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# OPERATOR METHODS, FRACTIONAL CALCULUS AND APPLICATIONS: ANALYTICAL AND COMPUTATIONAL PERSPECTIVES

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Abstract. Operator theory, symbolic calculus, umbral calculus, special functions, and integral transforms provide a unified and flexible framework for the study of analytical and computational problems arising in pure and applied mathematics, including systems involving fractional dynamics. In this work, a systematic operatorial umbral methodology is developed, aimed at representing and manipulating a wide class of special functions and integral kernels in a unified symbolic form. Starting from the notion of umbral images, classical transcendental functions are recast into compact operational structures, allowing algebraic treatment of Bessel, Hermite, Laguerre, and hypergeometric-type functions within a single formal setting. Within this framework, several examples are presented, including umbral representations of special functions, operational identities for polynomial families, generalized integral transforms, and fractional Hankel-type kernels involving Bessel functions and Mittag-Leffler propagators, showing how classical spectral representations can be extended through umbral techniques to fractional evolution settings. Finally, the same formalism is extended to fractional partial differential equations and operator evolution problems, illustrating the versatility of the proposed symbolic operatorial framework. The objective of the paper is to establish a coherent operational methodology that unifies umbral calculus, operator theory, and integral transform techniques for both analytical manipulation and formal solution construction across a broad class of mathematical problems.

1. **Premise.** Operator theory (OT) and symbolic calculus (SC) represent fundamental and versatile tools for the treatment of analytical and numerical problems arising in different fields of pure and applied mathematics [16, 13]. These methods allow complex operations to be handled in a structured and effective way, providing powerful techniques for the study of ordinary and partial differential equations, including models involving fractional dynamics, as well as applications in classic and quantum physics, biology, engineering, and advanced mathematical modeling [10]. Within the common framework of OT and SC, the umbral calculus (UC) [36, 35] has found applications in different fields of mathematics and physics, ranging from

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operator theory and special functions, to interpolation methods and computational approaches [10, 29, 17, 18, 34]. Originally developed as a formal technique for manipulating sequences and polynomials, it offers an elegant approach fo handling complex expressions. In the present article, a brief introduction to the umbral methodology will be provided, and its flexibility and applicability will be shown through numerous examples. In particular, the operational treatment of fractional differential equations and non-local evolution problems represents one of the areas in which symbolic and umbral techniques provide both analytical insight and computational advantages [25, 12].

2. **Introduction.** The term *umbra*[1] was used by Steven M. Roman and Giancarlo Rota [36] in the 20th century to emphasize the practice, in operational calculus, of recasting a given function $g(x)$ into a formal exponential series[2]. Previously, other scientists had addressed similar topics. John Blissard, in his "Theory of Generic Equations", formulated a theory for interpreting different functions as realizations of the same abstract entity [14]. This approach later inspired Edouard Lucas, in "Theorie nouvelle des nombres de Bernoulli et d'Euler", to treat Bernoulli and Euler polynomilas as ordinary monomials [31]. In the present methodology, similar foundations are adopted, but with strategies closer to the *Heaviside* viewpoint [27], by employing different techniques for constructing a unified mathematical language embedding symbolic calculus, umbral methods, integral transform theory, special functions, combinatorics, and generating functions theory in order to exploit them in analytical and numerical computations. From a computational perspective, indeed, the umbral framework provides a systematic way to derive closed-form representations and efficient algorithmic implementations of otherwise non-trivial special functions and operatorial expressions, similarly to other approaches that exploit integral- or transform-based representations for computational purposes [3]. This makes the approach particularly suitable for numerical evaluation and symbolic–numeric hybrid computations. A key element is the "umbral image", which provides the core mechanism for establishing rules which replace higher transcendental functions with elementary functions. Laplace and Borel transform theory, the following principle of permanence of the formal properties[3] [33], and the Ramanujian master theorem constitute the historical and methodological foundations of the theory [9, 24].

To introduce UC, some preliminary concepts need to be provided. The operator $\hat{c}$, called *umbral*, is a shift operator[4] $\hat{c} = e^{\partial_z}$ acting on an appropriate chosen initial function called "vacuum"[5]. It is a function that will change according to the problem to solve, as will be shown in the following examples. If the initial vacuum is, e.g., the function $\varphi(\nu) := \varphi_\nu = \frac{1}{\Gamma(\nu+1)}, \forall \nu \in \mathbb{R}$, the umbral operator

[1]It is derived from the Latin word "Ombra", meaning "shadow".

[2]The trasformation of the index $r$ in $c_r$, interpreted as the exponent of the umbral operator $\hat{c}$, is the essence of UC, since it corresponds to an exchange between coefficients and operators: $g(x) = \sum_{r=0}^{\infty} c_r \frac{x^r}{r!} \to \sum_{r=0}^{\infty} \hat{c}^r \frac{x^r}{r!} = e^{\hat{c}x}$.

[3]The principle of permanence of formal properties is a key concept in symbolic calculus and operator theory. It states that formal rules of algebraic manipulation, introduced in a restricted context, e.g. integers or polynomials, can be consistently extended to more general contexts, e.g. complex numbers, formal series, or analytical functions, while preserving their formal structure and validity.

[4]See [29] for the rigoruos details.

[5]The term vacuum is derived from the concept of vacuum in physics, see [29].

$\hat{c}^{\nu}$, $\forall \nu \in \mathbb{R}$, will be the action of the operator $\hat{c}$ on the vacuum $\varphi_0$ such that[6] $\hat{c}^{\nu}\varphi_0 := \varphi_\nu = \frac{1}{\Gamma(\nu+1)}$. By this definition, the operator $\hat{c}$ has the properties of the standard exponential $\hat{c}^{\,\pm\nu}\hat{c}^{\,\mu} = \hat{c}^{\,\pm\nu+\mu}$ and $(\hat{c}^{\,\pm\nu})^{\mu} = \hat{c}^{\,\pm\nu\,\mu}$, $\forall \nu,\mu \in \mathbb{R}$. In the following, several applications will be shown to explain the methodology.

## 3. Bessel-like functions umbral images: Application to fractional radial heat equation.

To clarify the strategy, the construction of the umbral image of the Bessel function is first introduced through the following example.

