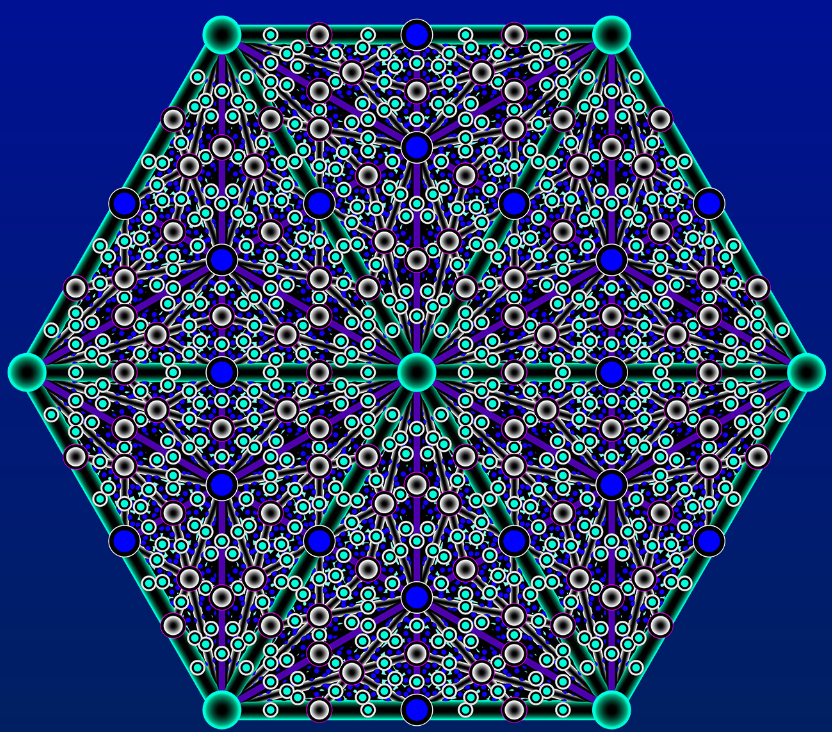
JOURNAL OF EXPERIMENTAL MATHEMATICS



ASSOCIATION FOR MATHEMATICAL RESEARCH

VOLUME 1 ISSUE 2

2025

JOURNAL OF EXPERIMENTAL MATHEMATICS

PUBLISHED BY

Association for Mathematical Research, amathr.org,
Davis, CA; Jenkintown, PA.

JOURNAL OF EXPERIMENTAL MATHEMATICS, 1(2):207–217, 2025
jexpmath.org

Greenberg's conjecture for real quadratic number fields

Pietro Mercuri * Maurizio Paoluzi René Schoof †

Received 24 Apr 2024; Revised 25 Dec 2024; Accepted 24 Mar 2025

Abstract:
We compute the 3-class groups A_n of the fields F_n in the cyclotomic $\mathbf{Z}_3$-extensions of the real quadratic fields of discriminant $f < 100{,}000$. In all cases the orders of A_n remain bounded as n goes to infinity. This is in agreement with Greenberg's conjecture.

Key words and phrases:
Iwasawa theory; Greenberg's conjecture; Real number fields; Algebraic number theory

1 Introduction

Let F be a totally real number field and let p be a prime. Let

$$F = F_0 \subset F_1 \subset F_2 \subset \dots$$

denote the cyclotomic $\mathbf{Z}_p$-extension of F. The p-class group A_n is the p-part of the ideal class group of the ring of integers of F_n. In his 1971 thesis, Ralph Greenberg conjectured that $\#A_n$ remains bounded as $n \to \infty$. See [Gre71, Gre76] and [Gre01, Conjecture (3.4)]. This is the so-called "$\lambda = 0$"-conjecture of Iwasawa theory. In this note we report on a computation for the prime $p = 3$ involving the 30394 real quadratic fields $\mathbf{Q}(\sqrt{f})$ of discriminant $f < 100{,}000$. See [FK86, IS97, IS96, KS95] for earlier computations. As a consequence we obtain the following result.

