its kinetic energy be $T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ (where dots indicate time derivatives) and its potential energy $U(q_1, \ldots, q_n)$. By hypothesis, the total energy $T + U$ is conserved; thus T, U, and all linear combinations $aT + bU$ with $a \neq b$ carry equivalent information. The difference, $L := T - U$, turns out to be particularly convenient to work with; it is known as the *Lagrangian*. A role analogous to that of Newton's second law of motion is taken by the *Euler–Lagrange equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, \ldots, n. \tag{1}$$

(More complete expositions of this approach to classical mechanics can be found in any upper-level textbook on classical mechanics, for instance, Goldstein [7] or Landau and Lifshitz [11]. Chapter 19 of Volume 2 of *The Feynman Lectures on Physics* [5] contains one of the few truly accessible descriptions.)

Dissipative forces can be introduced by including the *Rayleigh dissipation function*, a velocity-dependent pseudopotential suggested by Lord Rayleigh in 1873 [20]. Rayleigh introduced it for Stokes friction: $\mathcal{F} = 1/2 \sum k_i \|v_i\|^2$, where the sum runs over all frictional elements of the system. The idea can, however, be used with other friction models [15], letting $\mathcal{F} = \sum k_i f(v_i)$. For instance, for Coulomb friction, we have $\mathcal{F} = \sum k_i \|v_i\|$. (Nondifferentiability at $v_i = 0$ may require careful handling — see Example 4 .) The modified Lagrange equations are [7, p. 24]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0. \tag{2}$$

In the purely frictional case, we treat inertia (and hence kinetic energy) as negligible. The Lagrangian is thus just $-V$, where $V(q)$ is a scalar potential function giving the generalized forces driving the motion of the system. As this is independent of $\dot{q}$, the first term of (2) is zero, and the equations of motion become

$$\frac{\partial V}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0. \tag{3}$$

Example 1. (Perrault's tractrix.) Suppose a pocket watch, originally positioned at $(0, R)$ on a rough table top, is to be pulled by a chain of length R, the other end of which moves along the x-axis (Figure 3). The sliding friction of the watch is isotropic. What is the path followed by the watch? The coordinates will be x (the other end of the chain) and θ, the angle the chain makes with the x-axis. The position of the watch is $(x - R\cos(\theta), R\sin(\theta))$. The isotropic friction is described by a Rayleigh function dependent only on $\|v\|$ (or for simplicity on $\|v\|^2$), hence of the form

$$\mathcal{F} = \phi(\|v\|^2) = \phi(\dot{x}^2 + 2R\sin(\theta)\dot{x}\dot{\theta} + R^2\dot{\theta}^2) \text{ where } \phi' > 0.$$

If a force F, represented by a potential $V(x)$, is applied to the chain, the equations of motion are thus

$$\frac{\partial V}{\partial x} + \frac{\partial \mathcal{F}}{\partial \dot{x}} = -F + \phi'(\dot{x}^2 + 2R\sin(\theta)\dot{x}\dot{\theta} + R^2\dot{\theta}^2)(2\dot{x} + 2R\sin(\theta)\dot{\theta}) = 0$$

and

$$\frac{\partial V}{\partial \theta} + \frac{\partial \mathcal{F}}{\partial \dot{\theta}} = \phi'(\dot{x}^2 + 2R\sin(\theta)\dot{x}\dot{\theta} + R^2\dot{\theta}^2)(2R\sin(\theta)\dot{x} + 2R^2\dot{\theta}) = 0.$$

Assuming that the watch is moving, the second equation is satisfied if and only if $d\theta/dx = \dot{\theta}/\dot{x} = -\sin(\theta)/R$, which may be solved by separation of variables. The first equation relates F to the speed of the watch's motion; if we are only interested in the locus, we do not need to solve it.

We note that in this problem the locus is independent of the function ϕ. This is a feature of a significant class of such problems.