**Example 1.** By using the operator $\hat{c}$, the property of the $\Gamma$-function [1], and by exploiting the series representation of the real-order Bessel function $J_\nu(x)$ [1], the umbral image of the BF in terms of a Gaussian function is provided. Indeed, $\forall x,\nu \in \mathbb{R}$, the umbral $\nu$-order Bessel function can be written as [29]

$$J_\nu\left(x\right) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{k!\Gamma(\nu+k+1)} = \left(\hat{c}\frac{x}{2}\right)^{\nu} e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0. \tag{1}$$

Then, a trascendental function like the BF can be recast by downgrading it to a simple exponential form. In this way, the umbral operator $\hat{c}$ acts on $\varphi_0$ only, leaving $x$ unaffected. The operator $\hat{c}$ and variable $x$ can thus commute, and it is possible to handle $\hat{c}$ like an ordinary algebric quantity. This is the foundation of the formal calculus.

From this representation, several results follow, including the evaluation of the following integral, $\forall \nu,\alpha \in \mathbb{R} : Re(\nu+\alpha) > -1$, $\forall r \in \mathbb{R}^+$:

$$\int_0^\infty J_\nu(rx)x^\alpha dx = \left(\frac{\hat{c}r}{2}\right)^{\nu} \int_0^\infty e^{-\hat{c}\left(\frac{rx}{2}\right)^2} x^{\nu+\alpha} dx\varphi_0 = \frac{2^\alpha}{r^{\alpha+1}} \frac{\Gamma\left(\frac{\nu+\alpha+1}{2}\right)}{\Gamma\left(\frac{\nu-\alpha+1}{2}\right)}. \tag{2}$$

The above integral admits a spectral interpretation in terms of Bessel-type eigenfunctions associated with radial differential operators. Indeed, the functions $J_\nu(rx)$ arise as eigenfunctions of Bessel-type differential operators, namely

$$\left(\frac{1}{r}\partial_r r \partial_r - \frac{\nu^2}{r^2}\right) J_\nu(rx) = -x^2 J_\nu(rx), \tag{3}$$

which explains their natural appearance in radial spectral problems and Hankel-transform techniques. Specifically, Bessel kernels naturally arise in radial evolution problems, wave propagation, and diffusion models with cylindrical symmetry [24]. Within this framework, operational and umbral methods provide compact representations and significantly simplify the evaluation of otherwise non-trivial integral transforms. Moreover, in the symmetric case $\nu = 0$, the corresponding Bessel functions are directly connected with the radial part of the Laplace operator, which leads to the fractional radial heat equation of the form

$$\partial_t^\beta f(r,t) = \frac{1}{r}\partial_r\left(r\partial_r f(r,t)\right), \quad 0 < \beta \leq 1, \tag{4}$$

whose time evolution is governed by the Mittag-Leffler propagators in the corresponding Hankel space [24].

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[6] $\hat{c}^{\nu}\varphi_0 = e^{\nu\partial_z}\varphi_z\ |_{z=0} = \varphi_{z+\nu}|_{z=0} = \left.\frac{1}{\Gamma(z+\nu+1)}\right|_{z=0} = \frac{1}{\Gamma(\nu+1)}$. The power exponent $\nu$ may actually belong to the field of complex numbers $\mathbb{C}$, but in this context we limit our concepts to the field of real numbers $\mathbb{R}$.

To incorporate these dynamics, the Hankel-Bessel framework can be naturally extended to fractional evolution kernels through the Mittag-Leffler function in the following form:

$$l(r,t) = \int_0^\infty J_\nu(rx)\, x^\alpha\, E_\beta(-x^2 t^\beta)\, dx, \qquad 0 < \beta \leq 1. \tag{5}$$

In this context, the Mittag-Leffler factor acts as a fractional propagator, replacing the standard exponential semigroup associated with classical diffusion processes. Notably, the standard integral representation is recovered in the limit case $\beta = 1$, since

$$E_1(-x^2 t) = e^{-x^2 t}, \tag{6}$$

thus yielding the classical diffusion kernel.

Now, by introducing the umbral representations of the Mittag-Leffler function [29]

$$E_\beta(x) = e^{\,_\beta\hat{d}x}\psi_0, \qquad {}_\beta\hat{d}^k\psi_0 = \frac{\Gamma(k+1)}{\Gamma(\beta k+1)} \tag{7}$$

the integral (5) is recast as

$$\begin{aligned} l(r,t) &= \int_0^\infty \left(\hat{c}\frac{rx}{2}\right)^\nu e^{-\hat{c}\frac{r^2x^2}{4}} x^\alpha e^{-d_\beta x^2 t^\beta}\, dx\ \varphi_0\psi_0 \\ &= \left(\hat{c}\frac{r}{2}\right)^\nu \int_0^\infty x^{\nu+\alpha} \exp\left[-x^2\left(\hat{c}\frac{r^2}{4} + d_\beta t^\beta\right)\right] dx\varphi_0\psi_0. \end{aligned} \tag{8}$$

Finally, using the standard Gamma-type integral [1]

$$\int_0^\infty x^m e^{-Ax^2}\, dx = \frac{1}{2} A^{-\frac{m+1}{2}} \Gamma\left(\frac{m+1}{2}\right), \qquad \mathrm{Re}(A) > 0, \mathrm{Re}(m) > -1 \tag{9}$$

with $m = \nu+\alpha$ and $A = \hat{c}\frac{r^2}{4} + d_\beta t^\beta$, we obtain the closed-form umbral representation

$$l(r,t) = \frac{1}{2}\left(\hat{c}\frac{r}{2}\right)^\nu \left(\hat{c}\frac{r^2}{4} + d_\beta t^\beta\right)^{-\frac{\nu+\alpha+1}{2}} \Gamma\left(\frac{\nu+\alpha+1}{2}\right) \varphi_0\psi_0. \tag{10}$$

Then, by expanding with the generalized Newton formula

$$\left(\hat{c}\frac{r^2}{4} + d_\beta t^\beta\right)^{-\frac{\nu+\alpha+1}{2}} = \sum_{k=0}^\infty \binom{-\frac{\nu+\alpha+1}{2}}{k} \left(\hat{c}\frac{r^2}{4}\right)^{-\frac{\nu+\alpha+1}{2}-k} \left(d_\beta t^\beta\right)^k. \tag{11}$$

we get

$$l(r,t) = \frac{1}{2}\Gamma\left(\frac{\nu+\alpha+1}{2}\right) \sum_{k=0}^\infty \binom{-\frac{\nu+\alpha+1}{2}}{k} \hat{c}^{-\frac{\alpha+1}{2}-k} d_\beta^k \left(\frac{r}{2}\right)^{-\alpha-1-2k} t^{\beta k}\, \varphi_0\psi_0 \tag{12}$$

and making the umbral operators act on their respective vacua, we get

$$l(r,t) = \frac{1}{2}\Gamma\left(\frac{\nu+\alpha+1}{2}\right) \left(\frac{r}{2}\right)^{-\alpha-1} \sum_{k=0}^\infty \binom{-\frac{\nu+\alpha+1}{2}}{k} \frac{\Gamma(k+1)}{\Gamma!\left(\frac{1-\alpha}{2}-k\right)\Gamma(\beta k+1)} \left(\frac{4t^\beta}{r^2}\right)^k. \tag{13}$$

