The Intersection of 3-Maximal Submonids

Giuseppa Castiglione

Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, Italy

Štěpán Holub

Department of Algebra, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

Abstract

Very little is known about the structure of the intersection of two k-generated monoids of words, even for $k = 3$. Here we investigate the case of k-maximal monoids, that is, monoids whose basis of cardinality k cannot be non-trivially decomposed into at most k words. We characterize the intersection in the case of two 3-maximal monoids.

Keywords: 3-maximal monoids, intersection, free graph

1. Introduction

In this paper, we investigate the intersection of three-generated monoids of words in a special case when these monoids are 3-maximal. A monoid of words is k-maximal if its generating set cannot be non-trivially decomposed into at most k (shorter) words. Obviously, the intersection of two finitely generated monoids of words is regular. However, already in the case of free monoids generated by two words, the structure of the intersection can be quite complex as we recall in Theorem 8, see [13, 11]. While monoids generated by three words have been classified (by Budkina and Markov in [4] and by Spehner in [17], see also [9] for a comparison of the two classifications), there is no classification of their intersection. It is useful to note, and we shall use this fact in the paper, that the general question about the structure of the intersection of two k-generated monoids is in fact a question about maximal solvable systems of equations over $2k$ unknowns, where the left hand sides and right hand sides are formed from disjoint sets of k unknowns respectively. This indicates why the question is so difficult for $k = 3$, where we have to deal with six unknowns.

It turns out, however, that when the condition of being k-maximal is added, the problem simplifies considerably. In [5], a kind of defect theorem is shown for 2-maximal monoids, see Theorem 10 below. In case of 3-maximal monoids, studied in this paper, we encounter a situation which rather resembles the general case of two two-generated monoids. In fact, there is a close similarity to the related problem of binary equality sets. In [6], it was shown that the binary equality set is either generated by at most two words, or it is of the form $(uw^*v)^*$. While it was later shown in [10] that the latter possibility never takes place for binary equality words, we show in this paper that the set of possibilities given in the previous sentence is the exact description of

Email addresses: giuseppa.castiglione@unipa.it (Giuseppa Castiglione), holub@karlin.mff.cuni.cz (Štěpán Holub)

intersection of two 3-maximal monoids. This setting therefore fits, from the point of its complexity, somewhere between binary equality words, and the intersection of free two-generated monoids.

2. Preliminaries

Let Σ^* ($\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$ resp.) be the *free monoid* (*free semigroup* resp.) freely generated by a countable set Σ which will be fixed throughout the paper. As usually, we shall call the set Σ an *alphabet*, and understand elements of Σ^* (resp. Σ^+) as finite words (finite nonempty words resp.) over Σ with the monoid operation of concatenation. Note however, that Σ, understood as the set of generators satisfying $\Sigma \subseteq \Sigma^*$, is the set of words of length one, rather than a set of letters.

We say that a word u is a *prefix* (res. *proper prefix*) of w and we write $u \leq w$ (resp. $u < w$) if $w = uz$ for some $z \in \Sigma^*$ (resp. $z \in \Sigma^+$). We say that u is a suffix of w if $w = zu$ for some $z \in \Sigma^*$. Two words v and w are *prefix comparable* iff either $v \leq w$ or $w \leq v$. A word w is *primitive* if $w = v^n$ implies $n = 1$ and $w = v$, otherwise it is called *a power*. If we consider pairs of words, we see $\Sigma^* \times \Sigma^*$ as a direct product of monoids, that is, a concatenation is defined in the natural way on pairs. In particular, we say that $(u, v) \in \Sigma^* \times \Sigma^*$ is a prefix (resp. proper prefix)) of $(r, s) \in \Sigma^* \times \Sigma^*$, and we write $(u, v) \leq (r, s)$ (resp. $(u, v) < (r, s)$) if $u \leq r$ and $v \leq s$ (resp. $u < r$ and $v < s$).

Given $u, v \in \Sigma^*$, by $u \wedge v$ we denote the longest common prefix of u and v. Let $u \in \Sigma^*$, by *first*(u) we denote the first letter of u.

Given a subset X of Σ^*, by X^* we denote the submonoid of Σ^* generated by X. Conversely, given a submonoid M of Σ^*, there exists a unique minimal (w.r.t. the set inclusion) generating set $B(M)$ of M, that we call the *basis* of M, namely

$$B(M) = (M \setminus \{\varepsilon\}) \setminus (M \setminus \{\varepsilon\})^2. \tag{1}$$

That is, the basis of M is the set of all nonempty words of M that cannot be written as a concatenation of two nonempty words of M. For an arbitrary set $X \subseteq \Sigma^*$, we shall write $B(X)$ instead of $B(X^*)$. The cardinality of $B(X)$ is the *rank* of X, denoted $r(X)$.

A submonoid M of Σ^* with the basis B is said to be *free* if any word of M can be *uniquely* expressed as a product of elements of B. The basis of a free monoid is called a *code* (see [3]).

It is well-known (see [18]) that for any set $X \subseteq \Sigma^*$ there exists the smallest free submonoid $\langle X \rangle_f$ of Σ^* containing X. It is called the *free hull* of X. The basis of $\langle X \rangle_f$ is called the *free basis* of X, denoted by $B_f(X)$. The cardinality of $B_f(X)$ is called the *free rank* of X and denoted by $r_f(X)$. We refer to [8] for a procedure determining the free rank and the free basis of X.

For $w \in \langle X \rangle_f$, let $\mathit{first}_X(w) = b_1$ where $w = b_1 b_2 \cdots b_n$, $b_i \in B_f(X)$, be the unique factorization of w into elements of $B_f(X)$. The words $b_1, b_1 b_2, \ldots, b_1 b_2 \cdots b_n$ are called *X-prefixes* of w. We write $u <_X w$ if u is a X-prefix of w. Moreover, given $u, w \in \langle X \rangle_f$, by $u \wedge_X w$ we denote the *longest common X-prefix of u and w.*

Example 1. Let $X = \{abcac, bab, ab, cacabcacb, ca\}$. The free basis is $B = \{ab, b, ca, cac\}$, hence $r_f(X) = 4$. For $u = ab \cdot ca \cdot ca \cdot b \cdot cac \cdot b$ and $w = ab \cdot cac \cdot ca \cdot ca \cdot ab \in X^*$, we have $\mathit{first}_X(u) = ab$, $u \wedge w = abcac$ and $u \wedge_X w = ab$.

We have the following well-known lemma (see [2], Lemma 3.1).

Lemma 2. *Let X a finite set of Σ^* and B its free basis. Then for each $y \in B$ there exists $u \in X$ such that $\mathit{first}_X(u) = y$.*

In order to see the importance of the above lemma, let us define the *free graph* of a finite set $X \subset \Sigma^+$ as the undirected graph $G(X) = (X, E_X)$ without loops where $E_X = \{[u, w] \in X \times X \mid u \neq w \text{ and } \mathit{first}_X(u) = \mathit{first}_X(w)\}$. Let $c(X)$ be the number of connected components of $G(X)$. By Lemma 2, we now have that

$$r_f(X) = c(X).$$

Remark 3. The notion of the free graph is tightly related to the graph G_Q of the system Q of equations that is defined in [7, Section 3]. Namely, the graph of the system of all relations satisfied by the set X corresponds to a subgraph of the free graph of X. We remark that the two graphs would be equivalent if we considered the unique factorization hull instead of the free hull. We have chosen the free hull because it is simpler and sufficient for our purposes. In any case, the preceding equality immediately implies the Defect Theorem claiming that $r_f(X) < |X|$ if X is not a code (cf. [2] and [7]).

