SELF-SIMILARITY AND RESPONSE OF FRACTIONAL DIFFERENTIAL EQUATIONS UNDER WHITE NOISE INPUT

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Abstract

Self-similarity, fractal behavior and long-range dependence are observed in various branches of physical, biological, geological, socioeconomics and mechanical systems. Self-similarity, also termed self-affinity, is a concept that links the properties of a phenomenon at a certain scale with the same properties at different time scales as it happens in fractal geometry. The fractional Brownian motion (fBm), i.e. the Riemann-Liouville fractional integral of the Gaussian white noise, is self-similar; in fact by changing the temporal scale $t \to at$ ($a > 0$), the statistics in the new time axis (at) remain proportional to those calculated in the previous axis (t). The proportionality coefficient is a^{2H} being $H > 0$ the Hurst index. In the practical applications, the phenomena are usually ruled by fractional differential equations involving more terms. In this paper it is shown that the response of a multi-term fractional differential equation is a linear combination of self-similar processes with increasing order of Hurst exponent. The consequences of self-similarity are discussed in detail, closed forms of correlation and variance are presented for the general case and particularized for the cases of engineering interest.

Keywords: Self-similarity; Fractional differential equations; Stochastic dynamics; Correlation function.

1. Introduction

Fractional calculus is nowadays currently used to model several real phenomena because of its capability to properly take into account the "long-tail memory" effect experimentally observed. This effect cannot be predicted by classical differential equations involving integer order derivatives, since in this case the future state depends upon a number of previous state equal to the maximum order of derivatives in the differential equation. Considering a real experimental test, it follows that the response of a given system to a unit step function is a power law function rather than exponential one and the Boltzmann superposition principle leads to a fractional operator for which the future state depends upon the entire past history. Examples of these concepts arise in modelling real problems such as in fluid dynamics in non-Newtonian fluids [1-3], heat transportation [4-10], biomechanics [11-14], non-local continuum mechanics [15-17] and viscoelasticity [18-25]. Other applications and pertinent references can be found in literature [26-38].

In the second part of the last century, the self-similarity concept was introduced by B. Mandelbrot [39, 40] before its book on fractal geometry [41]. This concept may be summarized as follows: if changing the temporal scale ($t \to at$ with $a > 0$) some properties remain proportional to the originary scale t with a coefficient depending on the scale factor a^{pH}, being H the Hurst index and p the statistics order, then, the phenomenon is self-similar or self-affine. A relevant example of self-similarity is the *fractional Brownian motion* (fBm), that is the Riemann-Liouville fractional integral of the normal white noise process [42-46]. Attempts to connect fractional calculus and fractals can be found in the staggering number of publications devoted to this topic [47-51].
In the engineering problems, the equation of motion of viscoelastic materials is made by considering a summation of terms with one or more fractional derivatives. Such an example, for a viscoelastic Eulero-Bernoulli beam under time dependent actions, in modal coordinates the inertial term and the fractional viscoelastic term are present. In this case, the response to a normal white noise input is no-more self-similar. In this paper it is shown that, for the aforementioned case, the response process it is a linear combination of self-similar processes. Each of them has an increasing Hurst index with a law that depends on the maximum order of the fractional derivative in the equation of motion, and on the minimum order present in the various terms of the equation. Correlation and variance of the response is evaluated in closed form both in general and in particular cases for relevant engineering interest.

2. Preliminaries

In this section some preliminary concepts and definitions on self-similarity, fractional Brownian motion and fractional operators are reported for clarity sake as well as for introducing appropriate symbologies.

2.1 Self-similarity

The real stochastic process $X(t)$ $(t \in T;\ T = [0,\infty))$ is self-similar (ss) with *Hurst index* $H > 0$ if the finite dimensional distribution, labeled as $\{X(t),(t \in T)\}$, exhibits the following property:

$$\begin{aligned}&\{X(at),(t \in T)\} = \\ &= \{X(at_1), X(at_2),\ldots, X(at_n)\} \underline{\underline{d}} \{a^H X(t_1), a^H X(t_2),\ldots, a^H X(t_n)\} = \\ &= a^{nH}\{X(t_1), X(t_2),\ldots, X(t_n)\} \qquad \forall t_j \in T;\ a > 0;\ j = 1,2,\ldots,n\end{aligned} \tag{1}$$

where the symbol $\underline{\underline{d}}$ means *equal in distribution*. For Gaussian processes, in order to fulfill the self-similarity condition, it is sufficient that the finite dimensional distribution in Eq.(1) is verified for the correlation function $R_X(t_1,t_2)=E\left[X(t_1)X(t_2)\right]$ being $E[\bullet]$ the average of the term in parenthesis, that is

$$R_X(at_1,at_2)=E\left[X(at_1)X(at_2)\right]=E\left[a^H X(t_1)a^H X(t_2)\right]= \\ =a^{2H}R_X(t_1,t_2) \qquad \forall t_1,t_2\in T \quad . \tag{2}$$

Eq.(1) has the noticeable meaning that is by changing the temporal scale ($t_j \to at_j \forall j$) some properties remain unchanged as it happens in fractal geometry.

A relevant example of ss process is the (zero mean) normal Brownian motion $B(t)$ that is the solution of the simplest differential equation [52]:

$$\begin{cases} \dot{B}(t)=W(t) \\ B(t)=0 \qquad \forall t:\ -\infty<t\le 0 \end{cases} \tag{3}$$

where $W(t)$ is a normal white noise [53], fully described in probabilistic setting by the correlation function

$$E\left[W(t_1)W(t_2)\right]=q\delta(t_2-t_1)U(t); \qquad t_2>t_1\ge 0 \tag{4}$$

in which q is the constant strength of the white noise and $\delta(\bullet)$ is the Dirac's delta. Since $W(t)$ is a delta correlated process (zero-th order of Markovianity), the derivative in Eq.(3) has to be considered in the sense of the distribution theory because the process $B(t)$ is nowhere differentiable. From Eq.(3) it is apparent that as the initial condition in Eq.(3) is zero w.p.1, that is the system is quiescent in $t=0$, the process $B(t)$ may be obtained as $B(t)=\int_0^t W(\tau)d\tau$. It is widely known that the correlation function of $B(t)$, denoted as $R_B(t_1,t_2)$, is given as

$$R_B(t_1,t_2)=E\left[B(t_1)B(t_2)\right]=\min(t_1,t_2)E\left[B^2(1)\right] \tag{5}$$

being $E\left[B^2(1)\right]$ the variance at time $t=1$. It may be easily verified that $E\left[B^2(t)\right]=tq$ and thus, for $t=1$, $E\left[B^2(1)\right]=q$. From Eq.(5) immediately descends that

$$E\left[B(at_1)B(at_2)\right]=\min(at_1,at_2)q=E\left[a^{1/2}B(t_1)a^{1/2}B(t_2)\right]=qa\min(t_1,t_2) \tag{6}$$

and thus the Hurst index is $H = 1/2$. Since the variance of $B(t)$ increases for every value of *t*, it descends that the process is non-stationary. Moreover, the increments of the Brownian motion, namely $dB(t)$, are independent; it follows that $E\left[dB(t_j)dB(t_k)\right] = q\delta_{jk}dt$, being δ_{jk} the Kronecker delta ($\delta_{jk} = 0$ if $j \neq k$, $\delta_{jk} = 1$ if $j = k$), and thus $E\left[\left(dB(t)\right)^2\right] = qdt$. Since the variance $E\left[\left(dB(t)\right)^2\right]$ does not depend on *t*, $dB(t)$ is stationary. As a conclusion, $B(t)$ is self-similar with independent *stationary increments* (si) and Hurst index $H=1/2$ and thus it is labeled as H_{sssi}. It has been demonstrated that a wider and unique class of Gaussian H_{sssi} processes, termed as *Fractional Brownian* motion, are processes having correlation function [54]

$$\begin{aligned} &R_B(t_1,t_2) = E\left[B(t_1)B(t_2)\right] = \frac{1}{2}\left(t_1^{2H} + t_2^{2H} - (t_2 - t_1)^{2H}\right)E\left[B^2(1)\right] \\ &0 \leq H \leq 1; \quad 0 \leq t_1 \leq t_2 < T \end{aligned} \tag{7}$$

Fractional Brownian motion, that are Gaussian ones with correlation function expressed in Eq.(7), have the following proprieties: i) self-similarity with Hurst index *H*; ii) stationary increments; iii) variance $E\left[B^2(t)\right] = t^{2H}E\left[B^2(1)\right]$. Moreover, if $H = 1/2$ the fBm reverts into the Brownian motion.