Recalling the identity

$$\binom{-\gamma}{k} = \frac{(-1)^k}{k!}\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}, \qquad \gamma = \frac{\nu+\alpha+1}{2}, \tag{14}$$

and using the property of the Gamma function $\Gamma(k+1) = k!$, we finally obtain

$$l(r,t) = \frac{1}{2}\left(\frac{r}{2}\right)^{-\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\frac{\nu+\alpha+1}{2}+k\right)}{\Gamma\left(\frac{1-\alpha}{2}-k\right)\Gamma(\beta k+1)} \left(\frac{4t^\beta}{r^2}\right)^k, \quad \left|\frac{4t^\beta}{r^2}\right| < 1. \tag{15}$$

This series representation can be interpreted as a generalized Wright-type expansion of the fractional Hankel kernel.

**Observation 1.** It is important to emphasize that the umbral image is not unique; indeed, if another appropriate initial vacuum is chosen, the action of the umbral operator will yield different umbral image for the same function. If, for example, the pair umbral operator/vacuum is introduced such that $\hat{b}^k \psi_0 := \psi_k = \frac{1}{(\Gamma(k+1))^2}, \forall k \in \mathbb{R}$ ,you obtain $J_0(x) = \frac{1}{1+\hat{b}\left(\frac{x}{2}\right)^2}\psi_0, \forall x \in \mathbb{R}$ , thus expressing BF in a rational form, according to the operator $\hat{b}$ and the vacuum $\psi_0$. Obviously, the umbral images are "equivalent", meaning that the problems including the initial function (e.g., the BF) can be treated the same through the first of the second proposed option, as shown in the following example.

**Example 2.** The result of the integral $\int_{-\infty}^{\infty} J_0(x)dx$ is known, and it is 2. In the present example, the equivalence of the umbral transformations is shown, highlighting their simplicity of use. Indeed, we recast the 0-order BF through the Gaussian and the rational umbral images, and we obtain the same correct result:

$$\int_{-\infty}^{\infty} J_0(x)dx = \begin{cases} \int_{-\infty}^{\infty} e^{-\hat{c}\left(\frac{x}{2}\right)^2} dx \; \varphi_0 = 2\sqrt{\pi}\frac{1}{\Gamma\left(-\frac{1}{2}+1\right)} = 2 \\ \int_{-\infty}^{\infty} \dfrac{1}{1+\hat{b}\left(\frac{x}{2}\right)^2} dx \; \Phi_0 = 2\frac{\pi}{\left(\Gamma\left(\frac{1}{2}\right)\right)^2} = 2 \end{cases}. \tag{16}$$

We note that the power exponent of the umbral operator is real, so, like in the above example, it can be fractional $(-\frac{1}{2})$, and it will allow us to deal with fractional calculus too, as we shall see in the following sections.

This new interpretation of a certain function opens a wide range of possibilities. To this aim, we provide some examples including integrals and higher-order derivatives of BF [19].

**Example 3** (Integral involving the Bessel function)**.**

$$\int_{-\infty}^{\infty} \frac{J_0\left(\frac{2\sqrt{ax}}{\sqrt{1+bx^2}}\right)}{1+bx^2} dx = \sqrt{\frac{\pi}{b}} \sum_{r=0}^{\infty} \frac{\Gamma\left(r+\frac{1}{2}\right)}{r!^3} \left(-\frac{a}{b}\right)^r, \quad \forall a,b \in \mathbb{R}^+. \tag{17}$$

**Example 4** (High-order derivative)**.** The high-order derivative of a BF can be expressed in terms of Cebyshev-like polynomials ${}_2U_{n+m}(a,b)$,

$$\partial_x^m J_0\left(2\sqrt{ax+bx^2}\right) = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} x^n {}_2U_{n+m}(a,b), \quad \forall a,b,x \in \mathbb{R}, \forall m \in \mathbb{N} \tag{18}$$

where ${}_2U_k(a,b)$ is a generalization[7] of the classic Cebyshev polynomials[8] (CP) $U_k(a,b)$.

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[7] ${}_2U_k(a,b) = (-1)^k \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^r a^{k-2r} b^r}{(k-2r)!(k-r)!r!}, \quad \forall a,b \in \mathbb{R}, \forall k \in \mathbb{N}.$

[8] We recall that the classic CP in terms of series is $U_k(a,b) = (-1)^k \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^r (k-r)! a^{k-2r}(-b)^r}{(k-2r)!r!}$, $\forall a,b \in \mathbb{R}, \forall k \in \mathbb{N}$ [1]. These examples show that an SF can be expressed in terms of another SF.

**Observation 2.** It is important to recall that, in general, the operator is not commutative, so, if we want to calculate, e.g., $(J_0(x))^n$, we cannot use the same operator arised to the power $n$, but we have to define $n$ different operators acting on $n$ different vacua. For example, for $n = 2$, we obtain the umbral form $f(x; \alpha, \beta) := J_0(\alpha x)J_0(\beta x) = e^{-(\alpha^2 \hat{c}_1 + \beta^2 \hat{c}_2)\left(\frac{x}{2}\right)^2} \varphi_{0,1}\varphi_{0,2}$ (see [23] for the details), and through the same procedure illustrated so far, it is possible to treat the Weber-Schafheitlin integrals.

4. **Hermite functions.** There is a plethora of results regarding Hermite polynomials/functions (HP) and the operator's handling. We report only some example to give an idea.

