MULTIPLE SOLUTIONS WITH SIGN INFORMATION FOR A $(p,2)$-EQUATION WITH COMBINED NONLINEARITIES

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Abstract. We consider a parametric nonlinear Dirichlet problem driven by the sum of a p-Laplacian and of a Laplacian (a $(p,2)$-equation) and with a reaction which has the competing effects of two distinct nonlinearities. A parametric term which is $(p-1)$-superlinear (convex term) and a perturbation which is $(p-1)$-sublinear (concave term). First we show that for all small values of the parameter the problem has at least five nontrivial smooth solutions, all with sign information. Then by strengthening the regularity of the two nonlinearities we produce two more nodal solutions, for a total of seven nontrivial smooth solutions all with sign informations. Our proofs use critical point theory, critical groups and flow invariance arguments.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric $(p,2)$-equation:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) - \Delta u(z) = \lambda f(z,u(z)) + g(z,u(z)) & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad \lambda > 0,\ 2 < p < +\infty. \end{cases}$$

Here for $r \in (1,+\infty)$ by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

When $r = 2$, we have the usual Laplace differential operator denoted by Δu. In problem (P_λ) the differential operator is the sum of a p-Laplacian and a Laplacian and so it is not homogeneous. This is a source of difficulties in the analysis of (P_λ). The reaction (right hand side of (P_λ)) has the combined effects of two nonlinearities with different growth properties. The parametric term $\lambda f(z,x)$ with $f(z,x)$ a Carathéodory function (that is, for all $x \in \mathbb{R}$ $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$ $x \to f(z,x)$ is continuous), is $(p-1)$-superlinear in the x-variable, but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). Instead we employ a weaker condition which permits the consideration of $(p-1)$-superlinear maps which have "slower" growth near $\pm\infty$. The perturbation $g(z,x)$ is also a Carathéodory function, which is strictly $(p-1)$-sublinear near $\pm\infty$. So, we have a "concave-convex" problem, but in our case the parametric term is the "convex", in contrast to what we encounter in the literature, where the parametric term is the "concave". Moreover, usually in the literature the reaction of such problems has the special form $x \to \lambda|x|^{q-2}x + |x|^{r-2}x$ with $q < p < r$ (see Ambrosetti-Brezis-Cerami [1], who initiated the study of such problems and Garcia Azero-Manfredi-Peral Alonso [10]). Using variational tools and the

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theory of critical groups, we show that for all small parameter values $\lambda > 0$, problem (P_λ) has at least five nontrivial smooth solutions all with sign information (two positive, two negative and a nodal solution). If we strengthen the regularity of $f(z,\cdot)$ and $g(z,\cdot)$, we show that we can have two more nodal solutions, for a total of seven nontrivial smooth solutions, all with sign information.

Recently parametric $(p,2)$-equations were investigated by Papageorgiou-Rădulescu [17, 18], Papageorgiou-Winkert [25], He-Lei-Zhang-Sun [13], Papageorgiou-Scapellato [22], Papageorgiou-Vetro-Vetro [23]. Closer to our work, are those of He-Lei-Zhang-Sun [13], Papageorgiou-Scapellato [22]. In [13] the authors deal with a nonparametric equation with $(p-1)$-superlinear reaction and using the results of [18] and a flow invariance technique, produce seven nontrivial solutions with sign information. In [22] the authors consider a reaction of the form $\lambda|x|^{p-2}+f(z,x)$ with the perturbation $f(z,\cdot)$ being $(p-1)$-superlinear. They show that when the parameter $\lambda > 0$ is big the problem has three or four nontrivial smooth solutions with sign information.

We mention that $(p,2)$-equations arise in many mathematical models of physical processes. We mention problems in elasticity theory, where we encounter composites made of two different materials with distinct hardening exponents, see Zhikov [29]. But there are also other physical applications leading to $(p,2)$-equations. We mention the works of Bahrouni-Rădulescu-Repovš [2] on transonic flow problems, of Benci-D'Avenia-Fortunato-Pisani [4] on quantum physics and of Cherfils-Il'yasov [6] on reaction diffusion systems. Finally we mention the works on eigenvalue problems for such operators of Bhattacharaya-Emamizadeh-Farjudian [3], Bobkov-Tanaka [5], Chorfi-Rădulescu [7], Papageorgiou-Rădulescu-Repovš [20], Tanaka [28].

2. Mathematical Background

In the analysis of problem (P_λ) we will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \quad \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} < 0 \right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

We consider the following linear eigenvalue problem

$$-\Delta u(z) = \widehat{\lambda} u(z) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{1}$$

By $\widehat{\sigma}(2)$ we denote the set of distinct eigenvalues of (1). We know that $\widehat{\sigma}(2) = \{\widehat{\lambda}_k(2)\}_{k\geq 1}$ with $\widehat{\lambda}_k(2) \to +\infty$ as $k \to +\infty$. By $E(\widehat{\lambda}_k(2))$ $k \in \mathbb{N}$, we denote the corresponding eigenspace. Standard regularity theory (see Gilbarg-Trudinger [12]), implies that $E(\widehat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$. Also, we have the following orthogonal direct sum decomposition of $H_0^1(\Omega)$