Example 4. Consider X of the Example 1. The free graph $G(X)$ has a unique edge $[abcc, ab]$ connecting the only two words starting with $ab \in B$. Note that there is no edge between ca and $cacabcacb$, since $\mathit{first}_X(ca) = ca \neq cac = \mathit{first}_X(cacabcacb)$.

The free graph is a frequently used tool in the paper because it allows us to easily establish the free rank of a set by considering the properties of the edges of the associated free graph.

3. *k*-maximal Submonoids

In this section we study k-maximal submonioids introduced in [5]. With $\mathcal{M}_k$ we denote the family of submonoids M of Σ^* of rank at most k.

Definition 5. (cf. [5]) A submonoid $M \in \mathcal{M}_k$ is *k-maximal* if for every $M' \in \mathcal{M}_k$, $M \subseteq M'$ implies $M = M'$.

In other words, the elements of the basis of M cannot be nontrivially factored into at most k words.

Example 6. For every word $v \in \Sigma^+$, the submonoid $\{v\}^*$ (denoted simply by v^*) is 1-maximal if and only if v is a primitive word.

The submonoid $\{a, cbd, dbd\}^*$ is 3-maximal, whereas $\{a, cbd, dcbd\}^*$ is not 3-maximal since it is contained in $\{a, cb, d\}^*$.

Let $|X| = |\mathit{alph}(X)| = k$, where $\mathit{alph}(X)$ is the subset of letters of Σ occurring in the words in X. Then X^* is k-maximal if and only if $X^* = \mathit{alph}(X)^*$. Also, a k-maximal submonoid is obviously generated by primitive words. On the other hand, a finite set of k primitive words does not necessarily generate a k-maximal submonoid of Σ^* as we can see in Example 6.

In [5], it is proved that the basis of a k-maximal submonoid of Σ^* is a bifix code, in particular its free rank is k. We repeat the proof here.

Proposition 7. *Let X be the basis of a k-maximal submonoid. Then, X is a bifix code.*

Proof. If $uv, u \in X$, then $X^* \subseteq Y^*$ where Y is obtained from X by replacing uv with v. Hence X^* is not k-maximal, since $v \notin X$. Similarly for suffixes. □

The inverse is not true, see again Example 6 where $\{a, cbd, dcbd\}$ is a bifix code.

Submonoids generated by two words, i.e., elements of $\mathcal{M}_2$, have been extensively studied in the literature (cf. [14, 13, 15, 1]) and play an important role in many fundamental aspects of combinatorics on words. It is known (see [13] and [11]) that if X and U have free rank 2, then the intersection $X^* \cap U^*$ is a free monoid generated either by at most two words or by an infinite set of words. More formally, we have the following theorem.

Theorem 8. *Let $X = \{x, y\}$ and $U = \{u, v\}$ be two sets of Σ^* with free rank 2, then $X^* \cap U^*$ is one of the forms*

- *$X^* \cap U^* = \{\gamma, \beta\}^*$, for some $\gamma, \beta \in \Sigma^*$;*
- *$X^* \cap U^* = (\beta_0 + \beta(\gamma(1 + \delta + \cdots + \delta^t))^*\tau)^*$, for some $\beta_0, \beta, \gamma, \delta, \tau \in \Sigma^*$ and some $t \in N$.*

Example 9. Let $X_1 = \{abca, bc\}$ and $U_1 = \{a, bcabc\}$. One can verify that $X_1^* \cap U_1^* = \{abcabc, bcabca\}^*$. Let $X_2 = \{aab, aba\}$ and $U_2 = \{a, baaba\}$, then $X_2^* \cap U_2^* = (a(abaaba)^*baaba)^*$. Note that the submonoids here considered are not 2-maximal. Indeed, $X_1^*, U_1^* \subseteq \{a, bc\}^*$ and $X_2^*, U_2^* \subseteq \{a, b\}^*$.

To our knowledge nothing is proved in general for the intersection of two monoids of free rank 3. In [14] and [12], some properties of codes with three elements are studied.

Let us turn our attention to the intersection of k-maximal submonoids. For the intersection of 1-maximals, that is, for the submonoids in $\mathcal{M}_1$, we have the following important property: If x^* and u^* are 1-maximal submonoids (i.e., x and u are primitive words), then $x^* \cap u^* = \{\varepsilon\}$. A generalization of this result, given in [5], to the case of 2-maximal submonoids is the following.

Theorem 10. *Let $X = \{x, y\}$ and $U = \{u, v\}$, with $X \neq U$, be such that X^* and U^* are 2-maximal submonoids of Σ^*. If $X^* \cap U^* \neq \{\varepsilon\}$, then there exists a unique primitive word $z \in \Sigma^+$ such that $X^* \cap U^* = z^*$.*

Example 11. Let $X^* = \{abcab, cb\}$ and $U^* = \{abc, bcb\}^*$ be two 2-maximal submonoids of Σ^*, then their intersection is $\{abcabcbcb\}^*$.

The following example shows that Theorem 10 cannot be generalized to any $k > 2$.

Example 12. For $k = 4$ let $X = \{a, b, cd, ce\}$ and $U = \{ac, bc, da, ea\}$. It is easy to see that X^* and U^* are 4-maximal and $X^* \cap U^* = \{acda, acea, bcda, bcea\}^*$. For $k = 5$ the intersection can be generated by 6 elements, see for example $X = \{a, b, cd, ce, cf\}$ and $U = \{ac, bc, da, ea, fa\}$ are two 5-maximal submonoids and $X^* \cap U^* = \{acda, acea, acfa, bcda, bcea, bcfa\}^*$. Similar examples are easily found for $k > 5$.

In the following section, we characterize the intersection of two 3-maximal submonoids.

4. The Intersection of a Pair of 3-maximal Submonoids

In what follows, $X = \{x, y, z\}$ and $U = \{u, v, w\}$ will be two *distinct* three-element subsets of Σ^+ such that X^* and U^* are 3-maximal. Let also $Z = X \cup U$.

The following property holds:

Lemma 13. *The free rank of Z, that is, the number of connected components of $G(Z)$, is more than three. Formally, $3 < r_f(Z) = c(Z)$.*

Proof. If $r_f(Z) \leq 3$, then the inclusions $U^* \subseteq Z^*$ and $X^* \subseteq Z^*$ imply that X^* and U^* are not 3-maximal unless $X^* = U^* = Z^*$ which is excluded by the assumption that X and U are distinct. □

When we search for the elements of $X^* \cap U^*$ we are searching for those words that can be decomposed both into words x, y and z, and into words u, v, w. Consider, as an example, the sets $X = \{abbc, da, db\}$ and $U = \{abbca, b, cdad\}$. Then *abbcabbcdadb* is such a word as can be seen from its factorizations $abbc \cdot abbc \cdot da \cdot db = abbca \cdot b \cdot b \cdot cdad \cdot b$.

Double factorizations of this kind are best dealt with using two ternary morphisms as follows. We set $A = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ and define morphisms $g, h : A^* \to \Sigma^*$ by

$$\begin{array}{ll} g(\boldsymbol{a}) = x & h(\boldsymbol{a}) = u \\ g(\boldsymbol{b}) = y & h(\boldsymbol{b}) = v \\ g(\boldsymbol{c}) = z & h(\boldsymbol{c}) = w. \end{array}$$

For better readability, we use the boldface style for elements of A^*. The example above is then captured by the equality $g(\boldsymbol{aabc}) = h(\boldsymbol{abbcb}) = abbcabbcdadb$. That is, the word *abbcabbcdadb* has the structure $\boldsymbol{aabc}$ if considered in X^*, and $\boldsymbol{abbcb}$ if considered in U^*.

