fractal and fractional

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From Circular to Bessel Functions: A Transition through the Umbral Method

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Abstract: A common environment in which to place Bessel and circular functions is envisaged. We show, by the use of operational methods, that the Gaussian provides the umbral image of these functions. We emphasize the role of the spherical Bessel functions and a family of associated auxiliary polynomials, as transition elements between these families of functions. The consequences of this point of view and the relevant impact on the study of the properties of special functions is carefully discussed.

Keywords: Bessel functions; Hermite polynomials; umbral calculus

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1. Introduction

In abstract terms, Bessel and Gaussian functions are different manifestations of the same function, as has been shown recently [1] using concepts borrowed from umbral theory [2]. Aside from such an interesting, albeit academic statement, the practical outcome of this identification is a significant simplification of the formalism associated with the handling of Bessel functions, accordingly reduced to straightforward applications of the rules of the elementary calculus.

Such a point of view and the joint use of other tools, such as, for example, Ramanujan's master theorem (RMT) [3], allows for the evaluation of infinite integrals of Bessel functions in terms of ordinary Gaussian integrals [4]. Further computational technicalities, such as, for example, those involving the repeated derivatives of Bessel functions with respect to their variable or to their index, are indeed greatly simplified. In addition, the method suggests a new possibility for the introduction of auxiliary polynomials [5], allowing for significant progress in the study of the properties of Bessel functions and their link to other forms belonging to the Bessel-like family.

This paper is devoted to a further step in this direction. We discuss how new elements of speculation emerge from pure algebraic manipulations, as the possibility of framing trigonometric, Bessel and other special functions within the same context, by keeping the Gaussian (or moreover, the exponential) function as the reference pivot.

We introduce the topics discussed in this paper, and the formalism we exploit in the following, by showing how, by stretching the formalism itself, new results can be obtained; in particular we see that different families of Bessel functions are linked by straightforward Gaussian-like transforms.

According to [1], the 0th-order cylindrical Bessel function can be written by using the definition

$$J_0(x) = e^{-\hat{j}\left(\frac{x}{2}\right)^2}\varphi_0 \tag{1}$$

where $\hat{j}$ is an umbral operator whose action on the vacuum φ_0 is defined as follows [6].

Definition 1. *The function*

$$\varphi(\mu) := \varphi_\mu = \frac{1}{\Gamma(\mu+1)}, \quad \forall \mu \in \mathbb{C} \tag{2}$$

is called the umbral "vacuum".

This term, borrowed from Physical language, is used to stress that the action of the operators $\hat{j}$, raised to some power, is that of acting on an appropriate set of functions (in this case the Euler Gamma function), by "filling" the initial "state" $\varphi_0 = \dfrac{1}{\Gamma(1)}$.

Definition 2. *We define the Operator $\hat{j}$, called the umbral, by the vacuum shift operator:*

$$\hat{j} = e^{\partial_z} \tag{3}$$

where z is the domain's variable of the function on which the operator acts.

Theorem 1. *The umbral operator, $\hat{j}^{\,\mu}$, $\forall\ \mu \in \mathbb{C}$, is defined as the action of the operator $\hat{j}$ on the vacuum φ_0, such that*

$$\hat{j}^{\mu}\varphi_0 := \varphi_\mu = \frac{1}{\Gamma(\mu+1)} \tag{4}$$

Proof. $\forall\ \mu \in \mathbb{R}$, applying Equations (2) and (3), we obtain

$$\hat{j}^{\mu}\varphi_0 = e^{\mu\partial_z}\varphi_z\ |_{z=0} = \left.\varphi_{z+\mu}\right|_{z=0} = \left.\frac{1}{\Gamma(z+\mu+1)}\right|_{z=0} = \frac{1}{\Gamma(\mu+1)}$$

