CLASSIFYING G-GRADED ALGEBRAS OF EXPONENT TWO

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Abstract. Let F be a field of characteristic zero and let $\mathcal{V}$ be a variety of associative F-algebras graded by a finite abelian group G. If $\mathcal{V}$ satisfies an ordinary non-trivial identity, then the sequence $c_n^G(\mathcal{V})$ of G-codimensions is exponentially bounded. In [?, ?, ?], the authors captured such exponential growth by proving that the limit $\exp^G(\mathcal{V}) = \lim_{n\to\infty} \sqrt[n]{c_n^G(\mathcal{V})}$ exists and it is an integer, called the G-exponent of $\mathcal{V}$.

The purpose of this paper is to characterize the varieties of G-graded algebras of exponent greater then 2. As a consequence, we find a characterization for the varieties with exponent equal to 2.

1. Introduction

In 1999, a celebrated theorem of Giambruno and Zaicev proved the integrability of the exponential growth of any proper variety of associative algebras, confirming a famous conjecture posed by Amitsur in the early 1980's. More precisely, let F be a field of characteristic zero and let $\mathcal{V}$ be a variety of associative F-algebras. In 1972, Regev introduced the numerical sequence of codimensions of $\mathcal{V}$, $c_n(\mathcal{V})$, which measure, in some sense, the growth of the polynomial identities satisfied by the variety $\mathcal{V}$. He proved (see [?]) that such a sequence is exponentially bounded, provided that the variety $\mathcal{V}$ satisfies a non-trivial identity. Under the same hypothesis, Giambruno and Zaicev in [?, ?], showed that the limit

$$\exp(\mathcal{V}) = \lim_{n\to\infty} \sqrt[n]{c_n(\mathcal{V})}$$

exists and it is an integer, called the exponent of the variety $\mathcal{V}$. Moreover, in [?], the same authors characterized the varieties of exponent 2 by exhibiting a list of five suitable algebras of exponent 3 or 4.

In the last years, several extension of such results were proved in the setting of algebras with some additional structure. Between them, we recall the cases of algebras with involution ([?, ?, ?]), superalgebras ([?]) and more generally algebras graded by a group ([?, ?, ?]), algebras with a generalized H-action ([?]) and superalgebras with graded involution ([?]) or superinvolution ([?]).

In this paper we focus our attention on the so-called G-varieties, i.e., varieties of associative F-algebras graded by a finite abelian group G. As in the ordinary case, one can attach to a G-graded algebra A the numerical sequence of G-codimensions, $c_n^G(A)$, $n = 1, 2, \ldots$, which is the dimension of the space of multilinear G-graded polynomials in n variables in the corresponding relatively free G-graded algebra of countable rank. Such a sequence is exponentially bounded for any G-graded algebra A satisfying an ordinary non-trivial identity (see [?]). The growth of a G-variety $\mathcal{V}$ is defined as the growth of G-codimensions of any G-graded algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^G(A)$.

In order to capture this exponential growth of the G-codimensions, in [?, ?] for abelian groups and in [?] in general, the authors proved that, for any G-variety $\mathcal{V}$ satisfying an ordinary non-trivial identity, the limit

$$\exp^G(\mathcal{V}) = \lim_{n\to\infty} \sqrt[n]{c_n^G(\mathcal{V})}$$

exists and it is an integer, called the G-exponent of the G-variety $\mathcal{V}$. Such an integer can be explicitly computed and it turns out to be equal to the dimension of a suitable finite dimensional semisimple G-graded algebra over an algebraically closed field.

The purpose of this paper is to characterize the G-varieties having G-exponent greater than one. To this end, we shall explicitly exhibit a list of G-graded algebras A_i in order to prove the following result: a G-variety $\mathcal{V}$ has G-exponent greater than 2 if and only if $A_i \in \mathcal{V}$, for some i. By putting together this theorem with the results of Valenti concerning G-varieties of polynomial growth ([?]), we shall obtain a characterization of the G-varieties of exponent 2.

2. Preliminaries and basic results

Let F be a field of characteristic zero, G be a finite abelian group and A be a G-graded associative algebra over F, i.e., $A = \bigoplus_{g\in G} A_g$, where the A_g's are vector subspaces such that $A_g A_h \subseteq A_{gh}$, for all $g, h \in G$. We shall refer to such subspaces as the homogeneous components of A.

Let $F\langle X\rangle$ be the free associative algebra on a countable set X of non-commuting variables $x_1, x_2, \ldots$. One can define on such an algebra the following G-grading: write $X = \bigcup_{g\in G} X_g$, where $X_g = \{x_{1,g}, x_{2,g}, \ldots\}$ are disjoint sets and the elements of X_g have homogeneous degree g. If we denote by $\mathcal{F}_g$ the subspace of $F\langle X\rangle$ spanned by

2010 *Mathematics Subject Classification.* Primary 16R10, 16W50; Secondary 16W10.

Key words and phrases. Graded algebras, Exponent, Growth.

A. Ioppolo was partially supported by GNSAGA-INDAM.

F. Martino was supported by a postdoctoral grant PNPD from Capes, Brazil.

all monomials in the variables of X having homogeneous degree g, then $F\langle X\rangle = \bigoplus_{g\in G} \mathcal{F}_g$ is a G-graded algebra called the free G-graded algebra of countable rank over F. We shall denote it by $F\langle X, G\rangle$.

If $G = \{g_1, g_2, \ldots, g_m\}$, then a graded polynomial $f = f(x_{1,g_1}, \ldots, x_{t_1,g_1}, \ldots, x_{1,g_m}, \ldots, x_{t_m,g_m})$ of $F\langle X, G\rangle$ is a graded identity of A, and we write $f \equiv 0$, if

$$f(a_{1,g_1}, \ldots, a_{t_1,g_1}, \ldots, a_{1,g_m}, \ldots, a_{t_m,g_m}) = 0,$$

for all $a_{1,g_i}, \ldots, a_{t_i,g_i} \in A_{g_i}$, $i = 1, \ldots, m$.