2.2 Fractional operators

Fractional operators [55, 56] are neither else than convolution integrals with power law kernel and are the generalization of the classical derivatives and integrals of integer order to the order $\beta \in \mathbb{R}^+$ (or even to complex ones with positive real part). So the correct term for indicating such operators is *generalized* operators rather than *fractional*, because in the latter case β will belong to the rational realm instead of the real (or complex) one. However, since this nomenclature is widely used the term *fractional* is still used without any reference to fractals. There exist a lot of representations of fractional operators, among them, it will be mentioned the Riemann-Liouville (RL) fractional derivative and integrals, Caputo's, Marchaud, Rietz, Hadamard. All of them share a common point that is: they are convolution integrals with power law kernels. In the ensuing derivations the RL operators will be considered, in particular the RL fractional derivative, labeled as $\left({}_0D_t^\beta f\right)(t)$, and the RL fractional integral, labeled as $\left({}_0I_t^\beta f\right)(t)$, are introduced, respectively, in the form

$$\left({}_0D_t^\beta f\right)(t) = \frac{d^{n+1}}{dt^n}\int_0^t (t-\tau)^{n-\beta} f(\tau)d\tau; \quad \beta > 0 \tag{8}$$

$$\left({}_0I_t^\beta f\right)(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau)d\tau; \quad \beta>0 \tag{9}$$

where n is the integer part of β and $\Gamma(\bullet)$ is the Euler Gamma function $\left(\Gamma(\beta)=\int_0^\infty e^{-x}x^{\beta-1}dx;\ \Gamma(\beta+1)=\beta\Gamma(\beta)\right)$ interpolating all the factorials. In many applications of engineering interest, the Caputo's fractional derivative labeled as $\left({}_0^C D_t^\beta f\right)(t)$ is used, that is

$$\left({}_0^C D_t^\beta f\right)(t)=\frac{1}{\Gamma(n-\beta)}\int_0^t (t-\tau)^{n-\beta-1}\frac{d^n f(\tau)}{d\tau^n}d\tau; \ \beta>0\,. \tag{10}$$

The Caputo's fractional derivative in Eq.(10) coalesces with the RL fractional derivative in Eq.(8) provided $f(t)$ is continuous and differentiable up to the order n and the system in $t=0$ is such that $d^i f/dt^i=0 \ \forall i=0,1,\ldots n$. The use of definition of Eq.(10) is preferred (when $d^n f(\tau)/d\tau^n$ exists) for two reasons, the first one is that when the RL fractional derivative appears the initial conditions have to be defined as fractional derivative in $t=0$ that are seldom available, while in Caputo's one the initial conditions are the classical ones. The second motivation is that in some real problems, like in visco-elasticity, the Boltzmann superposition principle returns the Caputo's definition and not the RL one. For the purposes of the paper, since the initial condition in $t=0$ are zero (thus the RL and Caputo's fractional derivatives coalesce) and the integrals and derivatives of order β are always performed in the range $0-t$, in the following it will be denoted as $\left(D^\beta f\right)(t)$ and $\left(I^\beta f\right)(t)$ the two operators.

It is to be remarked that the representations of the fractional operators as well as the definition of the Euler Gamma function still hold when $\beta\in\mathbb{C}$ with $\mathrm{Re}(\beta)>0$. Now a question arises: why the fractional operators, that are the convolution integrals with power law kernel, are called fractional derivatives and integrals? The answer to this question is: all the properties of derivatives and integrals of integer order (linearity, Leibniz rule, semi-group property...) remain valid for RL operators. For the integration by parts and Leibniz rule see [57, 58]. Moreover for integer value of β, the classical derivatives and integrals are recovered. Further, it is worth to underline that the Laplace and Fourier transforms of fractional operators follow the same rule of the classical ones. Such an example by denoting as $\mathcal{F}\{f(t);\omega\}=\hat{f}(\omega)$, the Fourier transform of the fractional derivatives is

$$\mathcal{F}\left\{\left(D^{\beta} f\right)(t);\omega\right\}=(i\omega)^{\beta}\,\hat{f}(\omega);\ \ \mathcal{F}\left\{\frac{d^{n} f(t)}{dt^{n}};\omega\right\}=(i\omega)^{n}\,\hat{f}(\omega)\,. \tag{11}$$

However, the most important property for the ensuing derivations is the semi-group property, that is

$$D^{\beta_1}\left(D^{\beta_2} f\right)(t)=D^{\beta_2}\left(D^{\beta_1} f\right)(t)=\left(D^{\beta_1+\beta_2} f\right)(t);\ \ \beta_1,\beta_2>0 \tag{12a}$$

$$I^{\beta_1}\left(I^{\beta_2} f\right)(t)=I^{\beta_2}\left(I^{\beta_1} f\right)(t)=\left(I^{\beta_1+\beta_2} f\right)(t);\ \ \beta_1,\beta_2>0 \tag{12b}$$

$$D^{\beta_2}\left(I^{\beta_1} f\right)(t)=\left(D^{\beta_2-\beta_1} f\right)(t);\ \ \beta_2>\beta_1 \tag{12c}$$

Lastly, as it happens in classical differential calculus,

$$I^{\beta_1}\left(D^{\beta_2} f\right)(t)\neq D^{\beta_2}\left(I^{\beta_1} f\right)(t) \tag{13}$$

this is because, performing integration, a constant will appear at the left hand side of Eq.(13). However, if $f(t)$ and all the derivatives of $f(t)$ are zero in $t=0$ (namely $\left(d^{i} f/dt^{i}\right)_{t=0}=0\ \ \forall i=1,2,\ldots,n$) then the inequality in Eq.(13) becomes an identity. It follows that, under these conditions,

$$I^{\beta}\left(D^{\beta} f\right)(t)=f(t);\ \ \mathrm{D}^{\beta}\left(I^{\beta} f\right)(t)=f(t) \tag{14}$$

and thus the operators are invertible. More detailed informations may be found in the book Mainardi [59] and Atanacković et al [60]. These informations are sufficient for the development of the paper.

3. The one-term fractional differential equation driven by normal white noise

In this section some well-known results on the fBm are revisited for two main reasons: i) some concepts are well known and may be found in [61-63] therein after closed form solutions in terms of correlation function and variance of fBm are proposed; ii) these results are necessary to understand the next sections.