□

It satisfies the following:

$$\begin{aligned} \hat{j}^{\,\mu}\hat{j}^{\,\nu}\varphi_0 &= \hat{j}^{\,\mu+\nu}\varphi_0 \\ \left(\hat{j}^{\,\mu}\right)^r \varphi_0 &= \hat{j}^{\,r\,\mu}\varphi_0 \\ \hat{j}^{\,0} &= \frac{1}{\Gamma(1)} = 1 \end{aligned} \tag{5}$$

According to the previous definition and properties of the umbral operator, we obtain

$$e^{-\hat{j}\left(\frac{x}{2}\right)^2}\varphi_0 = \sum_{r=0}^{\infty}\frac{(-\hat{j})^{\,r}}{r!}\left(\frac{x}{2}\right)^{2\,r}\varphi_0 = \sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2\,r} = J_0(x) \tag{6}$$

Although we are borrowing terms, such as the vacuum, from Physics, the legitimacy of the above procedure has been justified by the use of methods based on the Borel transform [7]. The Gaussian integral identity

$$e^{-b^2} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2 - i\,2\,b\,\xi}\,d\,\xi \tag{7}$$

and the fact that we treat $\hat{j}$ as an ordinary algebraic quantity allows for the following conclusion:

$$\begin{aligned} J_0(x) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2 - i\,\hat{j}^{\,\frac{1}{2}}\,\xi\,x}\,d\xi\ \varphi_0 \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2}\left[e^{-i\,\hat{j}^{\,\frac{1}{2}}\,\xi\,x}\varphi_0\right]d\xi \end{aligned} \tag{8}$$

Regarding the term in the square brackets, it is almost straightforward to prove the following:

Theorem 2. *The 0th-order cylindrical Bessel function of the first kind can be expressed in terms of the following integral transform:*

$$J_0(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} W_0^{\left(\frac{1}{2}\right)}(i\,\xi\,x) d\,\xi, \quad \forall x \in \mathbb{R} \tag{9}$$

where $W_\mu^{(\nu)}(x)$ is the Bessel–Wright function [8].

Proof. Let

$$W_\mu^{(\nu)}(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{\Gamma(\nu\, r + \mu + 1)\, r!}, \quad \forall x \in \mathbb{R}, \forall \alpha, \beta \in \mathbb{R}_0^+ \tag{10}$$

be the Bessel–Wright function; then, we can use the umbral formalism of Equation (4) to recast the following:

$$W_\mu^{(\nu)}(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{\Gamma(\nu\, r + \mu + 1)\, r!} = \sum_{r=0}^{\infty} \frac{\hat{j}^{\,\nu\, r+\mu}}{r!}(-x)^r \varphi_0 = \hat{j}^{\,\mu} e^{-\hat{j}^{\,\nu} x} \varphi_0 \tag{11}$$

In particular, we obtain

$$W_0^{\left(\frac{1}{2}\right)}(x) = e^{-\hat{j}^{\,\frac{1}{2}} x} \varphi_0 = \sum_{r=0}^{\infty} \left(-\frac{\hat{j}^{\frac{r}{2}}}{r!} x^r \right) \varphi_0 = \sum_{r=0}^{\infty} \frac{(-x)^r}{\Gamma\left(\frac{r}{2}+1\right)\, r!} \tag{12}$$

which is recognized as a 0th-order Bessel–Wright function. By using this result, we can write

$$J_0(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} \left[e^{-i\,\hat{j}^{\,\frac{1}{2}}\xi\, x} \varphi_0 \right] d\xi = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} W_0^{\left(\frac{1}{2}\right)}(i\,\xi\,x) d\,\xi \tag{13}$$