$$H_0^1(\Omega) = \overline{\bigoplus_{k\geq 1} E(\widehat{\lambda}_k(2))}.$$

Moreover, each eigenspace $E(\widehat{\lambda}_k(2))$ has the "Unique Continuation Property" (UCP for short), namely if $u \in E(\widehat{\lambda}_k(2))$ and $u(\cdot)$ vanishes on a set of positive measure, then $u \equiv 0$. The first eigenvalue $\widehat{\lambda}_1(2) > 0$ is simple, that is, $\dim E(\widehat{\lambda}_1(2)) = 1$. The eigenvalues have the following variational characterizations:

$$\widehat{\lambda}_1(2) = \inf\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0\right], \tag{2}$$

$$\begin{aligned} k \geq 2, \quad \widehat{\lambda}_k(2) &= \sup\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \overline{H}_k = \bigoplus_{i=1}^{k} E(\widehat{\lambda}_i(2)), u \neq 0\right] \\ &= \inf\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \widehat{H}_k = \overline{\bigoplus_{i\geq k} E(\widehat{\lambda}_i(2))}, u \neq 0\right]. \end{aligned} \tag{3}$$

The infimum in (2) is attained at $E(\widehat{\lambda}_1(2))$ and it is easy to see that the elements of this eigenspace have fixed sign. In (3) both the supremum and the infimum are realized on $E(\widehat{\lambda}_k(2))$. The eigenfunctions corresponding to an eigenvalue $\widehat{\lambda}_k(2) > 0$, $k \geq 2$, are all nodal (sign changing). Using these facts, we can easily establish the following useful inequalities (see, for example, D'Agui-Marano-Papageorgiou [8]).

Lemma 1. *(a) If $k \in \mathbb{N}$, $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \widehat{\lambda}_k(2)$ for a.a. $z \in \Omega$, $\eta \not\equiv \widehat{\lambda}_k(2)$, then*

$$c_0\|u\|^2 \leq \|\nabla u\|_2^2 - \int_\Omega \eta(z)u^2 dz \quad \text{for some } c_0 > 0, \text{ all } u \in \widehat{H}_k.$$

(b) If $k \in \mathbb{N}$, $\eta \in L^\infty(\Omega)$, $\eta(z) \geq \widehat{\lambda}_k(2)$ for a.a. $z \in \Omega$, $\eta \not\equiv \widehat{\lambda}_k(2)$, then

$$\|\nabla u\|_2^2 - \int_\Omega \eta(z)u^2 dz \leq -c_1\|u\|^2 \quad \text{for some } c_1 > 0, \text{ all } u \in \overline{H}_k.$$

We will also consider a weighted version of (1). So, let $m \in L^\infty(\Omega)$, $m(z) \geq 0$ for a.a. $z \in \Omega$, $m \not\equiv 0$. We consider the following weighted linear eigenvalue problem

$$-\Delta u(z) = \widetilde{\lambda} m(z)u(z) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{4}$$

In the same way as for (1), using the spectral theorem for compact self-adjoint operators on a Hilbert space, we show that the spectrum $\widetilde{\sigma}(2,m)$ of (4) consists of a sequence $\{\widetilde{\lambda}_k(2,m)\}_{k\geq 1}$ with $\widetilde{\lambda}_k(2,m) \to +\infty$ as $k \to +\infty$. These eigenvalues have the same properties as $\widehat{\lambda}_k(2)$, $k \in \mathbb{N}$. Of course if $m(z) = 1$ for a.a. $z \in \Omega$, then $\widetilde{\lambda}_k(2,m) = \widehat{\lambda}_k(2)$. The map $m \to \widetilde{\lambda}_k(2,m)$ has the following monotonicity property (see [8]).

Lemma 2. *If $m_1, m_2 \in L^\infty(\Omega)$, $0 \leq m_1(z) \leq m_2(z)$ for a.a. $z \in \Omega$, $m_1 \not\equiv 0$, $m_1 \not\equiv m_2$, then $\widetilde{\lambda}_k(2,m_2) < \widetilde{\lambda}_k(2,m_1)$ for all $k \in \mathbb{N}$.*

For the second eigenvalue $\widehat{\lambda}_2 > 0$, we have also an alternative mini-max characterization (see, for example, Papageorgiou-Rădulescu [19]), which is more convenient for our arguments. So, let $\partial B_1^{L^2} = \{u \in L^2(\Omega) : \|u\|_2 = 1\}$ and $M = H_0^1(\Omega) \cap \partial B_1^{L^2}$. Also, let $\widehat{u}_1$ be the positive, L^2-normalized (that is, $\|\widehat{u}_1\|_2 = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(2) > 0$ (recall that the elements of $E(\widehat{\lambda}_1(2))$ have fixed sign). We know that $\widehat{u}_1 \in C_+ \setminus \{0\}$ and the Hopf maximum principle implies that $\widehat{u}_1 \in \operatorname{int} C_+$. We introduce the following set of paths in $H_0^1(\Omega)$

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1], M) : \widehat{\gamma}(-1) = -\widehat{u}_1, \widehat{\gamma}(1) = \widehat{u}_1\}.$$

Lemma 3. $\widehat{\lambda}_2(2) = \inf_{\widehat{\gamma}\in\widehat{\Gamma}} \max_{-1\leq t\leq 1} \|\nabla\widehat{\gamma}(t)\|_2^2$.

For $r \in (1, +\infty)$, let $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^* \left(\frac{1}{r} + \frac{1}{r'} = 1\right)$, we denote the nonlinear operator defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

If $r = 2$, then $A_2 = A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$. The operator $A_r(\cdot)$ has the following properties (see Papageorgiou-Rădulescu-Repovš [21]).