*PM is supported by GNSAGA - INdAM. Additionally, PM has also received support from the Dipartimento SBAI, Sapienza Università di Roma.
†RS is supported by GNSAGA - INdAM.

DOI: 10.56994/JXM.001.002.001

Theorem 1.1. *Greenberg's conjecture is true for $p = 3$ and the real quadratic fields of discriminant $f <$ 100,000.*

For each of the real quadratic fields with discriminant f in the range of our computation we have computed a certain Galois module $C(f)$, the finiteness of which is equivalent to Greenberg's conjecture. In this introduction we define the module $C(f)$. In the subsequent sections we explain our computation and its results. See [KS95] for the algebraic properties of $C(f)$ in the case $f \not\equiv 1 \pmod 3$. The slightly different case $f \equiv 1 \pmod 3$ is discussed in detail in [Pao02] and in Section 4 of this paper.

Let $F = \mathbf{Q}(\sqrt{f})$ be a real quadratic field of discriminant f. The Galois module $C(f)$ is defined in terms of cyclotomic units as follows. For $k \geq 1$ let ζ_k denote a primitive k-th root of unity. For $n \geq 0$ the n-th layer in the cyclotomic $\mathbf{Z}_3$-extension of F is

$$F_n = \mathbf{Q}(\sqrt{f}, \zeta_{3^{n+1}} + \zeta_{3^{n+1}}^{-1}).$$

The field F_n is a subfield of the cyclotomic field $\mathbf{Q}(\zeta_{3^{n+1}f})$. It is a cyclic degree 3^n extension of $F_0 = \mathbf{Q}(\sqrt{f})$. Its ring of integers O_n contains cyclotomic units. See [Sin80, Section 4]. The 3-part of the quotient of the unit group O_n^* by the subgroup generated by the cyclotomic units is a finite group denoted by B_n. It is known that the groups A_n and B_n have the same cardinality [Sin80, Theorems 4.1 and 5.3]. Therefore Greenberg's conjecture is true for the field F if and only if $\#B_n$ remains bounded as $n \to \infty$.

When the discriminant f is not congruent to $1 \pmod 3$, we let C_n denote the dual of the group B_n for $n \geq 0$. When $f \equiv 1 \pmod 3$, we let C_n denote the dual of the group $\tilde{B}_n$. Here $\tilde{B}_n$ sits in an exact sequence of the form

$$0 \longrightarrow \tilde{B}_n \longrightarrow B_n \xrightarrow{\phi_n} \mathbf{Z}_3/\log_3 \eta_0 \mathbf{Z}_3.$$

where for $\varepsilon \in O_n^*$ we put $\phi_n(\varepsilon) = \frac{1}{3^n}\log_3(N_n(\varepsilon))$. Here $N_n\colon F_n^* \to \mathbf{Q}(\sqrt{f})^*$ is the norm map. Since the 3-adic logarithm of a generator η_0 of the group of cyclotomic units in $\mathbf{Q}(\sqrt{f})$ is not zero, the rightmost group is a finite cyclic group. It follows that $[B_n : \tilde{B}_n]$ and hence the quotient $\#B_n/\#C_n$ is bounded independently of n. Therefore Greenberg's conjecture is true if and only if $\#C_n$ remains bounded as $n \to \infty$.

By [KS95, Lemma 2.1] and Section 4, the natural maps $B_m \to B_n$ are injective and the natural maps $C_n \to C_m$ are surjective for $n \geq m$. Let $C(f)$ denote the projective limit of the C_n. Then $C(f)$ is a Galois module and hence in the usual way a module over the Iwasawa algebra $\Lambda \cong \mathbf{Z}_3[[T]]$. It follows from the structure of the cyclotomic units that $C(f)$ is a cyclic Λ-module. See [KS95, Theorem 2.4] and Section 4. In other words, we have

$$C(f) = \varprojlim C_n \cong \Lambda/J, \qquad \text{for some ideal } J \subset \Lambda.$$

The vanishing of the Iwasawa μ-invariant of $\mathbf{Q}(\sqrt{f})$ means that J contains a monic polynomial and hence that $C(f)$ is a finitely generated $\mathbf{Z}_3$-module. See [FW79]. Greenberg's conjecture affirms that $C(f)$ is actually *finite*, so that $C_n = C(f)$ for sufficiently large n.