Considering Eq.(3) and generalising it to the case of fractional operators it is possible to obtain the following fractional differential equation, whose solution is the fBm $B_{\beta}(t)$

$$\begin{cases}\left(D^{\beta} B_{\beta}\right)(t)=W(t);\ \ \beta\in\mathbb{R}^{+}\\ B_{\beta}(t)=0 \qquad\qquad \forall t:\ -\infty<t\leq 0\end{cases}. \tag{15}$$

First of all, the response $B_\beta(t)$ coalesces with the Brownian motion only for $\beta=1$, that is $B_1(t)=B(t)$, while for $\beta\neq 1$ the process $B_\beta(t)$ strongly depends on the values of β. In order to check if $B_\beta(t)$ is a self-similar process or not, Eq.(15) is rewritten in the form

$$B_\beta(t)=\left(I^\beta W\right)(t). \tag{16}$$

Eq.(16) is valid since in Eq.(15) $B_\beta(t)$ is quiescent up to $t=0$ and thus, according to the RL fractional integral definition in Eq.(9), it leads to

$$B_\beta(t_j)=\frac{1}{\Gamma(\beta)}\int_0^{t_j}\left(t_j-\tau_j\right)^{\beta-1}W\left(\tau_j\right)d\tau_j;\ j=1,2;\ t_2\geq t_1>0. \tag{17}$$

It follows that the correlation function of $B_\beta(t)$ is given as

$$\begin{aligned}R_{B_\beta}(t_1,t_2)&=E\left[B_\beta(t_1)B_\beta(t_2)\right]=\\&=\frac{q}{\Gamma^2(\beta)}\int_0^{t_2}\int_0^{t_1}\left(t_1-\tau_1\right)^{\beta-1}\left(t_2-\tau_2\right)^{\beta-1}\delta\left(\tau_2-\tau_1\right)d\tau_1 d\tau_2\end{aligned} \tag{18}$$

having taken into account that $q\delta(\tau_2-\tau_1)=E\left[W(\tau_1)W(\tau_2)\right]$. Moreover, the double integral in Eq.(18) may be rewritten as

$$\begin{aligned}&\int_0^{t_2}\left(t_2-\tau_2\right)^{\beta-1}\left[\int_0^{t_1}\left(t_1-\tau_1\right)^{\beta-1}\delta\left(\tau_2-\tau_1\right)d\tau_1\right]d\tau_2=\\&\int_0^{t_2}\left(t_2-\tau_2\right)^{\beta-1}\left(t_1-\tau_2\right)^{\beta-1}U\left(t_1-\tau_2\right)d\tau_2=\Upsilon_\beta\left(t_1,t_2\right)\end{aligned} \tag{19}$$

where $U(\bullet)$ is the unit step function and $\Upsilon_\beta(t_1,t_2)$ is the following integral

$$\Upsilon_\beta(t_1,t_2)=\int_0^{t_1}\left(t_1t_2-(t_1+t_2)\tau_2+\tau_2^2\right)^{\beta-1}d\tau_2\ . \tag{20}$$

For $\beta=1$, $\Upsilon_\beta(t_1,t_2)=t_1$ that is the $\min(t_1,t_2)$ as already exploited in Eq.(5). For $\beta=2$ $\Upsilon_\beta(t_1,t_2)=\frac{t_1^2}{2}(t_2-t_1/3)$.

By changing the temporal scale $t_1 \to at_1$, $t_2 \to at_2$ the correlation function of $B_\beta(at)$ is given as

$$E\left[B_\beta(at_1)B_\beta(at_2)\right]=\frac{q}{\Gamma^2(\beta)}\int_0^{at_1}\left(a^2t_1t_2-a(t_1+t_2)\tau_2+\tau_2^2\right)^{\beta-1}d\tau_2 \tag{21}$$

and, by letting $\tau_2=\tau a$ in Eq.(21), it leads to

$$\begin{aligned}E\left[B_\beta(at_1)B_\beta(at_2)\right]&=\frac{q}{\Gamma^2(\beta)}a^{2\beta-1}\Upsilon_\beta(t_1,t_2)=\\&=\frac{q}{\Gamma^2(\beta)}E\left[a^{\beta-1/2}B(t_1)a^{\beta-1/2}B(t_2)\right]\end{aligned}. \tag{22}$$

This means that $B_\beta(t)$ is self-similar with Hurst index $(\beta-1/2)$, provided $\beta>1/2$ since the Hurst index must be positive. Closed form solution of $\Upsilon_\beta(t_1,t_2)$ for $t_1>t_2$ or $t_2>t_1$ is given as

$$\Upsilon_\beta(t_1,t_2)=(-1)^{-\beta}|t_2-t_1|^{2\beta-1}\mathrm{B}\left[-\frac{\min(t_1,t_2)}{|t_2-t_1|},\beta,\beta\right] \tag{23}$$

where $\mathrm{B}[\bullet,\beta,\beta]$ is the incomplete Euler Beta function. Notice that for $t_1=t_2$ Eq.(23) may not be directly used because Beta function attains infinite value and thus for $t_1=t_2=t$ the variance of $B_\beta(t)$ is evaluated trough Eq.(19) particularized as

$$E\left[B_\beta^2(t)\right]=\frac{q}{\Gamma^2(\beta)}\int_0^t\left(t^2-2t\tau+\tau^2\right)^{\beta-1}d\tau=\frac{q}{\Gamma^2(\beta)}\frac{t^{2(\beta-1/2)}}{2(\beta-1/2)}=\frac{q}{\Gamma^2(\beta)}\Upsilon_\beta(t,t). \tag{24}$$

Eq.(24) shows that the variance is a monotonic function (since $\beta>1/2$) and thus it does not attain stationarity conditions.

4. Self-similarity in two terms fractional differential equations

In this section the generalization of Eq.(15) for the more general case of equation of the kind

$$\begin{cases}\left(D^{\beta_2}X+\rho D^{\beta_1}X\right)(t)=W(t); & \beta_2>\beta_1;\ \beta_2>0.5\\ X(t)=0 \quad \forall t:\ -\infty<t\le 0\end{cases} \tag{25}$$

is reported being ρ a real positive constant. In order to make the paper self-consistent, some few remarks on the solution of Eq.(25) when the input is a deterministic function $f(t)$ are reported in appendix. In Eq.(25), $W(t)$ is a white noise process with strength $q(t)=qU(t)$. Because the system is quiescent up to $t=0$, the Riemann-Liouville derivative, the Caputo's fractional derivative and the Jumarie fractional derivative [64] coalesce each other. Herein the self-similarity of the response $X(t)$ of the Eq.(25) is discussed. By performing the Riemann-Liouville integral of order β_2 of both members of Eq.(25) it can be rewritten as

$$X(t)=B_{\beta_2}(t)-\rho\left(I^{\gamma}X\right)(t) \tag{26}$$

where $\gamma=\beta_2-\beta_1$. By using the iterate kernel method, the response process $X(t)$ is given as

$$X(t)=\sum_{k=0}^{\infty}(-\rho)^k B_{\nu_k}(t); \qquad \nu_k=\beta_2+k\gamma . \tag{27}$$

From Eq.(27) it may be recognized that $X(t)$ is a linear combination of self-similar processes with an increasing Hurst index defined as

$$H_k=\beta_2+k\gamma-1/2 \tag{28}$$

The corresponding cross-correlation function of $B_{\nu_k}(t_1)$ and $B_{\nu_r}(t_2)$ is given as

$$E\left[B_{\nu_k}(t_1)B_{\nu_r}(t_2)\right]=\frac{q}{\Gamma(\nu_k)\Gamma(\nu_r)}\int_0^{t_1}(t_2-\tau_2)^{\nu_r-1}(t_1-\tau_2)^{\nu_k-1}d\tau_2;\ t_2\geq t_1>0 \tag{29}$$

and the closed form of the integral in Eq.(29) is

$$t_1^{\nu_k}t_2^{\nu_r-1}\Gamma(\nu_k)\,{}_2\tilde{F}_1\left[1,1-\nu_r,1+\nu_k,t_1/t_2\right]. \tag{30}$$

being ${}_2\tilde{F}_1[\bullet,\bullet,\bullet,\bullet]$ the regularized hypergeometric function. By changing the temporal scale $t\to at$, from Eq.(29) it may be asserted that the cross Hurst index, denoted as H_{kr}, is given as

$$H_{kr}=(\nu_k+\nu_r-1)/2 \tag{31}$$

and the cross covariance of $B_{\nu_k}(t_1)$ and $B_{\nu_r}(t_2)$ is

$$E\left[B_{\nu_k}(t)B_{\nu_r}(t)\right]=\frac{q}{\Gamma(\nu_k)\Gamma(\nu_r)}\int_0^{t}(t-\tau)^{\nu_k+\nu_r-2}d\tau=\frac{qt^{2H_{kr}}}{\Gamma(\nu_k)\Gamma(\nu_r)(2H_{kr})}. \tag{32}$$