□

The umbral definition in Equation (11) provides a fairly useful tool to study the properties of the $W_\mu^{(\nu)}$ function. Regarding, for example, the relevant derivatives, we find

$$\begin{aligned}
\left(\frac{d}{dx}\right)^n W_\mu^{(\nu)}(x) &= (-1)^n \hat{j}^{\,n\,\nu+\mu} e^{-\hat{j}^{\,\nu} x} \varphi_0 \\
&= (-1)^n \sum_{r=0}^{\infty} \hat{j}^{\,(n+r)\,\nu+\mu} \frac{(-x)^r}{r!} \varphi_0 \\
&= (-1)^n \sum_{s=0}^{\infty} \frac{(-x)^r}{\Gamma(\nu\, r + \nu\, n + \mu + 1)\, r!} \\
&= (-1)^n W_{\mu+n\,\nu}^{(\nu)}(x)
\end{aligned} \tag{14}$$

which eventually yields

$$\left(\frac{d}{dx}\right)^n J_0(x) = \frac{(-i)^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} \xi^n W_{\frac{n}{2}}^{\left(\frac{1}{2}\right)}(i\,\xi\,x) d\xi \tag{15}$$

The formalism we have outlined suggests the existence of a thread, linking different families of Bessel functions. Although the same results can be obtained using a conventional (non-umbral) procedure, we stress that the protocol we have proposed is flexible, direct, straightforward and naturally suited for this type of problem.

The previous remarks have been aimed at both summarizing the few rules of the formalism we use in the paper and at further pushing the method to point out how the underlying formalism allows for a transparent link between different families of special functions.

2. The Umbral Version of the Trigonometric Functions

The cosine function, if written in umbral form, can also be considered a manifestation of the Gaussian function, if we take the freedom of writing

$$\cos(x) = e^{-\hat{c}\,x^2}\psi_0 \tag{16}$$

and define the umbral operator $\hat{c}$ through its vacuum ψ_0, such that

$$\hat{c}^{\nu}\psi_0 = \frac{\Gamma(\nu+1)}{\Gamma(2\,\nu+1)} \tag{17}$$

We recover the Taylor series expansion of the cos-function, as indicated below:

$$e^{-\hat{c}\,x^2}\psi_0 = \sum_{r=0}^{\infty}\frac{(-1)^r x^{2\,r}}{r!}\hat{c}^r\psi_0 = \sum_{r=0}^{\infty}\frac{(-1)^r x^{2\,r}}{(2\,r)!} \tag{18}$$

It is easy to check the consistency of the definition of Equation (16) with the elementary properties of the trigonometric functions; by indeed keeping the derivative with respect to x, we find

$$\begin{aligned}\frac{d}{dx}e^{-\hat{c}\,x^2}\psi_0 &= -2x\hat{c}e^{-\hat{c}\,x^2}\psi_0\\ &= -2x\sum_{r=0}^{\infty}(-1)^r\frac{(r+1)!}{(2r+2)!}\frac{x^{2r}}{r!}\\ &= -\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!}\\ &= -\sin(x)\end{aligned} \tag{19}$$

It is interesting to recover the cyclical law of the successive derivatives of the circular functions using the present formalism. To this aim, we recall the following identity [9]:

$$\left(\frac{d}{d\,x}\right)^n e^{-a\,x^2} = H_n(-2\,a\,x,\,-a)\,e^{-a\,x^2} = (-1)^n H_n(2\,a\,x,\,-a)\;e^{-a\,x^2} \tag{20}$$

where we have denoted by $H_n(x,y)$ the two variable Hermite Kampé de Fériét polynomials [10]:

$$H_n(x,\,y) = n!\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\frac{x^{n-2\,r}y^r}{(n-2\,r)!\,r!} \tag{21}$$

By keeping successive derivatives of both sides of Equation (16), we find

$$\begin{aligned}\left(\frac{d}{dx}\right)^n e^{-\hat{c}x^2}\psi_0 &= (-1)^n H_n(2\hat{c}x,-\hat{c})e^{-\hat{c}x^2}\psi_0\\ &= (-1)^n n!\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^r\frac{(2x)^{n-2r}}{(n-2r)!r!}\cos\left(x;n-r\right)\\ &= (-1)^n n!\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^r\frac{x}{(n-2\,r)!\,r!}\frac{j_{n-r-1}(x)}{(2x)^r}\\ &= \cos\left(x+n\,\frac{\pi}{2}\right)\end{aligned} \tag{22}$$