We say that a morphism $g : A^* \to \Sigma^*$ is *marked* if for each pair of letters $\boldsymbol{a}_1 \neq \boldsymbol{a}_2 \in A$ we have $\mathit{first}(g(\boldsymbol{a}_1)) \neq \mathit{first}(g(\boldsymbol{a}_2))$. Furthermore, if X is a finite set of Σ^* and B its free basis we say that the morphism g is X*-marked* if for each pair of letters $\boldsymbol{a}_1 \neq \boldsymbol{a}_2 \in A$ we have $\mathit{first}_X(g(\boldsymbol{a}_1)) \neq \mathit{first}_X(g(\boldsymbol{a}_2))$.

Let $g, h : A^* \to \Sigma^*$ be two morphisms. The *coincidence set* of g and h is the set defined as follows

$$C(g, h) = \{(\boldsymbol{r}, \boldsymbol{s}) \in A^+ \times A^+ | \; g(\boldsymbol{r}) = h(\boldsymbol{s})\}.$$

The elements of the coincidence set are called *solutions*. A solution is *minimal* if it cannot be written as the concatenation of other solutions. That is, if $(\boldsymbol{u}, \boldsymbol{v}) < (\boldsymbol{r}, \boldsymbol{s})$, then $(\boldsymbol{u}, \boldsymbol{v})$ is not a solution. Obviously, any solution has a unique decomposition into minimal solutions. That is, $C(g, h)$ is a free submonoid of the direct product $A^+ \times A^+$, freely generated by the set of minimal solutions.

The property of k-maximality guarantees (by Proposition 7) the following lemmas that are responsible for a relatively simple structure of the intersection. In particular, complications related to the second case of Theorem 8 are avoided.

Lemma 14. *$h(\boldsymbol{u}) \leq h(\boldsymbol{u}')$ iff $\boldsymbol{u} \leq \boldsymbol{u}'$.*

Proof. If $\boldsymbol{u} \leq \boldsymbol{u}'$, then $h(\boldsymbol{u}) \leq h(\boldsymbol{u}')$ holds trivially. Let now $h(\boldsymbol{u}) \leq h(\boldsymbol{u}')$, and proceed by contradiction. Assume that there exist $\boldsymbol{a} \neq \boldsymbol{a}' \in A$ such that $\boldsymbol{u} = \boldsymbol{pau}_1$ and $\boldsymbol{u}' = \boldsymbol{pa}'\boldsymbol{u}_1'$. Then $h(\boldsymbol{au}_1) < h(\boldsymbol{a}'\boldsymbol{u}_1')$ which implies that $h(\boldsymbol{a})$ and $h(\boldsymbol{a}')$ are prefix comparable, a contradiction with Proposition 7. □

Lemma 15. *Let $(\boldsymbol{r}, \boldsymbol{s})$ and $(\boldsymbol{r}', \boldsymbol{s}')$ be two distinct minimal solutions. Then $\boldsymbol{r}$ and $\boldsymbol{r}'$ are not prefix comparable, and $\boldsymbol{s}$ and $\boldsymbol{s}'$ are not prefix comparable.*

Proof. Assume that $\boldsymbol{r}$ and $\boldsymbol{r}'$ are prefix comparable, and assume, without loss of generality, that $\boldsymbol{r}' = \boldsymbol{rq}$, with $\boldsymbol{q} \in A^+$. Then $g(\boldsymbol{r}') = g(\boldsymbol{r})g(\boldsymbol{q}) = h(\boldsymbol{s})g(\boldsymbol{q}) = h(\boldsymbol{s}')$ and $h(\boldsymbol{s}) < h(\boldsymbol{s}')$. It follows by Lemma 14 that $\boldsymbol{s} < \boldsymbol{s}'$, hence $(\boldsymbol{r}', \boldsymbol{s}')$ is not minimal. Similarly, we prove that $\boldsymbol{s}$ and $\boldsymbol{s}'$ are not prefix comparable. □

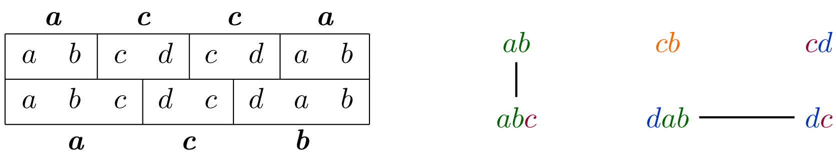

Figure 1: A representation of the solution $(\boldsymbol{acca}, \boldsymbol{acb})$ and the free graph of morphisms of Example 16.

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Example 16. Let

$$\begin{array}{ll} g(\boldsymbol{a}) = ab & h(\boldsymbol{a}) = abc \\ g(\boldsymbol{b}) = cb & h(\boldsymbol{b}) = dab \\ g(\boldsymbol{c}) = cd & h(\boldsymbol{c}) = dc. \end{array}$$

The pair $(\boldsymbol{acca}, \boldsymbol{acb})$ is a solution. Indeed $g(\boldsymbol{acca}) = abcdcdab = h(\boldsymbol{acb})$. See Figure 1 for a representation of the solution and the free graph. The free basis of $Z = X \cup U$ is $B = \{ab, c, cb, d\}$ and we highlight the decomposition into the free basis of Z by different colors. The edges of $G(Z)$ are $E_z = \{[g(\boldsymbol{a}), h(\boldsymbol{a})], [h(\boldsymbol{b}), h(\boldsymbol{c})]\}$. One can verify by a simple exhaustive search that the set of minimal solutions is $\{(\boldsymbol{ac}^i\boldsymbol{b}, \boldsymbol{ac}^{i+1}\boldsymbol{a}) \mid i \geq 0\}$ and the intersection is therefore $(abc(dc)^*dab)^*$.

This way, the problem of finding the intersection $X^* \cap U^*$ is reduced to the problem of finding minimal elements of the coincidence set of morphisms g and h. Indeed, when we find a minimal solution $(\boldsymbol{r}, \boldsymbol{s})$, with $\boldsymbol{r}, \boldsymbol{s} \in A^*$, then $g(\boldsymbol{r})$ (which is equal to $h(\boldsymbol{s})$) is an element of the minimal generating set of the intersection $X^* \cap U^*$. As we have seen in Example 16, the intersection of two 3-maximal submonoids can be infinitely generated. We shall see that in the case of a finite number of generators, the cardinality is at most two. A trivial example of a two generated intersection is $\{a, b, c\}^* \cap \{a, b, d\}^* = \{a, b\}^*$. A less trivial example is the following.

Example 17. Let

$$\begin{array}{ll} g(\boldsymbol{a}) = ab & h(\boldsymbol{a}) = abbc \\ g(\boldsymbol{b}) = bcdd & h(\boldsymbol{b}) = abcb \\ g(\boldsymbol{c}) = cbdd & h(\boldsymbol{c}) = ddab\,. \end{array}$$

There are only two minimal solutions, namely $(\boldsymbol{aba}, \boldsymbol{ac})$ and $(\boldsymbol{aca}, \boldsymbol{bc})$, hence the submonoid intersection is finitely generated by $\{abbcddab, abcbddab\}$. The free basis of $Z = X \cup U$ is $B = \{ab, ac, bd, cd, da\}$, see Figure 2 for a representations of the two solutions and the free graph.

