THE HK_r-INTEGRAL IS NOT CONTAINED IN THE P_r-INTEGRAL

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Dedicated to the memory of Norman Zarcone

Abstract. We compare a Perron-type integral with a Henstock-Kurzweil-type integral, both having been introduced to recover functions from their generalized derivatives defined in the metric L^r. We give an example of an HK_r-integrable function which is not P_r-integrable, thereby showing that the first integral is strictly wider than the second one.

1. History and aim

It is known that the Lebesgue integral does not integrate all derivatives. To solve the problem of recovering a function from its derivative Denjoy, at the beginning of the 1900s, introduced a very complicated process of integration that he called totalization and that became known as the special Denjoy integral (or D^*-integral). About the same time, Lusin gave a descriptive definition of the D^*-integral, having defined the class of ACG^*-functions which characterizes the D^*-primitives in the same way that the class of absolutely continuous functions characterizes the indefinite Lebesgue integrals. A little bit later Perron [15] solved the same problem of recovering a primitive by using a method based on approximation of the primitive by major and minor functions (see also [16]). At last in the 1960s Henstock and Kurzweil introduced a very simple Riemann-type integral to handle all derivatives (see [6],[7],[8],[9]). All those approaches turned out to lead to integrals equivalent to the D^*-integral (see [5]).

Similar problems of recovering primitives in terms of various generalized derivatives arise in many areas of analysis. For example, in harmonic analysis, a problem of recovering coefficients of series with respect to an orthogonal system from their sums can be reduced to the integration of an appropriate generalized derivative chosen in accordance with the considered system. In classical harmonic analysis, integration of the approximate symmetric derivative solves this problem of recovering the coefficients of trigonometric series (see [23]), while in the case of series with respect to characters of dyadic Cantor groups or of its generalizations the dyadic and p-adic derivatives and derivatives with respect to various derivate bases do the job (see [13], [14], [19], [20] and [21]). Generalizations of Denjoy, Perron and

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Henstock-Kurzweil integrals were introduced to integrate each of those generalized derivatives. It is remarkable that such generalized Perron-type integrals were, as a rule, equivalent to the respective Henstock-Kurzweil-type integrals as well as to descriptively defined Denjoy-Lusin-type integrals (see [19] [21] and [24]). All these integrals were shown to possess so called Hake property, i.e., cover a respectively defined improper integral (see [2], [18] and [22]). These definitions were extended also to functions defined or ranging in some kind of abstract spaces (see [1], [17]).

In this paper we consider integrals defined to deal with another type of derivative, the L^r-derivative, i.e., a derivative in the metric L^r. It was introduced by Calderón and Zygmund in 1961 in order to establish pointwise estimates for solutions of elliptic partial differential equation (see [3]). Gordon [4] in 1968 described a Perron-type integral, the P_r-integral, that recovers a function from its L^r-derivative, and considered an application of the L^r-derivative and the P_r-integral to Fourier series. In 2004, Musial and Sagher [10] defined the L^r-Henstock-Kurzweil integral, the HK_r-integral, that also recovers a function from its L^r-derivative, and showed that it is an extension of the P_r-integral. They also obtained a Lusin-type descriptive definition of the HK_r-integral in terms of ACG_r-functions. Some other properties of the HK_r-integral were investigated recently in [11] and [12]. But it has been an open problem since 2004 as to whether the P_r-integral integrates all HK_r-integrable functions.

Here we show that, in contrast to the classical case and to many other cases related to generalized derivatives mentioned above, the HK_r-integral is not equivalent to the P_r-integral. More precisely, we construct an example of an HK_r-integrable function which is not P_r-integrable and thereby show that the HK_r-integral is strictly wider than the P_r-integral.

2. Definitions and preliminary results

Throughout this paper we assume that $r \geq 1$ and we work on the closed interval $[a, b]$. We begin by giving the definitions of the L^r-derivates and the L^r-derivative.

Definition 2.1 ([4])**.** Let $f \in L^r[a, b]$. We define the upper right L^r-derivate of f at x, denoted by $D_r^+ f(x)$, to be the greatest lower bound of all α such that

$$\left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - \alpha t]_+^r \, dt\right)^{\frac{1}{r}} = o(h) \text{ as } h \to 0^+. \tag{2.1}$$

If no real number α satisfies (2.1), we set $D_r^+ f(x) = +\infty$. If (2.1) holds for every real number α, we set $D_r^+ f(x) = -\infty$.

We define the lower right L^r-derivate, $D_{+,r} f(x)$, the upper left L^r-derivate, $D_r^- f(x)$, and the lower left L^r-derivate, $D_{-,r} f(x)$, in a similar manner.