Within the present context, cos- and sin-functions are the 0th- and 1st-order cases of a more general class of functions, defined as

$$\cos(x;n) = \hat{c}^n e^{-\hat{c}\,x^2}\psi_0 = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{(n+r)!}{[2\,(n+r)]\,!} x^{2\,r} \tag{23}$$

These can be identified with the spherical Bessel functions [11] according to the identity

$$\cos(x;n+1) = \frac{j_n(x)}{2^{n+1}x^n} \tag{24}$$

This last result is an interesting and unexpected outcome of our formalism, indicating how the umbral procedure we have developed offers a natural way of connecting circular and Bessel-type functions, through the use of the exponential function.

The differential equation satisfied by the functions of Equation (23) can be derived from those of circular Bessel functions according to the identities

$$\begin{aligned} &Z_n(x) = \cos(x;n) \\ &j_{n-1}(x) = 2^n x^{n-1} Z_n(x) \\ &x\,Z_n''(x) + 2n\,Z_n'(x) + x\,Z_n(x) = 0 \end{aligned} \tag{25}$$

Regarding the integrals of the functions of Equation (23), we find

$$\begin{aligned} \int_{-\infty}^{+\infty} \cos(x;n)\,dx &= \int_{-\infty}^{+\infty} \left[\hat{c}^n e^{-\hat{c}x^2}\psi_0\right]\,dx \\ &= \hat{c}^n \int_{-\infty}^{+\infty} e^{-\hat{c}x^2}\,dx\,\psi_0 \\ &= \sqrt{\frac{\pi}{\hat{c}}}\,\hat{c}^{\,n}\psi_0 \\ &= \sqrt{\pi}\,\hat{c}^{\,n-\frac{1}{2}}\varphi_0 \\ &= \sqrt{\pi}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2\,n)} \end{aligned} \tag{26}$$

In deriving the previous identity, we have again taken the freedom, supposed as valid in [1,4,5] and then justified in [7], of treating the umbral operator as a standard algebraic quantity; such a point of view has been shown to be the leitmotiv underlying the umbral heuristic proof of the RMT outlined in [12].

Further insight into the "genesis" of the trigonmetric functions can be obtained by applying the Gauss transform method, as follows:

$$e^{-\hat{c}\,x^2}\psi_0 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2}\left[e^{-2i\,\hat{c}^{\frac{1}{2}}\,\xi\,x}\,\psi_0\right]d\,\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2}\,c_0^{\left(\frac{1}{2}\right)}\,(2\,i\,x\,\xi)\,d\,\xi \tag{27}$$

where

$$c_0^{\left(\frac{1}{2}\right)}(x) = e^{-\hat{c}^{\frac{1}{2}}\,x}\,\psi_0 = \sum_{r=0}^{\infty}\frac{(-x)^r}{r!}\hat{c}^{\frac{r}{2}}\psi_0 = \sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{r}{2}+1\right)}{(r!)^2}(-x)^r \tag{28}$$

is a Bessel trigonometric function whose properties are discussed elsewhere. To give a feeling of how the umbral formalism applies to the relevant study, we note that keeping the successive derivatives of the function defined in Equation (28) we find

$$\left(\frac{d}{dx}\right)^p c_{0,0}^{\left(\frac{1}{2}\right)}(x) = \left(\frac{d}{dx}\right)^p \left[e^{-\hat{c}^{\frac{1}{2}}x}\psi_0\right] = (-1)^p\left[\hat{c}^{\frac{p}{2}}e^{-\hat{c}^{\frac{1}{2}}x}\psi_0\right] = (-1)^p \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{r+p}{2}+1\right)}{r!\,(r+p)!}(-x)^r \tag{29}$$

which can be associated with the special function

$$c_{\mu,\alpha}^{(\nu)}(x) = \sum_{r=0}^{\infty} \frac{\Gamma(\nu r+\alpha+1)}{r!\,\Gamma(r+\mu+1)}(-x)^r \tag{30}$$

and therefore

$$\left(\frac{d}{dx}\right)^p c_{0,0}^{\left(\frac{1}{2}\right)}(x) = (-1)^p c_{p,\frac{p}{2}}^{\left(\frac{1}{2}\right)}(x) \tag{31}$$