For $n \geq 1$, the space of multilinear G-graded polynomials in the variables $x_{1,g_{i_1}}, \ldots, x_{n,g_{i_n}}$ is defined as

$$P_n^G = \operatorname{span}_F \left\{ x_{\sigma(1),g_{i_{\sigma(1)}}} \cdots x_{\sigma(n),g_{i_{\sigma(n)}}} \mid \sigma \in S_n,\ g_{i_1}, \ldots, g_{i_n} \in G \right\}.$$

The graded identities of A form a T_G-ideal

$$\mathrm{Id}^G(A) = \{ f \in F\langle X, G\rangle \mid f \equiv 0 \text{ on } A\}$$

which is an ideal of $F\langle X, G\rangle$ invariant under all graded endomorphisms of the free G-graded algebra.

Recall that two G-graded algebras A and B are said to be T_G-equivalent, and we write $A \sim_{T_G} B$, if and only if $\mathrm{Id}^G(A) = \mathrm{Id}^G(B)$.

It is well known that in characteristic zero every T_G-ideal is completely determined by its multilinear polynomials. Thus it is reasonable to consider the quotient space

$$P_n^G(A) = \frac{P_n^G}{P_n^G \cap \mathrm{Id}^G(A)}$$

and so one can define the n-th G-graded codimension of A as $c_n^G(A) = \dim_F P_n^G(A)$, $n \geq 1$. One important feature of the sequence of G-graded codimensions is given in the following.

Theorem 2.1. [?] *Let A be a G-graded algebra satisfying an ordinary non-trivial identity. Then the G-graded codimension sequence $c_n^G(A)$, $n = 1, 2, \ldots$, is exponentially bounded.*

Recall that if $\mathcal{V} = \mathrm{var}^G(A)$ is the variety generated by a G-graded algebra A (G-variety), we write $\mathrm{Id}^G(\mathcal{V}) = \mathrm{Id}^G(A)$, $c_n^G(\mathcal{V}) = c_n^G(A)$ and the growth of $\mathcal{V}$ is the growth of the sequence $c_n^G(\mathcal{V})$.

In [?, ?, ?], the authors captured the exponential growth of the G-graded codimension sequence of a G-graded algebra A satisfying an ordinary non-trivial identity by proving the existence and the integrability of the limit

$$\exp^G(A) = \lim_{n\to\infty} \sqrt[n]{c_n^G(A)},$$

called the G-exponent of A.

Let now introduce an useful tool in the theory of polynomial identities, the Grassmann envelope of an algebra. To this end, let $E = \langle e_1, e_2, \ldots \mid e_i e_j = -e_j e_i \rangle$ be the infinite dimensional Grassmann algebra over F and let $\mathbb{Z}_2 = \{0, 1\}$ be the cyclic group of order 2 in additive notation. It is well known that E has a natural $\mathbb{Z}_2$-grading, $E = E_0 \oplus E_1$, where E_0 is the span of all monomials in the e_i's of even length and E_1 is the span of all monomials in the e_i's of odd length.

We recall that if $A = \bigoplus_{(g,i)\in G\times\mathbb{Z}_2} A_{(g,i)}$ is a $G\times\mathbb{Z}_2$-graded algebra, then one can define the Grassmann envelope of A as

$$E(A) = \bigoplus_{g\in G} \left(E_0 \otimes A_{(g,0)} \oplus E_1 \otimes A_{(g,1)} \right).$$

The importance of the Grassmann envelope is highlighted in the following theorem proved separately by Aljadeff and Belov in [?] and Sviridova in [?].

Theorem 2.2. *Let A be a G-graded algebra satisfying an ordinary non-trivial identity. Then there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B such that $Id^G(A) = Id^G(E(B))$.*

Next we recall how to compute the G-exponent of a G-graded algebra A satisfying an ordinary non-trivial identity. According to Theorem **??**, there exists a finite dimensional $G\times\mathbb{Z}_2$-graded algebra B such that $\mathrm{Id}^G(A) = \mathrm{Id}^G(E(B))$.

By the Wedderburn-Malcev decomposition (see [?]), we write $B = B' + J$, where B' is a maximal semisimple subalgebra of B, which we may assume to be $G\times\mathbb{Z}_2$-graded by [?], and $J = J(B)$ is the Jacobson radical of B, which is a graded ideal (see [?]). Also we can write $B' = B_1 \oplus \cdots \oplus B_k$, where the B_j's are $G\times\mathbb{Z}_2$-graded simple algebras. The description of such algebras is given in the following theorem proved by Bahturin, Sehgal and Zaicev in [?].

Theorem 2.3. *Let A be a G-graded simple algebra. Then there exist a subgroup H of G, a 2-cocycle $\alpha : H \times H \to F^*$, where the action of H on F is trivial, an integer m and a m-tuple $(g_1 = e, g_2, \ldots, g_m) \in G^m$ such that A is G-graded isomorphic to $R = F^\alpha H \otimes M_m(F)$, where $R_g = span_F \left\{ b_h \otimes e_{ij} \mid g = g_i^{-1} h g_j \right\}$. Here $b_h \in F^\alpha H$ is a representative of $h \in H$.*

In [?] it was proved that

$$\exp^G(A) = \exp^G(E(B)) = \max \dim(C_1 \oplus \cdots \oplus C_h),$$

where $C_1, \dots, C_h \in \{B_1, \dots, B_k\}$ are distinct and $C_1 J C_2 J \cdots J C_h \neq 0$.