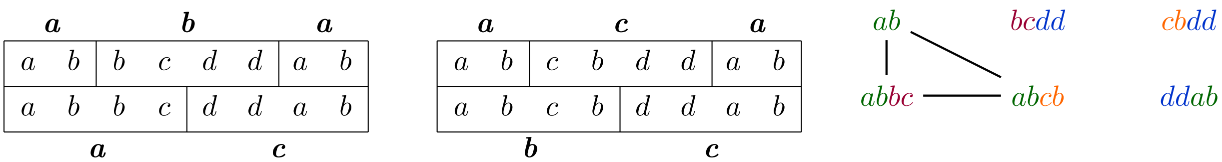

Figure 2: A representation of the solutions $(\boldsymbol{aba}, \boldsymbol{ac})$ and $(\boldsymbol{aca}, \boldsymbol{bc})$ and the free graph of morphisms of Example 17.

In what follows, we equivalently refer to X (resp. U) and $g(A)$ (resp. $h(A)$). Since $g(\boldsymbol{a})$, $g(\boldsymbol{b})$, $g(\boldsymbol{c}) \in \langle Z \rangle_f$ and $h(\boldsymbol{a})$, $h(\boldsymbol{b})$, $h(\boldsymbol{c}) \in \langle Z \rangle_f$ by definition, we have that $w \in \langle Z \rangle_f$ for any element w of the intersection.

As mentioned before, we often use the free graph of $G(Z)$ as the source of information about the free basis of $Z = X \cup U$. The set V_Z of nodes is the union of the images $g(A)$ and $h(A)$. In figures, we graphically arrange nodes in V_Z in two rows containing elements from $g(A)$ and $h(A)$ respectively. We know that the number of connected components is the free rank of Z, which is at least four. Moreover, we naturally distinguish two different kinds of edges. The edges that involve nodes in the same set, either $g(A)$ or $h(A)$, are *horizontal edges*, and the edges that involve one node of $g(A)$ and one of $h(A)$ are *vertical edges*.

The following two observations are immediate:

- A morphism g is Z-marked iff there are no horizontal edges in the corresponding row. Indeed, by definition, $[g(\boldsymbol{a}_1), g(\boldsymbol{a}_2)] \in E_Z$ iff $\mathit{first}_Z(g(\boldsymbol{a}_1)) = \mathit{first}_Z(g(\boldsymbol{a}_2))$. Analogously for the morphism h.
- A solution creates a vertical edge. Indeed, if $(\boldsymbol{r}, \boldsymbol{s}) \in C(g, h)$, then $\mathit{first}_Z(g(\boldsymbol{r}_1)) = \mathit{first}_Z(h(\boldsymbol{s}_1))$ and $[g(\boldsymbol{r}_1), g(\boldsymbol{s}_1)] \in E_Z$, where $\boldsymbol{r}_1 = \mathit{first}(\boldsymbol{r})$ and $\boldsymbol{s}_1 = \mathit{first}(\boldsymbol{s})$.

This implies the following property of our morphisms.

Lemma 18. *If $C(g, h) \neq \emptyset$, then either g or h is Z-marked. Moreover, if h is not marked, then there exist exactly two letters $\boldsymbol{a}_1, \boldsymbol{a}_2 \in A$ such that $\mathit{first}_Z(h(\boldsymbol{a}_1)) = \mathit{first}_Z(h(\boldsymbol{a}_2))$.*

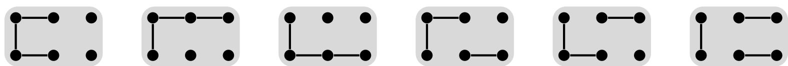

Figure 3: Free graphs with a nonempty coincidence set and two horizontal edges.

Proof. Since the set of solutions is nonempty, there is at least one vertical edge in the free graph of $G(Z)$. Since the free rank of $Z = X \cup U$ is at least four, according to Lemma 13, the free graph has at least four connected components. Consequently, as attested by Figure 3, the free graph of Z cannot contain two horizontal edges. The claim follows. □

Examples 16 and 17 shows two cases in which the morphism g is Z-marked and h is not. The following example shows two Z-marked morphisms g and h which have two minimal solutions.

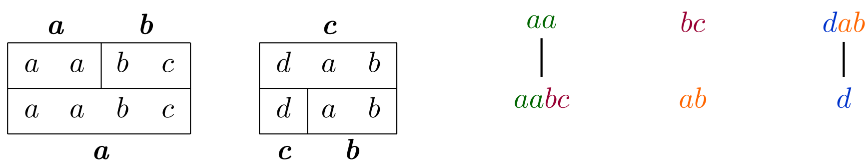

Figure 4: A representation of the solutions $(\boldsymbol{ab}, \boldsymbol{a}), (\boldsymbol{c}, \boldsymbol{cb})$ and the free graph of two Z-marked morphisms from Example 19.

Example 19. Let

$$\begin{array}{ll} g(\boldsymbol{a}) = aa & h(\boldsymbol{a}) = aabc \\ g(\boldsymbol{b}) = bc & h(\boldsymbol{b}) = ab \\ g(\boldsymbol{c}) = dab & h(\boldsymbol{c}) = d\,. \end{array}$$

The free basis of $Z = X \cup U$ is $B = \{aa, ab, bc, d\}$, g and h are both Z-marked, and the only two minimal solutions are $(\boldsymbol{ab}, \boldsymbol{a}), (\boldsymbol{c}, \boldsymbol{cb})$. In such a case each minimal solution introduces a vertical edge, which yields four connected components (cf. Figure 4).

Convention 20. By symmetry, we shall suppose in what follows that g is Z-marked, $\mathit{first}_Z(h(\boldsymbol{a})) \neq \mathit{first}_Z(h(\boldsymbol{c}))$ and $\mathit{first}_Z(h(\boldsymbol{b})) \neq \mathit{first}_Z(h(\boldsymbol{c}))$.

Now we introduce the key ingredient of the proof of our theorem, namely the definition of the critical overflow which was first introduced in [6] (see also [16, pp. 347–351]). We say that the word $o \in \Sigma^*$ is a *critical overflow* if $g(\boldsymbol{u}) = h(\boldsymbol{v})o$, for some $\boldsymbol{u}, \boldsymbol{v} \in A^*$, and there are pairs $(\boldsymbol{u}_1, \boldsymbol{u}_2)$, $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ in $A^+ \times A^+$ such that $\mathit{first}(\boldsymbol{u}_1) \neq \mathit{first}(\boldsymbol{u}_2)$, $\mathit{first}(\boldsymbol{v}_1) \neq \mathit{first}(\boldsymbol{v}_2)$ and both $g(\boldsymbol{u}\boldsymbol{u}_1) = h(\boldsymbol{v}\boldsymbol{v}_1)$ and $g(\boldsymbol{u}\boldsymbol{u}_2) = h(\boldsymbol{v}\boldsymbol{v}_2)$. Moreover, we say that o is *a critical overflow on* $(\boldsymbol{u}, \boldsymbol{v})$.