3. TRACTIONAL CONSTRUCTIONS: FROM MECHANICS TO GEOMETRY. Starting with the tractrix, what became known as "tractional motion" [2] was recognized as a method by which curves might be mechanically constructed. In the early modern period, the lack of a geometrical counterpart for algebraic objects was seen as a crisis for the earlier geometric foundation: a first widely accepted answer to such a problem is given by Descartes' 1637 *Géométrie*, which provided a general method to rephrase geometrical problems in the language of algebra and vice versa [3].

As a consequence, the Cartesian canon distinguished between *geometrical* (algebraic) curves and *mechanical* (transcendental) curves, with only the first considered to be within the subject matter of geometry. At the end of the 17th century, geometers like Huygens and Leibniz looked for a general class of constructions to justify the introduction of the transcendental curves that Descartes had excluded. For them, the ideas behind Perrault's construction constituted the key to solving inverse-tangent problems (i.e., finding the curve whose tangent satisfies certain conditions). For instance, in Perrault's formulation of the tractrix construction, the friction of the watch on the base plane ensures that the chain always remains straight and tangent to the curve defined by the watch. While Newton, as his correspondence shows, had already found the equation of the tractrix by 1676, he took the study of tractional motion no further.

However, as noted above, friction is not a single phenomenon, but a complicated class of phenomena, the study of which is a branch of physics (tribology) in itself. Therefore, to have any simple geometric model, tractional motion must be rephrased so as to avoid physical entities, as evident in Huygens' achievement of a "purely geometrical" principle [8, Lettre de Huygens à H. Basnage de Beauval, February 1693, pp. 407–422]:

> On a horizontal plane consider a physical point attached to an inextensible string or an inflexible rod. While pulling the other extremity (of the string or of the rod), if the point makes some resistance with the plane by its weight or other physical properties, this point moves along a trajectory in which the taut string or the rod are always tangent to the described curve.

This principle allows the frictional element of a simple tractional system to be treated as a "black box," allowing the mechanism to be analyzed geometrically with no knowledge of the form of (isotropic) friction involved.

This setting is the basis for some generalizations of the tractrix: by making one extremity of the string or rod move along an arbitrary curve called the directrix, one can find generalized tractrices; these were used in 1736 by Euler to solve quadratic first-order ODEs (Riccati's equation, named for Jacopo Riccati (1676–1754)). But even stronger generalizations had already been proposed by Leibniz in the same year as Huygens' publication: a universal machine to solve first-order ODEs [1, pp. 112–114] (an ideal device, not meant to be physically constructed) can be obtained by varying the length of the string. A general geometric theory of generalized tractrices by Vincenzo Riccati (1707–1775, the son of Jacopo) has only recently been re-evaluated [21].

However, Huygens' tractional principle requires that the geometry of the mechanism yields the direction of motion of the dragged point in a trivial fashion. There are problems for which the traction can still be given by a purely geometric analysis, without resorting to explicit frictional models and force calculations, even when the direction of motion cannot be read directly from any element of the mechanism. We will derive a generalized version of Huygens' tractional principle for a larger class of tractional mechanisms, which includes for instance the "gardener's hyperbola" device in Figure 7.

We note in passing that there are two rather trivial cases in which the tractional elements of a system may simply be ignored. If the system has only one degree of freedom, and the motive force is strong enough to overcome the combined static friction of all dragged points, then the locus of motion trivially follows that single degree of freedom. Thus, the friction of the pencil on the paper may be ignored when we consider the geometry of a compass.

Second, if some constraints of the system are inequalities (as in the inextensible chain of Perrault's watch), then we may have a "tensioning" force that acts to turn them into equalities. In the case of the gardener's ellipse (see Figure 2), frictional forces cannot do this; but the first construction in Section 4 (Figure 6) illustrates a case in which friction has such a role. Such forces are, again, usually ignored in a geometric analysis.