Lemma 4. *The map $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,r}(\Omega)$ and $\limsup_{n\to+\infty}\langle A_r(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,r}(\Omega)$).*

If $x \in \mathbb{R}$, then we set $x^\pm = \max\{\pm x, 0\}$. Given $u \in W_0^{1,p}(\Omega)$, we define

$$u^\pm(\cdot) = u(\cdot)^\pm.$$

We know that $u^\pm \in W_0^{1,p}(\Omega)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$. If $v, u \in W_0^{1,p}(\Omega)$ and $v \leq u$, then we define

$$[v, u] = \left\{y \in W_0^{1,p}(\Omega) : v(z) \leq y(z) \leq u(z) \quad \text{for a.a. } z \in \Omega\right\}.$$

By $\mathrm{int}_{C_0^1(\overline{\Omega})}[v, u]$, we denote the interior in the $C_0^1(\overline{\Omega})$-norm topology of $[v, u] \cap C_0^1(\overline{\Omega})$. In the sequel by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(\Omega), W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*) \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$.

We say that a set $S \subseteq W_0^{1,p}(\Omega)$ is downward (resp. upward) directed, if for all $u_1, u_2 \in S$ we can find $u \in S$ such that $u \leq u_1$, $u \leq u_2$ (resp. for all $v_1, v_2 \in S$ we can find $v \in S$ such that $v_1 \leq v$, $v_2 \leq v$).

If $h, \widehat{h} \in L^\infty(\Omega)$, then $h \prec \widehat{h}$ if for all $K \subseteq \Omega$ compact, we have

$$0 < c_K \leq (\widehat{h} - h)(z) \quad \text{for a.a. } z \in K.$$

Evidently if $h, \widehat{h} \in C(\Omega)$ and $h(z) < \widehat{h}(z)$ for all $z \in \Omega$, then $h \prec \widehat{h}$.

By $\delta_{k,m}$ we denote the Kronecker symbol, namely

$$\delta_{k,m} = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}, \quad k, m \in \mathbb{N}_0.$$

Given a measurable function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, we set

$$N_f(u)(\cdot) = f(\cdot, u(\cdot))$$

for all $u : \Omega \to \mathbb{R}$ measurable (the Nemytski map corresponding to $f(\cdot,\cdot)$).

As we already mentioned in the Introduction our approach involves also critical groups. So, let us recall some basic definitions and facts from the theory of critical groups, which we will need in the sequel.

Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \leq c\},$$

$$K_\varphi = \{u \in X : \varphi'(u) = 0\},$$
$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Given a topological pair (Y_1, Y_2) with $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}-relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. If $u \in K_\varphi$ is isolated and $c = \varphi(u)$ (that is, $u \in K_\varphi^c$), then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0.$$

Here U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the isolating neighborhood U of u.

Recall that $\varphi(\cdot)$ satisfies the "C-condition", if every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence.

Suppose that φ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see Papageorgiou-Rădulescu-Repovš [21], p. 386) implies that this definition is independent of the choice of $c < \inf \varphi(K_\varphi)$.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and that K_φ is finite. We define

$$M(t, u) = \sum_{k\in\mathbb{N}_0} \text{rank } C_k(\varphi, u)t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi,$$
$$P(t, \infty) = \sum_{k\in\mathbb{N}_0} \text{rank } C_k(\varphi, \infty)t^k \quad \text{for all } t \in \mathbb{R}.$$

Then the Morse relation says that

$$\sum_{u\in K_\varphi} M(t, u) = P(t, \infty) + (1 + t)Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{5}$$

where $Q(t) = \sum_{k\in\mathbb{N}_0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k.

3. Five Solutions

In this section we show that for $\lambda > 0$ small, problem (P_λ) has at least five nontrivial smooth solutions, all with sign information.

The hypotheses on the nonlinearities $f(z, x)$ and $g(z, x)$ are the following:

$H(f)_1$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$ $f(z, 0) = 0$, $f(z, x)x \geq 0$ for all $x \in \mathbb{R}$ and

(i) $|f(z, x)| \leq a(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$ and $p < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases}$;

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to\pm\infty} \frac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exist $q \in ((r-p)\max\{\frac{N}{p}, 1\}, p^*)$, $\beta_0 > 0$ such that

$$\beta_0 \leq \liminf_{x\to\pm\infty} \frac{f(z,x)x - pF(z,x)}{|x|^q} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) $\lim_{x\to 0} \frac{f(z,x)}{x} = 0$ uniformly for a.a. $z \in \Omega$.

Remark 1. Hypothesis $H(f)_1\,(ii)$ implies that $f(z,\cdot)$ is $(p-1)$-superlinear near $\pm\infty$. However, we do not use the AR-condition, a common hypothesis in the literature when dealing with superlinear problems. We recall that the AR-condition says that there exist $\tau > p$ and $M > 0$ such that

$$0 < \tau F(z,x) \leq f(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M, \tag{6a}$$

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, \pm M). \tag{6b}$$

Integrating (6a) and using (6b), we obtain the weaker condition

$$c_2|x|^\tau \leq F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M, \text{ some } c_2 > 0. \tag{7}$$

From (7) follows the much weaker condition

$$\lim_{x\to\pm\infty} \frac{F(z,x)}{|x|^p} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

In this paper, we replace the AR-condition (see (6a), (6b)), with the less restrictive condition $H(f)_1\,(ii)$. This way we incorporate in our framework $(p-1)$-superlinear nonlinearities with "slower" growth near $\pm\infty$. For example, consider the following two functions (for the sake of simplicity we drop the z-dependence):

$$f_1(x) = |x|^{p-2}x\ln(1+|x|) \quad \text{for all } x \in \mathbb{R},$$

$$f_2(x) = \begin{cases} x - |x|^{q-2}x & \text{if } |x| \leq 1 \\ |x|^{p-2}x\ln(|x|) & \text{if } 1 < |x| \end{cases} \quad \text{with } 2 < q < +\infty.$$

Both functions satisfy hypotheses $H(f)_1$ but fail to satisfy the AR-condition.