We have computed the Galois modules $C(f)$ for $f < 100{,}000$. It took about two weeks on a workstation with Intel processor i5. We found the following, which is equivalent to Theorem 1.1.

Theorem 1.2. *For $p = 3$ and for all discriminants $f < 100{,}000$ the module $C(f)$ is finite.*

In most cases the module $C(f)$ is actually zero. Indeed, for only 3359 out of the 30394 real quadratic fields considered, $C(f)$ is not zero and, equivalently, J is a proper Λ-ideal. This is about 11% of all cases. Of these, 2118 have J equal to the maximal ideal $(3,T)$ of Λ. In these cases $C(f)$ has order 3. For the remaining 1241 discriminants the module $C(f)$ is strictly larger. This is approximately 4% of all cases.

Rather than listing each ideal J, we indicate in Sections 3 and 5 how often ideals of a certain type appear in our computation. The full list of ideals may be of interest in itself and is available on GitHub [Iwa]. In Sections 3 and 5 we also single out some discriminants for which the ideal J has a remarkable shape.

2 The case $f \not\equiv 1 \pmod 3$

In this section we give a brief description of the algorithm in the case where the discriminant f is congruent to 0 or 2 modulo 3. This case is discussed in detail in [KS95]. Let $F = \mathbf{Q}(\sqrt{f})$ be a real quadratic field of discriminant f. Put $f' = f/3$ when $f \equiv 0 \pmod 3$ and $f' = f$ when $f \equiv 2 \pmod 3$. For $n \geq 0$ the n-th layer F_n in the cyclotomic tower of F is a subfield of $\mathbf{Q}(\zeta_{3^{n+1}f'})$. The cyclotomic unit $1 - \zeta_{3^{n+1}f'}$ is contained in $\mathbf{Q}(\zeta_{3^{n+1}f'})$. Put

$$\eta_n = \operatorname*{Norm}_{\mathbf{Q}(\zeta_{3^{n+1}f'})/F_n} (1 - \zeta_{3^{n+1}f'})^{\sigma - 1}.$$

Here σ is the non-trivial automorphism in $\mathrm{Gal}(F_n/\mathbf{Q}_n) \cong \mathrm{Gal}(\mathbf{Q}(\sqrt{f})/\mathbf{Q})$.

In [KS95] it is explained that the Galois module generated by η_n is free of rank 1 over $\mathbf{Z}[G_n]$. Here G_n denotes $\mathrm{Gal}(F_n/F_0)$. This implies that the Galois module $C(f)$ described in the introduction is a *cyclic* module over the Iwasawa algebra $\Lambda = \varprojlim \mathbf{Z}_3[G_n] \cong \mathbf{Z}_3[[T]]$. So we have $C(f) = \Lambda/J$ for some Λ-ideal J. For $n \geq 0$ we put $\omega_n(T) = (1+T)^{p^n} - 1$ and we write (ω_n) for the Λ-ideal generated by it. In [KS95] it is explained that in this case we have

$$C_n = C(f)/\omega_n C(f) = \Lambda/(J + (\omega_n)), \qquad \text{for all } n \geq 0.$$

The Galois module $C(f)$ is finite if and only if $\omega_n C(f) = 0$ and hence $C(f) = C_n$ for some $n \geq 0$. By Nakayama's lemma this happens if and only if $J + (\omega_n) = J + (\omega_{n+1})$ for some $n \geq 0$. This observation leads to the following algorithm. For $n = 0, 1, 2, \ldots$, we compute the shrinking ideals $J + (\omega_n)$ until we find that $J + (\omega_n) = J + (\omega_{n+1})$.

Our method for computing the ideals $J + (\omega_n)$ runs as follows. For a given n we first calculate a lot of elements in the ideal. As is explained in [KS95], this involves calculations with cyclotomic units modulo primes $r \equiv 1 \pmod{f'3^{n'}}$ for suitable $n' > n$. This leads to an *upper bound* for $\Lambda/(J + (\omega_n))$. To obtain a *lower bound* we employ a method due to G. and M.-N. Gras [GG77]. This involves calculations with high precision approximations of the cyclotomic units in $F_n \otimes \mathbf{R}$. See also [KS95, Section 4]. Clearly, when the upper and lower bounds agree, we have determined $J + (\omega_n)$ and hence $C_n = \Lambda/(J + (\omega_n))$.