3. Constructing G-graded algebras of exponent greater than 2

The purpose of this section is to construct some suitable G-graded algebras that will allow us to prove the main result of this paper. In what follows we shall denote by e the unit element of G.

In the group G, let $g \in G$ be an element of order n. We shall consider the cyclic subgroup $C_n = \langle g \rangle$ generated by g. The group algebra $A = FC_n$ of C_n over F has a natural C_n-grading $A = \bigoplus_{i=0}^{n-1} A_{g^i}$, where $A_{g^i} = Fg^i$, $0 \leq i \leq n-1$. It is clear that A can be regarded as a G-graded algebra by setting $A_{g'} = 0$, for all $g' \notin \langle g \rangle$. For all prime p greater than two, we denote by A_1^p the algebra FC_p. Here we want to highlight that in [?] it was proved that such an algebra generates a variety of almost polynomial growth, i.e., it grows exponentially but any proper subvariety has polynomial growth of the codimensions. Moreover, let $A_2 = FC_4$.

If $g \in G$ is an element of order 4 we consider $\overline{C_4}$ the cyclic subgroup of $G \times \mathbb{Z}_2$ generated by $(g, 1)$. We denote by $A_3 = E(F\overline{C_4})$ the Grassmann envelope of $F\overline{C_4}$, endowed with its natural C_4-grading.

Furthermore, if there exist $a, b \in G$ distinct elements of order 2, we let $K_{i,j} = \langle (a, i), (b, j) \rangle$ be the subgroup of $G \times \mathbb{Z}_2$ generated by (a, i) and (b, j). We denote by $A_4^{i,j}$ the Grassmann envelope of $F^\alpha K_{i,j}$, for some cocycle α.

Now let us consider $M_2(F)$ the algebra of 2×2 matrices endowed with an elementary $G \times \mathbb{Z}_2$-grading. It is clear that the homogeneous degree of e_{11} and e_{22} is always $(e, 0)$ whereas those of e_{12} and e_{21} are (g, i) and (g^{-1}, i), respectively, where $g \in G$ and $i \in \mathbb{Z}_2$. Thus

$$M_2(F)^{G \times \mathbb{Z}_2} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \oplus \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix},$$

if $g \neq g^{-1}$. Remark that, if $g = g^{-1}$, we have only the components $(e, 0)$ and (g, i) if $g \neq e$ or $i = 1$ and just one component (trivial grading) if $g = e$ and $i = 0$. Hence any elementary $G \times \mathbb{Z}_2$-grading is uniquely determined by the homogeneous degree of e_{12} and thus we denote by $M_2(F)_{g,i}$ the algebra of 2×2 matrices with elementary $G \times \mathbb{Z}_2$-grading induced by (g, i). Finally we define $A_5^{g,i}$ as the Grassmann envelope of $M_2(F)_{g,i}$.

Let G' be any finite abelian group. For all $k \in G'$ with $o(k) = 2$ and $h \in G'$, we consider the following subalgebra of $\begin{pmatrix} FG' & FG' \\ 0 & FG' \end{pmatrix}$:

$$A_{k,h} = \begin{pmatrix} F \oplus Fk & Fh \oplus Fkh \\ 0 & F \end{pmatrix},$$

with grading $(A_{k,h})_e = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$, $(A_{k,h})_k = \begin{pmatrix} Fk & 0 \\ 0 & 0 \end{pmatrix}$, $(A_{k,h})_h = \begin{pmatrix} 0 & Fh \\ 0 & 0 \end{pmatrix}$ and $(A_{k,h})_{kh} = \begin{pmatrix} 0 & Fkh \\ 0 & 0 \end{pmatrix}$. We denote by $A_6^{k,h}$ the Grassmann envelope of $A_{k,h}$, when $G' = G \times \mathbb{Z}_2$.

In a similar way, if we consider

$$A'_{k,h} = \begin{pmatrix} F & Fh \oplus Fkh \\ 0 & F \oplus Fk \end{pmatrix},$$

where k has order 2 and $h \in G \times \mathbb{Z}_2$, then we denote by $A_7^{k,h}$ the Grassmann envelope of $A'_{k,h}$.

Finally, let $UT_3(F)_{k,h}$ be the algebra of 3×3 upper-triangular matrices endowed with the elementary $G \times \mathbb{Z}_2$-grading uniquely determined by the triple (e, k, kh), $e = 1_{G \times \mathbb{Z}_2}$, $k, h \in G \times \mathbb{Z}_2$. We define $A_8^{k,h}$ as the Grassmann envelope of $UT_3(F)_{k,h}$.

We now collect the previous algebras in the following list, including also their corresponding exponents.

- $A_1^p = FC_p$, where $p > 2$ is prime; $\exp^G(A_1^p) = p$.
- $A_2 = FC_4$; $\exp^G(A_2) = 4$.
- $A_3 = E(F\overline{C_4})$; $\exp^G(A_3) = 4$.
- $A_4^{i,j} = E(F^\alpha K_{i,j})$, where $i, j \in \mathbb{Z}_2$; $\exp^G(A_4^{i,j}) = 4$.
- $A_5^{g,i} = E(M_2(F)_{g,i})$, where $g \in G$ and $i \in \mathbb{Z}_2$; $\exp^G(A_5^{g,i}) = 4$.
- $A_6^{k,h} = E(A_{k,h})$, where $k, h \in G \times \mathbb{Z}_2$ and $o(k) = 2$; $\exp^G(A_6^{k,h}) = 3$.
- $A_7^{k,h} = E(A'_{k,h})$, where $k, h \in G \times \mathbb{Z}_2$ and $o(k) = 2$; $\exp^G(A_7^{k,h}) = 3$.
- $A_8^{k,h} = E(UT_3(F)_{k,h})$, where $k, h \in G \times \mathbb{Z}_2$; $\exp^G(A_8^{k,h}) = 3$.