Consider a planar mechanism with two degrees of freedom, driven by a force that can be represented (at least locally) by a smooth potential V and with a *dragged point* subject to frictional forces. We find orthogonal coordinates (q_1, q_2) such that $V(k, q_2)$ is constant for each k; then $\partial V/\partial q_1 = F(t)$ and $\partial V/\partial q_2 = 0$. Suppose furthermore that the system is constrained by a single dragged point exerting an isotropic frictional force $-f(\|v\|)(v/\|v\|)$. This force is representable by a Rayleigh pseudopotential

$$\mathcal{F} = \Psi(\|v\|^2)$$

where Ψ is an increasing function: $\Psi(x) = x$ for Stokes's viscous friction, $\Psi(x) = \sqrt{x}$ for Coulomb's dry sliding friction, etc.

As the coordinate system is orthogonal, $\|v\|^2 = a(q_1, q_2)\dot{q_1}^2 + b(q_1, q_2)\dot{q_2}^2$. (If the coordinates are conformal, then $a = b$.) So the second equation of motion becomes

$$\partial V/\partial q_2 + \partial \mathcal{F}/\partial \dot{q_2} = 0 + 2\Psi'(\|v\|^2)b(q_1, q_2)\dot{q_2} = 0.$$

As Ψ' and b never vanish, we conclude that $\dot{q_2} = 0$, so that the locus of the system is a curve orthogonal to the level curves of q_1. Equivalently, *the instantaneous displacement of the dragged point is minimized, subject to the instantaneous change in* q_1. Note that this conclusion is independent of a, b, Ψ, and F; we will refer to it as the *dragged-point principle*. When applicable, it allows the locus of a dragged point to be computed without invoking dynamics.

Example 2. (The tractrix, redux.) We analyze Perrault's watch using the dragged-point principle. First, though it is on a flexible chain, friction acts as a tensioning force, so we may treat the chain as a rigid link of length R. Second, $\partial V/\partial\theta = 0$; so we take $q_2 = \theta$ and choose q_1 to make the coordinate system orthogonal. Then the watch moves along a trajectory orthogonal to the family of circles $(x - a)^2 + y^2 = R^2$ which may be found by integrating $y\,dx = (x - a)dy$ or, by eliminating a and separating variables:

$$\frac{\sqrt{R^2 - y^2}}{y} dy = dx.$$

Limitations of the principle. There are, however, limitations on this principle. It does not give the speed of motion; it cannot be applied if inertia is significant; and it fails if the frictional force is nonisotropic. Moreover, if there are two or more frictional elements, the principle is insufficient to predict the dynamics of the system: the details of the pseudopotentials Ψ_j become important.

Example 3. (Frictional Atwood's machine.) A person is snowshoeing at 1 m/s towing two children on sleds, using a single towrope passing through a frictionless carabiner (Figure 4). The coefficient of (Stokes) friction of one sled is k_1, that of the other is $k_2 = 2k_1$. How fast does each child move?

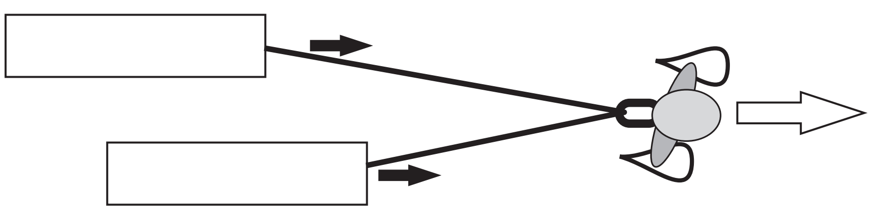

Figure 4. A frictional Atwood's machine, top view.

Let q_1, q_2 be the positions of the children, and F the force exerted by the snowshoer, whose location is $\frac{1}{2}(q_1 + q_2) + c$. Then $V = -(F/2)(q_1 + q_2) + c$ and $\partial V/\partial q_1 = \partial V/\partial q_2 = -F/2$. We also have $\mathcal{F} = \frac{1}{2}\left(k_1\dot{q}_1^2 + k_2\dot{q}_2^2\right)$. The equations of motion are thus

$$\frac{\partial V}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = -F/2 + k_j\dot{q}_j = 0$$

so that

$$\dot{q}_1 = 2\dot{q}_2 = -F/(2k_1).$$

So child 1 travels twice as fast as child 2; that is, at $4/3$ m/s and $2/3$ m/s, respectively.

