Analytically solvable Hamiltonian in invariant subspaces

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Abstract We study a generic time-dependent Jaynes–Cummings model, discovering universal features of its time evolution on a dynamical time scale which is here defined as an integral over time of the field-atom coupling strength modulus. Using the total excitation number as a constant of motion and the symmetry properties of the Hamiltonian, we decompose the infinite-dimensional Hilbert space of the system in a one-dimensional and an infinite number of two-dimensional dynamically invariant subspaces, in which the dynamical problem is solved using the Messina–Nakazato parameterization method. Therefore, we obtain a solution of the dynamics over the full Hilbert space. Its application to the evolution of the atomic population inversion over the dynamical time, under different initial conditions, highlights Rabi-like oscillations and revivals, which do not depend on the functional dependence of the Hamiltonian parameters on time and can be controlled by a crucial parameter in the solution. The universal dynamical features brought to light in this study assume particular significance in this quantum computing era, where general quantum simulators are of great use.

1 Introduction

Today's advances in science and technology aim to understand, use, and control quantum phenomena for a wide variety of applications that are expected to outperform current technology, including opportunities for opening new avenues in the processing of quantum information and in quantum computing [1–4]. A crucial prerequisite for such advances is our ability to control the dynamics of quantum systems (with particular interest for atoms interacting with radiation) [5–10], which, in turn, has further ignited interest in the evergreen goal of finding exact solutions for the time evolution of quantum models of radiation-matter systems [8]. In fact, the exact solutions of these analytical problems provide the analytic tools required to control the dynamics of the corresponding quantum systems, thus identifying the key control parameters and methods to be implemented in technological applications [9, 10].

In experiments, the desired control is often achieved by applying time-dependent external classical fields on the physical system of interest [8]. When the Hilbert space of the quantum system has a finite dimension (for example, this is the case for a spin subject to a time-dependent magnetic field [11]), the search for the exact matrix form of the time evolution operator leads to a non-autonomous system of coupled differential equations [12]. Unfortunately, there is no general procedure to solve in closed form even the structurally simple non-autonomous system of two linearly coupled homogeneous differential equations in normal form [13–15]. To address at least in part the dramatic impact of this physical–mathematical problem, it is sometimes convenient to resort to special parameterizations of the time evolution operator, exploiting constants of motion and, more generally, of symmetries that characterize the Hamiltonian model. It is well known that the concepts of symmetry group of a physical system and of dynamic symmetry associated with its Hamiltonian operator date back to the beginning of the development of Quantum Theory [16–18] and are of paramount importance to tackle the problem of its quantum dynamics.

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Quantum dynamics can also be viewed as a theory for the representation of the symmetry groups of physical systems [19]. This means that the Hilbert space of a system can be considered as nothing more than the representation space of the symmetry group of the given Hamiltonian model. This important interpretation of quantum theory [20, 21] can help us describe the quantum dynamics of a time-dependent Hamiltonian operator [22–26]. The symmetry group of the time-independent Hamiltonian operator of a quantum system plays a significant role in classifying its stationary states, describing its transition probabilities, and determining the pertinent selection rules [27–30].

Within this context, it could be advantageous to exploit the properties of the symmetry group of a given Hamiltonian operator to devise a systematic procedure to determine the time evolution of the quantum system. Refs. [31, 32] demonstrated, e.g., that the time evolution operator of an N-level quantum system whose Hamiltonian model has the dynamical symmetry $SU(2)$ can be obtained by exploiting different unitary representations of $SU(2)$. Later, this analytical treatment was extended to Hamiltonian operators with $SU(1,1)$ dynamical symmetry [33].

Determining the evolution operator for many models of stationary quantum systems is, in general, relatively simpler than for nonstationary quantum systems. The mathematical reason is that the corresponding Liouville–Cauchy problem determining the evolution operator takes the form of a system of ordinary differential equations with time-independent coefficients, which can be treated by applying standard methods. On the other hand, for nonstationary quantum systems, the procedure for finding exact solutions for the evolution operators is usually complicated and often requires the use of approximate methods. Nevertheless, in [14] a procedure was proposed to generate analytical solutions for the evolution operator of the general time-dependent Hamiltonian model with $SU(2)$ dynamical symmetry describing quantum two-level systems. Hereafter, we refer to this procedure as the MN ansatz. Different ways to parameterize the evolution operator matrix based on the MN ansatz were later derived and applied [34]. The approach was also extended to time-dependent Hamiltonians with an $SU(1,1)$ dynamical symmetry [35], arriving at solutions for the evolution operator parameterized in terms of hyperbolic functions. By exploiting the symmetry of the Hamiltonian in the parameterization, it is possible to explore different physical scenarios for two-level systems. In fact, the MN ansatz discloses all its potential when it is used, in combination with specific symmetry properties of Hamiltonian models, to investigate the dynamics of systems of two bilinearly-coupled spins (also greater than 1/2) [36–49]. Similarly, the approach allows to highlight collective properties of systems of N spins 1/2 under the action of local time-dependent magnetic fields, in which the internal couplings are dominated by trilinear terms [50].

The idea underlying the solution of all models mentioned is that the conserved physical quantities associated with their symmetry properties are the generators of dynamically invariant subspaces [51–53], where the restricted Hamiltonians of the interacting spins exhibit an $SU(2)$ symmetry which allows the application of the MN ansatz. Here we focus on a time-dependent Jaynes–Cummings (JC) model describing a two-level atom interacting with one quantum cavity mode, in the rotating wave approximation. Based on the symmetry properties of the Hamiltonian model, we present a systematic procedure to classify the corresponding set of dynamically invariant subspaces. As in the time-independent JC model, the quantum states of this bipartite system live in an infinite-dimensional Hilbert space. We assume that the mode frequency, the Bohr frequency of the two-level atom, and the coupling constant are all generic functions of time.

Time-dependent JC models have been considered in many studies [30, 54–57], where different approaches have been used to obtain exact solutions for their time evolutions. For example, Ref. [54] studies the two-photon and one-photon JC models with time-dependent atom-field coupling. Ref. [55] describes how the time-dependent JC model can be analytically solved at resonance for an arbitrary interaction coupling. Ref. [57] reports an algebraic treatment that highlights non-linear effects in the time-dependent JC model.