4. A characterization of G-varieties of exponent 2

The final goal of this section is to characterize the G-varieties of exponent 2. Such a result will be a corollary of another theorem concerning the G-varieties of exponent greater than 2. To this end, we need the following lemmas.

Lemma 4.1. *Let $A = B_{i_1} \oplus \cdots \oplus B_{i_k} + J$ be a finite dimensional G-graded algebra over an algebraically closed field F of characteristic zero with $\exp^G(A) > 2$. If there exist three distinct G-graded simple components $B_1 \cong B_2 \cong B_3 \cong F$ such that $B_1JB_2JB_3 \neq 0$, then $UT_3(F)_{k,h} \in var^G(A)$, where $UT_3(F)_{k,h}$ denote the algebra of 3×3 upper-triangular matrices endowed with the elementary G-grading uniquely determined by the triple (e, k, kh), $e = 1_G$, $k, h \in G$.*

Proof. Let e_1, e_2, e_3 be the unit elements of B_1, B_2, B_3, respectively. Then $e_n^2 = e_n \in (B_n)_e$ and $e_r e_s = \delta_{rs} e_r$, for $r, s = 1, 2, 3$, $n \in \{1, 2, 3\}$ and where δ_{rs} denotes the Kronecker delta. By standard arguments we may assume that in A we have $e_1Je_2Je_3 \neq 0$ and $Je_aJe_bJe_c = e_aJe_bJe_cJ = e_{n'}Je_nJe_{n'} = 0$, for all permutations (a, b, c) of $(1, 2, 3)$ and for $n, n' \in \{1, 2, 3\}$, $n \neq n'$. Hence there exist $j_1 = \sum_{g\in G} j_1^{(g)}$ and $j_2 = \sum_{g\in G} j_2^{(g)} \in J$ such that

$$e_1j_1e_2j_2e_3 = e_1 \sum_{g\in G} j_1^{(g)} e_2 \sum_{g\in G} j_2^{(g)} e_3 \neq 0.$$

Therefore at least one of the above summands must be non-zero, say $e_1j_1^{(k)}e_2j_2^{(h)}e_3$, for some $k, h \in G$. We consider the subalgebra U of A linearly generated by the elements

$$e_1, \quad e_2, \quad e_3, \quad e_1j_1^{(k)}e_2, \quad e_2j_2^{(h)}e_3, \quad e_1j_1^{(k)}e_2j_2^{(h)}e_3.$$

It is easily checked that U is a G-graded algebra with induced grading $U = U_e \oplus U_k \oplus U_h \oplus U_{kh}$, where

$$U_e = \mathrm{span}_F\{e_1, e_2, e_3\}, \quad U_k = \mathrm{span}_F\{e_1j_1^{(k)}e_2\}, \quad U_h = \mathrm{span}_F\{e_2j_2^{(h)}e_3\}, \quad U_{kh} = \mathrm{span}_F\{e_1j_1^{(k)}e_2j_2^{(h)}e_3\}.$$

Moreover, the linear map $\varphi : U \to UT_3(F)_{k,h}$, defined by

$$\varphi(e_1) = e_{11}, \quad \varphi(e_2) = e_{22}, \quad \varphi(e_3) = e_{33}, \quad \varphi(e_1j_1^{(k)}e_2) = e_{12}, \quad \varphi(e_2j_2^{(h)}e_3) = e_{23}, \quad \varphi(e_1j_1^{(k)}e_2j_2^{(h)}e_3) = e_{13},$$

is an isomorphism of G-graded algebras. Hence $UT_3(F)_{k,h} \in \mathrm{var}^G(A)$ and the proof is complete. □

Lemma 4.2. *Let $A = B_{i_1} \oplus \cdots \oplus B_{i_k} + J$ be a finite dimensional G-graded algebra over an algebraically closed field F of characteristic zero with $\exp^G(A) > 2$. If there exist two G-simple components $B_1 \cong Fe \oplus Fk$, where $e = 1_G$ and $o(k) = 2$, and $B_2 \cong Fe$ such that either $B_1JB_2 \neq 0$ or $B_2JB_1 \neq 0$, then $A_{k,h} \in var^G(A)$ or $A'_{k,h} \in var^G(A)$, for some $h \in G$.*

Proof. Suppose first that $B_1JB_2 \neq 0$. As in the previous lemma, there exists $j = \sum_{g\in G} j_g \in J$ such that $e_1je_2 \neq 0$, where e_1 and e_2 are the unit elements of B_1 and B_2, respectively. In particular, we must have $e_1j_he_2 \neq 0$, for some $h \in G$. Hence, let us consider the subalgebra D of A generated by the elements

$$e_1, \qquad e_2, \qquad ke_1, \qquad e_1j_he_2, \qquad ke_1j_he_2.$$

Clearly D is a G-graded algebra with induced grading $D = D_e \oplus D_k \oplus D_h \oplus D_{kh}$, where

$$D_e = \mathrm{span}_F\{e_1, e_2\}, \quad D_k = \mathrm{span}_F\{ke_1\}, \quad D_h = \mathrm{span}_F\{e_1j_he_2\}, \quad D_{kh} = \mathrm{span}_F\{ke_1j_he_2\}.$$

If one sets $\varphi : D \to A_{k,h}$ such that

$$\varphi(e_1) = e_{11}, \quad \varphi(e_2) = e_{22}, \quad \varphi(ke_1) = ke_{11}, \quad \varphi(e_1j_he_2) = he_{12}, \quad \varphi(ke_1j_he_2) = khe_{12},$$

then we get that $D \cong A_{k,h}$ as G-graded algebras. Thus $A_{k,h} \in \mathrm{var}^G(A)$.