The origin of the functions of Equation (30) can easily be traced back to the Bessel–Tricomi functions [11]:

$$C_\mu(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!\,\Gamma(r+\mu+1)} \tag{32}$$

and the functions of Equation (30) are recognized to be associated with the Borel transform of the functions of Equation (33), namely, the following [7]:

$$c_{\mu,\alpha}^{(\nu)}(x) = \int_0^\infty e^{-s}s^\alpha C_\mu(s^\nu x)\,ds \tag{33}$$

whose properties are explored elsewhere.

As for the other functions, we discuss the evaluation of the associated infinite integrals, by considering two paradigmatic examples.

The first is rather artificial and concerns the evaluation of the following integral:

$$\begin{gathered}\int_{-\infty}^{\infty} e^{-a x^2} c_{\mu,\alpha}^{(\nu)}(bx)dx = \int_{-\infty}^{\infty}\left[e^{-a x^2-\hat{\chi}_{\mu,\alpha}^{(\nu)}\,b\,x}\phi_0\right]dx = \sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}\left(\hat{\chi}_{\mu,\alpha}^{(\nu)}\right)^2}\phi_0, \\ \left(\hat{\chi}_{\mu,\alpha}^{(\nu)}\right)^r\phi_0 = \frac{\Gamma(\nu r+\alpha+1)}{\Gamma(r+\mu+1)}\end{gathered} \tag{34}$$

Accordingly, we eventually obtain

$$\int_{-\infty}^{\infty} e^{-a x^2} c_{\mu,\alpha}^{(\nu)}(b\,x)\,dx = \sqrt{\frac{\pi}{a}}\sum_{r=0}^{\infty}\frac{b^{2r}}{(4a)^r r!}\frac{\Gamma(2\,r\,\nu+\alpha+1)}{\Gamma(2\,r+\mu+1)} \tag{35}$$

A further and more familiar example, a naive consequence of this procedure, is the evaluation of the Fresnel integral:

$$C(x) = \int_x^{+\infty}\cos\left(\xi^2\right)d\,\xi \tag{36}$$

at $x=0$. The use of the previous identities yields

$$C(0) = \int_0^{+\infty}\left[e^{-\hat{c}\,x^4}\psi_0\right]dx = \left(\frac{1}{4}\int_0^\infty e^{-y}y^{\frac{1}{4}-1}dy\right)\hat{c}^{-\frac{1}{4}}\psi_0 = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{2}\sqrt{\frac{\pi}{2}} \tag{37}$$

The previous results have emerged in a fairly natural fashion from our formalism. Other means, of a more conventional nature, can be applied, albeit with more computational effort.

3. Final Comments

In the previous sections, we have developed a formalism capable of providing a unified point of view to apparently different functions; analogous ideas have also recently appeared in the literature [13], and the possibilities offered by the technique may be considered promising, if properly developed. To corroborate the previous statement, we recall that, aside from yielding a common environment for the Bessel congeries and their associates, the method allows for the derivation of previously unknown sum rules of lacunary Laguerre polynomials [14]. These results would have hardly been achieved by conventional means. Such a result is a consequence of the fact that the umbral image of the Laguerre polynomials, according to the method developed in Equation (6), is simply a Newton binomial [15], namely,

$$L_n(x,y)=\sum_{s=0}^{n}\frac{(-1)^s}{s!}\binom{n}{s}y^{n-s}x^s=\sum_{s=0}^{n}(-1)^s\binom{n}{s}y^{n-s}x^s\hat{\jmath}^s\varphi_0=\left[(y-\hat{\jmath}\,x)^n\varphi_0\right] \tag{38}$$