In this study, we present a procedure to obtain the exact solution for the evolution operator of the complete system (in its infinite-dimensional Hilbert space) in terms of an infinite number of evolution operators each describing a fictitious system made of a qubit subject to a magnetic field. First, the Hilbert space is split into infinitely many dynamically invariant subspaces. Then, we demonstrate that it is possible to determine the overall dynamics of the system by applying the MN ansatz to a single subspace only.

In sect. 2, we introduce the time-dependent JC model under investigation and demonstrate that its Hilbert space can be partitioned into a one-dimensional and infinitely many two-dimensional dynamically-invariant subspaces. In sect. 3, we present the general procedure to determine the evolution operator of a general system whose Hilbert space is a direct sum of infinitely-many finite-dimensional dynamically-invariant subspaces. In sect. 4, we use the approach to explore the quantum dynamics of a generic time-dependent JC model, generating an exact solution for the evolution operator of the whole system. A concrete application is given in sect. 5. Concluding remarks and discussion of future prospects follow in sect. 6.

2 Invariant subspaces in the time-dependent Jaynes–Cummings model

We begin our analysis by examining some general aspects of the conditions for the exact solution of the time-dependent JC model described by the Hamiltonian operator (hereafter we assume $\hbar = 1$)

$$\mathrm{H}_{\mathrm{JC}}(t) = \varpi(t)a^{\dagger}a + \omega_0(t)\mathrm{J}_z + l(t)a\mathrm{J}_+ + l^*(t)a^{\dagger}\mathrm{J}_-, \tag{1}$$

In (1), a and $a^\dagger$ are the bosonic annihilation and creation operators, respectively, and $\mathrm{J}_\pm = \mathrm{J}_x \pm i\mathrm{J}_y$ are two of the spherical components of the spin angular momentum operator $\mathbf{J} \equiv (\mathrm{J}_x, \mathrm{J}_y, \mathrm{J}_z)$. The Hamiltonian model (1) is a sufficiently regular function of time for any of our analytical purposes, as it represents physical systems which in principle can be implemented using suitable experimental setups.

The Hamiltonian operator $\mathrm{H}_{\mathrm{JC}}(t)$ describes a time-dependent JC model with a rotating wave coupling between a $(2j+1)$-level system (j denotes the main quantum number associated with the spin angular momentum operator) and a bosonic mode. It is easy to verify that the time-independent total excitation number operator

$$\mathrm{N}_{\mathrm{JC}} = a^\dagger a + \mathrm{J}_z + j\mathrm{I}, \tag{2}$$

(I is the identity operator, and it may be omitted following common shorthand notation), commutes with $\mathrm{H}_{\mathrm{JC}}(t)$ at any time. This means that N_{JC} is a constant of the motion, that is, while the Hamiltonian depends on time, the Schrödinger equation gives

$$\frac{d}{dt}\langle\psi(t)|\mathrm{N}_{\mathrm{JC}}|\psi(t)\rangle = 0, \quad \forall\ |\psi(t)\rangle \in \mathcal{H}, \tag{3}$$

at any time, whence

$$\langle\psi(t)|\mathrm{N}_{\mathrm{JC}}|\psi(t)\rangle = \langle\psi(0)|\mathrm{N}_{\mathrm{JC}}|\psi(0)\rangle, \quad \forall\ |\psi(t)\rangle \in \mathcal{H}, \forall t\ \geq 0. \tag{4}$$

In Eqs. (3) and (4), $\mathcal{H}$ is the Hilbert space of the composite system. The state vector of the system evolves over time as

$$|\psi(t)\rangle = \mathrm{U}(t)|\psi(0)\rangle. \tag{5}$$

$\mathrm{U}(t)$ is the evolution operator, which is determined by solving the Cauchy problem

$$i\frac{d\,\mathrm{U}(t)}{dt} = \mathrm{H}_{\mathrm{JC}}(t)\,\mathrm{U}(t), \quad \mathrm{U}(0) = \mathrm{I}. \tag{6}$$

The set $\mathcal{N}_0$ of all nonnegative integers contains all eigenvalues of N_{JC}. Let us denote by N a generic eigenvalue of N_{JC}. While $N = 0$ corresponds to a singlet, all other eigenvalues exhibit degeneracies. Next, we identify the number of two-dimensional dynamically invariant subspaces of N_{JC}.

The total excitation number is $N = n + (-j + r) + j = n + r$, where $n = 0,\ 1,\ ...,\ r = 0,\ 1,\ ...,\ 2j$, and j is a positive half-integer. The eigenvalue of j_z is here written as $(-j + r)$. When $j = 1/2$, it is $r = 0,\ 1$ and all invariant subspaces determined by N_{JC} with $N \geq 1$ are two-dimensional, since each value of N can be realized in two ways, $N = n + r = N + 0 = (N - 1) + 1$, which correspond to the spin-down and spin-up states of the atom. When $j > 1/2$ the dimension of all subspaces with $N \geq 2j$ is $2j + 1$, while for $N < 2j$ their dimension grows from 1 for $N = 0$ to $2j$ for $N = 2j$. Thus, for $j > 1/2$, there is only one two-dimensional dynamically invariant subspace of N_{JC} in $\mathcal{H}$ corresponding to $N = 1$. The significance of such subspaces in the solution of the system dynamics will be examined in the next section.

3 Evolution operator in a dynamically invariant subspace

A symmetry property of a quantum system can help construct exact analytical solutions for its time evolution operator. To this end, it is necessary to grasp how the conserved (measurable) quantity associated with the given symmetry restricts the dynamical problem at hand. With reference to the present investigation, we now consider a Hamiltonian model $\mathrm{H}(t)$ (in particular, this Hamiltonian can be independent of time) and discuss in detail how to benefit from the knowledge of the structure of the invariant subspaces resulting from the symmetry properties of $\mathrm{H}(t)$.