**Example 5** (Umbral HP)**.** The HP can be expressed in umbral terms by binomial representations [29] . Indeed, $\forall x, y, z \in \mathbb{R}, \forall n \in \mathbb{N}$, we get

$$H_n(x, y) = \left(x + {}_y\hat{h}\right)^n \theta_0, \quad \theta(z) := \theta_z = y^{\frac{z}{2}} \left( \frac{\Gamma(z+1)}{\Gamma\left(\frac{1}{2}z + 1\right)} \left|\cos\left(\frac{\pi}{2}z\right)\right| \right), \tag{19}$$

and obtained, $\forall r, s \in \mathbb{R}$, by the umbral operator ${}_y\hat{h}^r\ \theta_0 := \theta_r$, with

$$\theta_r = \frac{y^{\frac{r}{2}} r!}{\Gamma\left(\frac{r}{2} + 1\right)} \left|\cos\left(r\frac{\pi}{2}\right)\right| = \begin{cases} 0 & r = 2s + 1 \\ y^s \dfrac{(2s)!}{s!} & r = 2s \end{cases} . \tag{20}$$

The proposed umbral HP allows us to find the new result of the integral involving a product of two variables Hermite Kampè dè Fèrièt functions [6] shown in the following lemma.

**Lemma 4.1** (Integral of product of two variables Hermite functions)**.** $\forall x, y, z \in \mathbb{R}$ *and* $\forall n, m \in \mathbb{N}$, *the following integral result holds:*

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} H_n(x, y)H_m(x, z)dx = \sqrt{\frac{\pi}{\alpha}} H_{n,m}\left(0, z + \frac{1}{4\alpha}, 0, w + \frac{1}{4\alpha} \mid \frac{1}{2\alpha}\right), \tag{21}$$

*where*

$$H_{n,m}(x, a, y, b \mid \tau) = n!m! \sum_{r=0}^{\min(n,m)} \frac{H_{n-r}(x, a)H_{m-r}(y, b)\tau^r}{(n-r)!(m-r)!r!} \tag{22}$$

*or in* $H_n(x)$ *base,*

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\alpha x^2} H_n(x)\, H_m(x)\, dx = & \sqrt{\frac{\pi}{\alpha}} n!m! \sum_{r=0}^{\min(n,m)} \frac{\left(\frac{2}{\alpha}\right)^r \left(-1 + \frac{1}{\alpha}\right)^{\frac{n-r}{2}} \left(-1 + \frac{1}{\alpha}\right)^{\frac{m-r}{2}}}{\Gamma\left(\frac{n-r}{2} + 1\right) \Gamma\left(\frac{m-r}{2} + 1\right) r!} \cdot \\ & \cdot \left|\cos\left(\frac{n-r}{2}\pi\right)\right| \left|\cos\left(\frac{m-r}{2}\pi\right)\right| . \end{aligned} \tag{23}$$

*Proof.* In [10] (Eqs. 7.3.5-7.3.6), it was proved that

$$\begin{aligned} I_{n,m}(\alpha, \delta_1, \delta_2, z, w) &= \int_{-\infty}^{\infty} e^{-\alpha x^2} H_n(x + \delta_1, z)H_m(x + \delta_2, w)dx \\ &= \sqrt{\frac{\pi}{\alpha}} H_{n,m}\left(\delta_1, z + \frac{1}{4\alpha}, \delta_2, w + \frac{1}{4\alpha} \mid \frac{1}{2\alpha}\right), \end{aligned} \tag{24}$$

with $H_{n,m}(x,a,y,b \mid \tau)$ is polynomial in (22). By applying the HP umbral image in (19), we provide the umbral polynomial family

$$H_{n,m}(x,a,y,b \mid \tau) = n!m! \sum_{r=0}^{\min(n,m)} \frac{(x + {}_a\hat{h})^{n-r}(y + {}_b\hat{h})^{m-r}\tau^r}{(n-r)!(m-r)!r!} \theta_{0,a}\theta_{0,b} \, . \tag{25}$$

If we consider the limit for $x, y \to 0$, we get

$$\begin{aligned} H_{n,m}(0,a,0,b \mid \tau) &= n!m! \sum_{r=0}^{\min(n,m)} \frac{({}_a\hat{h})^{n-r}({}_b\hat{h})^{m-r}\tau^r}{(n-r)!(m-r)!r!} \theta_{0,a}\theta_{0,b} =_{n-r=2s,\; m-r=2k}= \\ &= n!m! \sum_{r=0}^{\min(n,m)} \frac{\tau^r a^{\frac{n-r}{2}} b^{\frac{m-r}{2}}}{\Gamma\left(\frac{n-r}{2}+1\right)\Gamma\left(\frac{m-r}{2}+1\right) r!} \left|\cos\left(\frac{n-r}{2}\pi\right)\right| \left|\cos\left(\frac{m-r}{2}\pi\right)\right| . \end{aligned} \tag{26}$$

Coming back to Eq (24), we write

$$\begin{aligned} I_{n,m}(\alpha,\delta_1=0,\delta_2=0,z,w) &= \int_{-\infty}^{\infty} e^{-\alpha x^2} H_n(x+0,z) H_m(x+0,w) dx \\ &= \sqrt{\frac{\pi}{\alpha}} H_{n,m}\left(0, z+\frac{1}{4\alpha}, 0, w+\frac{1}{4\alpha} \mid \frac{1}{2\alpha}\right) \\ &= \sqrt{\frac{\pi}{\alpha}} n!m! \sum_{r=0}^{\min(n,m)} \frac{\left(\frac{1}{2\alpha}\right)^r \left(z+\frac{1}{4\alpha}\right)^{\frac{n-r}{2}} \left(w+\frac{1}{4\alpha}\right)^{\frac{m-r}{2}}}{\Gamma\left(\frac{n-r}{2}+1\right)\Gamma\left(\frac{m-r}{2}+1\right) r!} \cdot \\ &\quad \cdot \left|\cos\left(\frac{n-r}{2}\pi\right)\right| \left|\cos\left(\frac{m-r}{2}\pi\right)\right| \end{aligned} \tag{27}$$

or, in $H_n(x)$ base,

$$\begin{aligned} I_{n,m}(\alpha,\delta_1=0,\delta_2=0,z=-1,w=-1) &= \int_{-\infty}^{\infty} e^{-\alpha x^2} H_n\left(\frac{x}{2}\right) H_m\left(\frac{x}{2}\right) dx \\ &= \sqrt{\frac{\pi}{\alpha}} n!m! \sum_{r=0}^{\min(n,m)} \frac{\left(\frac{1}{2\alpha}\right)^r \left(-1+\frac{1}{4\alpha}\right)^{\frac{n-r}{2}} \left(-1+\frac{1}{4\alpha}\right)^{\frac{m-r}{2}}}{\Gamma\left(\frac{n-r}{2}+1\right)\Gamma\left(\frac{m-r}{2}+1\right) r!} \cdot \\ &\quad \cdot \left|\cos\left(\frac{n-r}{2}\pi\right)\right| \left|\cos\left(\frac{m-r}{2}\pi\right)\right| \end{aligned} \tag{28}$$

and seeing as

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} H_n\left(\frac{x}{2}\right) H_m\left(\frac{x}{2}\right) dx = 2\int_{-\infty}^{\infty} e^{-4\alpha\xi^2} H_n(\xi) H_m(\xi) d\xi, \tag{29}$$

finally (if n and m are of the same parity, otherwise it is zero) we obtain the solution in (23). □

It is important to note that the proof just illustrated allows the result to be extended to integrals of multiple products, both to multi-variable Hermite functions and to other special functions. For the most well-known families of SP, in fact, we have formalized one or more umbral images, as shown in Table 1 below [1, 6, 4, 20].