If $B_2JB_1 \neq 0$, with similar arguments, we get $A'_{k,h} \in \mathrm{var}^G(A)$ and we are done. □

Now we are in a position to characterize the G-varieties of exponent greater than 2.

Theorem 4.1. *Let F be an algebraically closed field of characteristic zero and let $\mathcal{V} = var^G(A)$ be a G-variety generated by the G-graded algebra A. Then $\exp^G(\mathcal{V}) > 2$ if and only if at least one of the G-graded algebras $A_1^p, A_2, A_3, A_4^{i,j}, A_5^{g,i}, A_6^{k,h}, A_7^{k,h}, A_8^{k,h}$ belongs to $\mathcal{V}$.*

Proof. It is clear that if at least one among $A_1^p, A_2, A_3, A_4^{i,j}, A_5^{g,i}, A_6^{k,h}, A_7^{k,h}, A_8^{k,h}$ belongs to $\mathcal{V}$, then $\exp^G(\mathcal{V}) > 2$, since the G-exponents of the previous algebras are all greater than 2.

Let us suppose that $\exp^G(\mathcal{V}) > 2$. By Theorem **??**, there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B such that $\mathcal{V} = \mathrm{var}^G(A) = \mathrm{var}^G(E(B))$. Moreover, let $B = B_1 \oplus \cdots \oplus B_m + J$ be the Wedderburn-Malcev decomposition of B. By Theorem **??**, we have that for all $i = 1, \dots, m$, $B_i \cong M_n(F^\alpha H)$ as $G \times \mathbb{Z}_2$-graded algebras, for some $n \geq 1$, $H \leq G \times \mathbb{Z}_2$ and α 2-cocycle of H.

Suppose first that $B_i \cong M_n(F^\alpha H) = F^\alpha H \otimes M_n(F)$, $n > 1$. Consider the subalgebra $M_n(F)$ with induced $G \times \mathbb{Z}_2$-grading. Notice that e_{11} and e_{22} have homogeneous degree $(e, 0)$ whereas e_{12} and e_{21} have homogeneous degree (g, i) and (g^{-1}, i), for some $(g, i) \in G \times \mathbb{Z}_2$, respectively. Thus the algebra $M_2(F)_{g,i}$ defined in the previous section is a $G \times \mathbb{Z}_2$-graded subalgebra of $M_n(F) \subseteq M_n(F^\alpha H)$. As a consequence, $A_5^{g,i} = E(M_2(F)_{g,i}) \in \mathrm{var}^G(E(B)) = \mathrm{var}^G(A)$.

Hence we can suppose that $n = 1$ and so $B_i \cong F^\alpha H$.

If $p \mid |H|$, where p is a prime number greater than 2, then there exists $g' \in H$ of order p and so we must have $g' = (g, 0)$, with $o(g) = p$. Hence being $C_p = \langle (g, 0) \rangle$ a cyclic group of order p, we may assume that α is trivial on it. Thus $E(FC_p) = E_0 \otimes FC_p$ has the same G-identities of FC_p. It follows that $A_1^p = FC_p \in \mathrm{var}^G(A)$.

Otherwise, $|H| = 2^k$, with $k > 1$. If there exists $g' \in H$ of order 4, then we have two possibilities: or $g' = (g, 0)$ or $g' = (g, 1)$, with $o(g) = 4$. In the first case we get $A_2 \in \mathrm{var}^G(A)$ whereas in the second one we have $A_3 \in \mathrm{var}^G(A)$. On the other hand, if there are no elements of order 4 in H, then

$$H = \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{k-\text{times}}.$$

Hence there exist distinct elements $g', h' \in H$ of order 2 such that $g' = (a, i)$ and $h' = (b, j)$, with $o(a) = o(b) = 2$ and $i, j \in \mathbb{Z}_2$. It is easily checked that $F^\alpha K_{i,j}$ is a subalgebra of B_i and therefore $A_4^{i,j} \in \mathrm{var}^G(A)$.

Since $\exp^G(A) > 2$, by the basic property of the G-exponent seen in Section 2, it follows that there exist distinct G-simple components $B_{i_1}, \dots, B_{i_l}$ such that $B_{i_1} J \cdots J B_{i_l} \neq 0$ and $\dim_F(B_{i_1} + \cdots + B_{i_l}) > 2$. Therefore, we may assume that one of the following possibilities occurs:

1. there exist distinct C_1, C_2, C_3 such that $C_1 J C_2 J C_3 \neq 0$ and $C_1 \cong C_2 \cong C_3 \cong Fe$,
2. for some $i_1 \neq i_2$, $B_{i_1} J B_{i_2} \neq 0$ and $B_{i_1} \cong Fe$ and $B_{i_2} \cong Fe \oplus Fk$,
3. for some $i_1 \neq i_2$, $B_{i_1} J B_{i_2} \neq 0$ and $B_{i_1} \cong Fe \oplus Fk$ and $B_{i_2} \cong Fe$.

If (1) holds, by Lemma **??**, $UT_3(F)_{k,h} \in \mathrm{var}^G(B)$, for some $k, h \in G \times \mathbb{Z}_2$ and so $A_8^{k,h} \in \mathrm{var}^G(A)$. Instead, if (2) or (3) holds, then Lemma **??** applies and so $A_{k,h}$ or $A'_{k,h} \in \mathrm{var}^G(B)$. Therefore $A_6^{k,h}$ or $A_7^{k,h} \in \mathrm{var}^G(A)$ and the proof is complete. $\square$

Next proposition proves that the list of algebras $A_1^p, A_2, A_3, A_4^{i,j}, A_5^{g,i}, A_6^{k,h}, A_7^{k,h}, A_8^{k,h}$ cannot be reduced. Recall that we denote by x_g or $x_{i,g}$ a variable of homogeneous degree g and by x_i a generic variable.