Let us assume that a Hermitian operator $\mathrm{A} = \mathrm{A}^\dagger$ associated with a certain symmetry property is a constant of the motion for the system described by the Hamiltonian $\mathrm{H}(t)$. $\mathcal{S}_n$ be the *invariant* subspace spanned by all eigenstates of A associated with the n-th eigenvalue, which, in principle, can also be degenerate. Accordingly, we introduce the *time-independent* projection operator $\mathrm{P}_n = \mathrm{P}_n^2$ onto $\mathcal{S}_n$. Since $\sum_n \mathrm{P}_n = \mathrm{I}$, we obtain

$$\mathrm{H}(t) = \left(\sum_n \mathrm{P}_n\right)\mathrm{H}(t)\left(\sum_{n'} \mathrm{P}_{n'}\right) = \sum_{n,n'} \mathrm{P}_n\mathrm{H}(t)\mathrm{P}_{n'} = \sum_n \mathrm{P}_n\mathrm{H}(t)\mathrm{P}_n = \sum_n \mathrm{H}_n(t), \tag{7}$$

where we used the fact that $[\mathrm{P}_n,\ \mathrm{H}(t)] = 0\ \ \forall\, n$. In Eq. (7), we defined the Hamiltonian $\mathrm{H}_n(t) = \mathrm{P}_n\mathrm{H}(t)\mathrm{P}_n$, which acts on subspace $\mathcal{S}_n$. $\mathrm{H}(t)$ is the generator of the evolution operator of the system. In fact, the dynamics of $\mathrm{U}(t)$ is ruled by the Schrödinger equation

$$i\frac{d\,\mathrm{U}(t)}{dt} = \mathrm{H}(t)\,\mathrm{U}(t), \quad \mathrm{U}(0) = \mathrm{I}. \tag{8}$$

Note that here $\mathrm{U}(t)$ is the evolution operator acting on the total Hilbert space $\mathcal{H}$ of the system. Remarkably, it is possible to find the system evolution within each subspace $\mathcal{S}_n$. To this end, we project Eq. (8) as follows:

$$i\frac{d\,\mathrm{U}_n(t)}{dt} = \mathrm{P}_n\mathrm{H}(t)\mathrm{U}(t)\mathrm{P}_n = \mathrm{P}_n\mathrm{H}(t)\mathrm{P}_n^2\mathrm{U}(t)\mathrm{P}_n = \mathrm{H}_n\mathrm{U}_n, \tag{9}$$

where $\mathrm{U}_n(t) \equiv \mathrm{P}_n\mathrm{U}(t)\mathrm{P}_n$ and $\mathrm{U}_n(0) = \mathrm{I}_n$, namely, the identity operator in $\mathcal{S}_n$. Each operator $\mathrm{U}_n(t)$ acts as the evolution operator in $\mathcal{S}_n$ and as the null operator outside this subspace. Therefore, similarly to the Hamiltonian, the evolution operator $\mathrm{U}(t)$ can be expressed as the summation $\mathrm{U}(t) = \sum_n \mathrm{U}_n(t)$. In fact, the evolution operator $\mathrm{U}(t)$, which is constructed from the Hamiltonian $\mathrm{H}(t)$, also commutes with the projection P_n. Therefore, all off-diagonal terms of the evolution operator in the full Hilberst space are zero, and $\mathrm{U}(t) = \sum_{n,n'} \mathrm{P}_n\mathrm{U}(t)\mathrm{P}_{n'} = \sum_n \mathrm{P}_n\mathrm{U}(t)\mathrm{P}_n = \sum_n \mathrm{U}_n(t)$.

At this point, we can state that, if a physical system is described by a time-dependent Hamiltonian model with an invariant two-dimensional subspace, the time evolution within this subspace can be reduced to the relatively simpler one of a two-level system described by a time-dependent Hamiltonian acting only in such a subspace, regardless of the knowledge of other possibly existing invariant subspaces and whatever their dimension. When the symmetry properties of the Hamiltonian $\mathrm{H}(t)$ of a given system enable a partition of its Hilbert space into (even infinitely many) two-dimensional dynamically invariant subspaces in which the restricted dynamical problem is exactly solvable, we can rebuild the total evolution operator and thus derive the time evolution of the system with Hamiltonian $\mathrm{H}(t)$ from any initial condition.

4 The evolution operator of time-dependent JC models.

We now apply the decomposition scheme presented in the previous section to a time-dependent JC model $\mathrm{H}_{\mathrm{JC}}(t)$ that describes a two-level atom coupled to a cavity quantum field mode. For $s = 1/2$, the Hamiltonian operator $\mathrm{H}_{\mathrm{JC}}(t)$ given in Eq. (1) reads

$$\mathrm{H}_{\mathrm{JC}}(t) = \varpi(t)a^\dagger a + \frac{\omega_0(t)}{2}\sigma_z + l(t)a\sigma_+ + l^*(t)a^\dagger\sigma_-, \tag{10}$$

where $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ are ladder operators constructed from Pauli operators, $\varpi(t)$ and $\omega_0(t)$ are analytical real functions of time, and $l(t) = |l(t)|\exp[i\phi_l(t)]$ is a generally complex coupling strength, which is a function of time. The interaction term in (10) only couples states $|g,\, n+1\rangle$ and $|e,\, n\rangle$. Here, $|g\rangle$ and $|e\rangle$ denote the ground and excited states of the atom, respectively, while $|n\rangle$ represents a Fock state of the cavity quantum field. Differently from the time-independent JC model, the time dependence of $\varpi(t)$ and $\omega_0(t)$ opens the question whether the conclusions reached using the model (10), based on the rotating wave approximation, are experimentally realistic over any interval of time. The time dependence of $\varpi(t)$ can describe a parametric quantum excitation, while $\omega_0(t)$ expresses the time modulation of the atomic two level energies, which can be used, e.g., to investigate the dynamics of the spontaneous emission by a giant two-level atom [58].

In the following, we focus on the solution of the time-dependent Schrödinger equation in the full Hilbert space of the system with Hamiltonian (10), by exploiting the analysis in the previous section and placing no restrictions on the time variations of the model parameters. In the two-dimensional subspace $\mathcal{S}_n$, ($n = 0,\ 1,\ 2,\ 3,\ ...$) generated by the basis $\mathcal{B}_n = \{|e,\, n\rangle; |g,\, n+1\rangle\}$, $\mathrm{H}_{\mathrm{JC}}(t)$ is dynamically invariant and the matrix representation of its restriction $\mathrm{H}_n(t)$ reads

$$\mathrm{H}_n(t) = \begin{pmatrix} \Omega_+^{(n)}(t) & \omega_n(t) \\ \omega_n^*(t) & \Omega_-^{(n)}(t) \end{pmatrix}, \tag{11}$$

with $\Omega_+^{(n)}(t) = n\varpi(t) + \omega_0(t)/2$, $\Omega_-^{(n)}(t) = (n+1)\varpi(t) - \omega_0(t)/2$ and $\omega_n(t) = \sqrt{n+1}\ |l(t)|\ \exp[i\phi_l(t)]$.