In Table 1, only gives an idea of the flexibility of the method. It is important to note that new integral transformations, such as the transform based on the Voigt function [30], have also been formalized (so not only functions or polynomials). In addition, the umbral image of hypergeometric functions through the use of one or more combined umbral operators offers multiple developments [21], as we will show in the following section.

TABLE 1. Special Functions and Umbral Images

| Family | Umbral Image |
|---|---|
| Laguerre | $\mathbf{L_n(x,y) = (y - \hat{c}x)^n \varphi_0}$ |
| Jacobi | $\mathbf{P_n^{(\alpha,\beta)}(z) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (z-1)^{n-k} (z+1)^k}$ |
| Legendre | $\mathbf{P_n(x) = R_n^{(0,0)}(\xi,\eta) = n! \left[\hat{c}_1 \frac{x-1}{2} + \hat{c}_2 \frac{x+1}{2}\right]^n \varphi_{1,0}\varphi_{2,0}}$ <br> $R_n^{(\alpha,\beta)}(\xi,\eta) = n! \hat{c}_1^{\alpha} \hat{c}_2^{\beta} [\hat{c}_1\xi + \hat{c}_2\eta]^n \varphi_{1,0}\varphi_{2,0}$ |
| Tricomi-Bessel | $\mathbf{C_\nu(x) = \left(\frac{1}{x}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{x})}$ |
| Chebyshev | $\mathbf{U_n(x,y) = \frac{1}{n!} \int_0^\infty e^{-s} \left(-x\,s + {}_{(-y\,s)}\hat{h}\right)^n \theta_0 \, ds}$ |
| Gegenbauer | $\mathbf{C_n^{(\nu)}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{\Gamma(n-k+\nu)}{\Gamma(\nu) k! (n-2k)!} (2x)^{n-2k}}$ <br> $K_n^{(\nu)}(2x,-1) = n! C_n^{(\nu)}(x)$ |
| Hypergeometric | $\mathbf{{}_2F_1(a,b;c;x) = \exp\{{}^a\hat{\chi}^b \hat{\chi}_c \hat{\chi} x\} p_{0,a} p_{0,b} \varepsilon_0}$ |
| Voigt Transform | $\mathbf{{}_V\hat{f}(x,y;z) = \int_0^\infty e^{-xt-yt^2} f(zt)\, dt}$ |

5. **Hypergeometric functions.** The Gaussian hypergeometric function (HF) is defined by the series[9] [1, 2]

$$ {}_2F_1[\alpha,\beta;\gamma;x] = \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r} \frac{x^r}{r!}, \quad |\, x \,| < 1, \tag{30} $$

where $(d)_r$ denotes the Pochhammer symbol $(d)_r = \frac{\Gamma(d+r)}{\Gamma(d)}$ [1]. In umbral terms, we can fix $\hat{d}^r \Phi_0 = (d)_r$ [21] and recast the HF in terms of the action of 3 operators on 3 vacua,

$$ {}_2F_1[\alpha,\beta;\gamma;x] = \sum_{r=0}^{\infty} \frac{\hat{\alpha}^r \hat{\beta}^r}{\hat{\gamma}^r} \frac{x^r}{r!} \Phi_0^{\alpha} \Phi_0^{\beta} \Phi_0^{\gamma} = e^{\frac{\hat{\alpha}\hat{\beta}}{\hat{\gamma}} x} \Phi_0^{\alpha} \Phi_0^{\beta} \Phi_0^{\gamma}, \tag{31} $$

or, recalling that the umbral image is not unique, we can write

$$ {}_2F_1[\alpha,\beta;\gamma;x] = e^{{}_2\hat{\chi}_1 x} \phi_0, \tag{32} $$

through the operator ${}_2\hat{\chi}_1^r \phi_0 = \frac{(\alpha)_r(\beta)_r}{(\gamma)_r}$. Then, we can provide different results as shown in the examples below [21].

**Example 6** (Derivative calculus)**.** Simplifying the notation ${}_2\hat{\chi}_1 \to \hat{\chi}$, we get

$$ \frac{d}{dx}\, {}_2F_1[\alpha,\beta;\gamma;x] = \hat{\chi} e^{\hat{\chi} x} \phi_0 = \sum_{r=0}^{\infty} \frac{(\alpha)_{r+1}(\beta)_{r+1}}{(\gamma)_{r+1}} \frac{x^r}{r!} \tag{33} $$

which, in view of the identity $(d)_{n+m} = (d)_m\,(d+m)_n$, yields the well-known recurrence formula

$$ \frac{d}{dx}\, {}_2F_1[\alpha,\beta;\gamma;x] = \frac{\alpha\beta}{\gamma}\, {}_2F_1(\alpha+1,\beta+1;\gamma+1;x). \tag{34} $$

**Example 7** (Integral calculus)**.** Similarly, by using the identity $(d)_{-r} = \frac{(-1)^r}{(1-d)_r}$, it is easy to calculate

$$ \int {}_2F_1[\alpha,\beta;\gamma;x]\, dx = \int e^{\hat{\chi} x} dx \phi_0 = \frac{(\gamma-1)}{(\alpha-1)(\beta-1)}\, {}_2F_1[\alpha-1,\beta-1;\gamma-1;x]. \tag{35} $$

[9] We will treat also the confluent HF with $|\, x \,| \le \infty$.

or more complicated forms $(0 < Re(\nu) < \min(Re(\alpha), Re(\beta)),\ \mu \neq 0), \nu \in \mathbb{R}$

$$\int_0^\infty t^{\nu-1}\,{}_2F_1[\alpha,\beta;\gamma;-t^\mu]\,dt = \frac{1}{\mu}\Gamma\left(\frac{\nu}{\mu}\right)\hat{\chi}^{-\left(\frac{\nu}{\mu}\right)}\phi_0 = \frac{1}{\mu}\Gamma\left(\frac{\nu}{\mu}\right)\frac{(\alpha)_{-\left(\frac{\nu}{\mu}\right)}(\beta)_{-\left(\frac{\nu}{\mu}\right)}}{(\gamma)_{-\left(\frac{\nu}{\mu}\right)}}. \tag{36}$$