The calculation of the lower bound for C_n becomes very time consuming and takes a lot of memory as n grows. This is caused by the high precision computations with units in cyclotomic fields of seven digit conductors and degrees in the hundreds. In fact, for most discriminants f it becomes infeasible when n exceeds 2. Fortunately, for most f we find that $J + (\omega_n) = J + (\omega_{n+1})$ and hence $C(f) = C_n$ for $n \leq 2$.

In the rare cases where we need to consider $J + (\omega_n)$ for $n \geq 3$, it is still feasible to compute the upper bound in the sense that we can easily calculate a lot of elements in the ideal $J + (\omega_n)$. An application of the Cebotarev density theorem suggests that these elements probably *generate* $J + (\omega_n)$, so that our upper bound is actually *equal* to the lower bound, but we have no rigorous proof of this.

Fortunately, we can still rigorously prove that $C(f) = \Lambda/J$ is finite and thus confirm Greenberg's conjecture even when we cannot use our algorithm to compute lower bounds for $\Lambda/(J + (\omega_n))$. It suffices to have an upper bound for n and a lower bound for *some* $m \leq n$ to which the following lemma applies. In the range of our computations this always works out with $n \geq m = 2$.

Lemma 2.1. *Let M be a finitely generated Λ-module. Suppose that for certain integers $n \geq m \geq 0$ and $b \geq a \geq 0$ we have*

$$\#M/\omega_m M \geq p^a \quad \textit{and} \quad \#M/\omega_n M \leq p^b.$$

If $b - a < n - m$, then $\omega_n M = 0$. In particular, if $M/\omega_n M$ is finite, so is M.

Proof. In the filtration

$$\omega_n M \subset \omega_{n-1} M \subset \ldots \subset \omega_{m+1} M \subset \omega_m M$$

there are $n - m$ inclusions. We have inequalities

$$\#(\omega_m M/\omega_n M) = \frac{\#M/\omega_n M}{\#M/\omega_m M} \leq p^{b-a} < p^{n-m}.$$

It follows that one of the inclusions must be an equality. So we have $\omega_{k+1} M = \omega_k M$ for some $k = m, \ldots, n-1$. Then $x = \omega_{k+1}/\omega_k$ is an element of the maximal ideal of Λ that has the property that $x\omega_k M = \omega_k M$. Nakayama's lemma implies then $\omega_k M = 0$. It follows that $\omega_n M$ is zero, as required. □

3 Numerical data for discriminants $f \not\equiv 1 \pmod 3$

3.1 Case $f \equiv 0 \pmod 3$

There are 7606 real quadratic fields with discriminant $f \equiv 0 \pmod 3$ and $f < 100{,}000$. For precisely 769 of them the Galois module $C(f) = \Lambda/J$ is not zero. This is approximately 10%. For 513 discriminants J is equal to the maximal ideal $(3, T)$ of Λ. For the remaining 256 discriminants J is strictly smaller. Section 3.1 contains some data.

The rows of Section 3.1 correspond to the *level of stabilization n*. This means that n is the smallest integer for which the ideals $J + (\omega_n)$ and $J + (\omega_{n+1})$ are equal and hence $J = J + (\omega_n)$. In particular, we have $\Lambda/J = C(f) = C_n$. The number n is also the smallest for which $\omega_n = (1+T)^{3^n} - 1$ is in J. Equivalently, 3^n is the order of $1 + T$ in the multiplicative group $(\Lambda/J)^*$.

The columns are indexed by the symbols T^k for $k = 1, 2, \ldots$. The entry in the n-th row and the T^k-column is the number of discriminants for which the level of stabilization is n, and the image of J in the ring $\mathbf{F}_3[[T]]$ is the ideal (T^k). Since ω_n is congruent to T^{3^n} modulo 3, the (n, T^k)-entry is zero whenever $k > 3^n$. In particular, in the row corresponding to $n = 0$, all entries with $k > 1$ are zero.