It follows that, according to Eq.(27), the variance of the response process $X(t)$ is evaluated in the form

$$E\left[X^2(t)\right]=q\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-\rho)^{k+r}\frac{t^{2H_{kr}}}{\Gamma(\nu_k)\Gamma(\nu_r)(2H_{kr})}= \sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-\rho)^{k+r}E\left[B_{\nu_k}(t)B_{\nu r}(t)\right] \tag{33}$$

It is to be stressed that $X(t)$ is not self-similar; in fact, by changing the temporal scale $t\to at$ in Eq.(33), the variance becomes

$$E\left[X^2(at)\right]=q\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-\rho)^{k+r}\frac{a^{2H_{kr}}t^{2H_{kr}}}{\Gamma(\nu_k)\Gamma(\nu_r)(2H_{kr})}= \sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-\rho)^{k+r}a^{2H_{kr}}E\left[B_{\nu_k}(t)B_{\nu_r}(t)\right] \tag{34}$$

and the correlation function of $X(t)$ at the time $t\to at$ is

$$E\left[X(at_1)X(at_2)\right]=\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-\rho)^{k+r}a^{2H_{kr}}E\left[B_{\nu_k}(t_1)B_{\nu_r}(t_2)\right]. \tag{35}$$

Since the Hurst index is not unique the process $X(t)$ is not self-similar. However, from Eq.(35) it is clear that by knowing $E\left[B_{\nu_k}(t_1)B_{\nu_r}(t_2)\right]$ for a couple of instants $0<t_1\le t_2$, the correlation for at_1 and at_2 of $X(t)$ is readily computed as a linear combination of the correlations already evaluated in the time instants t_1 and t_2. This property drastically reduces the computational burden of the correlation function since by computing the correlation at time t_1 and t_2 all the values of the correlation along the lines with slope t_2/t_1 or t_1/t_2 are computed by means of Eq.(35).

5. Applications

In this section the reliability of the proposed approach is asserted through two different examples of engineering interest. Specifically, the variance of the response, calculated with Eq.(33), is particularized for each case and it is compared with the steady state variance. The latter, obtained by taking into account Eq.(11), is calculated as

$$\sigma_X^2=E\left[X^2(t)\right]=2\int_0^{\infty}S_X(\omega)d\omega=\frac{q}{\pi}\int_0^{\infty}\frac{1}{\left|(i\omega)^{\beta_2}+\rho(i\omega)^{\beta_1}\right|^2}d\omega \tag{36}$$

in which $S_X(\omega)$ is the Power Spectral Density of the response $X(t)$. It is to be emphasized that for $0<\beta_1<1$ there are two different cases: for $0<\beta_1<0.5$ the system attains a steady state value, while for $0.5\leq\beta_1<1$ the variance increases without limit. This fact can be easily asserted considering the integral Eq.(36) that, for $0.5\leq\beta_1<1$ does not converge. The physical explanation of this behaviour is due to the fact that when $0<\beta_1<0.5$ the elastic phase prevails [65]; by contrast for $0.5\leq\beta_1<1$ the fluid phase prevails and thus the elastic component has not the strength for moderating the displacements. This effect is evidenced by increasing the temperature of the beam because in this case every material goes towards the fluid and β_1 increases reaching $\beta_1=1$ when the solid phase totally disappears.

5.1 Massless mechanical system

In this application it is assumed that in Eq.(25) $\beta_2=1$, $0\leq\beta_1<0.5$ and $q(t)=qU(t)$. In this case, Eq.(25) represents the massless system depicted in Fig.1. It is a dashpot in parallel with the so-called *springpot* that is a fractional viscoelastic device characterized by parameters ρ and β_1. The system is excited by a normal white noise $W(t)$ with uniform strength $qU(t)$.

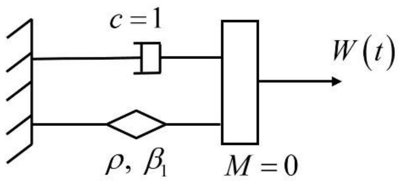

Fig. 1 – The massless mechanical system for $\beta_2=1$ and $0\leq\beta_1<0.5$

In the case depicted in Fig.1, according to Eq.(33), $\nu_k=1+k(1-\beta_1)$ and $2H_{kr}=(k+r)(1-\beta_1)+1$ and thus the variance of the response is particularized as

$$E\left[X^2(t)\right]=q\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-\rho)^{k+r}\frac{t^{2H_{kr}}}{\Gamma(k(1-\beta_1)+1)\Gamma(r(1-\beta_1)+1)(2H_{kr})}. \tag{37}$$

The variance in Eq.(37) is depicted in Fig.2 for different values of ρ and β_1 and it is compared with the steady state value (horizontal line) assuming $q=0.3$.

From this figure it may be stated that the solution in terms of variance attains the steady state response as expected. Moreover, the lower the values of ρ, the higher the values of time to reach the steady state response. This is due to the fact that the elastic part in the springpot becomes negligible when ρ decreases and/or β_1 increases. Particular cases of this application are: $\beta_1 = 1$ and $\beta_1 = 0$. In the first case, the springpot reverts into the dashpot instead in the second case the springpot reverts into the elastic spring (Fig.2a).