**Example 8** (Generalized HF)**.** $\forall p \in \mathbb{R} : p \leq q+1$,

$$\begin{aligned}&\int_{-\infty}^\infty e^{-\alpha x^2}\,{}_pF_q[a_1,...,a_p;b_1,...,b_q;\varepsilon x]\,dx = \\ &= \sqrt{\frac{\pi}{\alpha}}\,{}_{2p}F_{2q}\left[\frac{a_1}{2},\frac{a_1+1}{2},...,\frac{a_p}{2},\frac{a_p+1}{2};\frac{b_1}{2},\frac{b_1+1}{2},...,\frac{b_q}{2},\frac{b_q+1}{2};2^{2(p-q)}\left(\frac{\varepsilon^2}{4\alpha}\right)\right].\end{aligned} \tag{37}$$

**Example 9** (Differential equation)**.** The $\nu$-order Tricomi functions [1, 4]

$$C_\nu(x) = \sum_{r=0}^\infty \frac{(-x)^r}{r!\,\Gamma(\nu+r+1)} \tag{38}$$

satisfy the eigenvalue equation

$$\left(\frac{d}{dx}x\frac{d}{dx} + \nu\frac{d}{dx}\right)C_\nu(\lambda x) = -\lambda C_\nu(\lambda x). \tag{39}$$

Via the link between the Tricomi and hypergeometric functions

$$C_{b-1}(x) = \frac{1}{\Gamma(b)}\,{}_0F_1[;b;-x]. \tag{40}$$

we can write

$$\left(x\frac{d}{dx}+b\right)\frac{d}{dx}\,{}_0F_1[;b+1;-\lambda x] = \lambda\,{}_0F_1[;b+1;-\lambda x]. \tag{41}$$

By using the umbral point of view ${}_0\hat{\chi}_1^n\ \varepsilon(b)_0 := {}_b\hat{\chi}^n\varepsilon_0 = \frac{1}{(b)_n}$ and introducing the differential operator

$$\frac{d}{d\,{}_b\hat{\chi}\,x} := {}_b\overline{D}_x \to \left(x\frac{d}{dx}+b\right)\frac{d}{dx}, \tag{42}$$

the eigenvalue problem (41) can be simply expressed as

$${}_b\overline{D}_x\ e^{\lambda\,{}_b\hat{\chi}\,x}\varepsilon_0 = \lambda e^{\lambda\,{}_b\hat{\chi}\,x}\varepsilon_0. \tag{43}$$

The given examples show the considerable flexibility and transversality of the techniques presented. To finish this paper, we present in the following section the operative methodology applied to the fractional PDE solution.

6. **Fractional partial differential equations.** Fractional PDEs extend the classical concept of partial derivatives by replacing integer differential operators with fractional ones. They are applied in many fields, including quantum physics, economics, biology, energy context, or the theory of anomalous stochastic processes. They are able to capture anomalous diffusive or sub-diffusive behavior. Among the most widely used definitions of fractional derivatives are those of Riemann-Liouville [32], Caputo [15], or Atangana [8], each with specific properties useful for different application contexts. Theoretical and numerical analysis of fractional PDEs is an expanding field of research, with solution methods ranging from integral transforms to finite difference-based numerical schemes and spectral methods. From an operator-theoretic point of view, fractional operators are interpreted as

pseudo-differential or sectorial operators acting on suitable functional spaces, ensuring a well-posed framework for the associated evolution problems. In particular, the Hilbert space setting $L^2(\mathbb{R})$ provides the natural framework when Fourier-based representations are employed. Below we will show the operative approach.

**Example 10** (Lamb-Bateman equation)**.** The integral equation

$$\int_0^\infty u(x-y^2)dy = f(x) \tag{44}$$

was proposed by Lamb [28] in his study of the diffraction of a solitary wave. The integral can be approached through operatorial methods [11, 5]. In particular, we want to find $u(x)$ in terms of $f(x)$, so we get

$$\hat{L}\, u(x) = f(x), \qquad \hat{L} = \int_0^\infty e^{-y^2\partial_x} dy = \frac{1}{2}\sqrt{\frac{\pi}{\partial_x}}, \tag{45}$$

giving

$$\hat{L}^{-1} = \frac{2}{\sqrt{\pi}}\,\partial_x^{\frac{1}{2}} \;\rightarrow\; u(x) = \frac{2}{\sqrt{\pi}}\,\partial_x^{\frac{1}{2}} f(x). \tag{46}$$

We have accordingly expressed the solution of the Lamb-Bateman equation in terms of the derivative of order $\frac{1}{2}$ of the function $f(x)$, but what is the action of a fractional derivative on a given function?

To answer this question, we use the Laplace transform (LT) $Re(a), Re(\nu) > 0$, $a^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-as}s^{\nu-1}ds$ [1]. The LT, by using the shift operator, allows to write

$$\partial_x^{-\frac{1}{2}} f(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty e^{-s\partial_x} s^{-\frac{1}{2}} ds\, f(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty s^{-\frac{1}{2}} f(x-s)ds. \tag{47}$$

Changing the integration variable and arranging the limits of integration, we eventually find

$${}_a\partial_x^{-\frac{1}{2}} f(x) = \frac{1}{\sqrt{\pi}}\int_a^x (x-\xi)^{-\frac{1}{2}} f(\xi)d\xi\ . \tag{48}$$

The lowest integration limit, following from (47), is $-\infty$; we have put a generic $a$ to be consistent with the current definition of differintegral operator. If $a = 0$, the sub-index on the left of the derivative operator will be neglectd. The semigroup property of the fractional differintegral operator yields $\partial_x^{\frac{1}{2}} = \partial_x^{-\frac{1}{2}}\partial_x$. Finally, from (47), we have

$$\partial_x^{\frac{1}{2}} f(x) = \frac{1}{\sqrt{\pi}}\int_0^x (x-\xi)^{-\frac{1}{2}} f'(\xi)d\xi. \tag{49}$$

More generally, any derivative of order $\alpha$ can be written as

$$\partial_x^\alpha\ f(x) = \partial_x^{(\alpha-1)} f'(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-\xi)^{-\alpha} f'(\xi)d\xi\ , \quad 0<\alpha<1, \tag{50}$$

which is the definition of the Caputo fractional derivative.