In the previous sections, we have seen that by interchanging $\hat{\jmath}$ and $\hat{c}$ operators into the argument of the exponential, we have realized different forms of special functions. By replacing, in Equation (38), $\hat{\jmath}$ with $\hat{c}$, we can define the further family of polynomials umbrally equivalent to $L_n(x,y)$:

$$\begin{aligned}\lambda_n(x,y)&=\left[(y-\hat{c}\,x)^n\psi_0\right]\\&=\sum_{s=0}^{n}(-1)^s\binom{n}{s}y^{n-s}x^s\hat{c}^s\psi_0\\&=\sum_{s=0}^{n}\frac{(-1)^s s!}{(2s)!}\binom{n}{s}y^{n-s}x^s\\&=n!\sum_{s=0}^{n}\frac{(-1)^s}{(2s)!(n-s)!}y^{n-s}x^s\end{aligned} \tag{39}$$

A straightforward application of our procedure yields for the relevant generating functions the following:

$$\begin{aligned}\sum_{n=0}^{\infty}t^n\lambda_n(x,y)&=\sum_{n=0}^{\infty}t^n\left[(y-\hat{c}\,x)^n\psi_0\right]\\&=\frac{1}{(1-y\,t)\left[1+\dfrac{\hat{c}\,x\,t}{1-yt}\right]}\psi_0\\&=\frac{1}{1-y\,t}\left[\sum_{r=0}^{\infty}\left(-\frac{\hat{c}xt}{1-yt}\right)^r\psi_0\right]\\&=\frac{1}{1-yt}e_0\left(\frac{x\,t}{1-y\,t}\right),\\e_0(x)&=\sum_{r=0}^{\infty}(-1)^r\frac{r!}{(2\,r)!}x^r\end{aligned} \tag{40}$$

The generating function (Equation (41)) indicates that the $\lambda_n(x,y)$ belong to the family of Appel polynomials in the y variable; this is a characteristic shared with the $L_n(x,y)$. In this paper we are not interested in this aspect of the problem, which is discussed elsewhere.

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\lambda_n(x,y)=e^{y\,t}\left[e^{-x\,\hat{c}\,t}\psi_0\right]=e^{yt}\cos(\sqrt{xt}) \tag{41}$$

Furthermore, the polynomials of Equation (39) are easily shown to satisfy the recurrences

$$\begin{aligned} &\partial_y \lambda_n(x, y) = n\,\lambda_{n-1}(x,y) \\ &\hat{\Delta}\,\lambda_n(x, y) = n\,\lambda_{n-1}(x, y) \\ &\hat{\Delta} = -4x^{\frac{1}{2}}\partial_x x^{\frac{1}{2}}\partial_x = -2\,(1 + 2\,x\,\partial_x)\;\partial_x \end{aligned} \tag{42}$$

which, once combined, yield the differential equation:

$$\begin{aligned} \partial_y \lambda_n(x, y) &= \hat{\Delta}\,\lambda_n(x, y) \\ \lambda_n(x, 0) &= (-1)^n \frac{n!}{(2\,n)!} x^n \end{aligned} \tag{43}$$

This, accordingly, suggests the following operational definition:

$$\lambda_n(x, y) = e^{y\,\hat{\Delta}} \lambda_n(x, 0) \tag{44}$$

which can be further handled to obtain

$$e^{y\,\hat{\Delta}} e_0(x) = \frac{1}{1-y} e_0\left(\frac{x}{1-y}\right) \tag{45}$$

Equations (40)–(45) are very similar to analogous identities satisfied by the Laguerre polynomials [10,16]. In particular, Equation (43) is a kind of heat equation involving the differential operator $\hat{\Delta}$. To complete the analogy with Laguerre polynomials, we introduce the associated $\lambda-$ polynomials, specified by

$$\begin{aligned} \lambda_n^{(\nu)}(x, y) &= \hat{c}^{\nu}\,(y - \hat{c}\,x)^n \psi_0 \\ &= \sum_{s=0}^{n} (-1)^s \binom{n}{s}\, y^{n-s} x^s \hat{c}^{\nu+s} \psi_0 \\ &= n! \sum_{s=0}^{n} (-1)^s \frac{\Gamma(\nu + s + 1)}{s!\,(n-s)!\Gamma(2\,(\nu + s) + 1)}\, y^{n-s} x^s \end{aligned} \tag{46}$$