$H(g)_1$: $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that $|g(z,x)| \leq a_\rho(z)$ for a.a. $z \in \Omega$, all $|x| \leq \rho$;

(ii) $\lim_{x\to\pm\infty} \frac{g(z,x)}{|x|^{p-2}x} = 0$ uniformly for a.a. $z \in \Omega$;

(iii) there exist a function $\eta \in L^\infty(\Omega)$ and $\beta_1 > 0$ such that

$$\widehat{\lambda}_1(2) \leq \eta(z) \quad \text{for a.a. } z \in \Omega,\ \eta \not\equiv \widehat{\lambda}_1(2),$$

$$\eta(z) \leq \liminf_{x\to 0} \frac{g(z,x)}{x} \leq \limsup_{x\to 0} \frac{g(z,x)}{x} \leq \beta_1 \text{ uniformly for a.a. } z \in \Omega.$$

Remark 2. Evidently hypothesis $H(g)_1\,(ii)$ implies that for a.a. $z \in \Omega$, $g(z,\cdot)$ is strictly $(p-1)$-sublinear. Hypothesis $H(g)_1\,(iii)$ permits at zero partial interaction with the principal eigenvalue $\widehat{\lambda}_1(2) > 0$ (nonuniform nonresonance). The following function satisfies hypotheses $H(g)_1$ above

$$g(z,x) = \begin{cases} \eta(z)x & \text{if } |x| \le 1 \\ \eta(z)|x|^{q-2}x & \text{if } 1 < |x| \end{cases} \quad \text{with } 1 < q < p.$$

Using these conditions on the nonlinearities of the reaction of (P_λ) we will be able to produce constant sign smooth solutions for (P_λ) when $\lambda > 0$ is small. We will also show that there exist extremal constant sign solutions, that is, a smallest positive solution and a biggest negative solution for each problem (P_λ) with $\lambda > 0$ small. These extremal solutions will then lead us to a nodal solution.

We introduce the following C^1-functional $\varphi_\lambda^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda^\pm(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \lambda\int_\Omega F(z,\pm u^\pm)dz - \int_\Omega G(z,\pm u^\pm)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 1. *If hypotheses $H(f)_1$, $H(g)_1$ hold and $\lambda > 0$, then the functionals $\varphi_\lambda^\pm$ satisfy the C-condition.*

Proof. We do the proof for φ_λ^+, the proof for φ_λ^- being similar. Let $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi_\lambda^+(u_n)| \le M_1 \quad \text{for some } M_1, \text{ all } n \in \mathbb{N}, \tag{8}$$

$$(1 + \|u_n\|)(\varphi_\lambda^+)'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to +\infty. \tag{9}$$

From (9) we have

$$\left|\langle(\varphi_\lambda^+)'(u_n), h\rangle\right| \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+,$$

$$\Rightarrow \quad \left|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \lambda\int_\Omega f(z,u_n^+)hdz - \int_\Omega g(z,u_n^+)hdz\right| \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{10}$$

for all $n \in \mathbb{N}$. In (10) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ and we obtain

$$\|\nabla u_n^-\|_p^p + \|\nabla u_n^-\|_2^2 \le \varepsilon_n \quad \text{for all } n \in \mathbb{N},$$

$$\Rightarrow \quad u_n^- \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to +\infty. \tag{11}$$

From (8) and (11) we have

$$\|\nabla u_n^+\|_p^p + \frac{p}{2}\|\nabla u_n^+\|_2^2 - \lambda\int_\Omega pF(z,u_n^+)dz - \int_\Omega pG(z,u_n^+)dz \le M_2 \tag{12}$$

for some $M_2 > 0$, all $n \in \mathbb{N}$. Next in (10) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$-\|\nabla u_n^+\|_p^p - \|\nabla u_n^+\|_2^2 + \lambda\int_\Omega f(z,u_n^+)u_n^+dz + \int_\Omega g(z,u_n^+)u_n^+dz \le \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

We add (12) and (13) and recall that $2 < p$. We obtain

$$\lambda\int_\Omega [f(z,u_n^+)u_n^+ - pF(z,u_n^+)]dz + \int_\Omega [g(z,u_n^+)u_n^+ - pG(z,u_n^+)]dz \le M_3 \tag{14}$$

for some $M_3 > 0$, all $n \in \mathbb{N}$.