Since $\mathrm{H}_n(t)$ is not a traceless matrix, the corresponding evolution operator $\mathrm{U}_n(t)$ must be written in terms of the complex Cayley–Klein parameters in the general form

$$\mathrm{U}_n(t) = \begin{pmatrix} \alpha_n(t) & \beta_n(t) \\ \gamma_n(t) & \delta_n(t) \end{pmatrix}, \tag{12}$$

with $\alpha_n\alpha_n^* + \beta_n\beta_n^* = \gamma_n\gamma_n^* + \delta_n\delta_n^* = 1$ and $\alpha_n\gamma_n^* + \beta_n\delta_n^* = 0$. However, the form of $\mathrm{U}_n(t)$ can be remarkably restricted and simplified after decomposing the Hamiltonian (11) into the sum of the time-dependent diagonal operator $\mathrm{h}_d(t) = (n+1/2)\varpi(t)\mathrm{I}_2$ and the traceless Hamiltonian operator

$$\mathrm{h}_n(t) = \begin{pmatrix} \Omega(t) & \omega_n(t) \\ \omega_n^*(t) & -\Omega(t) \end{pmatrix}, \tag{13}$$

where $\Omega(t) = (\omega_0(t) - \varpi(t))/2$ and therefore the diagonal elements do not depend on n. In fact, the evolution operator (12) can be written in the form

$$\mathrm{U}_n(t) = c_n(t)\mathrm{u}_n(t), \tag{14}$$

provided that

$$c_n(t) = \exp[-i\varphi_n(t)], \quad \varphi_n(t) = (2n+1)\varphi_0(t), \quad \varphi_0(t) = \frac{1}{2}\int_0^t \varpi(t')dt'. \tag{15}$$

The function $c_n(t)$ is the Cayley–Klein parameter corresponding to the diagonal evolution operator associated with the time-dependent Hamiltonian $\mathrm{h}_d(t)$, which possesses a $U(1) \subseteq SU(2)$ dynamical symmetry. $\mathrm{u}_n(t)$ is instead the time evolution operator

corresponding to the traceless Hamiltonian operator $\mathrm{h}_n(t)$. Since $\mathrm{h}_n(t)$ describes a system with $SU(2)$ dynamical symmetry, the evolution operator for the time-dependent JC model system is finally written as

$$\mathrm{U}_n(t) = c_n(t)\begin{pmatrix} a_n(t) & b_n(t) \\ -b_n^*(t) & a_n^*(t) \end{pmatrix}. \tag{16}$$

It is worth noting that $c_n(t)$ is a functional of the frequency $\varpi(t)$ of the field mode, scaled by $(n + 1/2)$ (hence, large values of n entail fast oscillations in the dynamic evolution within subspace $\mathcal{S}_n$), and introduces only a global time-dependent phase factor in the evolved state belonging to $\mathcal{S}_n$.

Now we focus on the two-dimensional dynamical problem in each given subspace $\mathcal{S}_n$, leaving out the global phase factor of the system state. Clearly, these phase factors must be taken into account to describe the evolution of the system starting from an initial state which does not belong to a single dynamically invariant subspace. $a_n(t) \equiv |a_n(t)|\exp\{i\phi_{a_n}(t)\}$ and $b_n(t) \equiv |b_n(t)|\exp\{i\phi_{b_n}(t)\}$ satisfy the condition $|a_n(t)|^2 + |b_n(t)|^2 = 1$ and must obey the Cauchy-Liouville problem

$$i\frac{d\mathrm{u}_n(t)}{dt} = \mathrm{h}_n(t)\mathrm{u}_n(t), \quad \mathrm{u}_n(0) = \mathrm{I}_2. \tag{17}$$

Eq. (17) implies that the parameters $a_n(t)$ and $b_n(t)$ satisfy the linear system of non-autonomous first-order differential equations

$$\mathcal{L}_n = \begin{cases} \dot{a}_n(t) = -i\,\Omega(t)a_n(t) + i\sqrt{n+1}\,l(t)b_n^*(t), \\ \dot{b}_n(t) = -i\sqrt{n+1}\,l(t)a_n^*(t) - i\,\Omega(t)b_n(t), \end{cases} \tag{18}$$

with the initial value condition $a_n(0) = 1$ and $b_n(0) = 0$. The longitudinal field $\Omega(t)$ is the same for all invariant subspaces of the system Hilbert space, as it does not depend on n, while the n-dependent transverse field is $\omega_n(t) = \sqrt{n+1}l(t)$.

Solving the system of Eq. (18) with arbitrary time-dependent coefficients $\Omega(t)$ and $l(t)$ is clearly not an easy task. Ref. [14] presents a systematic procedure for unveiling analytically solvable Hamiltonians describing quantum two-level systems controlled by classical fields. While we refer to [35, 37, 38] for a detailed analysis of the method and its previous applications, we show here that this procedure can also be successfully applied to treat time-dependent JC models. The key idea of the method consists in finding parametric forms of the complex quantities $a_n(t)$ and $b_n(t)$ of $\mathrm{u}_n(t)$ that identically satisfy the system $\mathcal{L}_n$. The parameterization is based on the introduction of the real function

$$\Phi_n(\mu_n; t) = \sqrt{(n+1)(\mu_n^2+1)}\int_0^t |l(t')|dt', \tag{19}$$

with the real parameter $\mu_n \in [0, \infty)$. Then, using the procedure in [14], the moduli of $a_n(\mu_n; t)$ and $b_n(\mu_n; t)$ (we modified the notation for these quantities to stress their parametric dependencies on μ_n) are written as

$$|a_n(\mu_n; t)| = \sqrt{\frac{\mu_n^2 + \cos^2\Phi_n(\mu_n; t)}{1+\mu_n^2}}, \tag{20}$$

$$|b_n(\mu_n; t)| = \frac{1}{\sqrt{1+\mu_n^2}}\sqrt{\sin^2\Phi_n(\mu_n; t)}, \tag{21}$$

while their respective phases $\phi_{a_n}(\mu_n; t)$ and $\phi_{b_n}(\mu_n; t)$ are given by