The relevant generating function writes

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda_n^{(\nu)}(x, y) = e^{y\,t} \cos\left(\sqrt{x\,t}; \nu\right) \tag{47}$$

We believe it is important to comment on the link between lambda polynomials and other polynomial forms playing an important role in analysis. We note therefore, that

$$h_n(x) = \sum_{k=0}^{n} \left(\begin{array}{c} n+k \\ n-k \end{array} \right) (-x)^k \tag{48}$$

are orthogonal polynomials with the weight function

$$\rho(x) = \frac{1}{2\,\pi}\sqrt{\frac{4-x}{x}} \tag{49}$$

playing an important role in the theory of Catalan numbers and in the solution of the Hausdorff moment problem [17].

The $h_n(x)$ can be readily written in terms of the integral transform of the previously introduced polynomials $\lambda_n(x, y)$, according to the identity

$$h_n(x) = \frac{1}{n!}\int_0^{\infty} e^{-\xi}\xi^n \lambda_n(x\,\xi, 1)\,d\xi \tag{50}$$

In a forthcoming investigation we will take further advantage of the previous restyling of the polynomials $h_n(x)$ by exploring in greater depth the relevant properties.

We have shown that Bessel and Wright functions can be linked through appropriate Gaussian transforms, as reported in Equation (9). It is interesting to further extend such a concept by considering, for example, the following Theorem.

Theorem 3. *Let*

$$_HJ_n(x, y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}(x,y)}{2^{n+2r} r!\,(n+r)!}, \quad \forall x, y \in \mathbb{R}, \forall n \in \mathbb{N} \tag{51}$$

be the Hermite–Bessel functions (HBF) [9,15], with generating function

$$\sum_{n=-\infty}^{+\infty} t^n\, {_HJ_n}(x, y) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{4}\left(t-\frac{1}{t}\right)^2} \tag{52}$$

Then, we can provide the integral identity between ordinary and Hermite–Bessel functions through the identity

$$_HJ_n(x, y) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2} J_n(x + 2\sqrt{y}\,\xi)\,d\xi \tag{53}$$

Proof. The use of the Gauss transform (Equation (7)) and the Bessel generating function [6]:

$$\sum_{n=-\infty}^{+\infty} t^n J_n(x) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} \tag{54}$$

yields

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} t^n\, {_HJ_n}(x, y) &= e^{\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{4}\left(t-\frac{1}{t}\right)^2} \\
&= e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2} e^{\sqrt{y}\,\left(t-\frac{1}{t}\right)\,\xi} d\xi \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2} e^{\left(\frac{(x+2\sqrt{y}\,\xi)}{2}\left(t-\frac{1}{t}\right)\right)} d\xi \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2} \sum_{n=-\infty}^{+\infty} t^n J_n\left(x + 2\sqrt{y}\,\xi\right)\,d\xi
\end{aligned} \tag{55}$$

which implies the hypothesis. □

The usefulness of HBF arises in the study of electromagnetic processes involving the emission of radiation by charges constrained on trajectories that do not allow dipole approximation (namely, the charge motion cannot be reduced to purely harmonic motion) [18,19]. Identities of the type of Equation (53) could be useful in particular to study the relevant Kapteyn series, for example, by taking advantage of the following (see [20] and references therein):

$$\sum_{n=1}^{+\infty} \frac{J_{2n}(2\,n\,z)}{n^2} = \frac{z^2}{2}, \qquad z \in \mathbb{C},\ 0 \le z < 1 \tag{56}$$

The extension of the previous identity to Hermite–Bessel functions can be done using Equation (53), which yields

$$\sum_{n=1}^{+\infty} \frac{{}_HJ_{2n}(2\,n\,x,\,4\,n^2 y)}{n^2} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2}\,(x - 2\sqrt{y}\,\xi)^2\,d\xi = \frac{1}{2}\left(x^2 + \frac{y}{2}\right) \tag{57}$$

We have checked numerically that the range of validity of Equation (57) is rather limited: $x \leq 0.5$ and $y \ll x$.