Informally, if o is a critical overflow on a pair $(\boldsymbol{u}, \boldsymbol{v})$, then $(\boldsymbol{u}, \boldsymbol{v})$ is a prefix of at least two distinct minimal solutions $(\boldsymbol{u}\boldsymbol{u}_1, \boldsymbol{v}\boldsymbol{v}_1)$ and $(\boldsymbol{u}\boldsymbol{u}_2, \boldsymbol{v}\boldsymbol{v}_2)$. It represents the situation when the continuation of $(\boldsymbol{u}, \boldsymbol{v})$ is not given uniquely during the construction of the minimal solution neither for $\boldsymbol{u}$ nor for $\boldsymbol{v}$.

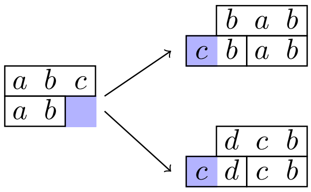

Figure 5: A critical overflow of morphisms of Example 21.

Example 21. Let $g(\boldsymbol{a}) = abc, g(\boldsymbol{b}) = bab$ and $g(\boldsymbol{c}) = dcb$, $h(\boldsymbol{a}) = ab, h(\boldsymbol{b}) = cb$ and $h(\boldsymbol{c}) = cd$. Then c is a critical overflow on $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{a}, \boldsymbol{a})$ and two minimal solutions are $(\boldsymbol{ab}, \boldsymbol{aba})$ and $(\boldsymbol{ac}, \boldsymbol{acb})$. See Figure 5 for a representation.

Remark 22. Since g and h are morphism and $\langle Z \rangle_f$ is free, it follows that the critical overflows belongs to $\langle Z \rangle_f$.

The previous remark is a basic trivial property of free monoids and its free basis but it is fundamental for the proof of the following results that characterize the critical overflows in our setting and the corresponding properties of the free graph.

Remark 23. For sake of completeness, we should also consider the case when o is nonempty and $g(\boldsymbol{u})o = h(\boldsymbol{v})$ in the definition of the critical overflow. Note however, that such a situation is excluded by the assumption that g is marked in Convention 20.

Proposition 24. *If $(\boldsymbol{r}, \boldsymbol{s})$ and $(\boldsymbol{r}', \boldsymbol{s}')$ are two distinct minimal solutions, then there is a critical overflow o on $(\boldsymbol{u}, \boldsymbol{v})$, with $\boldsymbol{u} = \boldsymbol{r} \wedge \boldsymbol{r}'$ and $\boldsymbol{v} = \boldsymbol{s} \wedge \boldsymbol{s}'$. Therefore, h is Z-marked iff o is an empty overflow.*

Proof. By lemma 15, the components of the two minimal solutions are not prefix comparable respectively. Therefore $\boldsymbol{r} = \boldsymbol{u}\boldsymbol{u}_1$, $\boldsymbol{r}' = \boldsymbol{u}\boldsymbol{u}_2$, $\boldsymbol{s} = \boldsymbol{v}\boldsymbol{v}_1$ and $\boldsymbol{s}' = \boldsymbol{v}\boldsymbol{v}_2$ where $\boldsymbol{u} = \boldsymbol{r} \wedge \boldsymbol{r}'$, $\boldsymbol{v} = \boldsymbol{s} \wedge \boldsymbol{s}'$, and all $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are nonempty. Let $\boldsymbol{a} = \mathit{first}(\boldsymbol{u}_1)$, $\boldsymbol{a}' = \mathit{first}(\boldsymbol{u}_2)$, $\boldsymbol{b} = \mathit{first}(\boldsymbol{v}_1)$ and $\boldsymbol{b}' = \mathit{first}(\boldsymbol{v}_2)$ where $\boldsymbol{a} \neq \boldsymbol{a}'$ and $\boldsymbol{b} \neq \boldsymbol{b}'$. The case $\boldsymbol{u} = \varepsilon$ and $\boldsymbol{v} \neq \varepsilon$ is excluded by the assumption that g is marked. The remaining possibilities are as follows:

- If $\boldsymbol{u} = \boldsymbol{v} = \varepsilon$, then the empty word is a critical overflow on $(\varepsilon, \varepsilon)$. Since $G(Z)$ has two vertical edges $[g(\boldsymbol{a}), h(\boldsymbol{b})]$ and $[g(\boldsymbol{a}'), h(\boldsymbol{b}')]$, it cannot have an horizontal edge, hence h is Z-marked.

- If $\boldsymbol{u} \neq \varepsilon$ and $\boldsymbol{v} = \varepsilon$, then $g(\boldsymbol{u})$ is a nonempty critical overflow on $(\boldsymbol{u}, \varepsilon)$. We have $h(\boldsymbol{b}) \neq h(\boldsymbol{b}')$, but $\mathit{first}_Z(h(\boldsymbol{b})) = \mathit{first}_Z(h(\boldsymbol{b}'))$ i.e., h is not Z-marked.
- Finally, if both $\boldsymbol{u} \neq \varepsilon$ and $\boldsymbol{v} \neq \varepsilon$, then we have $h(\boldsymbol{v}) < g(\boldsymbol{u})$ because g is marked and moreover there is a nonempty critical overflow o with $g(\boldsymbol{u}) = h(\boldsymbol{v})o$. By Remark 22 we have that $\mathit{first}_Z(o) = first_Z(h(\boldsymbol{b})) = first_Z(h(\boldsymbol{b}'))$, hence h is not Z-marked.

□

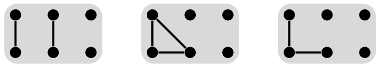

Figure 6: Free graphs in the three cases of critical overflows.

Remark 25. We can reformulate the three cases of critical overflows of the previous proof in terms of properties of $G(Z)$ as follows. Let $(\boldsymbol{r}, \boldsymbol{s})$ and $(\boldsymbol{r}', \boldsymbol{s}')$ be two distinct minimal solutions and $\boldsymbol{u} = \boldsymbol{r} \wedge \boldsymbol{r}'$, $\boldsymbol{v} = \boldsymbol{s} \wedge \boldsymbol{s}'$. Let $\boldsymbol{r} = \boldsymbol{u}\boldsymbol{u}_1$, $\boldsymbol{r}' = \boldsymbol{u}\boldsymbol{u}_2$, $\boldsymbol{s} = \boldsymbol{v}\boldsymbol{v}_1$ and $\boldsymbol{s}' = \boldsymbol{v}\boldsymbol{v}_2$, $\boldsymbol{a} = \mathit{first}(\boldsymbol{r}_1)$, $\boldsymbol{a}' = \mathit{first}(\boldsymbol{r}'_1)$, $\boldsymbol{b} = \mathit{first}(\boldsymbol{s}_1)$ and $\boldsymbol{b}' = \mathit{first}(\boldsymbol{s}'_1)$ where $\boldsymbol{a} \neq \boldsymbol{a}'$ and $\boldsymbol{b} \neq \boldsymbol{b}'$. Then,

1. If $\boldsymbol{u} = \boldsymbol{v} = \varepsilon$, then $G(Z)$ has two vertical edges $[g(\boldsymbol{a}), h(\boldsymbol{b})]$ and $[g(\boldsymbol{a}'), h(\boldsymbol{b}')]$ (see the first case in Figure 6). Note that Example 19 with Figure 4 show such a situation.
2. If $\boldsymbol{u} \neq \varepsilon$ and $\boldsymbol{v} = \varepsilon$, then $G(Z)$ has two vertical edges $[g(\mathit{first}(\boldsymbol{u})), h(\boldsymbol{b})]$ and $[g(\mathit{first}(\boldsymbol{u})), h(\boldsymbol{b}')]$ and an horizontal edge $[h(\boldsymbol{b}), h(\boldsymbol{b}')]$ creating a connected component of three nodes (see the second case in Figure 6). Example 17 with Figure 2 illustrate such a case.
3. Finally, if both $\boldsymbol{u} \neq \varepsilon$ and $\boldsymbol{v} \neq \varepsilon$, then $G(Z)$ has a vertical edge $[g(\mathit{first}(\boldsymbol{u})), h(\mathit{first}(\boldsymbol{v}))]$ and a horizontal edge $[h(\boldsymbol{b}), h(\boldsymbol{b}')]$ (see the first case in Figure 6). Note that Example 16 and Figure 1 show such a situation.