$$\phi_{a_n}(\mu_n; t) = \frac{\phi_l(t)}{2} - \tan^{-1}\left[\frac{\mu_n}{\sqrt{\mu_n^2+1}}\tan\Phi_n(\mu_n; t)\right], \tag{22}$$

$$\phi_{b_n}(\mu_n; t) = \frac{\phi_l(t)}{2} - \mathrm{sgn}[\sin\Phi_n(\mu_n; t)]\frac{\pi}{2}, \tag{23}$$

in which, without loss of generality, we set $\phi_l(0) = 0$. Each invariant subspace $\mathcal{S}_n$ has its own solutions (20)–(23) parameterized in terms of the function $\Phi_n(\mu_n, t)$. However, the solutions in different subspaces are not entirely independent, because (20)–(23) satisfy the Cauchy-Liouville system $\mathcal{L}_n$ whatever n provided that the n-dependent condition

$$\mu_n\sqrt{n+1}|l(t)| = \Omega(t) + \frac{\dot{\phi}_l(t)}{2}, \tag{24}$$

is satisfied. This condition can be verified, e.g., by adding the first Eq. (18) multiplied by $a_n^*(t)$ to the second Eq. (18) multiplied by $b_n^*(t)$, using the condition $|a_n(\mu_n; t)|^2 + |b_n(\mu_n; t)|^2 = 1$, and inserting the expression (20) for $|a_n(\mu_n; t)|$ into the resulting equation. Since the right-hand side of Eq. (24) does not depend on n, the dependence of the parameter μ_n on n is necessarily given by

$$\mu_n = (n+1)^{-1/2}\,\mu_0, \quad n \geq 0, \tag{25}$$

where μ_0 is the real number that satisfies condition (24) for $n = 0$. Then, Eq. (19) yields

$$\Phi_n(\mu_0; t) = \sqrt{\frac{n+\mu_0^2+1}{\mu_0^2+1}}\Phi_0(\mu_0; t), \quad n \geq 0, \tag{26}$$

where

$$\Phi_0(\mu_0;t)=\sqrt{\mu_0^2+1}\int_0^t|l(t')|dt'. \tag{27}$$

Therefore, for an arbitrary time control through the time-dependent fields $\Omega(t)$ and $l(t)$, the solution of the Cauchy-Liouville problem (18) for $n=0$ determines the dynamics of the JC model in all invariant subspaces $\mathcal{S}_n$ via the constraint (25).

It is worth emphasizing that, whatever the value of $\Omega(t)$, the solution of the dynamical problem on the complete Hilbert space (namely, the solutions in all invariant subspaces) depends on our ability to find the solution to the Cauchy problem $\mathcal{L}_0$, which provides a straightforward mathematical tool to build the evolution operator $\mathrm{U}(t)$ generated by $\mathrm{H_{JC}}(t)$. We therefore devised an approach to describe exactly the quantum dynamics of the time-dependent JC model for arbitrary initial quantum states. In the next section, we present a concrete example in which the full dynamics of the system is described in terms of the solution obtained for one invariant subspace.

5 Universal dependence of system evolution on phase $\phi_l(\kappa)$

Now we show how the parameterization (20)–(23) can be used to determine the time evolution operator and to grasp universal aspects of the system time evolution that do not depend on the specific expression for the time-dependent field-atom coupling strength. To this end, we define a dimensionless dynamical time scale κ as

$$\kappa(t)=\int_0^t\big|l(t')\big|dt', \tag{28}$$

for an arbitrary coupling $l(t)$. Condition (24) implies the dynamic constraint

$$\mu_0\kappa=\int_0^t\Omega(t')\,dt'+\frac{\phi_l(t)}{2}, \tag{29}$$

which must be verified at any time. All functions $\Phi_n(\mu_0,\kappa)$ depend linearly on the dynamical time scale:

$$\Phi_n(\mu_0;\kappa)=\sqrt{\mu_0^2+n+1}\,\kappa,\quad n\geq 0. \tag{30}$$

For the sake of simplicity, the quantities Φ_n are denoted by the same symbols in Eq. (30), although their functional dependence on κ is different from that on t. The same choice is made for the notation of the other functions.

In terms of κ and using Eq. (25), the moduli of the quantities a_n and b_n, which determine the time evolution operator, are written

$$|a_n(\mu_0;\kappa)|=\sqrt{\frac{\mu_0^2+(n+1)\cos^2\left(\sqrt{\mu_0^2+n+1}\,\kappa\right)}{\mu_0^2+n+1}}, \tag{31}$$

$$|b_n(\mu_0;\kappa)|=\sqrt{\frac{(n+1)\sin^2\left(\sqrt{\mu_0^2+n+1}\,\kappa\right)}{\mu_0^2+n+1}}, \tag{32}$$

where μ_0 and κ are related by the field and atomic terms of the Hamiltonian model through Eq. (29). The phases of a_n and b_n also depend on ϕ_l, which in turn depends on κ because of Eq. (29). This dependence becomes particularly simple when the resonance condition $\omega_0(t)=\varpi(t)$ is satisfied and therefore $\Omega(t)=0$. More precisely, in this case, for any functional form of $\omega_0(t)$ or $\varpi(t)$, condition (29) yields a linear dependence of ϕ_l on κ:

$$\phi_l(\kappa)=2\mu_0\kappa. \tag{33}$$

Basically, Eq. (33) expresses condition (24) for the validity of (20)–(23) in terms of κ, whatever $l(t)$, and μ_0 determines the change rate of the phase of the field-atom coupling strength as a function of κ. Using (25), (30), and (33), the phases of a_n and b_n read

$$\phi_{a_n}(\mu_0;\kappa)=\mu_0\kappa-\tan^{-1}\left[\frac{\mu_0}{\sqrt{\mu_0^2+n+1}}\tan\left(\sqrt{\mu_0^2+n+1}\,\kappa\right)\right], \tag{34}$$

$$\phi_{b_n}(\mu_0;\kappa)=\mu_0\kappa-\mathrm{sgn}\big[\sin[\sqrt{\mu_0^2+n+1}\,\kappa]\big]\frac{\pi}{2}, \tag{35}$$

thus only depending on time through κ. Once the resonance condition is fulfilled, Eq. (33) still corresponds to a class of analytical models, since the time dependence of $l(t)$ was not specified.