In the first column we count the discriminants for which the ideal J is of the form $J = (T - a, b)$ for certain $a, b \in \mathbf{Z}$. For 536 discriminants we have $a = 0$ and there is stabilization at level $n = 0$. This means

n	T	T^2	T^3	Total
0	536	0	0	536
1	112	50	2	164
2	35	7	2	44
3	15*	0	0	15
4	5*	1*	0	6
5	2*	0	0	2
6	2*	0	0	2
	707	58	4	769

Table 3.1: The modules Λ/J for $f \equiv 0 \pmod 3$.

that $\#C_0 = \#C_1$ or, equivalently $\#A_0 = \#A_1$. The discriminants for which J is equal to the maximal ideal of Λ are included here. This entry was checked by computing the class numbers of the fields F_0 and F_1 of degrees 2 and 6 respecively using a few lines of PARI/GP [The20] code. For the other entries in the first column, we have $a \notin b\mathbf{Z}_3$ and stabilization occurs at level $n = v_3(b/a)$.

An asterisk indicates that we do not have a rigorous lower bound for $C(f)$ for some of the discriminants appearing in this entry. However, our upper bound is very likely to be sharp, so that almost certainly $C(f)$ is isomorphic to Λ/J. In each case Lemma 2.1 was applied to prove Greenberg's conjecture. The 62 cases appearing in the second and third columns were dealt with using the polynomial arithmetic of Magma [BCP97]. We single out nine discriminants f for special mention.

f	J	n	T^k
31989	$(T-996, 2187)$	6	T
38424	$(T+261, 2187)$	5	T
59061	$(T^2+3T-9, 81)$	4	T^2
60513	$(T^3+3, 3T, 9)$	2	T^3
61629	$(T^3, 3)$	1	T^3
69117	$(T+69, 729)$	5	T
71049	$(T^3, 3)$	1	T^3
76584	$(T^3+3, 3T, 9)$	2	T^3
95385	$(T-2988, 6561)$	6	T

Table 3.2: Exotic Galois modules for $f \equiv 0 \pmod 3$.

3.2 Case $f \equiv 2 \pmod 3$

There are 11394 real quadratic fields with discriminant $f \equiv 2 \pmod 3$ and $f < 100{,}000$. For precisely 1250 of them the Galois module $C(f) = \Lambda/J$ is not zero. This is approximately 11% of all discriminants. For 781 discriminants J is equal to the maximal ideal $(3, T)$ of Λ. For the remaining 469 discriminants J is strictly smaller. This is about 4% of all cases. Section 3.2 contains some data.

n	T	T^2	T^3	T^4	Total
0	827	0	0	0	827
1	158	87	8	0	253
2	101	7	4	1	113
3	36*	2*	0	0	38
4	13*	1*	0	0	14
5	4*	0	0	0	4
6	1*	0	0	0	1
	1140	97	12	1	1250

Table 3.3: The modules Λ/J for $f \equiv 2 \pmod 3$.

The interpretation of the entries of the table is the same as in the case $f \equiv 0 \pmod 3$. The 781 discriminants with $J = (3, T)$ are included in the entry with $n = 0$ of the first column. Also in this case the discriminants in the first column were taken checked using a few lines of PARI/GP code. The other 110 cases were dealt with using the polynomial arithmetic of Magma. We single out nine discriminants for special mention.

f	J	n	T^k
14165	$(T-255, 729)$	5	T
16673	$(T+462, 2187)$	6	T
29165	$(T-282, 729)$	5	T
47633	$(T^2-9, 3T-90, 243)$	4	T^2
51809	$(T^2+18, 3T-18, 81)$	3	T^2
71921	$(T^2+18, 3T+18, 81)$	3	T^2
76604	$(T+294, 729)$	5	T
90005	$(T+15, 729)$	5	T
98105	$(T^4+3, 3T, 9)$	2	T^4

Table 3.4: Exotic Galois modules for $f \equiv 2 \pmod 3$.