Note that in any of this cases, the graph $G(Z)$ cannot have further edges.

Lemma 26. *Let o be a critical nonempty overflow such that $g(\boldsymbol{u}) = h(\boldsymbol{v})o$ with $\boldsymbol{u}, \boldsymbol{v} \in A^*$. Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in A^+$ be such that $g(\boldsymbol{u}\boldsymbol{u}_1) = h(\boldsymbol{v}\boldsymbol{v}_1)$ and $g(\boldsymbol{u}\boldsymbol{u}_2) = h(\boldsymbol{v}\boldsymbol{v}_2)$ and*

$$\boldsymbol{a} = \mathit{first}(\boldsymbol{u}_1) \neq \mathit{first}(\boldsymbol{u}_2) = \boldsymbol{a}', \quad \boldsymbol{b} = \mathit{first}(\boldsymbol{v}_1) \neq \mathit{first}(\boldsymbol{v}_2) = \boldsymbol{b}' .$$

Then

$$o = h(\boldsymbol{b}) \wedge_Z h(\boldsymbol{b}').$$

Proof. Note that o is prefix comparable with both $h(\boldsymbol{b})$ and $h(\boldsymbol{b}')$. If $h(\boldsymbol{b}) \leq o$ or $h(\boldsymbol{b}') \leq o$, then also $h(\boldsymbol{b})$ and $h(\boldsymbol{b}')$ are prefix comparable, contradicting Proposition 7. Moreover $o \leq h(\boldsymbol{b}) \wedge_Z h(\boldsymbol{b}')$, by Remark 22.

Let $o < h(\boldsymbol{b}) \wedge_Z h(\boldsymbol{b}')$ and let $o = (h(\boldsymbol{b}) \wedge_Z h(\boldsymbol{b}'))o'$. Then $\mathit{first}_Z(g(\boldsymbol{a})) = \mathit{first}_Z(g(\boldsymbol{a}')) = \mathit{first}_Z(o')$, a contradiction with g being Z-marked. Therefore $o = h(\boldsymbol{b}) \wedge_Z h(\boldsymbol{b}')$. □

We have the following corollary.

Corollary 27. *The critical overflow is unique.*

Proof. The critical overflow is empty if and only if h is marked by Proposition 24. If h is not marked, then the critical overflow is given uniquely by Lemma 26 and Lemma 18. □

We prove the following useful lemma.

Lemma 28. *Let $g(\boldsymbol{u}) = h(\boldsymbol{v})o$ and $g(\boldsymbol{u}') = h(\boldsymbol{v}')o$. If $\boldsymbol{u}' \leq \boldsymbol{u}$, then $\boldsymbol{v}' \leq \boldsymbol{v}$.*

Proof. If $\boldsymbol{u}' \leq \boldsymbol{u}$, then also $g(\boldsymbol{u}') \leq g(\boldsymbol{u})$, hence $h(\boldsymbol{v}') \leq g(\boldsymbol{u}') \leq g(\boldsymbol{u})$. Since both $h(\boldsymbol{v})$ and $h(\boldsymbol{v}')$ are prefixes of $g(\boldsymbol{u})$, they are prefix comparable. Lemma 14 now implies that also $\boldsymbol{v}$ and $\boldsymbol{v}'$ are prefix comparable. A length argument concludes the proof. □

The following lemma constitutes an important part of the proof of the main theorem.

Lemma 29. *Let $(\boldsymbol{r}_1, \boldsymbol{s}_1)$, $(\boldsymbol{r}_2, \boldsymbol{s}_2)$ and $(\boldsymbol{r}_3, \boldsymbol{s}_3)$ be three distinct minimal solutions. Then the critical overflow is nonempty. Moreover, let*

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{r}_1 \wedge \boldsymbol{r}_2, & \boldsymbol{v} &= \boldsymbol{s}_1 \wedge \boldsymbol{s}_2, \\ \boldsymbol{u}' &= \boldsymbol{r}_2 \wedge \boldsymbol{r}_3, & \boldsymbol{v}' &= \boldsymbol{s}_2 \wedge \boldsymbol{s}_3, \\ \boldsymbol{u}'' &= \boldsymbol{r}_1 \wedge \boldsymbol{r}_3, & \boldsymbol{v}'' &= \boldsymbol{s}_1 \wedge \boldsymbol{s}_3\,. \end{aligned}$$

be such that $\boldsymbol{u}'' \leq \boldsymbol{u} \leq \boldsymbol{u}'$. Then

$$\boldsymbol{u}'' = \boldsymbol{u} < \boldsymbol{u}', \qquad \boldsymbol{v}'' = \boldsymbol{v} < \boldsymbol{v}',$$

and there are nonempty words $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ such that

$$g(\boldsymbol{u}) = h(\boldsymbol{v})o, \qquad og(\boldsymbol{p}) = h(\boldsymbol{q})o, \qquad og(\boldsymbol{u}_1) = h(\boldsymbol{v}_1),$$

and

$$\mathit{first}(\boldsymbol{v}_1) \neq \mathit{first}(\boldsymbol{q}).$$

Proof. We first show that the critical overflow is nonempty. Indeed, assume it is not empty. Then, by the uniqueness of the critical overflow, we have $\boldsymbol{r}_1 \wedge \boldsymbol{r}_2 = \boldsymbol{r}_1 \wedge \boldsymbol{r}_3 = \boldsymbol{r}_2 \wedge \boldsymbol{r}_3 = \varepsilon$. By Remark 25 case 1, the graph $G(Z)$ has three vertical edges, a contradiction.

By Proposition 24 and Corollary 27, we have

$$g(\boldsymbol{u}) = h(\boldsymbol{v})o, \qquad g(\boldsymbol{u}') = h(\boldsymbol{v}')o, \qquad g(\boldsymbol{u}'') = h(\boldsymbol{v}'')o. \tag{2}$$

where o is the (unique) nonempty critical overflow. From $\boldsymbol{u}'' \leq \boldsymbol{u} \leq \boldsymbol{u}'$, we have $\boldsymbol{v}'' \leq \boldsymbol{v} \leq \boldsymbol{v}'$ by Lemma 28. Let

$$(\boldsymbol{r}_1, \boldsymbol{s}_1) = (\boldsymbol{u}\boldsymbol{u}_1, \boldsymbol{v}\boldsymbol{v}_1), \qquad (\boldsymbol{r}_2, \boldsymbol{s}_2) = (\boldsymbol{u}\boldsymbol{u}_2, \boldsymbol{v}\boldsymbol{v}_2) = (\boldsymbol{u}'\boldsymbol{u}_2', \boldsymbol{v}'\boldsymbol{v}_2'), \qquad (\boldsymbol{r}_3, \boldsymbol{s}_3) = (\boldsymbol{u}'\boldsymbol{u}_3, \boldsymbol{v}'\boldsymbol{v}_3), \tag{3}$$