Example 4. (Frictional Atwood's machine, continued.) Suppose instead that the sleds are on a rough sidewalk, subject to Coulomb friction. How fast does each child move?

The Rayleigh function $\mathcal{F} = \sum k_j \|\dot{q}_j\|$ for Coulomb friction is nondifferentiable when $\dot{q}_j = 0$; to avoid problems, we will replace $\|\dot{q}_j\|$ on $[-\epsilon, \epsilon]$ by a continuously differentiable concave function g with $g'(-\epsilon) = -k_j$ and $g'(\epsilon) = k_j$ as in Figure 5. By the intermediate value theorem, g' takes every value between $-k_j$ and k_j. We now

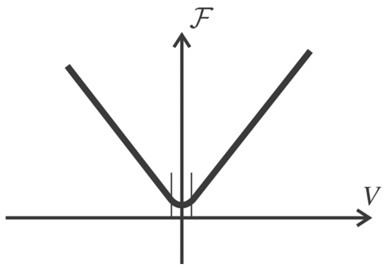

Figure 5. Modification of the Coulomb friction pseudopotential.

have $\mathcal{F} = k_1|\dot{q}_1| + k_2|\dot{q}_2|$ unless $|\dot{q}_j| < \epsilon$. For $\dot{q}_j > \epsilon$, the equations of motion are

$$\frac{\partial V}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = -F/2 + k_j = 0.$$

But this implies $k_1 = F/2 = k_2 = 2k_1$, which is inconsistent. The only solution (if the adult is moving) puts $|\dot{q}_2| < \epsilon$, in which range $\partial\mathcal{F}/\partial\dot{q}_2$ can take smaller values. We interpret this as saying that child 2 does not move, while child 1 travels twice as fast as the adult. Note in particular that the locus of child 2 is not the same as in the previous example.

4. AN APPLICATION: GARDENER'S HYPERBOLAS. We present two tractional constructions for a hyperbola. The mechanism in Figure 6 is in fact only a very minor variation on Kepler's method. The segments $\overline{FP}$ and $\overline{PC}$ are shortened simultaneously from the points F and C, via rings or pulley blocks, and an object at P slides tractionally. The sum $T_1 + T_2$ of the tensions at P lies within the angle $\angle FPC$; the velocity vector is parallel to that, and so long as the stabilizing force vector is within the opposite angle, it will always keep the lines taut, so we do not need to assume exactly isotropic friction. Neither the magnitude nor the exact direction of the frictional force is critical; any minor variation will be compensated for by changed tension in one or both lines. As with the gardener's ellipse of Figure 2, this is a static construction; individual points may be verified without drawing the curve.

While it looks superficially similar, the mechanism in Figure 7 is a true tractional device (and does require that friction be isotropic, though the force law still does not need to be specified). Here there is a frictionless ring at P, allowing the tension in the lines to equalize. The lines could run as in Figure 6, or for simplicity we can make the end at F fast, stand at C, and pull the cord.

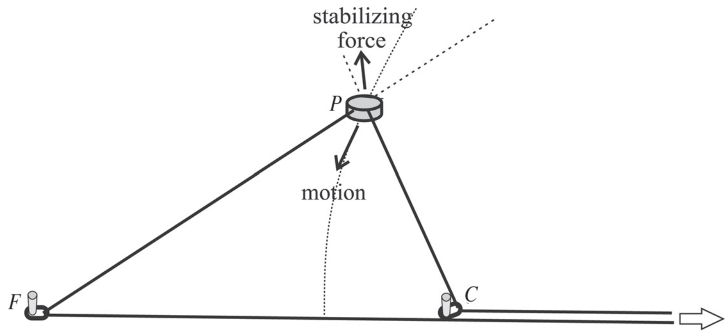

Figure 6. A gardener's hyperbola: the force at P is approximate and may be provided by friction.