Proposition 4.1. *Let A and B be distinct G-graded algebras among $\{A_1^p, A_2, A_3, A_4^{i,j}, A_5^{g,i}, A_6^{k,h}, A_7^{k,h}, A_8^{k,h}\}$. Then we have that $\mathrm{var}^G(A) \nsubseteq \mathrm{var}^G(B)$.*

Proof. Since all the above algebras have exponential growth of the codimensions and since the algebra A_1^p generates a variety of almost polynomial growth, we immediately get that $A \notin \mathrm{var}^G(A_1^p)$, for any $A \in \{A_1^q, A_2, A_3, A_4^{i,j}, A_5^{g,i}, A_6^{k,h}, A_7^{k,h}, A_8^{k,h}\}$, $p \neq q$.

Notice that, given two algebras A and B, the condition $\mathrm{var}^G(A) \subseteq \mathrm{var}^G(B)$ implies $\exp^G(A) \leq \exp^G(B)$. Hence we get that $A_1^p \notin \mathrm{var}^G(B)$, for $p > 3$ and $B \in \{A_1^q, A_2, A_3, A_4^{i,j}, A_5^{g,i}, A_6^{k,h}, A_7^{k,h}, A_8^{k,h}\}$, $q < p$. For the same reason, we get that $C \notin \mathrm{var}^G(D)$, for $C \in \{A_2, A_3, A_4^{i,j}, A_5^{g,i}\}$ and $D \in \{A_6^{k,h}, A_7^{k,h}, A_8^{k,h}\}$.

The algebras A_2, A_3 and $A_4^{i,j}$ have more non-zero homogeneous components than the algebra $A_5^{g,i}$. It follows that $A_2, A_3, A_4^{i,j} \notin \mathrm{var}^G(A_5^{g,i})$.

In what follows, we shall consider the C_3-graded algebra A_1^3 with homogeneous degrees e, g_1, g_1^2 ($o(g_1) = 3$) and the C_4-graded algebras A_2 and A_3 with homogeneous degrees e, g_2, g_2^2, g_2^3 ($o(g_2) = 4$) and e, g_3, g_3^2, g_3^3 ($o(g_3) = 4$), respectively. The $C_2 \times C_2$-graded algebra $A_4^{i,j}$ has homogeneous degrees e, a, b, ab with $o(a) = o(b) = o(ab) = 2$ whereas those ones of the algebra $A_5^{g,i}$ are e, g, g^{-1}. Moreover, we shall consider the algebras $A_6^{k,h}$ and $A_7^{k,h}$ with homogeneous degrees $e, k_1, h_1, k_1 h_1$ ($o(k_1) = 2$) and $e, k_2, h_2, k_2 h_2$ ($o(k_2) = 2$), respectively. Finally the algebra $A_8^{k,h}$ has homogeneous degrees e, k, h, kh.

Now we want to highlight that if two algebras are graded by different non-isomorphic groups then there is nothing to prove. This is the case of the algebras A_2 and $A_4^{i,j}$, A_3 and $A_4^{i,j}$, A_1^3 and B, where $B \in \{A_2, A_3, A_4^{i,j}, A_6^{k,h}, A_7^{k,h}\}$.

In order to complete the proof we need to notice the following facts.

- $A_1^3 \notin \mathrm{var}^G(A_5^{g,i})$. In fact, if $A_5^{g,i}$ has trivial grading, then A_1^3 has more non-zero homogeneous components and we are done. Now we have to consider only the cases in which $g = g_1$ or $g = g_1^2$. In both, it is clear that $x_{1,g} x_{2,g} \equiv 0$ in $A_5^{g,i}$ but not in A_1^3.
- $A_1^3 \notin \mathrm{var}^G(A_8^{k,h})$. In fact, if $g_1 \notin \{k, h, kh\}$ the result follows. Otherwise, $x_{g_1}^3 \equiv 0$ on $A_8^{k,h}$ but not on A_1^3.
- $A_2 \notin \mathrm{var}^G(A_3)$. In fact, $x_{1,g_1} x_{2,g_1} + x_{2,g_1} x_{1,g_1} \equiv 0$ on A_3 but not in A_2 ($g_1 = g_2$).
- $A_3 \notin \mathrm{var}^G(A_2)$. In fact, $[x_{1,g_1}, x_{2,g_1}] \equiv 0$ on A_2 but not in A_3 ($g_1 = g_2$).
- $B \notin \mathrm{var}^G(A_2)$, $B \in \{A_5^{g,i} A_6^{k,h}, A_7^{k,h}, A_8^{k,h}\}$. In fact, $[x_1, x_2] \equiv 0$ on A_2 but not in B.
- $B \notin \mathrm{var}^G(A_3)$, $B \in \{A_6^{k,h}, A_7^{k,h}\}$. In fact, if $k_1 \neq g_3^2$ then the result is obvious. Let now $k_1 = g_3^2$. In any case, $[x_{h_1}, x_{k_1}] \equiv 0$ on A_3 but not in $A_6^{k,h}$, $h_1 \in \{e, g_3, g_3^3\}$. For the algebra $A_7^{k,h}$ the proof is analogous.
- $A_8^{k,h} \notin \mathrm{var}^G(A_3)$. In fact, if $k, h, kh \neq e$, then $[x_e, x_{g_3^2}] \equiv 0$ on A_3 but not in $A_8^{k,h}$, $g_3^2 \in \{k, h, kh\}$. Otherwise we have to consider $[x_{1,e}, x_{2,e}]$.