Hypotheses $H(f)_1\,(i),(ii)$ imply that we can find $\widehat{\beta}_0 \in (0,\beta_0)$ and $c_3 > 0$ such that

$$f(z,u_n^+)u_n^+ - pF(z,u_n^+) \ge \widehat{\beta}_0(u_n^+)^q - c_3 \quad \text{for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}. \tag{15}$$

Note that by taking $r < p^*$ close to p^* (see hypothesis $H(f)_1\,(i)$), we can have $(r-p)\max\{\frac{N}{p}, 1\} > p$ and so $q > p$ (see hypothesis $H(f)_1\,(ii)$). Then on account of hypotheses $H(g)_1\,(i), (ii)$, given $\varepsilon > 0$, we can find $c_4 = c_4(\varepsilon) > 0$ such that

$$g(z, u_n^+)u_n^+ - pG(z, u_n^+) \geq -\varepsilon (u_n^+)^q - c_4 \quad \text{for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}. \tag{16}$$

Returning to (14) and using (15) and (16), we obtain that

$$[\widehat{\beta}_0 - \varepsilon]\|u_n^+\|_q^q \leq M_4 \quad \text{for some } M_4 > 0, \text{ all } n \in \mathbb{N}.$$

Choosing $\varepsilon \in (0, \widehat{\beta}_0)$, we infer that

$$\{u_n^+\}_{n\geq 1} \subseteq L^q(\Omega) \text{ is bounded.} \tag{17}$$

<u>Case 1</u>: $N \neq p$.

It is clear from hypothes $H(f)_1\,(ii)$ that without any loss of generality, we may assume that $q \leq r < p^*$. So, we can find $t \in [0, 1)$ such that

$$\frac{1}{r} = \frac{1-t}{q} + \frac{t}{p^*}.$$

Invoking the interpolation inequality (see Papageorgiou-Winkert [24], p 116), we have

$$\begin{aligned} &\|u_n^+\|_r \leq \|u_n^+\|_q^{1-t}\|u_n^+\|_{p^*}^t, \\ \Rightarrow\ &\|u_n^+\|_r^r \leq c_5\|u_n^+\|^{tr} \quad \text{for some } c_5 > 0, \text{ all } n \in \mathbb{N}. \end{aligned} \tag{18}$$

(see (17) and use the Sobolev embedding theorem).

Hypothesis $H(f)_1\,(i)$ implies that

$$0 \leq f(z, x)x \leq c_6[1 + |x|^r] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_6 > 0. \tag{19}$$

Similarly hypotheses $H(g)_1\,(i), (ii)$ and the fact that $p < r$ imply that given $\varepsilon > 0$, we can find $c_7 = c_7(\varepsilon) > 0$ such that

$$g(z, x)x \leq \varepsilon|x|^r + c_7 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{20}$$

In (10) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ and use (19), (20). Then choosing $\varepsilon \in (0, c_6)$, we obtain

$$\begin{aligned} \|\nabla u_n^+\|_p^p &\leq c_8[1 + \|u_n^+\|_r^r] \quad \text{for some } c_8 > 0 \\ &\leq c_9[1 + \|u_n^+\|^{tr}] \quad \text{for some } c_9 > 0, \text{ all } n \in \mathbb{N}. \end{aligned} \tag{21}$$

The hypothesis on q (see $H(f)_1\,(ii)$) implies that $tr < p$. Hence from (21) we infer that

$$\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{22}$$

<u>Case 2</u>: $N = p$.

In this case $p^* = +\infty$ and by the Sobolev embedding theorem (see for example [21], p. 56), we have $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \leq s < +\infty$. So, in the previous argument (Case 1), we replace p^* by $p_0 > r$ big so that

$$tr = \frac{p_0(r-q)}{p_0 - q} \quad \text{(recall that } r - p < q).$$

Then again we have (22). From (22) and (11) it follows that

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega). \tag{23}$$

In (10) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (23). Then

$$\begin{aligned}
&\lim_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle] = 0,\\
\Rightarrow\quad &\limsup_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle] \leq 0 \quad (\text{since } A(\cdot) \text{ is monotone}),\\
\Rightarrow\quad &\limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \leq 0,\\
\Rightarrow\quad &u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Lemma 4).}
\end{aligned}$$

This proves that φ_λ^+ satisfies the C-condition. In a similar fashion we show that φ_λ^- satisfies the C-condition. ☐

Next using purely variational methods based on the critical point theory, we will show that for all $\lambda > 0$ small, problem (P_λ) has multiple constant sign solutions.

Proposition 2. *If hypotheses $H(f)_1$, $H(g)_1$ hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least four nontrivial solutions of constant sign $u_0, \widehat{u} \in \operatorname{int} C_+$ and $v_0, \widehat{v} \in -\operatorname{int} C_+$.*

Proof. First we produce the positive solutions. We work with the functional φ_λ^+.

Hypotheses $H(f)_1\,(i), (ii)$ imply that given $\varepsilon > 0$ we can find $c_{10} = c_{10}(\varepsilon) > 0$ such that

$$F(z,x) \leq \varepsilon x^2 + c_{10}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{24}$$

Similarly on account of hypotheses $H(g)_1$, we see that given $\varepsilon > 0$ we can find $c_{11} = c_{11}(\varepsilon) > 0$ such that

$$G(z,x) \leq c_{11} x^2 + \varepsilon|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{25}$$

Then for all $u \in W_0^{1,p}$, we have

$$\begin{aligned}
\varphi_\lambda^+(u) &\geq \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \lambda\varepsilon\|u^+\|_2^2 - \lambda c_{10}\|u^+\|_r^r - c_{11}\|u^+\|_2^2 - \varepsilon\|u^+\|_p^p \,(\text{see (24), (25)})\\
&\geq \left[\frac{1}{p} - \varepsilon\right]\|u\|^p + \left[\frac{1}{2} - \lambda\varepsilon\right]\|u\|^2_{H_0^1(\Omega)} - c_{12}[\lambda\|u\|^r + \|u\|^2] \quad \text{for some } c_{12} > 0.
\end{aligned}$$