4 The case $f \equiv 1 \pmod 3$

As before we write $F = \mathbf{Q}(\sqrt{f})$ and F_n for the n-th layer in the cyclotomic $\mathbf{Z}_3$-extension of $F = F_0$. When the discriminant f is congruent to $1 \pmod 3$, our method to compute the Galois module $C(f)$ is the same, but the details are slightly different. See [Pao02]. The differences are caused by the fact that the Galois module generated by the cyclotomic unit η_n is *not free* over the ring $\mathbf{Z}[G_n]$ when $f \equiv 1 \pmod 3$. Here η_n is defined in Section 2 and G_n denotes $\mathrm{Gal}(F_n/F_0)$. Indeed, in this case we have $N_n\eta_n = 1$, where N_n is the norm map $F_n^* \to F^*$. When $f \equiv 1 \pmod 3$, the Galois module Cyc_n of cyclotomic units in F_n on which σ acts as -1, is a direct product of the submodules generated by η_n and η_0. Here η_0 is the cyclotomic unit in F_0. It generates a group isomorphic to $\mathbf{Z}$ with trivial Galois action. On the other hand, the Galois module $\langle\eta_n\rangle$ generated by η_n is free of rank 1 over the ring $\mathbf{Z}[G_n]/(N_n)$. See [Sin80].

The submodule $\tilde{B}_n$ of B_n that was defined in the introduction, is isomorphic to $O_{n,1}^*/\langle\eta_n\rangle$. Here $O_{n,1}^*$ denotes the part of the kernel of the norm map $N_n : O_n^* \to O_0^*$ on which σ acts as -1. The Galois modules $O_{n,1}^*$, $\langle\eta_n\rangle$ and $\tilde{B}_n$ are killed by the norm N_n and are hence $\mathbf{Z}[G_n]/(N_n)$-modules. Since $\langle\eta_n\rangle$ is free of rank 1, it is more convenient to deal with $\tilde{B}_n$ rather than with B_n itself. For instance, from the exactness of the sequence of $\mathbf{Z}[G_n]/(N_n)$-modules

$$0 \longrightarrow \langle\eta_n\rangle \longrightarrow O_{n,1}^* \longrightarrow \tilde{B}_n \longrightarrow 0$$

one deduces that the natural map $\tilde{B}_m \to \tilde{B}_n$ identifies $\tilde{B}_m$ with the kernel of the endomorphism ω_m' of $\tilde{B}_n$ for $m \le n$. Here we put $\omega_m' = \omega_m/T$. In particular, we have $\omega_0' = 1$ and $C_0 = 0$. It follows that the Galois module $C(f)$ is isomorphic to Λ/J for some ideal J and $C_n = C(f)/\omega_n' C(f) = \Lambda/(J + (\omega_n'))$ for all $n \ge 0$.

Our strategy is the one explained in Section 2: for each $n = 1, 2, \ldots$, we compute the shrinking ideals $J + (\omega_n')$ until we find $J + (\omega_n') = J + (\omega_{n+1}')$, in which case Nakayama's lemma implies that $J = J + (\omega_n')$ and hence $C(f) = C_n$ and we are done. When $f \equiv 1 \pmod 3$ the issues with upper bounds and lower bounds are similar to the ones described in Section 2 for $f \not\equiv 1 \pmod 3$. In particular, we can still prove that $C(f) = \Lambda/J$ is finite in each case in the range of our computations. When the lower bound is not available for some $n \ge 3$, we invoke Lemma 2.1 with ω_m and ω_n replaced by ω_m' and ω_n' respectively.

It is not relevant for our algorithm and computations, but in the rest of this section we analyze the cokernel of the inclusion map $\tilde{B}_n \hookrightarrow B_n$. For $n \ge 0$, let U_n denote the part of the unit group $(O_n \otimes \mathbf{Z}_3)^*$ on which σ acts as -1. Since $\tilde{B}_n$ is the kernel of the map $B_n \longrightarrow U_0/\langle\eta_0\rangle$ induced by $\varepsilon \mapsto \sqrt[p^n]{N_n\varepsilon}$ for $\varepsilon \in O_n^*$, the quotient $B_n/\tilde{B}_n$ is isomorphic to a subgroup of the cyclic group $U_0/\langle\eta_0\rangle$ and is hence bounded independently of n. This can be made more precise.