(cf. Figure 7). From Lemma 15 we deduce that $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_2'$ and $\boldsymbol{v}_3$ are nonempty, and

$$\boldsymbol{b}_1 = \mathit{first}(\boldsymbol{v}_1) \neq \mathit{first}(\boldsymbol{v}_2) = \boldsymbol{b}_2, \qquad \boldsymbol{b}_2' = \mathit{first}(\boldsymbol{v}_2') \neq \mathit{first}(\boldsymbol{v}_3) = \boldsymbol{b}_3. \tag{4}$$

We show that $\boldsymbol{u}'' = \boldsymbol{u} < \boldsymbol{u}'$ and $\boldsymbol{v}'' = \boldsymbol{v} < \boldsymbol{v}'$. Assume $\boldsymbol{u} = \boldsymbol{u}'$, hence, by (2) and 3, also $\boldsymbol{v} = \boldsymbol{v}'$ and $\boldsymbol{v}_2 = \boldsymbol{v}_2'$. This implies $\boldsymbol{b}_1 \neq \boldsymbol{b}_2 = \boldsymbol{b}_2' \neq \boldsymbol{b}_3$. Note also that $\boldsymbol{b}_1 \neq \boldsymbol{b}_3$ follows from $\boldsymbol{v}'' \leq \boldsymbol{v}$. By Lemma 26, we have $o = h(\boldsymbol{b}_1) \wedge_Z h(\boldsymbol{b}_2) = h(\boldsymbol{b}_2') \wedge_Z h(\boldsymbol{b}_3)$, i.e., we have three different elements of $h(A)$ having a nonempty common prefix. Therefore $G(Z)$ has two distinct horizontal

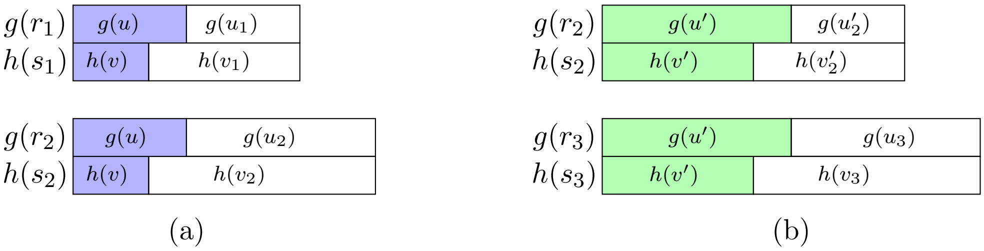

Figure 7: The representation of three minimal solutions.

edges, a contradiction. We therefore have $\boldsymbol{u} < \boldsymbol{u}'$, hence also $\boldsymbol{v} < \boldsymbol{v}'$. From (3), we now obtain $\boldsymbol{u} = \boldsymbol{r}_1 \wedge \boldsymbol{r}_2 = \boldsymbol{r}_1 \wedge \boldsymbol{u}' = \boldsymbol{r}_1 \wedge \boldsymbol{r}_3 = \boldsymbol{u}''$. Similarly, we have $\boldsymbol{v} = \boldsymbol{v}''$.

Suppose that $\boldsymbol{u} < \boldsymbol{u}'$ and $\boldsymbol{v} < \boldsymbol{v}'$, we obtain nonempty words $\boldsymbol{p}$ and $\boldsymbol{q}$ such that $\boldsymbol{u}' = \boldsymbol{u}\boldsymbol{p}$, and $\boldsymbol{v}' = \boldsymbol{v}\boldsymbol{q}$, and

$$g(\boldsymbol{u}) = h(\boldsymbol{v})o, \qquad og(\boldsymbol{p}) = h(\boldsymbol{q})o, \qquad og(\boldsymbol{u}_1) = h(\boldsymbol{v}_1).$$

Note that $(\boldsymbol{r}_2, \boldsymbol{s}_2) = (\boldsymbol{u}\boldsymbol{p}\boldsymbol{u}_2', \boldsymbol{v}\boldsymbol{q}\boldsymbol{v}_2')$. In particular, $\boldsymbol{v}_2 = \boldsymbol{q}\boldsymbol{v}_2'$, hence (4) implies

$$\mathit{first}(\boldsymbol{v}_1) \neq \mathit{first}(\boldsymbol{q}).$$

□

We can now prove the main result of the paper.

Theorem 30. *Let $X = \{x, y, z\}^*$, $U = \{u, v, w\}^*$ be different 3-maximal submonoids of Σ^*. Then $X^* \cap U^* = \{\alpha, \beta\}^*$, for some $\alpha, \beta \in \Sigma^*$*
or
$X^* \cap U^* = \{\alpha\gamma^*\beta\}^*$, for some $\alpha, \beta, \gamma \in \Sigma^+$.

Proof. If there are at most two minimal solutions, then the first option takes place. In the rest of the proof we shall assume that there are at least three distinct minimal solutions.

Let $(\boldsymbol{r}_1, \boldsymbol{s}_1)$, $(\boldsymbol{r}_2, \boldsymbol{s}_2)$ and $(\boldsymbol{r}_3, \boldsymbol{s}_3)$ be three distinct minimal solutions, and let $\boldsymbol{u}$, $\boldsymbol{u}'$ and $\boldsymbol{u}''$ be as in Lemma 29. Since $\boldsymbol{u}$, $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are prefix comparable, we can assume, without loss of generality, that $\boldsymbol{u}'' \leq \boldsymbol{u} \leq \boldsymbol{u}'$. Using Lemma 29 and canceling, if necessary, superfluous factors $(\boldsymbol{w}, \boldsymbol{w}')$ satisfying $og(\boldsymbol{w}) = h(\boldsymbol{w}')o$, we can now choose $\boldsymbol{u}_b$, $\boldsymbol{v}_b$, $\boldsymbol{u}_m$, $\boldsymbol{v}_m$, $\boldsymbol{u}_e$ and $\boldsymbol{v}_e$ such that

$$g(\boldsymbol{u}_b) = h(\boldsymbol{v}_b)o, \qquad og(\boldsymbol{u}_m) = h(\boldsymbol{v}_m)o, \qquad og(\boldsymbol{u}_e) = h(\boldsymbol{v}_e), \tag{5}$$

and

$$\mathit{first}(\boldsymbol{v}_e) \neq \mathit{first}(\boldsymbol{v}_m), \tag{6}$$

with the following additional minimality properties:

1. If $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{v}}) < (\boldsymbol{u}_b, \boldsymbol{v}_b)$, then $g(\overline{\boldsymbol{u}}) \neq h(\overline{\boldsymbol{v}})o$.
2. If $(\varepsilon, \varepsilon) \neq (\overline{\boldsymbol{u}}, \overline{\boldsymbol{v}}) < (\boldsymbol{u}_m, \boldsymbol{v}_m)$, then $og(\overline{\boldsymbol{u}}) \neq h(\overline{\boldsymbol{v}})o$.
3. If $(\varepsilon, \varepsilon) \neq (\overline{\boldsymbol{u}}, \overline{\boldsymbol{v}}) < (\boldsymbol{u}_e, \boldsymbol{v}_e)$, then $og(\overline{\boldsymbol{u}}) \neq h(\overline{\boldsymbol{v}})o$.

We call the pairs $(\boldsymbol{u}_b, \boldsymbol{v}_b)$ and $(\boldsymbol{u}_m, \boldsymbol{v}_m)$ and $(\boldsymbol{u}_e, \boldsymbol{v}_e)$ the *beginning*, *middle* and *end block* respectively, which motivates the notation.