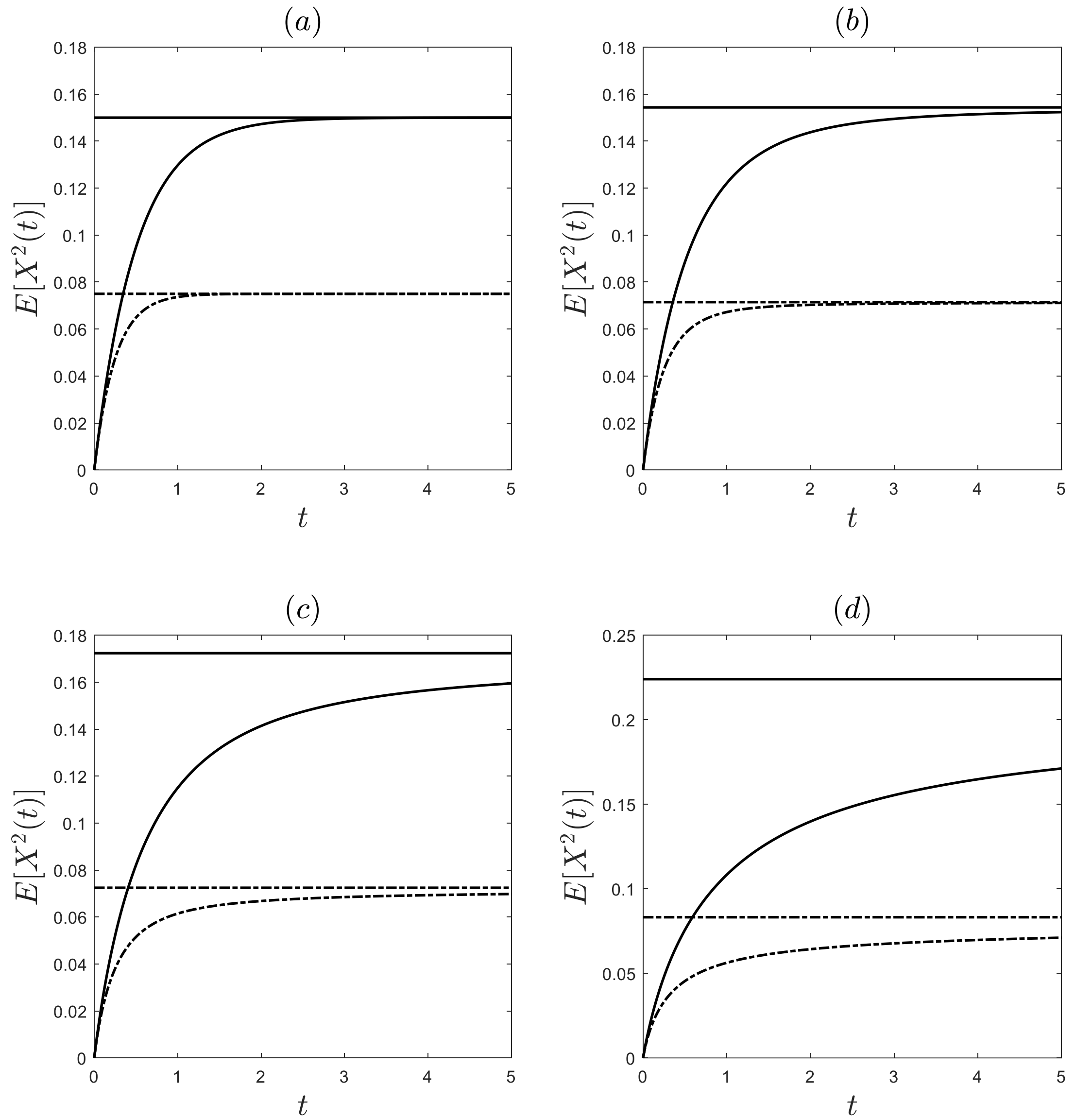

Fig. 2 – Variance of the response for $\beta_2 = 1$ *and* $q = 0.3$. $\rho = 1$ *(continuous line),* $\rho = 2$ *(dash dotted line):* $\beta_1 = 0$ *(2a);* $\beta_1 = 0.1$ *(2b);* $\beta_1 = 0.2$ *(2c);* $\beta_1 = 0.3$ *(2d).*

5.2 Fractional viscoelastic Euler-Bernoulli beam

In this application a fractional viscoelastic Euler-Bernoulli simply supported beam excited by a white noise is considered. In order to prove the reliability of the proposed approach, the variance of the response is compared with both the steady state response in terms of variance by Eq.(36) and the results obtained in [66] with the aid of Grünwald Letnikov (GL) integration method [56]. The equation of motion for this kind of structural system can be written as

$$\tilde{m}\frac{\partial^2 X(z,t)}{\partial t^2}+E_{\beta_1}I_x\frac{\partial^4}{\partial z^4}\left(\left(D_t^{\beta_1}X\right)(z,t)\right)=p(z,t) \tag{38}$$

in which $X(z,t)$ is the displacement of the beam, $\tilde{m}$ is the mass per unit length, I_x is the moment of inertia of the cross section with respect to the *x* axis, $p(z,t)$ is the structural excitation and E_{β_1} is the (anomalous) viscoelastic coefficient ($\left[E_{\beta_1}\right]=\left[FL^{-2}T^{\beta_1}\right]$) that depends on the material at hands. In this application $p(z,t)=W(t)\left(U(z)-U(z-L)\right)$ being $W(t)$ a white noise with constant strength $q(t)=qU(t)$. The solution $X(z,t)$ is a linear combination of the eigenfunctions $\phi_v(z)$ and thus it can be expressed as

$$X(z,t)=\sum_{v=1}^{\infty}Y_v(t)\phi_v(z) \tag{39}$$

in which $Y_v(t)$ are the modal coordinates. The eigenfunctions $\phi_v(z)$ depend on the constrain and for the simply supported beam $\phi_v(z)=\sqrt{2/L}\,\sin(v\pi z/L)$. Inserting Eq.(39) in Eq.(38), multiplying by $\phi_s(z)$ and integrating from zero to *L* with respect to *z*, Eq.(38) can be written as

$$\sum_{v=1}^{\infty}\ddot{Y}_v(t)\int_0^L\phi_v(z)\phi_s(z)dz+\frac{E_{\beta_1}I_x}{\tilde{m}}\sum_{v=1}^{\infty}\left(\left(D^{\beta_1}Y_v\right)(t)\int_0^L\frac{\partial^4\phi_v(z)}{\partial z^4}\phi_s(z)dz\right)=\frac{1}{\tilde{m}}\int_0^L p(z,t)\phi_s(z)dz\,. \tag{40}$$

By using the relations of orthogonality (see [66, 67]), the s-th equation can be written in the form

$$\ddot{Y}_s(t)+\rho_s\left(D^{\beta_1}Y_s\right)(t)=\bar{W}_s(t) \tag{41}$$

where $\rho_s=E_{\beta_1}I_x\pi^4s^4/\left(\tilde{m}L^4\right)$ and $\bar{W}_s(t)$ is related to the white noise $W(t)$ in the form

$$\bar{W}_s(t)=W(t)\frac{\sqrt{2L}}{\tilde{m}\pi s}\left(1-\cos(s\pi)\right). \tag{42}$$

It is to be stressed that Eq.(25) reverts into Eq.(41) when $\beta_2=2$ and thus it is a particular case of the systems reported in [68,69]. Because the system is quiescent up to $t=0$, it is clear from Eq.(42) that for even values of s it remains at rest. In order to calculate the variance of the response $X(z,t)$, it is necessary to compute the cross covariance of $Y_u(t)$ and $Y_v(t)$ that can be obtained as

$$E\left[Y_u(t)Y_v(t)\right]=q_{uv}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-\rho_u)^k(-\rho_v)^r\frac{t^{2H_{kr}}}{\Gamma(\nu_k)\Gamma(\nu_r)(2H_{kr})} \tag{43}$$

being

$$q_{uv}=\frac{q}{\tilde{m}^2}\left(\frac{2L(\cos(u\pi)-1)(\cos(v\pi)-1)}{uv\pi^2}\right). \tag{44}$$

The variance of the response $X(z,t)$ can be obtained as

$$E\left[X^2(z,t)\right]=\sum_{u=1}^{\infty}\sum_{v=1}^{\infty}\phi_u(z)\phi_v(z)E\left[Y_u(t)Y_v(t)\right]. \tag{45}$$

In order to assert the reliability of the proposed approach, a numerical simulation was performed considering the following parameters: $E_{\beta_1}=29.566\cdot10^6\,N\,s^{\beta_1}/m$, $L=2m$, $I_x=1.333\cdot10^4 m^4$, $\tilde{m}=43.2\;kg/m$, $\beta_1=0.279$ and $q=10^4N^2s$. The numerical simulation was conducted considering 10^5 samples of $X(z,t)$ with a time sampling step $\Delta t=0.002$. The results obtained in terms of variance of the displacement in the middle point through the use of Eq.(45) are depicted in Fig.3. In the same figure, the variance of the steady state solution evaluated in Eq.(36), is also reported and the results in the transient zone evaluated with the GL approach [66].

From Fig.3 it is clear that the proposed approach can estimate the response in terms of variance better than the GL approach; in fact, the variance estimated with the proposed approach tends asymptotically to the steady state value unlike the GL approach. This is due to the fact that the results obtained using the GL approach vary significantly with the variation of the time sampling step Δt. In particular, the smaller Δt, the closer the variance approaches the steady state value. This means that in order to obtain accurate results using the GL approach, a very small time sampling step must be chosen. Furthermore, a very large number of samples $X(z,t)$ are required to obtain good results and therefore the computational burden required is very high. Contrary to the GL approach, the results obtained using the proposed approach do not depend on the choice of Δt and the response in terms of variance can be directly calculated using Eqs. (43-45) without the need to generate samples of the structural response. Due to the above reasons the choice of the proposed approach is preferable with respect to the choice of a GL approach.