Choose $\lambda_0 > 0$ big so that $\frac{1}{2\lambda_0} < \frac{1}{p}$. Then for all $0 < \lambda \leq \lambda_0$ we have

$$\begin{aligned}
\varphi_\lambda^+(u) &\geq c_{13}\|u\|^p - c_{12}[\lambda\|u\|^r + \|u\|^2] \quad \text{for some } c_{13} > 0\\
&= [c_{13} - c_{12}(\lambda\|u\|^{r-p} + \|u\|^{2-p})]\|u\|^p.
\end{aligned} \tag{26}$$

Consider the function

$$k_\lambda^+(t) = \lambda t^{r-p} + t^{2-p} \quad \text{for } t > 0.$$

Evidently $k_\lambda \in C^1(0, +\infty)$ and since $2 < p < r$, we see that

$$k_\lambda^+(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and as } t \to +\infty.$$

Therefore we can find $t_0 \in (0, +\infty)$ such that

$$k_\lambda^+(t_0) = \inf_{t>0} k_\lambda^+,$$

$$\begin{aligned} &\Rightarrow \quad (k_\lambda^+)'(t_0) = 0, \\ &\Rightarrow \quad \lambda(r-p)t_0^{r-p-1} = (p-2)t_0^{2-p-1}, \\ &\Rightarrow \quad t_0 = t_0(\lambda) = \left[\frac{p-2}{\lambda(r-p)}\right]^{\frac{1}{r-2}}. \end{aligned}$$

Then we have

$$k_\lambda^+(t_0) = \lambda\left[\frac{p-2}{\lambda(r-p)}\right]^{\frac{r-p}{r-2}} + \left[\frac{\lambda(r-p)}{p-2}\right]^{\frac{p-2}{r-2}}.$$

Note that $\dfrac{r-p}{r-2} < 1$. Therefore we see that

$$k_\lambda^+(t_0) \to 0^+ \quad \text{as } \lambda \to 0^+.$$

So, we can find $\lambda_+^* \in (0, \lambda_0]$ such that

$$k_\lambda^+(t_0) < \frac{c_{13}}{c_{12}} \quad \text{for all } \lambda \in (0, \lambda_+^*).$$

Then from (26) we have

$$\inf[\varphi_\lambda^+(u) \,:\, \|u\| = t_0(\lambda)] = d_\lambda^+ > 0 \quad \text{for all } \lambda \in (0, \lambda_+^*). \tag{27}$$

On account of hypothesis $H(g)_1\,(iii)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$G(z,x) \geq \frac{1}{2}[\eta(z) - \varepsilon]x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{28}$$

Recall that $\widehat{u}_1 \in \operatorname{int} C_+$ is the positive, L^2-normalized eigenfunction corresponding to $\widehat{\lambda}_1(2) > 0$. We choose $t \in (0,1)$ small such that

$$t\widehat{u}_1(z) \in [0, \delta] \quad \text{for all } z \in \overline{\Omega}. \tag{29}$$

The sign condition on $f(z,\cdot)$ implies that $F \geq 0$. Then using (28), (29) we have

$$\begin{aligned} \varphi_\lambda^+(t\widehat{u}_1) &\leq \frac{t^p}{p}\|\nabla \widehat{u}_1\|_p^p + \frac{t^2}{2}\left[\widehat{\lambda}_1(2) - \int_\Omega \eta(z)\widehat{u}_1^2 dz\right] \\ &= \frac{t^p}{p}\|\nabla \widehat{u}_1\|_p^p + \frac{t^2}{2}\int_\Omega \left[\widehat{\lambda}_1(2) - \eta(z)\right]\widehat{u}_1^2 dz \quad (\text{recall that } \|\widehat{u}_1\|_2 = 1). \end{aligned}$$

Since $\eta(z) \geq \widehat{\lambda}_1(2)$ for a.a. $z \in \Omega$ and $\eta(z) \not\equiv \widehat{\lambda}_1(2)$, we have that

$$\int_\Omega \left[\widehat{\lambda}_1(2) - \eta(z)\right]\widehat{u}_1^2 dz < 0.$$

Therefore

$$\varphi_\lambda^+(t\widehat{u}_1) \leq c_{14}t^p - c_{15}t^2 \quad \text{for some } c_{14}, c_{15} > 0.$$

Since $2 < p$, taking $t \in (0,1)$ even smaller if necessary, we will have

$$\varphi_\lambda^+(t\widehat{u}_1) < 0. \tag{30}$$

Consider the following minimization problem

$$\inf\left[\varphi_\lambda^+(u) \,:\, u \in \overline{B}_{t_0(\lambda)}\right] = m_\lambda^+ \quad (\lambda \in (0, \lambda_+^*)),$$

with $B_{t_0(\lambda)} = \left\{u \in W_0^{1,p}(\Omega) \,:\, \|u\| < t_0(\lambda)\right\}$.

Since $\varphi_\lambda^+(\cdot)$ is sequentially weakly semicontinuous and $B_{t_0(\lambda)} \subseteq W_0^{1,p}(\Omega)$ is sequentially weakly compact (from the reflexivity of $W_0^{1,p}(\Omega)$ and the Eberlein-Smulian theorem), we can apply the Weierstrass-Tonelli theorem and find $u_0 \in W_0^{1,p}(\Omega)$ such that $\varphi_\lambda^+(u_0) = m_\lambda^+$.