From (5) it follows that all elements of the set $\{(\boldsymbol{u}_b\boldsymbol{u}_m^i\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m^i\boldsymbol{v}_e) \mid i \geq 0\}$ are solutions. We prove that it is in fact the set of all minimal solutions, i.e., we prove that a pair $(\boldsymbol{r}, \boldsymbol{s})$ is a minimal solution iff $(\boldsymbol{r}, \boldsymbol{s}) = (\boldsymbol{u}_b\boldsymbol{u}_m^i\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m^i\boldsymbol{v}_e)$ for a certain $i \geq 0$. First, we prove that each minimal solution is of the form $(\boldsymbol{r}, \boldsymbol{s}) = (\boldsymbol{u}_b\boldsymbol{u}_m^i\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m^i\boldsymbol{v}_e)$ for a certain $i \geq 0$. We have that $(\boldsymbol{u}_b\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_e)$ and $(\boldsymbol{u}_b\boldsymbol{u}_m\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m\boldsymbol{v}_e)$ are minimal and have the desired form. We proceed by induction on the length of $\boldsymbol{r}$. Assume that $\boldsymbol{r} \neq \boldsymbol{u}_b\boldsymbol{u}_e$, $\boldsymbol{r} \neq \boldsymbol{u}_b\boldsymbol{u}_m\boldsymbol{u}_e$, and that any minimal solution shorter than $|\boldsymbol{r}|$ is of the desired form.

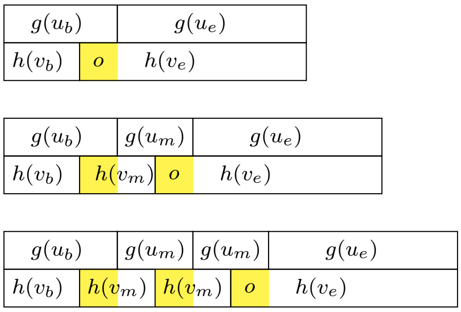

Figure 8: The block decomposition of minimal solutions $(\boldsymbol{u}_b\boldsymbol{u}_m^i\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m^i\boldsymbol{v}_e)$, with $i = 0, 1, 2$.

From Proposition 24, we deduce

$$g(\boldsymbol{u}_b\boldsymbol{u}_e \wedge \boldsymbol{r}) = h(\boldsymbol{v}_b\boldsymbol{v}_e \wedge \boldsymbol{s})o. \tag{7}$$

A length argument implies that pairs $(\boldsymbol{u}_b\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_e)$ and $(\boldsymbol{u}_b\boldsymbol{u}_e \wedge \boldsymbol{r}, \boldsymbol{v}_b\boldsymbol{v}_e \wedge \boldsymbol{s})$ are prefix comparable. Minimality of blocks, that is, properties 1, 2 and 3, then imply $\boldsymbol{u}_b\boldsymbol{u}_e \wedge \boldsymbol{r} = \boldsymbol{u}_b$ and $\boldsymbol{v}_b\boldsymbol{v}_e \wedge \boldsymbol{s} = \boldsymbol{v}_b$. Similarly, we obtain from the minimality of blocks that $\boldsymbol{u}_b\boldsymbol{u}_m\boldsymbol{u}_e \wedge \boldsymbol{r}$ is either $\boldsymbol{u}_b$ or $\boldsymbol{u}_b\boldsymbol{u}_m$. The first possibility is excluded by Lemma 29, since $\boldsymbol{u}_b\boldsymbol{u}_e \wedge \boldsymbol{u}_b\boldsymbol{u}_m\boldsymbol{u}_e = \boldsymbol{u}_b$, and we would have $\boldsymbol{u} = \boldsymbol{u}' = \boldsymbol{u}''$ for the triple $(\boldsymbol{u}_b\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_e)$, $(\boldsymbol{u}_b\boldsymbol{u}_m\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m\boldsymbol{v}_e)$, $(\boldsymbol{r}, \boldsymbol{s})$. From 7, we obtain $\boldsymbol{v}_b\boldsymbol{v}_m\boldsymbol{v}_e \wedge \boldsymbol{s} = \boldsymbol{v}_b\boldsymbol{v}_m$. Therefore, we have that $(\boldsymbol{r}, \boldsymbol{s}) = (\boldsymbol{u}_b\boldsymbol{u}_m\boldsymbol{t}, \boldsymbol{v}_b\boldsymbol{v}_m\boldsymbol{w})$, where $\boldsymbol{t}, \boldsymbol{w} \neq \varepsilon$. Then $og(\boldsymbol{t}) = h(\boldsymbol{w})$, and $(\boldsymbol{u}_b\boldsymbol{t}, \boldsymbol{v}_b\boldsymbol{w})$ is a solution. Furthermore, $(\boldsymbol{u}_b\boldsymbol{t}, \boldsymbol{v}_b\boldsymbol{w})$ is minimal, indeed, let $(\boldsymbol{r}', \boldsymbol{s}')$ be the minimal solution such that $(\boldsymbol{r}', \boldsymbol{s}') \leq (\boldsymbol{u}_b\boldsymbol{t}, \boldsymbol{v}_b\boldsymbol{w})$. By the induction assumption, we have $(\boldsymbol{r}', \boldsymbol{s}') = (\boldsymbol{u}_b\boldsymbol{u}_m^i\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m^i\boldsymbol{v}_e)$ for some i. Hence $(\boldsymbol{u}_b\boldsymbol{u}_m^{i+1}\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m^{i+1}\boldsymbol{v}_e) \leq (\boldsymbol{u}_b\boldsymbol{u}_m\boldsymbol{t}, \boldsymbol{v}_b\boldsymbol{v}_m\boldsymbol{w}) = (\boldsymbol{r}, \boldsymbol{s})$, which implies $(\boldsymbol{u}_b\boldsymbol{u}_m^{i+1}\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m^{i+1}\boldsymbol{v}_e) = (\boldsymbol{r}, \boldsymbol{s})$ since $(\boldsymbol{r}, \boldsymbol{s})$ is minimal.

It remains to show that $(\boldsymbol{u}_b\boldsymbol{u}_m^i\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m^i\boldsymbol{v}_e)$ is minimal for each $i \geq 0$. Consider the minimal solution $(\boldsymbol{r}, \boldsymbol{s}) \leq (\boldsymbol{u}_b\boldsymbol{u}_m^i\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m^i\boldsymbol{v}_e)$. Then, as we have just shown, we have $(\boldsymbol{r}, \boldsymbol{s}) = (\boldsymbol{u}_b\boldsymbol{u}_m^j\boldsymbol{u}_e, \boldsymbol{v}_b\boldsymbol{v}_m^j\boldsymbol{v}_e)$ for some $0 \leq j \leq i$. Then $\boldsymbol{v}_e \leq \boldsymbol{v}_m^{i-j}\boldsymbol{v}_e$, and we deduce $i = j$ from (6).

The second option of the theorem now takes place with $\alpha = g(\boldsymbol{u}_b), \gamma = g(\boldsymbol{u}_m)$ and $\beta = g(\boldsymbol{u}_e)$. $\square$

5. Conclusions

The hypothesis of k-maximality considerably simplifies the structure of the intersection of monoids and gives an interesting connection with the generation of binary equality set, in the case $k = 3$. Our proof also shows the importance of the free graph in this context. Together with combinatorial properties of k-maximal monoids investigated in [5], this is promising for further investigation of cases with arbitrary k.

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