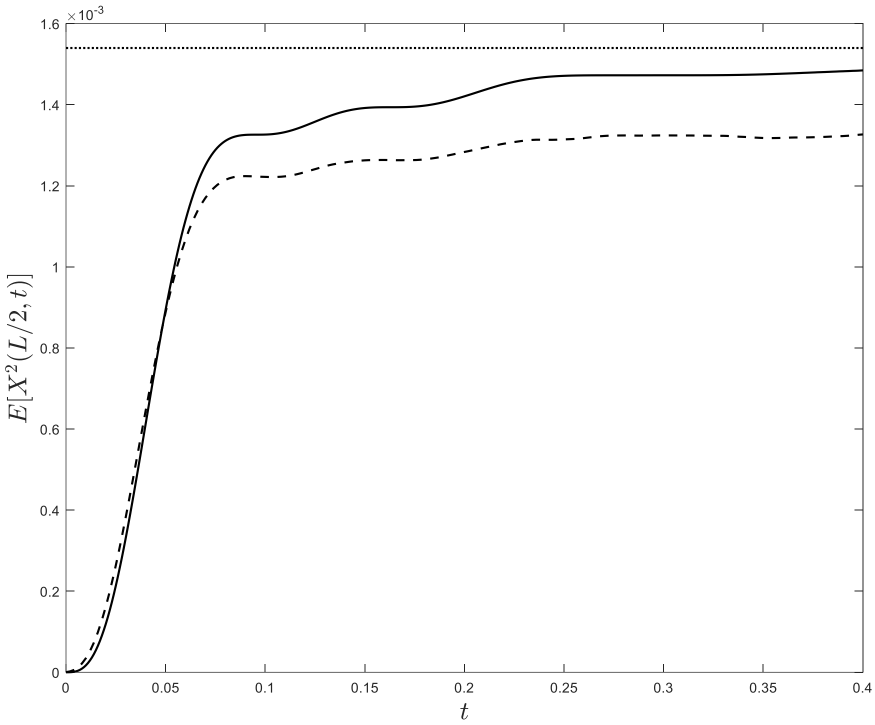

Fig. 3 – Variance of the response in the middle point of the simply supported viscoelastic Euler-Bernoulli beam: results of GL integration for $\Delta t = 0.002$ *(dashed line), results of Eq.(45) (continuous line), steady state value (dotted line).*

Conclusions

A review of the self-similarity concepts of fractional Brownian motion (fBm) has been presented, finding exact solution of correlation function and variance of the response. It is shown that the one term fractional differential equation excited by a stationary white noise has the fBm as solution that is self-similar with Hurst index H=1/2. Moreover, it is demonstrated that the response of a multi-term fractional differential equation excited by a stationary white noise is not self-similar but it is a linear combination of self-similar processes with increasing Hurst index. Variance and correlation of the response are calculated for the general form of the multi-term fractional differential equation and particularized for different cases of engineering interest showing that the proposed approach is preferable over the GL approach.

REFERENCES

[1] del-Castillo-Negrete, D. (2006). Fractional diffusion models of nonlocal transport. Physics of Plasmas, 13(8), 082308.

[2] Yang, X., Liang, Y., Chen, W. (2018). A spatial fractional seepage model for the flow of non-Newtonian fluid in fractal porous medium. Communications in Nonlinear Science and Numerical Simulation, 65, 70–78.

[3] Ionescu, C. M., Birs, I. R., Copot, D., Muresan, C. I., Caponetto, R. (2020). Mathematical modelling with experimental validation of viscoelastic properties in non-Newtonian fluids. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 378(2172), 20190284.

[4] Želi, V., Zorica, D. (2018). Analytical and numerical treatment of the heat conduction equation obtained via time-fractional distributed-order heat conduction law. Physica A: Statistical Mechanics and Its Applications, 492, 2316–2335.

[5] Povstenko, Y. (2012). Theories of thermal stresses based on space–time-fractional telegraph equations. Computers & Mathematics with Applications, 64(10), 3321–3328.

[6] Hua, Y.-C., Cao, B.-Y. (2017). Slip Boundary Conditions in Ballistic–Diffusive Heat Transport in Nanostructures. Nanoscale and Microscale Thermophysical Engineering, 21(3), 159–176.

[7] Zingales, M. (2014). Fractional-order theory of heat transport in rigid bodies. Communications in Nonlinear Science and Numerical Simulation, 19(11), 3938–3953.

[8] Atanacković, T., Konjik, S., Oparnica, L., Zorica, D. (2012). The Cattaneo type space-time fractional heat conduction equation. Continuum Mechanics and Thermodynamics, 24(4–6), 293–311.

[9] Mongioví, M. S., Zingales, M. (2013). A non-local model of thermal energy transport: The fractional temperature equation. International Journal of Heat and Mass Transfer, 67, 593–601.

[10] Povstenko, Y. Z. (2013). Fractional Heat Conduction in Infinite One-Dimensional Composite Medium. Journal of Thermal Stresses, 36(4), 351–363.

[11] Ionescu, C., Lopes, A., Copot, D., Machado, J. A. T., Bates, J. H. T. (2017). The role of fractional calculus in modeling biological phenomena: A review. Communications in Nonlinear Science and Numerical Simulation, 51, 141–159.

[12] Failla, G., Zingales, M. (2020). Advanced materials modelling via fractional calculus: challenges and perspectives. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 378(2172), 20200050.

[13] Bologna, E., Di Paola, M., Dayal, K., Deseri, L., Zingales, M. (2020). Fractional-order nonlinear hereditariness of tendons and ligaments of the human knee. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 378(2172), 20190294.

[14] Ramírez-Torres, A., Penta, R. Grillo, A. Two-scale, non-local diffusion in homogenised heterogeneous media. Arch Appl Mech (2021).

[15] Carpinteri, A., Mainardi, F. (Eds.). (1997). Fractals and Fractional Calculus in Continuum Mechanics.

[16] Di Paola, M., Zingales, M. (2008). Long-range cohesive interactions of non-local continuum faced by fractional calculus. International Journal of Solids and Structures, 45(21), 5642–5659.

[17] Alotta, G., Di Paola, M., Pinnola, F.P., Zingales, M. A fractional nonlocal approach to nonlinear blood flow in small-lumen arterial vessels. Meccanica 55, 891–906 (2020).

[18] Di Paola, M., Pirrotta, A., Valenza, A. (2011). Viscoelastic behavior through fractional calculus: an easier method for best fitting experimental results. Mechanics of Materials, 43, 799–806.

[19] Di Paola, M., Failla, G., Pirrotta, A. (2012). Stationary and non-stationary stochastic response of linear fractional viscoelastic systems. Probabilistic Engineering Mechanics, 28, 85–90.

[20] Di Paola, M., Heuer, R., Pirrotta, A. (2013). Fractional Visco-Elastic Euler-Bernoulli Beam. International Journal of Solids and Structures, 50(22-23), 3505-3510.

[21] Pirrotta, A., Cutrona, S., Di Lorenzo, S. (2015). Fractional visco-elastic Timoshenko beam from elastic Euler-Bernoulli beam. Acta Mechanica, 226, 179-189.

[22] Cataldo, E., Di Lorenzo, S., Fiore, V., Maurici, M., Nicoletti, F., Pirrotta, A., Scaffaro, R., Valenza, A. (2015). Bending test for capturing the vivid behavior of giant reeds, returned through a proper fractional visco-elastic model. Mechanics of Materials, 89, 159-168.

[23] Pirrotta, A., Cutrona, S., Di Lorenzo, S., Di Matteo, A. (2015). Fractional visco-elastic Timoshenko beam deflection via single equation. International Journal for Numerical Methods in Engineering, 104(9), 869-886.

[24] Di Lorenzo, S., Di Paola, M., La Mantia, F., Pirrotta, A. (2017). Non-linear viscoelastic behavior of polymer melts interpreted by fractional viscoelastic model. Meccanica, 52(8), 1843-1850.