From (30) and (27) we see that

$$\begin{aligned} & \varphi_\lambda^+(u_0) = m_\lambda^+ < \varphi_\lambda^+(0) = 0 < d_\lambda^+, \\ \Rightarrow \quad & 0 < \|u_0\| < t_0(\lambda). \end{aligned}$$

So, we have

$$\begin{aligned} & \widehat{\varphi}'_+(u_0) = 0, \\ \Rightarrow & \langle A_p(u_0), h\rangle + \langle A(u_0), h\rangle = \lambda \int_\Omega f(z, u_0^+) h dz + \int_\Omega g(z, u_0^+) h dz \text{ for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{31}$$

In (31) we choose $h = -u_0^- \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} & \|\nabla u_0^-\|_p^p + \|\nabla u_0^-\|_2^2 = 0, \\ \Rightarrow \quad & u_0 \geq 0,\ u_0 \neq 0. \end{aligned}$$

From (31) it follows that

$$-\Delta_p u_0(z) - \Delta u_0(z) = \lambda f(z, u_0(z)) + g(z, u_0(z)) \quad \text{for a.a. } z \in \Omega, \quad u_0\big|_{\partial\Omega} = 0. \tag{32}$$

From (32) and Theorem 7.1, p. 286, of Ladyzhenskaya-Ural′tseva [14], we have that $u_0 \in L^\infty(\Omega)$. Then Theorem 1 of Lieberman [15] implies that $u_0 \in C_+ \setminus \{0\}$.

Let $\rho = \|u_0\|_\infty$. On account of hypotheses $H(g)_1\,(i), (iii)$, we see that we can find $\widehat{\xi}_\rho > 0$ such that

$$g(z, x)x + \widehat{\xi}_\rho |x|^p \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{33}$$

The sign condition on $f(z, \cdot)$ implies that

$$f(z, u_0(z)) \geq 0 \quad \text{for a.a. } z \in \Omega.$$

So, from (32) and (33) we have

$$\begin{aligned} & -\Delta_p u_0(z) - \Delta u_0(z) + \widehat{\xi}_\rho u_0(z)^{p-1} \geq g(z, u_0(z)) + \widehat{\xi}_\rho u_0(z)^{p-1} \geq 0 \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow \quad & \Delta_p u_0(z) + \Delta u_0(z) \leq \widehat{\xi}_\rho u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow \quad & u_0 \in \operatorname{int} C_+ \quad \text{(see Pucci-Serrin [27], pp. 111, 120).} \end{aligned}$$

Let $\widetilde{u} \in \operatorname{int} C_+$. On account of hypotheses $H(f)_1\,(ii)$, $H(g)_1\,(ii)$, we see that

$$\varphi_\lambda^+(s\widetilde{u}) \to -\infty \quad \text{as } s \to +\infty. \tag{34}$$

From Proposition 1 we know that

$$\varphi_\lambda^+(\cdot) \text{ satisfies the } C\text{-condition.} \tag{35}$$

Then (27), (34), (35) permit the use of the mountain pass theorem (see, for example, Papageorgiou-Rădulescu-Repovš [21], p. 401). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\varphi_\lambda^+} \quad \text{and} \quad d_\lambda^+ \leq \varphi_\lambda^+(\widehat{u}). \tag{36}$$

As we did for u_0, from (36) we infer that $\widehat{u} \in \operatorname{int} C_+$ is a solution of (P_λ) $(\lambda \in (0, \lambda_+^*))$ and $\widehat{u} \neq u_0$.

Similarly, working this time with the functional $\varphi_\lambda^-(\cdot)$, we produce a $\lambda_-^* > 0$ such that for all $\lambda \in (0, \lambda_-^*)$ problem (P_λ) has at least two negative smooth solutions

$$v_0 \in -\operatorname{int} C_+,\ \widehat{v} \in -\operatorname{int} C_+,\ v_0 \neq \widehat{v}.$$

Let $\lambda^* = \min\{\lambda_+^*, \lambda_-^*\}$ to finish the proof of the proposition. $\square$

We can also produce extremal constant sign solutions for problem (P_λ), $\lambda \in (0, \lambda^*)$. That is, there is a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ and a biggest negative solution $v_\lambda^* \in -\operatorname{int} C_+$ for $\lambda \in (0, \lambda^*)$. In the sequel we will use these extremal solutions in order to generate a nodal one. The idea is simple. Using truncations, we focus on the order interval $[v_\lambda^*, u_\lambda^*]$. Then any nontrivial solution in this interval, distinct from v_λ^* and u_λ^*, is necessarily nodal on account of the extremality of v_λ^* and of u_λ^*.

To implement this plan, we proceed as follows. Hypotheses $H(f)_1$, $H(g)_1$ imply that given $\varepsilon > 0$, we can find $c_{16} = c_{16}(\varepsilon) > 0$ such that

(37)
$$[\lambda f(z,x) + g(z,x)]x \geq [\eta(z) - \varepsilon]x^2 - c_{16}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ all } \lambda \in (0, \lambda^*).$$

This unilateral growth estimate on the reaction leads to the following auxiliary $(p,2)$-equation

$$-\Delta_p u(z) - \Delta u(z) = [\eta(z) - \varepsilon]u(z) - c_{16}|u(z)|^{r-2}u(z) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{38}$$

From Proposition 3.3 of Papageorgiou-Rădulescu [17], we have:

Proposition 3. *Problem* (38) *admits a unique positive solution* $\widetilde{u} \in \operatorname{int} C_+$ *and since the problem is odd* $\widetilde{v} = -\widetilde{u} \in -\operatorname{int} C_+$ *is the unique negative solution of* (38).