Definition 2.2 ([4])**.** We define the upper (two-sided) L^r derivate as follows:

$$\overline{D}_r f(x) = \max\left\{D_r^+ f(x), D_r^- f(x)\right\}.$$

Similarly we define the lower (two-sided) L^r-derivate as follows:

$$\underline{D}_r f(x) = \min\left\{D_{+,r} f(x), D_{-,r} f(x)\right\}.$$

Definition 2.3 ([4])**.** Let $f \in L^r[a, b]$. If $\overline{D}_r f(x)$ and $\underline{D}_r f(x)$ are the same real number, i.e., if all four L^r-derivates are equal and finite, then we say that f is

L^r-differentiable at x. The common value, denoted by $f_r'(x)$, is the L^r-derivative of f at x.

If f is L^r-differentiable at x, then $f_r'(x)$ is the unique real number α such that

$$\left(\frac{1}{h}\int_{-h}^{h}\left|f\left(x+t\right)-f\left(x\right)-\alpha t\right|^{r}dt\right)^{\frac{1}{r}}=o\left(h\right).$$

It is clear that if a function f is differentiable at a point x then it is also L^r-differentiable at the same point and $f_r'(x) = f'(x)$.

To define L^r-major functions and L^r-minor functions, we need a notion of L^r-continuity.

Definition 2.4 ([4])**.** A function $F \in L^r[a,b]$ is said to be L^r-continuous at $x \in [a,b]$ if

$$\lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h}\left|F\left(y\right)-F\left(x\right)\right|^{r}dy=0.$$

If F is L^r-continuous for all $x \in E$, we say that F is L^r-continuous on E.

Definition 2.5 ([4])**.** Suppose f is a function defined on $[a,b]$. A finite-valued function $\psi \in L^r[a,b]$ is said to be an L^r-major function of f if

(1) $\psi(a) = 0$,
(2) ψ is L^r-continuous on $[a,b]$,
(3) except for at most a denumerable subset of $[a,b]$ we have

$$-\infty \neq \underline{D}_r\psi(x) \geq f(x). \tag{2.2}$$

A function ϕ is an L^r-minor function of f if $-\phi$ is an L^r-major function of $-f$.

It was proved in [4] that for any L^r-major function ψ and any L^r-minor function ϕ of f, the function $\psi - \phi$ is non-decreasing on $[a,b]$. This property allows us to define the Perron-type P_r-integral in a standard way:

Definition 2.6 ([4])**.** Suppose f is a function defined on $[a,b]$. If $\inf\psi(b)$ taken over all L^r-major functions of f equals $\sup\phi(b)$ taken over all L^r-minor functions of f, then the common value, denoted by

$$(P_r)\int_a^b f$$

is called the P_r-integral of f on $[a,b]$, and f is said to be P_r-integrable on $[a,b]$.

Remark 2.7. We cannot avoid the exceptional set for the inequality (2.2) in the definition of L^r-major and L^r-minor functions without losing the so-called Hake property of the P_r integral (see Example 1, §7 in [4]). This fact shows also that, in contrast to the classical case (see [5]), the requirement that the inequality (2.2) must hold everywhere leads to an integral that is more narrow than the original L. Gordon integral.

Now we recall the definition of L^r-Kurzweil-Henstock-type integral given in [10]. In what follows a *tagged interval* is a pair $(x,[c,d])$ where $x \in [c,d]$ is a *tag*, $[c,d] \subset [a,b]$, and a *gauge* is a strictly positive function δ on $[a,b]$. We say that $(x,[c,d])$ is *δ-fine* if $[c,d] \subset [a,b] \cap [x-\delta(x), x+\delta(x)]$.

Definition 2.8. A function $f : [a,b] \to \mathbb{R}$ is L^r-Henstock-Kurzweil integrable (HK_r-integrable) on $[a,b]$ if there exists a function $F \in L^r[a,b]$ so that for any $\varepsilon > 0$ there exists a gauge δ so that for any finite collection of nonoverlapping δ-fine tagged intervals $\mathcal{Q} = \{(x_i, [c_i, d_i]), 1 \le i \le q\}$ we have

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} < \varepsilon.$$

By Theorem 5 in [10], the function F in Definition 2.8 is unique up to an additive constant, so we can consider the indefinite HK_r-integral

$$F(x) = (HK_r) \int_a^x f, \text{ for each } x \in (a,b].$$

It was also proved in [10] that the indefinite HK_r-integral F is L^r-continuous on $[a,b]$ with $F_r'(x) = f(x)$ a.e. on $[a,b]$.

One of the main results in [10] was the following

Theorem 2.9. *If $f : [a,b] \to R$ is P_r-integrable then it is HK_r-integrable and the values of integrals coincide.*

An equivalent descriptive definition of the HK_r-integral was obtained in [10] using following absolute continuity condition.