[25] Colinas-Armijo, N., Cutrona, S., Di Paola, M., Pirrotta, A. (2017). Fractional Viscoelastic beam under torsion. Communications in Nonlinear Science and Numerical Simulation, 48, 278-287.

[26] Di Paola, M., Zingales, M. (2012). Exact mechanical models of fractional hereditary materials. Journal of Rheology, 56(5), 983-1004.
[27] Celauro, C., Fecarotti, C., Pirrotta, A., Collop, A.C. (2012). Experimental validation of a fractional model for creep/recovery testing of asphalt mixtures. Construction and Building materials, 36, 458-466.
[28] Failla, G., Pirrotta, A. (2012). On the Stochastic Response of a Fractionally-damped Duffing Oscillator. Communications in Nonlinear Science and Numerical Simulation, 17(12), 5131-5142.
[29] Fecarotti, C., Celauro, C., Pirrotta, A. (2012). Linear ViscoElastic (LVE) Behaviour of Pure Bitumen via Fractional Model. Procedia - Social and Behavioral Sciences, 53, 450-461.
[30] Di Lorenzo, S., Di Paola, M., Pinnola, F.P., Pirrotta, A. (2014). Stochastic Response Of Fractionally Damped Beams. Probabilistic Engineering Mechanics, 35, 37-43.
[31] Di Matteo, A., Kougioumtzoglou, I.A., Pirrotta, A., Spanos, P.D., Di Paola, M. (2014). Stochastic response determination of nonlinear oscillators with fractional Derivatives Elements via the wiener path integral. Probabilistic Engineering Mechanics, 38, 127-135.
[32] Di Paola, M., Pirrotta, A. (1999). Non-linear systems under impulsive parametric input. International Journal of Non-Linear Mechanics, 34(5), 843–851.
[33] Bucher, C., Pirrotta, A. (2015). Dynamic Finite Element analysis of fractionally damped, structural systems in the time domain, Acta Mechanica, 226(12), 3977-3990.
[34] Spanos, P.D., Di Matteo, A., Cheng, Y., Pirrotta, A., Li, J. (2016). Galerkin Scheme-Based Determination of Survival Probability of Oscillators with Fractional Derivative Elements. Journal of Applied Mechanics, Transactions ASME, 83(12), 121003.
[35] Yurchenko, D., Burlon, A., Di Paola, M., Failla, G., Pirrotta, A. (2017). Approximate analytical mean-square response of an impacting stochastic system oscillator with fractional damping. ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part B: Mechanical Engineering 3(3), 030903.
[36] Spanos, P.D., Di Matteo, A., Pirrotta, A. (2019). Steady-state dynamic response of various hysteretic systems endowed with fractional derivative elements. Nonlinear Dynamics, 1-12.
[37] Atanacković, T., Janev, M., Pilipović, S. (2020). On the thermodynamical restrictions in isothermal deformations of fractional Burgers model. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 378(2172), 20190278.
[38] Metzler, R., Klafter, J. (2000). The random walk's guide to anomalous diffusion: a fractional dynamics approach. Physics Reports 339(1): 1-77.
[39] Mandelbrot, B. B., Wallis, J. R. (1967). How long is the coast of Britain? Statistical self-similarity and fractional dimension. Science 156(3775): 636-638.
[40] Mandelbrot, B. B., Van Ness, J. H. (1968). Fractional Brownian motions, fractional noises and applications. Siam Review 10(4): 422-437.
[41] Mandelbrot, B. B. (1982). The Fractal Geometry of Nature. W. H. Freeman and co., San Francisco.
[42] Kaplan, L. M., Kuo, C.-C. J. (1994). Extending self-similarity for fractional Brownian motion. IEEE Transactions on Signal Processing, 42(12), 3526–3530.
[43] Gefferth, A., Veitch, D., Maricza, I., Molnár, S., Ruzsa, I. (2003). The nature of discrete second-order self-similarity. Advances in Applied Probability, 35(02), 395–416.
[44] Taqqu, M. S. (2003). Fractional Brownian motion and long-range dependence. InP. Doukhan, G. Oppenheim, & M. S. Taqqu (Eds.), Theory and Applications of Long-Range Dependence (5–38). Boston: Birkhäuser.
[45] Allegrini, P., Buiatti, M., Grigolini, P., West, B. J. (1998). Fractional Brownian motion as a nonstationary process: An alternative paradigm for DNA sequences. Physical Review E, 57(4), 4558–4567.
[46] West, B. J., Grigolini, P. (2000). Fractional Differences, Derivatives and Fractal Time Series. Applications of Fractional Calculus in Physics, 171–201.
[47] Balankin, A. S. (2020). Fractional space approach to studies of physical phenomena on fractals and in confined low-dimensional systems. Chaos, Solitons & Fractals, 132, 109572.
[48] Alaimo, G., Zingales, M. (2015). Laminar flow through fractal porous materials: the fractional-order transport equation. Communications in Nonlinear Science and Numerical Simulation, 22(1-3), 889–902.
[49] Butera, S., Di Paola, M. (2014). A physically based connection between fractional calculus and fractal geometry. Annals of Physics, 350, 146–158.
[50] Deseri, L., Di Paola, M., Zingales, M., Pollaci, P. (2013). Power-law hereditariness of hierarchical fractal bones. International Journal for Numerical Methods in Biomedical Engineering, 29(12), 1338–1360.
[51] Zhang X, Ostoja-Starzewski, M. (2020). Impact force and moment problems on random mass density fields with fractal and Hurst effects. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 378: 20190591.

[52] Pirrotta, A. (2005). Non-linear systems under parametric white noise input: Digital simulation and response. International Journal of Non-Linear Mechanics 40(8): 1088-1101.
[53] Di Paola, M., Falsone, G., Pirrotta, A. (1992). Stochastic response analysis of nonlinear systems under Gaussian inputs. Probabilistic Engineering Mechanics 7(1): 15-21.
[54] Samorodnitsky, G., Taqqu, M. S. (1994). Stable non-Gaussian Random processes. New York: Chapman and Hall/CRC.
[55] Samko, S. G., Kilbas, A. A., Marichev, O. I. (1993). Fractional Integrals and derivatives. Gordon and Breach Science Publishers.
[56] Podlubny, I. (1999). Fractional Differential Equations. An Introduction to fractional derivatives, fractional differential equations their solution and some of their applications. Academic Press, San Diego.
[57] Tarasov, V. E. (2013). No Violation of the Leibniz Rule. No Fractional Derivative. Communication in Nonlinear Science and Numerical Simulation 18(11): 2945-2948.
[58] Tarasov, V. E. (2016). Leibniz Rule and Fractional Derivatives of Power Functions. Journal of Computational and Nonlinear Dynamics 11(3). ID. 031014.
[59] Mainardi, F. (2010). Fractional Calculus and Waves in linear viscoelasticity. London: Imperial College press.
[60] Atanacković, T. M., Pilipović, S., Stanković, B., Zorica, D. (2014). Fractional Calculus with Applications in Mechanics. London: Wiley.
[61] Hurst, H. E. (1951). Long-term storage capacity of reservoirs. Transactions of the American Society of Civil Engineers 116: 770-799.
[62] Nigmatullin, R. R., Baleanu, D. (2012). The derivation of the Generalized fractional equations describing self-similar processes. Fractional Calculus & Applied Analysis 15(4): 718-740.
[63] Jumarie, G. (2013). Stochastic differential equations with fractional Brownian motion input. International Journal of System Science 24(6): 1113-1131.
[64] Jumarie, G. (2006). Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. Computers & Mathematics with Applications 51(9-10): 1367-1376.
[65] Di Paola, M., Zingales, M. (2009). Exact mechanical models of fractional hereditary materials. Journal of Rheology 56(5): 983-1004.
[66] Burlon, A., Sucato, V., Failla, G., Di Paola, M. (2022) Nonstationary stochastic analysis of fractional viscoelastic Euler-Bernoulli beams. Springer Proceedings in Mathematics & Statistics, Under Review.
[67] Di Lorenzo, S., Pinnola, F. P., Pirrotta, A. (2012). On the dynamics of fractional visco-elastic beams. Proceedings of the ASME 2012 International Mechanical Engineering Congress and Exposition. Volume 4: Dynamics, Control and Uncertainty, Parts A and B. Houston, Texas, USA. November 9-15: 1273-1281.
[68] Di Matteo, A., Pirrotta, A. (2015). Generalized differential transform method for nonlinear boundary value problem of fractional order. Communications in Nonlinear Science and Numerical Simulation 29(1-3): 88-101.
[69] Di Paola, M., Heuer, R., Pirrotta, A. (2013). Fractional visco-elastic Euler-Bernoulli beam. Int. J. Solids Struct. 50: 3505-3510.