Further comments on this point, as well as the derivation of further identities of this type, are discussed elsewhere.

Analogous considerations can be developed for the study of Laguerre–Bessel functions [9,15], defined through the generating function:

$$\sum_{n=-\infty}^{+\infty} t^n\,{}_LJ_n(x,\,y) = e^{\frac{(y-\hat{j}\,x)}{2}\left(t-\frac{1}{t}\right)}\varphi_0 = e^{\frac{y}{2}\left(t-\frac{1}{t}\right)} C_0\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] \tag{58}$$

The methods we have outlined allow for significant simplification of the relevant properties through an appropriate "translation", namely,

$${}_LJ_n(x,\,y) = J_n(y - \hat{j}\,x)\,\varphi_0 \tag{59}$$

thus finding, for example, the following (see [20]):

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{{}_LJ_{2n}(2\,n\,x,\,2\,n\,y)}{n^{2p}} &= \sum_{n=1}^{\infty} \frac{J_{2n}\left[2\,n\,\left(y - \hat{j}\,x\right)\right]}{n^{2\,p}}\varphi_0 \\
&= \sum_{k=1}^{p} A_k\,\left(y - \hat{j}\,x\right)^{2\,k}\,\varphi_0 \\
&= \sum_{k=1}^{p} A_k L_{2\,k}(x, y) \\
A_k &= \sum_{r=1}^{k} \frac{(-1)^{r+k} r^{2\,(k-p)}}{(k-r)!(k+r)!}
\end{aligned} \tag{60}$$

At this point, it is fairly natural to include in this gallery of Bessel "Bestiario" their λ-counterpart, defined (see Equation (39)) as

$${}_\lambda J_n(x,\,y) = J_n(y - \hat{c}\,x)\,\psi_0 \tag{61}$$

and through the generating function:

$$\sum_{n=-\infty}^{+\infty} t^n\,{}_\lambda J_n(x,\,y) = e^{\frac{(y-\hat{c}\,x)}{2}\left(t-\frac{1}{t}\right)}\psi_0 = e^{\frac{y}{2}\left(t-\frac{1}{t}\right)} \cos\left(\sqrt{\frac{x}{2}\left(t - \frac{1}{t}\right)}\right) \tag{62}$$

Before closing the paper, we consider the function ${}_\lambda J_n(x,\,0)$, which can be cast in the following series form:

$${}_\lambda J_n(x,\,0) = \sum_{r=0}^{+\infty} \frac{(-1)^{3r+n}(2r+n)!}{r!(r+n)![2(2r+n)]!}\left(\frac{x}{2}\right)^{2r+n} \tag{63}$$

In the case of $n = 0$, abusing our umbral notation and by recalling the rule of the Gaussian successive derivatives, we write

$$\partial_x^n\,{}_\lambda J_0(x,\,0) = (-1)^n H_n\left(\hat{j}\,\hat{c}^2\,\frac{x}{2},\,-\frac{\hat{j}\,\hat{c}^2}{4}\right)\,e^{-\hat{j}\left(-\hat{c}\frac{x}{2}\right)^2}\varphi_0\psi_0 \tag{64}$$

The Bessel and circular umbral operators ($\hat{j}$ and $\hat{c}$) act on the "vacua" (φ_0, ψ_0).

The final examples, regarding the artificial construction of Bessel-type functions, have been aimed at further stressing that, despite being complicated in their explicit representation in terms of series, the operational method greatly simplifies the study of the relevant properties.

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