- $B \notin \mathrm{var}^G(A_4^{i,j})$, $B \in \{A_6^{k,h}, A_7^{k,h}\}$. In fact, $[x_e, x_{k_i}] \equiv 0$ on $A_4^{i,j}$ but not in B, $k_i \in \{a, b, ab\}$, $i = 1, 2$.
- $A_8^{k,h} \notin \mathrm{var}^G(A_4^{i,j})$. In fact, if $k, h, kh \neq e$, then $[x_e, x_k] \equiv 0$ on $A_4^{i,j}$ but not in $A_8^{k,h}$, where $k \in \{a, b, ab\}$. Otherwise, we have to consider $[x_{1,e}, x_{2,e}]$.
- $A_4^{i,j} \notin \mathrm{var}^G(A_4^{i',j'})$, $(i', j') \neq \{(0,0), (i,j)\}$. In fact, let $\bar{g} \in \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $(A_4^{i,j})_{\bar{g}} = E_0 \otimes F^\alpha(\bar{g}, 0)$ and $(A_4^{i',j'})_{\bar{g}} = E_1 \otimes F^\alpha(\bar{g}, 1)$. Then $x_{1,\bar{g}}x_{2,\bar{g}} + x_{2,\bar{g}}x_{1,\bar{g}} \equiv 0$ in $A_4^{i',j'}$ but not in $A_4^{i,j}$.
- $A_4^{i,j} \notin \mathrm{var}^G(A_4^{0,0})$, $(i,j) \neq (0,0)$. In fact, let $\bar{g}, \bar{h} \in \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $(A_4^{i,j})_{\bar{g}} = E_1 \otimes F^\alpha(\bar{g}, 1)$ and $(A_4^{i,j})_{\bar{h}} = E_1 \otimes F^\alpha(\bar{h}, 1)$. Then $\alpha(\bar{g}, \bar{h})x_{\bar{h}}x_{\bar{g}} - \alpha(\bar{h}, \bar{g})x_{\bar{g}}x_{\bar{h}} \equiv 0$ in $A_4^{0,0}$ but not in $A_4^{i,j}$.
- $A_5^{g,i} \notin \mathrm{var}^G(A_3)$. If $g = e$ then $[x_{1,g}, x_{2,g}] \equiv 0$ on A_3 but not on $A_5^{g,i}$ whereas if $g = g_3^2$, then $[x_{1,e}, x_{2,e}] \equiv 0$ on A_3 but not on $A_5^{g,i}$. Otherwise, we have to consider $x_{g_3}x_{g_3^3} + x_{g_3^3}x_{g_3}$.
- $A_5^{g,i} \notin \mathrm{var}^G(A_4^{i,j})$. As before, if $g = a$ or $g = b$ or $g = ab$, then $[x_e, x_g] \equiv 0$ on $A_4^{i,j}$ but not on $A_5^{g,i}$. Otherwise, we have to consider $[x_{1,e}, x_{2,e}]$.
- $A_6^{k,h}, A_7^{k,h} \notin \mathrm{var}^G(A_5^{g,i})$. In fact, if $h_1 \neq e$ and $h_1 \neq k$ (resp. h_2), then $A_6^{k,h}$ and $A_7^{k,h}$ have more non-zero homogeneous components than $A_5^{g,i}$ and we are done. Otherwise we have to distinguish two cases. If $g = e$, then $A_6^{k,h}$ and $A_7^{k,h}$ have again more non-zero homogeneous components than $A_5^{g,i}$. If $g \neq e$, we have that $[x_{1,e}, x_{2,e}] \equiv 0$ on $A_5^{g,i}$ but not on $A_6^{k,h}, A_7^{k,h}$.
- $A_8^{k,h} \notin \mathrm{var}^G(A_5^{g,i})$. Let $g \neq e$. If $h \neq e$, $k \neq e$, $kh \neq e$ and $k \neq h$, then $A_8^{k,h}$ has more non-zero homogeneous components than $A_5^{g,i}$ and we are done. Let now consider the cases $k = e$ or $h = e$ or $kh = e$. We have that $[x_{1,e}, x_{2,e}] \equiv 0$ on $A_5^{g,i}$ but not in $A_8^{k,h}$. Otherwise $k = h$ and we have to distinguish two cases. If $g \neq g^{-1}$, we have that $[x_{1,k}, x_{2,k}] \equiv 0$ on $A_5^{g,i}$ but not in $A_8^{k,h}$ (here $g = k$). If $g = g^{-1}$, then we have to consider $[x_{1,e}, x_{2,e}]$ (notice that $o(k) = 2$ since $g = k$). Finally, let $g = e$. If $k, h, kh \neq e$, then $A_8^{k,h}$ has more non-zero homogeneous components than $A_5^{g,i}$. If $k = h = e$ the polynomial $[[x_1, x_2]^2, x_1] \equiv 0$ on $A_5^{g,i}$ but not in $A_8^{k,h}$.
- $A_6^{k,h} \notin \mathrm{var}^G(A_7^{k,h})$ since $x_{k_1}[x_e, x_{h_1}] \equiv 0$ on $A_7^{k,h}$ but not in $A_6^{k,h}$.
- $A_6^{k,h} \notin \mathrm{var}^G(A_8^{k,h})$ since $x_{1,k_1}x_{2,k_1}x_{3,k_1} \equiv 0$ on $A_8^{k,h}$ but not in $A_6^{k,h}$.
- $A_7^{k,h} \notin \mathrm{var}^G(A_6^{k,h})$ since $[x_e, x_{h_2}]x_{k_2} \equiv 0$ on $A_6^{k,h}$ but not in $A_7^{k,h}$.
- $A_8^{k,h} \notin \mathrm{var}^G(A_6^{k,h})$ since $[x_e, x_h]x_k \equiv 0$ on $A_6^{k,h}$ but not in $A_8^{k,h}$.
- $A_7^{k,h} \notin \mathrm{var}^G(A_8^{k,h})$ since $x_{1,k_2}x_{2,k_2}x_{3,k_2} \equiv 0$ on $A_8^{k,h}$ but not in $A_7^{k,h}$.
- $A_8^{k,h} \notin \mathrm{var}^G(A_7^{k,h})$ since $x_{1,k}[x_{2,e}, x_{3,h}] \equiv 0$ on $A_7^{k,h}$ but not in $A_8^{k,h}$.