Appendix: Solution of two terms fractional differential equations

In order to find the statistics of the solution of Eq.(25), a brief insight on the solution in deterministic setting is necessary. It is obvious that many papers and textbooks are available in literature [56,68]. In this appendix the formulations are reproposed with the aim to give a unified treatment of the fractional differential equations. Let be the equation of motion of the given systems in the form

$$\begin{cases} \left(D^{\beta_2} X + \rho D^{\beta_1} X\right)(t) = f(t)\mathrm{U}(t);\ \beta_2 > \beta_1 \geq 0 \\ X(t) = 0 \ \text{ w.p.1 } \ \forall t \leq 0 \end{cases} \tag{A.1a-A.1b}$$

where $f(t)$ is a deterministic forcing function and Eq.(A.1b) indicates that the system is quiescent up to $t=0$. The latter condition is essential because, in general, $\left(D^{\beta}\left(I^{\beta} X\right)\right)(t) = X(t)$ but

$\left(I^{\beta}\left(D^{\beta}X\right)\right)(t)$ is not $X(t)$ as it happens in classical differentials and integrals. The initial condition in Eq.(A.1b) ensures that fractional derivative and integrals are invertible. As it happens in classical differential equations, the first step is to solve Eq.(A.1a) for the case $f(t)=1$, i.e. calculate $X(t)$ that is the response to the unit step function, in the following denoted as $C_{\beta_1\beta_2}(t)$. In this case, Eq.(A.1a) can be rewritten as

$$X(t)=\left(I^{\beta_2}U\right)(t)-\rho\left(I^{\gamma}X\right)(t) \tag{A.2}$$

and, by using the iterate kernel method

$$X^{(j)}(t)=\left(I^{\beta_2}U\right)(t)-\rho\left(I^{\gamma}X^{(j-1)}\right)(t). \tag{A.3}$$

Since $\lim_{j\to\infty}X^{(j)}(t)\to X(t)$, then the solution $X(t)$, labeled as $C_{\beta_1\beta_2}(t)$, is given in the form

$$C_{\beta_1\beta_2}(t)=X(t)=\sum_{k=0}^{\infty}(-\rho)^k I^{\nu_k}U(t)=\sum_{k=0}^{\infty}(-\rho)^k\frac{t^{\nu_k}}{\Gamma(\nu_k+1)}=t^{\beta_2}E_{\gamma,\beta_2+1}\left(-\rho t^{\gamma}\right) \tag{A.4}$$

where $\nu_k=\beta_2+k\gamma$ and the fractional derivative of $C_{\beta_1\beta_2}(t)$ is given as

$$\left(D^{\beta_2}C_{\beta_1\beta_2}\right)(t)=\sum_{k=0}^{\infty}\frac{\left(-\rho t^{\gamma}\right)^k}{\Gamma(1+\gamma k)}=E_{\gamma}\left(-\rho t^{\gamma}\right) \tag{A.5}$$

where $E_{\gamma}\left(-\rho t^{\gamma}\right)$ is the one parameter Mittag-Leffler function that is the *fractional exponential* since

$$D^{n\gamma}E_{\gamma}\left(-\rho t^{\gamma}\right)=(-\rho)^n E_{\gamma}\left(-\rho t^{\gamma}\right);\ n\in\mathbb{N}. \tag{A.6}$$

It is obvious that for $\gamma=1$, $E_{\gamma}\left(-\rho t^{\gamma}\right)=\exp(-\rho t)$. Once the function $C_{\beta_1\beta_2}(t)$ is evaluated, the solution of Eq.(A.1a) is readily obtained by using the Boltzmann superposition principle, that is

$$X(t)=\int_0^t C_{\beta_1\beta_2}(t-\tau)\,\dot{f}(\tau)\,d\tau=\int_0^t h_{\beta_1\beta_2}(t-\tau)\,f(\tau)\,d\tau \tag{A.7}$$

where $h_{\beta_1\beta_2}(t)$ is the impulse response function $h_{\beta_1\beta_2}(t)=\dot{C}_{\beta_1\beta_2}(t)$; thus

$$h_{\beta_1\beta_2}(t)=\sum_{k=0}^{\infty}(-\rho)^k\frac{t^{\nu_k-1}}{\Gamma(\nu_k)}=t^{\beta_2-1}\sum_{k=0}^{\infty}\frac{\left(-\rho t^{\gamma}\right)^k}{\Gamma(\beta_2+\gamma k)}=t^{\beta_2-1}E_{\gamma,\beta_2}\left(-\rho t^{\gamma}\right). \tag{A.8}$$

For the case of engineering interest $\beta_2 = 2$ and the impulse response function is given as

$$h_{\beta_1 2}(t) = t\sum_{k=0}^{\infty} \frac{(-\rho t^{\gamma})^k}{\Gamma(\gamma k + 2)} = tE_{\gamma,2}(-\rho t^{\gamma}) \tag{A.9}$$

where $E_{\gamma,2}(\bullet)$ is the two parameters Mittag-Leffler function. By directly approaching through the iterative kernel method applied to the originary Eq.(A.1a), being $f(t)$ an assigned function, the response is readily computed as

$$X(t) = \sum_{k=0}^{\infty} (-\rho)^k \left(I^{\nu_k} f\right)(t) \tag{A.10}$$

that exactly coalesces with the results obtained by Eq.(A.7).

As a conclusion:

1) For any fractional (or as particular case classical) differential equation the first step is to solve the case of the response to a unit step function that is the kernel of the correspondent integral form and then it is possible to use the Boltzmann or the Duhamel integral to find the solution;
2) The fractional derivative of the maximum order of the response to a unit step function is a fractional exponential;
3) The first derivative of the response $C_{\beta_1\beta_2}(t)$ is not exactly a fractional exponential, unless $\beta_{\max} = 1$.

With these informations, the variance of the response $X(t)$ of Eq.(25) is readily found for the general case of nonstationary white noise $W(t) = q(t)U(t)$. As in fact, according to the Boltzmann superposition principle, the variance $E\left[X^2(t)\right]$ for the general case of nonstationary white noise can be calculated as

$$\begin{aligned} E\left[X^2(t)\right] &= \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-\rho)^{k+r}}{\Gamma(\nu_k)\Gamma(\nu_r)2H_{kr}} \int_0^t (t-\tau)^{2H_{kr}} \dot{q}(\tau)\,d\tau \\ &= \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-\rho)^{k+r}}{\Gamma(\nu_k)\Gamma(\nu_r)} \int_0^t (t-\tau)^{2H_{kr}-1} q(\tau)\,d\tau \end{aligned} \tag{A.11}$$

The second equality at the right hand side of Eq.(A.11) is the Duhamel integral obtained integrating by parts the Boltzmann superposition principle. Eq.(A.11) remains valid since the system under study is linear and then the equations in terms of moments are linear too and thus the Boltzmann or the Duhamel superposition principle still holds.