**Example 11** (Heat-like equation)**.** The solution of evolution-like equations involving partial fractional derivatives as, e.g., a kind of heat equation

$$1)\ \begin{cases} \dfrac{\partial}{\partial t}F(x,t) = -\dfrac{\partial^\alpha}{\partial x^\alpha}F(x,t) \\ F(x,t)|_{t=0} = f(x), \qquad 0<\alpha<1 \end{cases} \tag{51}$$

or more complicated forms like

$$2)\ \begin{cases} \partial_t F(x,t) = -\sqrt{1+\partial_x^{\nu}}\ F(x,t), \quad \forall x \in \mathbb{R}, \forall t \in \mathbb{R}^+ \\ F(x,0) = f(x) \end{cases} \tag{52}$$

can be solved by the use of an appropriate transform, so, according to our formalism, we obtain

$$\begin{aligned} &1)\ F(x,t) := F_\alpha(x,t) = \hat{U}_\alpha(t) f(x), \qquad \hat{U}_\alpha(t) = e^{-t\frac{\partial^\alpha}{\partial x^\alpha}}, \\ &2)\ F(x,t) := F_\nu(x,t) = \hat{U}_\nu f(x), \qquad \hat{U}_\nu = e^{-t\sqrt{1+\partial_x^\nu}}. \end{aligned} \tag{53}$$

The action of the evolution operator $\hat{U}(t)$ on the initial function $f(x)$ can be obtained via the so-called Lévy transform method or the ”stretching integral”

$$e^{-\tau x^\gamma} = \int_0^\infty e^{-(\tau x^\gamma)^{\gamma^{-1}}\eta}\ g_\gamma(\eta)\ d\eta = \int_0^\infty e^{-\tau^{\gamma^{-1}} x\ \eta}\ g_\gamma(\eta)\ d\eta. \tag{54}$$

We can therefore manage to transform the exponential evolution operator

$$1)\ \hat{U}_\alpha(t) = \int_0^\infty e^{-t^{\alpha^{-1}}\eta\frac{\partial}{\partial x}} g_\alpha(\eta) d\eta \ \to F_\alpha(x,t) = \int_0^\infty g_\alpha(\eta) f\left(x - t^{\alpha^{-1}}\eta\right) d\eta. \tag{55}$$

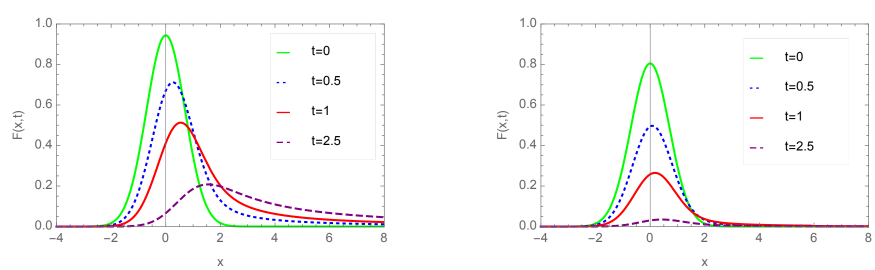
and

$$\begin{aligned} 2)\ F_\nu(x,t) &= \int_0^\infty e^{-t^2(1+\partial_x^\nu)\eta} g_{\frac{1}{2}}(\eta) d\eta\ f(x) \\ &= \int_0^\infty g_{\frac{1}{2}}(\eta) e^{-\eta t^2} \left( \int_0^\infty g_\nu(\varepsilon) f\left(x - \left(\eta t^2\right)^{\frac{1}{\nu}} \varepsilon\right) d\varepsilon \right) d\eta\ . \end{aligned} \tag{56}$$

in terms of the Lévy functions $g_{\frac{1}{2}}(\eta)$

$$g_{\frac{1}{2}}(\eta) = \frac{1}{2\sqrt{\pi}\ \eta^{\frac{3}{2}}} e^{-\frac{1}{4\eta}}. \tag{57}$$

In Figure 1, the solutions $F_\alpha(x,t)$ of the problem (51) and $F_\nu(x,t)$ of the problem (52), with $\alpha = \frac{1}{2}$, is shown for different times. It is evident there is anomalous behavior in functions of time in both solutions, although in $F_\alpha(x,t)$ it is more prominent.




(A) $F_\alpha(x,t)$ for $\alpha = \frac{1}{2}$ and $t = 0, 0.5, 1, 2.5$. (B) $F_\nu(x,t)$ for $\nu = \frac{1}{2}$ and $t = 0, 0.5, 1, 2.5$.

FIGURE 1. Solutions A) $F_\alpha(x,t)$ of problem (51) and B) $F_\nu(x,t)$ of problem (52), with $\alpha = \nu = \frac{1}{2}$ for different times and Gaussian initial function $f(x)$.

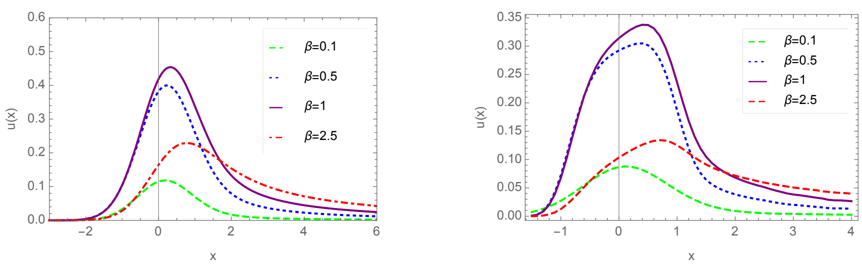
**Example 12.** We can extend the previous procedure to multiple variables. Let, $\forall \nu, \beta \in \mathbb{R} : 0 < \nu < 1$, and $\beta > 0$,

$$\hat{D}_x^{(\nu,\beta)}\, u(x) = f(x), \qquad\qquad D_x^{(\nu,\beta)} = (1+\partial_x^\nu)^\beta, \tag{58}$$

where the unknown function $u(x)$ is expressed in terms of $f(x)$ as

$$\begin{aligned} u(x) := u_{(\nu,\beta)}(x) = \hat{D}_x^{(\nu,-\beta)} f(x) &= \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-s} s^{\beta-1} e^{-s\partial_x^\nu} ds\; f(x) \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-s} s^{\beta-1} \left( \int_0^\infty g_\nu(\eta) f\left(x - \eta\, s^{\nu^{-1}}\right) d\eta \right) ds. \end{aligned} \tag{59}$$

In Figure 2, solutions $u(x)$ are shown for different values of $\nu$ and $\beta$ and Gaussian or super-Gaussian initial function $f(x)$.