The proof is now complete. □

The following table summarizes the contents of the previous proposition.

	A_1^p	A_1^3	A_2	A_3	$A_4^{i,j}$	$A_5^{g,i}$	$A_6^{k,h}$	$A_7^{k,h}$	$A_8^{k,h}$
A_1^p	$\times$	EXP	EXP	EXP	EXP	EXP	EXP	EXP	EXP
A_1^3	APG	$\times$	GR	GR	GR	CMP or f_8	GR	GR	$x_{g_1}^3$
A_2	APG	APG	$\times$	f_2	GR	CMP	EXP	EXP	EXP
A_3	APG	APG	f_1	$\times$	GR	CMP	EXP	EXP	EXP
$A_4^{i,j}$	APG	APG	GR	GR	$\times$	CMP	EXP	EXP	EXP
$A_5^{g,i}$	APG	APG	$[x_1, x_2]$	f_3 or f_4	f_4 or f_6	$\times$	EXP	EXP	EXP
$A_6^{k,h}$	APG	APG	$[x_1, x_2]$	$[x_{h_1}, x_{k_1}]$	$[x_e, x_{k_1}]$	CMP or f_4	$\times$	f_{13}	f_{15}
$A_7^{k,h}$	APG	APG	$[x_1, x_2]$	$[x_{h_2}, x_{k_2}]$	$[x_e, x_{k_2}]$	CMP or f_4	f_{11}	$\times$	f_{16}
$A_8^{k,h}$	APG	APG	$[x_1, x_2]$	f_4 or f_5	f_4 or f_7	CMP or f_4, f_9, f_{10}	f_{12}	f_{14}	$\times$

Let $i, j = 1, \ldots, 9$, $i \neq j$. In the box of $(i+1, j+1)$-position we explain why the algebra in the $(i+1)$-th row of the first column does not belong to the variety generated by the algebra in the $(j+1)$-th column of the first row.

APG : the algebra in the first row generates a variety of almost polynomial growth which cannot contain a proper subvariety of exponential growth, like the algebra in the first column.

EXP : the algebra in the first row has exponent less than the algebra in the first column.

GR : the algebras are graded by distinct non-isomorphic groups.

CMP : the algebra in the first column has more non-zero components than the algebra in the first row.

f : The polynomial in the box is an identity for the corresponding algebra in the first row but not for the corresponding algebra in the first column. We shall use the following notation for the polynomials inside the table:

$f_1 = [x_{1,g_1}, x_{2,g_1}]$

$f_2 = x_{1,g_1}x_{2,g_1} + x_{2,g_1}x_{1,g_1}$

$f_3 = [x_{1,g}, x_{2,g}]$

$f_4 = [x_{1,e}, x_{2,e}]$

$f_5 = [x_e, x_{g_3^2}]$

$f_6 = [x_e, x_g]$

$f_7 = [x_e, x_k]$

$f_8 = x_{1,g}x_{2,g}$

$f_9 = x_{1,k}x_{2,k}$

$f_{10} = \big[[x_1, x_2]^2, x_1\big]$

$f_{11} = [x_e, x_{h_2}]x_{k_2}$

$f_{12} = [x_e, x_h]x_k$

$f_{13} = x_{k_1}[x_e, x_{h_1}]$

$f_{14} = x_k[x_e, x_h]$

$f_{15} = x_{1,k_1}x_{2,k_1}x_{3,k_1}$

$f_{16} = x_{1,k_2}x_{2,k_2}x_{3,k_2}$

In the last theorem we finally give the characterization of the G-graded algebras with G-exponent equal to 2. We denote by $UT_2^G(F)$ the algebra of 2×2 upper-triangular matrices over F endowed with an elementary G-grading, by E the infinite dimensional Grassmann algebra with trivial grading and by $E^{\mathbb{Z}_2}$ the Grassmann algebra with natural $\mathbb{Z}_2$-grading.

Theorem 4.2. *A G-graded algebra A has $\exp^G(A) = 2$ if and only if $A_1^p, A_2, A_3, A_4^{i,j}, A_5^{g,i}, A_6^{k,h}, A_7^{k,h}, A_8^{k,h} \notin$ var$^G(A)$ and at least one algebra among $UT_2^G(F)$, E and $E^{\mathbb{Z}_2}$ belongs to var$^G(A)$.*

Proof. Let us suppose that $\exp^G(A) = 2$. Since $\exp^G(B) > 2$, $B \in \{A_1^p, A_2, A_3, A_4^{i,j}, A_5^{g,i}, A_6^{k,h}, A_7^{k,h}, A_8^{k,h}\}$, we get that such algebras do not belong to the variety $\mathrm{var}^G(A)$. By [**?**, Theorem 9], we get that at least one algebra among $UT_2^G(F)$, E and $E^{\mathbb{Z}_2}$ belongs to $\mathrm{var}^G(A)$ and we are done.

On the other hand, since $A_1^p, A_2, A_3, A_4^{i,j}, A_5^{g,i}, A_6^{k,h}, A_7^{k,h}, A_8^{k,h} \notin \mathrm{var}^G(A)$, we have that $\exp^G(A) \le 2$. Moreover, since at least one algebra among $UT_2^G(F)$, E and $E^{\mathbb{Z}_2}$ belongs to $\mathrm{var}^G(A)$, by [**?**, Theorem 9], we get that $\exp^G(A) > 1$ and the proof is complete. ☐

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