The group N_nU_n is equal to the subgroup $U_0^{p^n}$ of U_0. It follows that $N_nO_n^*$ is contained in $U_0^{p^n}$. Put

$$\sqrt[p^n]{N_nO_n^*} = \{u \in U_0 : u^{p^n} \in N_nO_n^*\}.$$

For every $n \ge 0$ we have inclusions

$$N_nO_n^{*p} = N_{n+1}O_n^* \subset N_{n+1}O_{n+1}^* \subset N_nO_n^*.$$

It follows that we have a filtration

$$O_0^* \subset \ldots \subset \sqrt[p^n]{N_nO_n^*} \subset \sqrt[p^{n+1}]{N_{n+1}O_{n+1}^*} \subset \ldots \subset U_0,$$

with successive subquotiens of order 1 or p. The fact that $N_n\mathrm{Cyc}_n$ is equal to $\langle \eta_0^{p^n}\rangle$ gives rise to the isomorphisms

$$B_n/\tilde{B}_n \cong N_n O_n^*/\langle \eta_0^{p^n}\rangle \cong \sqrt[p^n]{N_n O_n^*}/\langle \eta_0\rangle.$$

This leads to the following filtration

$$O_0^*/\langle \eta_0\rangle \subset \ldots \subset B_n/\tilde{B}_n \subset B_{n+1}/\tilde{B}_{n+1} \subset \ldots \subset U_0/\langle \eta_0\rangle.$$

with successive subquotiens of order 1 or p. The leftmost group is cyclic of order $h_0 = \#A_0$ and the rightmost group has order $\log_3 \eta_0$. Writing ε_0 for a fundamental unit of $F = \mathbf{Q}(\sqrt{f})$, there are $v_3 \log_3 \varepsilon_0$ distinct steps in this filtration. By Nuccio [Nuc10] we have $B_n/\tilde{B}_n = U_0/\langle \eta_0\rangle$ when n is sufficiently large.

5 Numerical data for discriminants $f \equiv 1 \pmod 3$.

There are 11394 real quadratic fields with discriminant $f \equiv 1 \pmod 3$ and $f < 100{,}000$. For precisely 1340 of them the module $C(f)$ is not zero. This is approximately 12% of all discriminants. For 824 discriminants J is equal to the maximal ideal $(3,T)$ of Λ. For the remaining 516 discriminants the ideal J is strictly smaller. This is 4.5% of all cases.

The mathematics is a bit different in this case. First of all, the groups A_0, B_0 are irrelevant for our computations and we have $C_0 = 0$. In addition, every module C_n is a cyclic module over the ring $\Lambda/(\omega_n)$ that is killed by ω_n'. In particular, C_1 is a cyclic module over the discrete valuation ring $\Lambda/(\omega_1')$, where $\omega_1' = \omega_1/T = T^2+3T+3$. Since T is a uniformizer of the ring $\Lambda/(\omega_1')$, the module C_1 is isomorphic to $\Lambda/(T^2+3T+3, T^k)$ for some $k \geq 0$.

n	T	T^2	T^3	T^4	T^5	Total
1	824	79	0	0	0	903
2	249	18	8	1	0	276
3	88	7	1	0	1	97
4	47*	3*	0	0	0	50
5	9*	0	1*	0	0	10
6	2*	0	0	0	0	2
7	2*	0	0	0	0	2
	1221	107	10	1	1	1340

Table 5.1: The modules Λ/J for $f \equiv 1 \pmod 3$.

We single out eleven discriminants for special mention.