In the general off-resonance case, while Eqs. (31) and (32) still hold (as they were derived without using the resonance condition), to determine the dependence of the phases ϕ_{a_n} and ϕ_{b_n} on κ it is necessary to obtain t as a function of κ from Eq. (28), substitute it into the function of time Ω in Eq. (29), and hence find the dependence of $\phi_l(t)$ on κ.

At this point, we have all necessary tools to describe the quantum dynamics of the atom-field system. We consider an initial condition (in the full Hilbert space of the system) in which the electromagnetic field is in one of the coherent states $|\pm\alpha\rangle$ and the atom can be in either its ground state $|g\rangle$ or its excited state $|e\rangle$ [59]:

$$|\Psi_{r,\pm\alpha}(0)\rangle \equiv |r,\pm\alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\pm\alpha|^2}\sum_{n=0}^{\infty}\frac{(\pm\alpha)^n}{\sqrt{n!}}|r,n\rangle, \quad r=g,e. \tag{36}$$

The quantum states evolve in time as follows:

$$\left|\Psi_{g,\pm\alpha}(t)\right\rangle = \mathrm{e}^{-\frac{1}{2}|\pm\alpha|^2}\left[\mathrm{e}^{i\theta_0(t)}|g,0\rangle+\sum_{n=0}^{\infty}\frac{(\pm\alpha)^{n+1}\mathrm{e}^{-i\varphi_n(t)}}{\sqrt{(n+1)!}}\left[b_n(t)|e,n\rangle+a_n^*(t)|g,n+1\rangle\right]\right], \tag{37}$$

$$\left|\Psi_{e,\pm\alpha}(t)\right\rangle = \mathrm{e}^{-\frac{1}{2}|\pm\alpha|^2}\sum_{n=0}^{\infty}\frac{(\pm\alpha)^{n}}{\sqrt{n!}}\mathrm{e}^{-i\varphi_n(t)}\left[a_n(t)|e,n\rangle-b_n^*(t)|g,n+1\rangle\right], \tag{38}$$

where the phase

$$\theta_0(t) = \frac{1}{2}\int_0^t \omega_0(t')dt', \tag{39}$$

is easily obtained from the Schrödinger equation considering that

$$\mathrm{H_{JC}}(t)|g,0\rangle = \frac{\omega_0(t)}{2}\sigma_z|g,0\rangle = -\frac{\omega_0(t)}{2}|g,0\rangle. \tag{40}$$

Even under resonance conditions, where $a_n(t)$ and $b_n(t)$ can be written as functions of κ, the quantum states also depend explicitly on t through the phases $\varphi_n(t)$, which need to be determined from the time-dependent field frequency $\varpi(t)$ appearing in the given Hamiltonian model. On the contrary, and irrespective of whether the resonance condition is satisfied or not, the expression for the atomic population inversion (which describes the difference between the occupation probabilities of the excited and ground atomic states) depends on time exclusively through κ, as it only depends on the absolute values of $a_n(\mu_0;\kappa)$ and $b_n(\mu_0;\kappa)$. In fact, the population inversions for the system starting from states $|g,\pm\alpha\rangle$ and $|e,\pm\alpha\rangle$ are respectively given by

$$\mathcal{D}_{g,\pm\alpha}(\mu_0,\kappa) = \langle\Psi_{g,\pm\alpha}(t)|\sigma_z|\Psi_{g,\pm\alpha}(t)\rangle = -\mathrm{e}^{-\bar{n}}\left(1+\sum_{n=0}^{\infty}\frac{\bar{n}^{n+1}}{(n+1)!}\frac{\mu_0^2+(n+1)\cos\left(2\sqrt{\mu_0^2+n+1}\,\kappa\right)}{\mu_0^2+n+1}\right), \tag{41}$$

and

$$\mathcal{D}_{e,\pm\alpha}(\mu_0,\kappa) = \langle\Psi_{e,\pm\alpha}(t)|\sigma_z|\Psi_{e,\pm\alpha}(t)\rangle = \mathrm{e}^{-\bar{n}}\sum_{n=0}^{\infty}\frac{\bar{n}^{n}}{n!}\frac{\mu_0^2+(n+1)\cos\left(2\sqrt{\mu_0^2+n+1}\,\kappa\right)}{\mu_0^2+n+1}, \tag{42}$$

where we inserted the expressions (37) and (38) for the system state, respectively. In the expressions (41) and (42) for the population inversions there is no explicit dependence on t (the dependence on t only appears through κ) and $\bar{n} = |\alpha|^2$ is the mean photon number in the initial state of the field. μ_0 emerges from (41) and (42) as the crucial parameter determining the evolution of the atomic population inversion in the dynamic time scale κ starting from one of the states (36).

Eqs. (41) and (42) show that the population inversion in a generic time-dependent Jaynes-Cumming model is a universal function of the here defined dynamical time scale κ. That is, any choice of the time-dependent parameters in the Hamiltonian leads to the same functions of κ in Eqs. (41) and (42). In addition, insofar as different field-atom coupling strengths lead to the same κ (for example, if they have the same moduli but their phases have a different time evolution), the corresponding population inversions evolve in the same way over time. On a coarse-grained timescale, this is also true for coupling strengths with different variations over shorter timescales but the same instantaneous average value on the coarse-grained scale.

Applications of Eqs. (41) and (42) are shown in Figs. 1, 2 and 3, which were generated by using a mean photon number $\bar{n} = 10$ and truncating the series (41) and (42) at Fock state $n = 500$. The evolution of the atomic population inversion $\mathcal{D}_{g,\pm\alpha}(\mu_0,\kappa)$ when the atom is initially prepared in its ground state is shown in Fig. 1. The red line was obtained for $\mu_0 = 0$ and is characterized by Rabi oscillations of the population inversion around zero.