(A) $u(x)$ for $\nu = \frac{1}{2}$, different $\beta$ values, and Gaussian initial function $f(x)$.

(B) $u(x)$ for $\nu = \frac{1}{3}$, different $\beta$ values, and super-Gaussian initial function $f(x)$.

Figure 2. Solutions $u(x)$ of the problem (59) for different $\beta$ values with A) $\nu = \frac{1}{2}$ and Gaussian initial function $f(x)$ and B) $\nu = \frac{1}{3}$ and super-Gaussian initial function $f(x)$.

## 7. Computational perspectives.

Although the present work is primarily focused on symbolic calculus, special functions, and operator theory, the proposed operatorial and umbral framework also exhibits significant computational advantages. In contrast with standard discretization-based methods for fractional differential equations, the present approach reformulates fractional operators and transcendental kernels in terms of integral transforms and operational representations involving elementary functions. This allows analytical manipulations such as composition of operators and evaluation of special functions to be performed in a systematic way, often leading to computationally tractable integral representations.

To illustrate this aspect, we consider the fractional heat-type evolution problem in (51) with initial datum $F(x,0) = e^{-x^2}$. Within the proposed formalism, the solution can be represented as

$$F_\alpha(x,t) = \int_0^\infty g_\alpha(\eta)\, e^{-\left(x - t^{1/\alpha}\eta\right)^2} d\eta, \tag{60}$$

where $g_\alpha(\eta)$ denotes the Lévy stable density introduced in the previous sections. This representation provides a direct computational strategy for evaluating the solution. Indeed, instead of discretizing the fractional derivative operator in (51), the problem reduces to the numerical evaluation of a one-dimensional integral involving the initial condition and the Lévy kernel. Standard quadrature techniques can therefore be applied directly to the integral representation.

In general, explicit closed-form expressions of $g_\alpha$ in terms of classical special functions are available only in special cases, while in the general setting such representations are not available in closed form, particularly when combined with additional kernel structures. This motivates the numerical evaluation of (60). As an illustrative example, we consider $\alpha = \frac{1}{3}$ and compute numerically the solution for different values of the time variable.

The numerical procedure relies on the observation that the fractional dynamics is fully encoded in the Lévy kernel $g_\alpha(\eta)$, while the spatial dependence is retained in the initial datum. The integral representation (60) is evaluated over a truncated domain $[0, \eta_{\max}]$, where the truncation is justified by the algebraic decay of the Lévy density $g_\alpha(\eta) \simeq \eta^{-1-\alpha}, \eta \to \infty$ [7, 26]. Consequently, the contribution of the far-field region becomes negligible, and $\eta_{\max}$ is chosen such that the truncation error satisfies $\int_{\eta_{\max}}^{\infty} g_\alpha(\eta)\, d\eta < \varepsilon$ for a prescribed tolerance $\varepsilon > 0$. In practice, the exponential factor in (60) further improves convergence. The truncated integral is then approximated via a quadrature rule on a finite grid $\{\eta_k\}_{k=1}^{N}$:

$$F_\alpha(x,t) \approx \sum_{k=1}^{N} w_k\, g_\alpha(\eta_k)\, e^{-(x - t^{1/\alpha}\eta_k)^2}, \tag{61}$$

where the $w_k$ denote the quadrature weights associated with the chosen integration scheme on $[0, \eta_{\max}]$. The resulting approximation is evaluated for each selected value of $(x, t)$ in the computational domain. A numerical consistency check was performed by evaluating, in Mathematica, the integral representation with increasingly stringent numerical parameters (integration domain, accuracy goals, and working precision). In all tested cases, the computed values remained stable within numerical precision. Representative results differed by an error less than $10^{-8}$, indicating that the numerical evaluation of the integral representation is robust and reliable for the considered range of parameters. This procedure avoids any discretization of the fractional derivative operator itself, reducing the computation to the evaluation of a weighted integral involving the initial datum. The resulting profiles are reported in Figure 3, highlighting the characteristic anomalous propagation associated with fractional dynamics and confirming the effectiveness of the proposed operational framework.

8. **Conclusion.** The proposed methodology, based on operatorial methods, umbral calculus, and special function manipulation proved to be a powerful tool for dealing with non-trivial integrals including products of special functions or for the study of fractional PDEs. The use of formal operators and umbral techniques has made it possible to obtain compact symbolic representations and to simplify otherwise complex calculations, providing a unified theoretical framework and explicit solutions in closed form, establishing direct connections between differential operators and integral transforms. In particular, the use of umbral images has facilitated the manipulation of special functions such as Hermite polynomials, Wright or Bessel functions, or other families of equations closely related to fractional PDEs. The given examples show that the methods discussed so far are not just a technical refinement, but a conceptual tool capable of unifying many mathematical structures

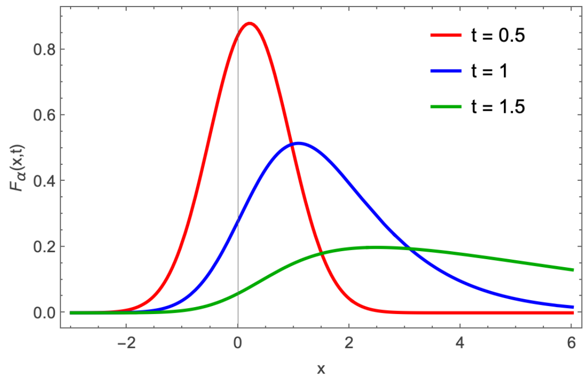



FIGURE 3. Numerical evaluation of the solution (60) for $f(x) = e^{-x^2}$ and $\alpha = \frac{1}{3}$ at different times.

under one single formalism, improving the understanding of anomalous diffusive phenomena.

Future prospects include the extension of these methods to wider classes of fractional PDEs, inclusion with advanced integral transforms, and the development of new representations based on pseudo-differential operators. In addition, the intersection of umbral calculus, integral kernel theory, and numerical techniques represents a promising field for applications in numerical analysis, quantum physics, engineering, and finance. The implementation of these methodologies in symbolic computing and artificial intelligence environments can find wide research applications, e.g. by automating the analysis and classification of solutions for differential systems of increasing complexity. The work currently under way is specifically concerned with the development of the described methodologies in the methods for optimization of neural networks in deep learning.

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