Definition 2.10 ([10])**.** Let $E \subset [a,b]$. We say that $F \in AC_r(E)$ if for all $\varepsilon > 0$ there exist $\eta > 0$ and a gauge δ defined on E so that for any finite collection of nonoverlapping δ-fine tagged intervals $\{(x_i, [c_i, d_i]), 1 \le i \le q\}$ having tags in E and such that $\sum_{i=1}^{q}(d_i - c_i) < \eta$ we have

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r dy \right)^{1/r} < \varepsilon. \tag{2.3}$$

Definition 2.11. We say that $F \in ACG_r(E)$ if E can be written as $E = \cup_{n=1}^{\infty} E_n$ where $F \in AC_r(E_n)$ for all n.

Now a descriptive characterization of the HK_r-integral is given by the following result in [10].

Theorem 2.12. *A function f is HK_r-integrable on $[a,b]$ if and only if there exists $F \in ACG_r[a,b]$ such that $F_r' = f$ a.e.; the function $F(x) - F(a)$ being the indefinite HK_r-integral of f.*

Remark 2.13. It is an open problem whether the class $ACG_r[a,b]$ coincides with the class of HK_r-primitives. This problem is equivalent to the question of whether each function in the class $ACG_r[a,b]$ is L^r-differentiable a.e.

3. Main result

We show here that the converse of Theorem 2.9 is not true, i.e., the class of P_r-integrable functions is strictly included in the class of HK_r-integrable functions. Namely we prove the following

Theorem 3.1. *There exists a function which is HK_r-integrable on $[a,b]$ but which is not P_r integrable on $[a,b]$.*

Proof. We construct a function $F \in ACG_r[0,1]$ such that F'_r exists a.e., and so by Theorem 2.12, F is the HK_r integral of its L^r-derivative $f := F'_r$. We then show that f is not P_r integrable.

Let $P \subset [0,1]$ be a symmetric perfect Cantor-type set with contiguous intervals u_n of rank $n = 1, 2, \ldots$ each having length $|u_n| = 3^{-n}2^{-n+2}$. The set which is left after removing all contiguous intervals up to rank n from $[0,1]$ is constituted by 2^n segments (closed intervals) r_n of total length 3^{-n} which are called *segments of rank* n. So $|P| = 0$. Note that each u_n is the interval concentric with some interval r_{n-1} (we put $r_0 = [0,1]$).

Let v_n be the interval concentric with u_n such that

$$|v_n| = 6^{-rn}|u_n| = 3^{-(r+1)n}2^{-(r+1)n+2}. \tag{3.1}$$

Now we define a function F which will serve as the indefinite HK_r-integral for its derivative. We put $F(x) = 0$ outside of the union of intervals v_n of all rank, i.e., on the set P and on each set $u_n \setminus v_n$. We put $F(x) = 2^{n/r}$ if $x \in v_n$. We want the function F to be differentiable on u_n. It is clear how to make it smooth changing it in small neighborhoods of endpoints of v_n, without influencing further estimations. We keep the same notation F for the modified function, but to simplify computation we shall allow ourselves to treat it as if it has its original constant values on all v_n and $u_n \setminus v_n$. So we have

$$\int_{u_n} F^r = \int_{v_n} F^r = \frac{4}{3^{(r+1)n}2^{rn}} \tag{3.2}$$

and

$$\frac{1}{|u_n|}\int_{u_n} F^r = \frac{1}{3^{rn}2^{(r-1)n}}. \tag{3.3}$$

Summing (3.2) over all intervals u_n we get

$$\int_0^1 F^r = \sum_{n=1}^{\infty} 2^{n-1}\frac{4}{3^{(r+1)n}2^{rn}} = \frac{2}{3^{r+1}2^{r-1} - 1}.$$

Hence $F \in L^r[0,1]$

Note that F is differentiable a.e. on $[0,1]$. To show that

$$f(x) = \begin{cases} F'(x) & \text{at } x \in [0,1] \setminus P, \\ 0 & \text{at } x \in P \end{cases}$$

is HK_r-integrable with F being its indefinite integral, we check that $F \in ACG_r[0,1]$. In turn, this is reduced to checking that $F \in AC_r(P)$.

We have to show that for any $\varepsilon > 0$ we can find $\eta > 0$ (and a gauge δ, but in fact our gauge can be arbitrary) so that for any partition $\{(x_i, I_i)\}$ tagged in P the inequality $\sum_i |I_i| < \eta$ implies $\sum_{i=1}\left(|I_i|^{-1}\int_{I_i} F^r\right)^{1/r} < \varepsilon$.