In the resonance case, $\mu_0 = 0$ also implies $\phi_l(\kappa) = 0$ according to Eq. (33). Moreover, using Equations (28) and (33) to express (18) in terms of κ, it is easy to see that, at resonance, $\mu_0 = 0$ corresponds to the case in which the Liouville problem in $\mathcal{S}_n$ is defined by a set of differential equations with constant coefficients and can thus be solved through standard procedures. The coupling

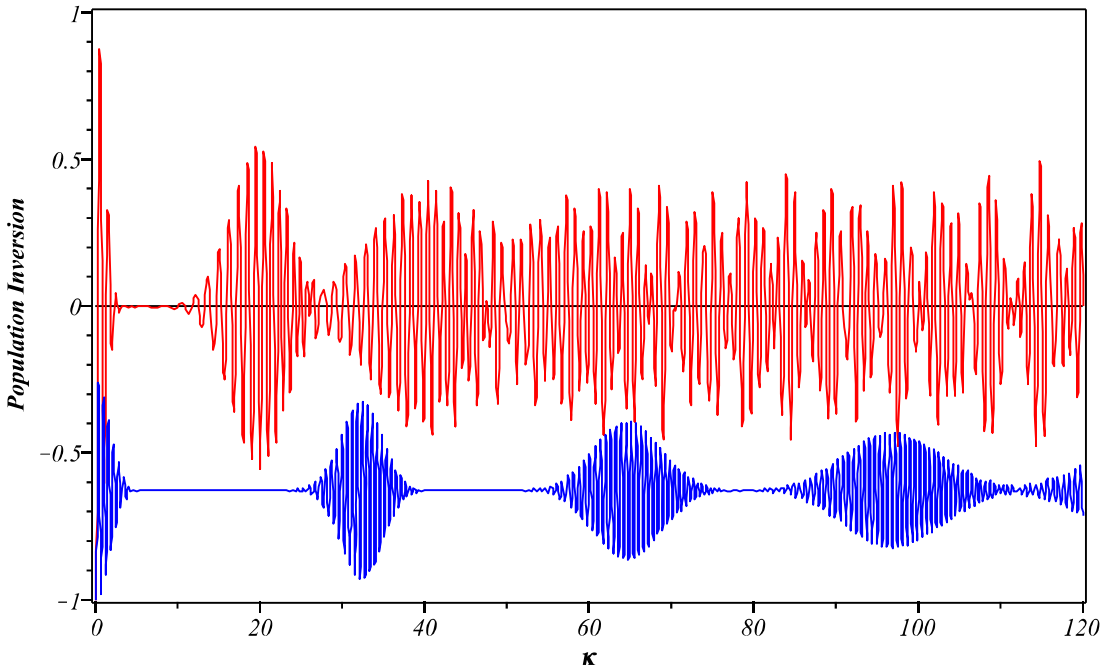

Fig. 1 Atomic population inversion as a function of the dimensionless dynamical time scale κ, for the atom initially prepared in its ground state. Both lines correspond to a mean photon number $\bar{n} = 10$. The red line corresponds to $\mu_0 = 0$, while the blue line is obtained for $\mu_0 = 4$

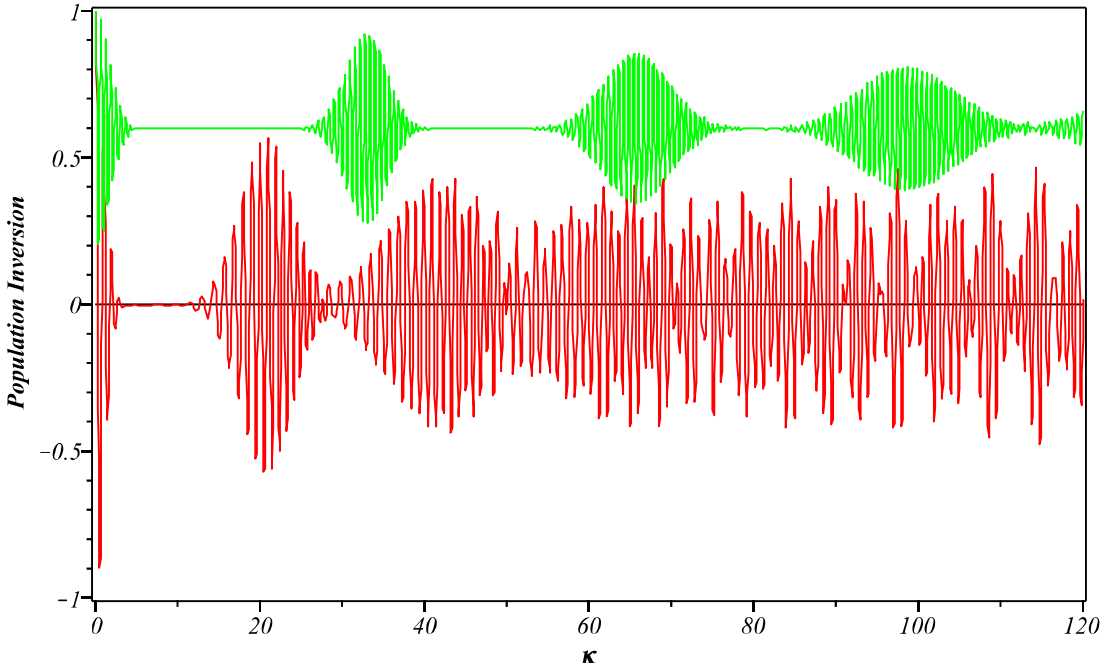

Fig. 2 Atomic population inversion as a function of the dimensionless dynamical time scale κ, for the atom initially prepared in its excited state. The mean photon number used is $\bar{n} = 10$. The red line is obtained for $\mu_0 = 0$, while the green line corresponds to $\mu_0 = 4$

constant is a real number and the dynamical behavior over the κ scale is quite similar to that of the resonant time-independent JC model. Therefore, the envelope of the sinusoidal Rabi oscillations collapses to zero, followed by a revival at a later times which is a pure quantum phenomenon.