By Nakayama's lemma the ideal J contains a monic polynomial of degree 1 if and only if the ideal (T^2+3T+3, T^k) does. If J is a proper ideal, this happens precisely when $k = 1$, in which case C_1 is isomorphic to the order 3 module $\Lambda/(3,T)$. These cases appear in the first column of Section 5 and were computed using PARI/GP. Their ideals J are of the form $(T-a, b)$ with level of stabilization equal to

f	J	n	T^k
15217	$(T^4+3,3T,9)$	2	T^4
30904	$(T^3-27,3T-63,243)$	5	T^3
39256	$(T+621,2187)$	7	T
40441	$(T^2,9T-27,81)$	4	T^2
44053	$(T+348,729)$	6	T
57832	$(T^2+27,3T-27,81)$	4	T^2
71821	$(T^3+18,3T+9,27)$	3	T^3
78037	$(T-849,2187)$	7	T
80056	$(T^5+9T+9,3T^2+18,27)$	3	T^5
81769	$(T^2+18,3T+9,81)$	4	T^2
96712	$(T-30,729)$	6	T

Table 5.2: Exotic Galois modules for $f \equiv 1 \pmod 3$.

$v_3(b)$. In particular, the first entry contains the 824 discriminants for which J is equal to the ideal $(3,T)$. The 119 entries in the remaining columns of Section 5 were taken care of using Magma's polynomial arithmetic.

Acknowledgments

References

[BCP97] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24:235–265, 1997. 211

[FK86] T. Fukuda and K. Komatsu. On $\mathbb{Z}_p$-extensions of real quadratic fields. *J. Math. Soc. Japan*, 38:95–102, 1986. 207

[FW79] B. Ferrero and L. C. Washington. The Iwasawa invariant μ_p vanishes for abelian number fields. *Annals of Math.*, 109:77–395, 1979. 208

[GG77] G. Gras and M.-N. Gras. Calcul du nombre de classes et des unités des extensions abéliennes réelles de $\mathbb{Q}$. *Bulletin des Sciences Math.*, 101:97–129, 1977. 209

[Gre71] R. Greenberg. *On some questions concerning the Iwasawa invariants*. PhD thesis, Princeton University, 1971. 207

[Gre76] R. Greenberg. On the Iwasawa invariants of totally real number fields. *Amer. J. Math.*, 98:263–284, 1976. 207

[Gre01] Ralph Greenberg. Iwasawa theory—past and present. In *Class field theory—its centenary and prospect (Tokyo, 1998)*, volume 30 of *Adv. Stud. Pure Math.*, pages 335–385. Math. Soc. Japan, Tokyo, 2001. 207

[IS96] H. Ichimura and H. Sumida. On the Iwasawa invariants of certain real abelian fields. II. *Internat. J. Math.*, 7(6):721–744, 1996. 207

[IS97] H. Ichimura and H. Sumida. On the Iwasawa invariants of certain real abelian fields. *Tohoku Math. J.*, 49:203–215, 1997. 207

[Iwa] Iwasawa modules repository. https://github.com/mercuri-pietro/Iwasawa-modules. 209

[KS95] J. S. Kraft and R. Schoof. Computing Iwasawa modules of real quadratic number fields. *Compositio Math.*, 97:135–155, 1995. Erratum: Compositio Math. **103** (1996), 241. 207, 208, 209

[Nuc10] F. Nuccio. Cyclotomic units and class groups in $\mathbb{Z}_p$-extensions of real abelian fields. *Math. Proc. Cambridge Phil. Soc.*, 148:93–106, 2010. 214

[Pao02] M. Paoluzi. *La congettura di Greenberg per campi quadratici reali*. PhD thesis, Sapienza Università di Roma, 2002. 208, 213

[Sin80] W. Sinnott. On the Stickelberger ideal and the circular units of an abelian field. *Invent. Math.*, 62:181–234, 1980. 208, 213

[The20] The PARI Group. PARI/GP 2.13.0, 2020. Univ. Bordeaux. 211

AUTHORS

Pietro Mercuri

Università di Trento
Dipartimento di matematica
Trento 38122
Italy

mercuri.ptr@gmail.com
https://sites.google.com/view/mercuriptr/home

Maurizio Paoluzi

Via Mariano Rampolla 24
00168 Rome
Italy

mauriziopaoluzi@gmail.com

René Schoof

Università di Roma, Tor Vergata
Via della Ricerca Scientifica
00133 Rome
Italy

schoof.rene@gmail.com
https://www.mat.uniroma2.it/~schoof/