Choose n such that $(2/3)^n < \varepsilon/8$ and take $\eta = \frac{1}{2}(|u_n| - |v_n|)$. Let $\{(x_i, I_i)\}$ be any partition tagged in P with $\sum_i |I_i| < \eta$. Then $|I_i| < \frac{1}{2}(|u_n| - |v_n|)$ and so $I_i \cap v_k = \emptyset$ and $\int_{I_i \cap u_k} F = 0$ for any $k \leq n$ and each i. At the same time if $I_i \cap v_k \neq \emptyset$ for $k > n$, then $|I_i \cap u_k| > \frac{1}{2}(|u_k| - |v_k|) > \frac{1}{4}|u_k|$ and by (3.3) we have

$$\frac{1}{|I_i \cap u_k|}\int_{I_i \cap u_k} F^r \leq \frac{4}{|u_k|}\int_{u_k} F^r = \frac{4}{3^{rk}2^{(r-1)k}} \leq \frac{4}{3^{rk}}. \tag{3.4}$$

Note that an interval I_i can be represented, up to a set of measure zero, as the union of its non-void intersections with intervals u_k of various ranks. If $k \leq n$, then $\int_{I_i \cap u_k} F = 0$, while if $k > n$ then (3.4) holds.

So we can estimate $\frac{1}{|I_i|}\int_{I_i} F^r$ for each i using the following obvious inequality for positive numbers

$$\frac{\sum_k a_k}{\sum_k b_k} < \sum \frac{a_k}{b_k} \tag{3.5}$$

(this is true for infinite sums provided the series are convergent).

We note that there are 2^{k-1} intervals of rank k. We also note that an interval u_k with $k > n$ can have non-empty intersection with no more than two different I_i because each such interval must contain either an interval to the right of the left endpoint of u_k or an interval to the left of the right endpoint of u_k and note that for each of these intervals (3.4) applies. Now summing up over i, using (3.4) and (3.5), and with the assumption that the average value of F over an empty interval is zero, we finally obtain

$$\sum_i \left(\frac{1}{|I_i|}\int_{I_i} F^r\right)^{1/r} \leq \sum_i \sum_{u_k : I_i \cap u_k \neq \emptyset} \left(\frac{1}{|I_i \cap u_k|}\int_{I_i \cap u_k} F^r\right)^{1/r} \leq$$
$$\leq \sum_{k=n+1}^{\infty} 2^k \left(\frac{4}{3^{rk}}\right)^{1/r} \leq 8\left(\frac{2}{3}\right)^n < \varepsilon.$$

So f is HK_r-integrable with F being its indefinite integral.

We show now that f has no L^r-minor (as well as no L^r-major) functions. Assuming that such a minor function m exists, the difference $R := F - m$ is non-decreasing on $[0, 1]$ (see [4]). To show that this assumption leads to a contradiction, it is enough to prove that for any non-decreasing function R with $R(0) = 0$ we have $D_r^+(F(x) - R(x)) = +\infty$ on an uncountable set, and so the function $F - R$ cannot be an L^r-minor function. In fact the above equality holds at any point of the set P which is not a left endpoint of any contiguous interval to P. Let x be such a point. Then for any N we can find $n > N$ such that $x \in r_n \subset r_{n-1}$ and r_n is the left of two segments of rank n which are subsets of r_{n-1}. Let u_n be the contiguous interval of rank n which is concentric with r_{n-1}. Note that $|u_n|$ is four times the length of each of the two segments of rank n that straddle it and that u_n is to the right of x. Take h_n so that $u_n \subset (x, x + h_n)$ and so that

$$h_n < 2|u_n| = 3^{-n}2^{-n+3}. \tag{3.6}$$

We are going to show that for a chosen x and for any real α

$$\limsup_{n\to\infty} \frac{1}{h_n^{r+1}} \int_0^{h_n} [(F(x+t) - R(x+t) + R(x) - \alpha t]_+^r \, dt = +\infty. \tag{3.7}$$

We can assume that n is chosen so that $2^{\frac{n}{r}-1} > R(1) + |\alpha|$. Then $[(F(x+t) - R(x+t) + R(x) - \alpha t]_+ > 2^{\frac{n}{r}-1}$ if $x + t \in v_n \subset u_n$ and using (3.1) and (3.6) we finally obtain

$$\frac{1}{h_n^{r+1}} \int_0^{h_n} [(F(x+t) - R(x+t) + R(x) - \alpha t]_+^r \, dt >$$
$$\frac{|v_n| 2^{n-r}}{h_n^{r+1}} \geq \frac{3^{-(r+1)n}2^{-(r+1)n+2}2^{n-r}}{3^{-(r+1)n}2^{-(r+1)n+3(r+1)}} = 2^{n-4r-1}.$$

This shows that (3.7) holds and that no real α satisfies (2.1) for the function $F-R$ at the considered x. Therefore $\overline{D}_r(F-R)(x) = +\infty$ on an uncountable set and $F-R$ is a L^r-minor function for no R. This proves that f has no L^r-minor function for any $r \geq 1$ and so f is not P_r-integrable. The theorem is proved. $\square$

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