Returning to the general situation (that is, generally not at resonance and for nonzero μ_0), the occupation probability difference $\mathcal{D}_{e,\pm\alpha}(\mu_0, \kappa)$ for the case in which the atom is initially prepared in its excited state evolves as in Fig. 2. Figure 3 compares the evolutions of the atomic population inversion for both initial conditions of the atomic state and different values of μ_0. As the value of μ_0 increases, the atomic population inversion evolves around an increasing non-zero value, remaining closer to its initial value. For example, when the atom is initially prepared in its ground state, for larger μ_0 the difference in the occupation probabilities of the excited and ground states remains closer to -1 during the system evolution, and the revivals become sparser and characterized by Rabi oscillations of smaller amplitude. In other words, the state of the atom is less influenced by the interaction with the field for larger μ_0. These results can be explained by rewriting Eqs. (41) and (42) as follows:

$$\begin{aligned}\mathcal{D}_{g,\pm\alpha}(\mu_0,\kappa) &= -\,\mathrm{e}^{-\bar{n}}\sum_{n=0}^{\infty}\frac{\bar{n}^n}{n!}\frac{\mu_0^2+n\cos\left(2\sqrt{\mu_0^2+n}\,\kappa\right)}{\mu_0^2+n}\\ &= -\,1+2\mathrm{e}^{-\bar{n}}\sum_{n=1}^{\infty}\frac{\bar{n}^n}{n!}\frac{1}{\frac{\mu_0^2}{n}+1}\sin^2\left(\sqrt{\mu_0^2+n}\,\kappa\right),\end{aligned} \tag{43}$$

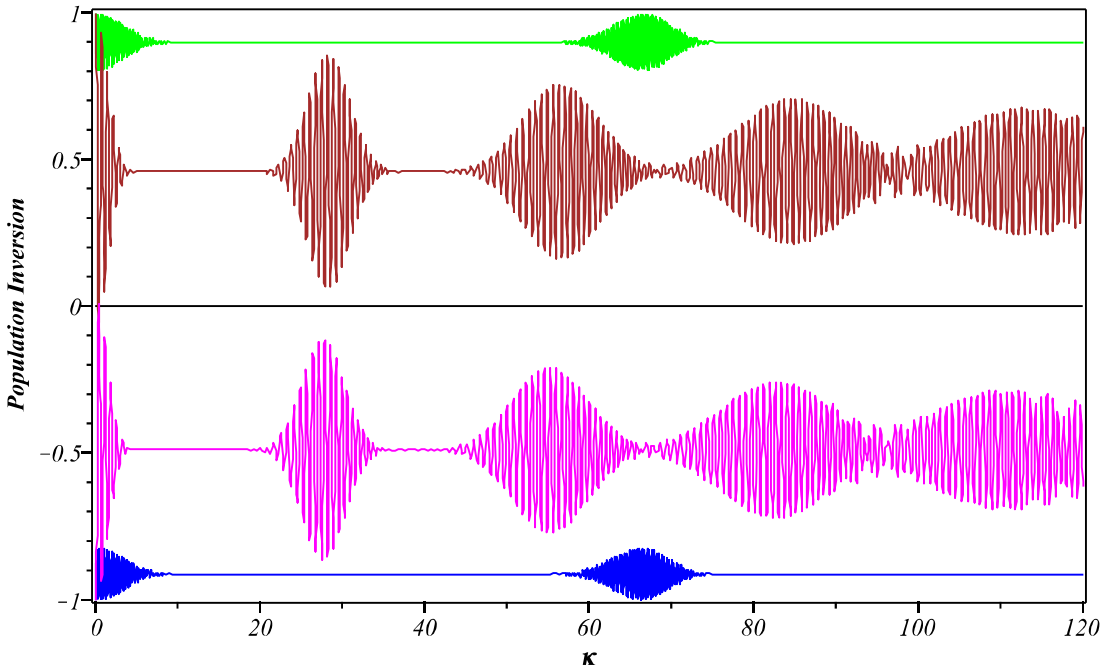

Fig. 3 Evolution of the atomic population inversion as a function of the dimensionless dynamical time scale κ from the initial states (37) (top lines) and (38) (bottom lines). The mean photon number is $\bar{n} = 10$. The green and blue lines correspond to $\mu_0 = 10$, while the brown and magenta lines are obtained for $\mu_0 = 3$. Increasing the value of μ_0, the initial population of the atomic state remains protected from the influence of the cavity field

and

$$\mathcal{D}_{e,\pm\alpha}(\mu_0,\kappa) = 1 - 2\mathrm{e}^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \frac{1}{\frac{\mu_0^2}{n+1}+1} \sin^2\left(\sqrt{\mu_0^2 + n}\,\kappa\right), \tag{44}$$

where we used some simple algebraic manipulation together with the series expansion of the exponential and the trigonometric double angle formulas. These expressions clearly show that values of μ_0 on the order of $\sqrt{\bar{n}}$ or larger abate the main terms in the summation, thus leading to values of $\mathcal{D}_{g,\pm\alpha}(\mu_0, \kappa)$ and $\mathcal{D}_{e,\pm\alpha}(\mu_0, \kappa)$ close to -1 and 1, respectively, as is shown in the figures.

6 Concluding remarks

We studied a generic time-dependent JC model (namely, its parameters were arbitrary functions of time), highlighting universal features of its physical behavior that do not depend on the specific functional dependence of the model parameters on time. Considering that the total excitation number is a constant of motion and using the symmetry properties of the Hamiltonian operator, we decomposed the infinite-dimensional Hilbert space of the system into a one-dimensional and infinitely many two-dimensional dynamically invariant subspaces in which the dynamics is ruled by an effective two-dimensional $SU(2)$ Hamiltonian.

Apart from the trivial solution in the one-dimensional subspace, this approach allowed us to solve the Liouville–Cauchy dynamical problem in the two-dimensional subspaces using the MN ansatz [14], which was developed to treat time-dependent Hamiltonians with $SU(2)$ and $SU(1, 1)$ dynamical symmetries and has also been applied to coupled quantum systems [42–44, 60–62]. We showed that the JC dynamics can be mapped to that of a semiclassical Rabi model (like that describing a fictitious spin 1/2 in a fictitious magnetic field; see Eq. (13)); hence, we used the MN ansatz to describe the system dynamics in the generic subspace and exploited the dynamical invariance of the subspaces to formally solve the system dynamics in the full Hilbert space.

As a significant application, the parameterization inherent in the MN method allowed us to explore the evolution of the atomic population inversion starting from different initial conditions over the dynamical time scale κ defined in Eq. (28). Our analysis shows that the dynamics of the atomic inversion in terms of κ is the same for any time-dependent JC model, regardless of the specific time dependence of the model parameters. Moreover, we identified a parameter (μ_0) that rules the represented dynamics, making the Rabi oscillations over the dynamical time scale become sparser and ultimately disappear as its value grows. The universal features of this study can usefully find application in the new quantum computing era, where a crucial role is played by quantum simulators, namely, controllable quantum systems that mimic quantum processes, by emulating or simulating other quantum systems.

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