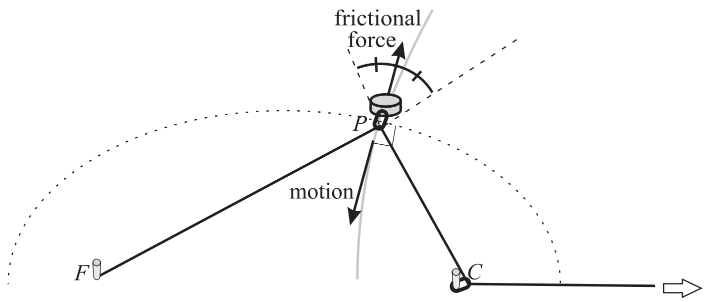

Figure 7. Another gardener's hyperbola: the force at P must be parallel to that of friction.

Let the foci be $(-R, 0)$ and $(R, 0)$, so the string lengths are given by $l_1^2 = (x + R)^2 + y^2$ and $l_2^2 = (x - R)^2 + y^2$ (where x and y are the Cartesian coordinates of the point). We adopt *elliptic-hyperbolic coordinates* with $q_1 = l_1 + l_2$ and $q_2 = l_1 - l_2$; this is an orthogonal coordinate system [**12**]. The force F applied to the cord is represented by a potential W with $\partial W/\partial q_1 = -F$ and $\partial W/\partial q_2 = 0$. By the dragged-point principle, the equations of motion are satisfied if and only if the point moves so as to minimize displacement for a given change in q_1. Clearly such motion is along a level curve of q_2: thus the difference between the string lengths is constant, and the locus is an arc of a hyperbola.

As in Example 1, this construction is independent of the friction model; we merely require that the frictional force be parallel in a negative sense to the motion vector, with friction large enough that inertia may be ignored. (Given the complex mechanical properties of turf, this is a good thing for outdoor constructions!) Another surprising feature is that because we idealize the line as perfectly flexible, so that the tensions in the two segments are equal, there is no requirement that it be inextensible! In a real-world application, light polypropylene or nylon cord (both notoriously stretchy), or even shock cord, will serve. In contrast, it is easily seen that the mechanisms of Figures 2 and 6 will not work unless the lines are inextensible.

5. CONCLUSIONS. We have shown that if an inertialess planar device with two degrees of freedom and a single frictional element traces a curve, that curve is independent of the (isotropic) friction model. This is not true in general for devices with more than one such element. As an application of this, we presented two constructions for the hyperbola, one a variation on the classic string constructions given by Ibn Sahl and Kepler, the other more essentially tractional.

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Pollock's Generalized Tetrahedral Numbers Conjecture

The nth tetrahedral number $Te_n = \binom{n+2}{3}$ represents the sum of the first n triangular numbers. In the song "The Twelve Days of Christmas," Te_n counts the total number of gifts received after day n.

A longstanding conjecture of Pollock (from [4]) is that every positive integer may be expressed as the sum of at most five tetrahedral numbers. To date, only 241 positive integers have been found requiring five tetrahedral numbers (see [3]). Recently, progress has been made (in [1]) on a related conjecture of Pollock from the same 19th century paper.

Here we instead consider generalized tetrahedral numbers $Te_n = \frac{(n+2)(n+1)n}{6}$, defined for all integers n. These are the generalized binomial coefficients $\binom{n+2}{3}$, as popularized in [2]. With these we can prove the following.

Theorem. *Every integer may be expressed as the sum of at most four generalized tetrahedral numbers.*

Proof. For arbitrary $n \in \mathbb{Z}$, we have $Te_n + Te_{n-2} + Te_{-n-1} + Te_{-n-1} = \frac{1}{6}((n+2)(n+1)n + n(n-1)(n-2) + 2(-n+1)(-n)(-n-1)) = n$. ∎

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