We introduce the following two sets

$$\begin{aligned} S_\lambda^+ &= \text{ set of positive solutions of } (P_\lambda), \\ S_\lambda^- &= \text{ set of negative solutions of } (P_\lambda). \end{aligned}$$

From Proposition 2 and its proof, we know that for all $\lambda \in (0, \lambda^*)$

$$\emptyset \neq S_\lambda^+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad \emptyset \neq S_\lambda^- \subseteq -\operatorname{int} C_+.$$

Moreover, from Filippakis-Papageorgiou [9], we know that

$$\begin{aligned} S_\lambda^+ &= \text{ is downward directed,} \\ S_\lambda^- &= \text{ is upward directed.} \end{aligned}$$

The next proposition produces a lower bound for S_λ^+ and an upper bound for S_λ^-.

Proposition 4. *If hypotheses* $H(f)_1$, $H(g)_1$ *hold and* $\lambda \in (0, \lambda^*)$, *then* $\widetilde{u} \leq u$ *for all* $u \in S_\lambda^+$ *and* $v \leq \widetilde{v}$ *for all* $v \in S_\lambda^-$.

Proof. Let $u \in S_\lambda^+$. We introduce the following Carathéodory function

$$k_+(z,x) = \begin{cases} [\eta(z) - \varepsilon]x^+ - c_{16}(x^+)^{r-1} & \text{if } x \leq u(z), \\ [\eta(z) - \varepsilon]u(z) - c_{16}u(z)^{r-1} & \text{if } u(z) < x. \end{cases} \tag{39}$$

We set $K_+(z,x) = \int_0^x k_+(z,s)ds$ and consider the C^1-functional $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega K_+(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (39) it is clear that $\sigma_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\widetilde{u}_0) = \inf[\sigma_+(u) : u \in W_0^{1,p}(\Omega)]. \tag{40}$$

Since $u \in \operatorname{int} C_+$ we can find $t \in (0,1)$ small such that $t\widehat{u}_1 \leq u$ (see Papageorgiou-Rădulescu-Repovš [21], Proposition 4.1.22, p. 274). From (39) we have

$$\begin{aligned}
\sigma_+(t\widehat{u}_1) &= \frac{1}{p}\|\nabla\widehat{u}_1\|_p^p + \frac{t^2}{2}\widehat{\lambda}_1(2) - \frac{t^2}{2}\int_\Omega[\eta(z)-\varepsilon]\widehat{u}_1^2dz + \frac{c_{16}}{r}t^r\|\widehat{u}_1\|_r^r \\
&\leq c_{17}t^p - c_{18}t^2 \quad \text{for some } c_{17}, c_{18} > 0 \\
&\qquad\qquad\qquad\qquad \text{(choose } \varepsilon > 0 \text{ small and recall that } p < r,\ 0 < t < 1), \\
\Rightarrow&\quad \sigma_+(t\widehat{u}_1) < 0 \quad \text{(choosing } t < 1 \text{ even smaller if necessary)}, \\
\Rightarrow&\quad \sigma_+(\widetilde{u}_0) < 0 = \sigma_+(0) \quad \text{(see (40))}, \\
\Rightarrow&\quad \widetilde{u}_0 \neq 0.
\end{aligned}$$

From (40) we have

$$\sigma_+'(\widetilde{u}_0) = 0,$$

$$\Rightarrow \quad \langle A_p(\widetilde{u}_0), h\rangle + \langle A(\widetilde{u}_0), h\rangle = \int_\Omega k_+(z, \widetilde{u}_0)hdz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{41}$$

In (41), first we choose $h = -\widetilde{u}_0^- \in W_0^{1,p}(\Omega)$. Using (39), we obtain

$$\widetilde{u}_0 \geq 0,\ \widetilde{u}_0 \neq 0.$$

Next in (41) we choose $h = (\widetilde{u}_0 - u)^+ \in W_0^{1,p}(\Omega)$. Using (39), we have

$$\begin{aligned}
&\langle A_p(\widetilde{u}_0), (\widetilde{u}_0 - u)^+\rangle + \langle A(\widetilde{u}_0), (\widetilde{u}_0 - u)^+\rangle \\
&= \int_\Omega \left([\eta(z)-\varepsilon]u - c_{16}u^{r-1}\right)(\widetilde{u}_0 - u)^+dz \\
&\leq \int_\Omega [\lambda f(z,u) + g(z,u)](\widetilde{u}_0 - u)^+dz \quad \text{(see (37))} \\
&= \langle A_p(u), (\widetilde{u}_0 - u)^+\rangle + \langle A(u), (\widetilde{u}_0 - u)^+\rangle \quad (\text{since } u \in S_\lambda^+), \\
\Rightarrow&\quad \widetilde{u}_0 \leq u.
\end{aligned}$$

So, we have proved that

$$\widetilde{u}_0 \in [0, u],\ \widetilde{u}_0 \neq 0. \tag{42}$$

From (39), (41), (42) and Proposition 3 it follows that

$$\begin{aligned}
&\widetilde{u}_0 = \widetilde{u} \in \operatorname{int} C_+, \\
\Rightarrow&\quad \widetilde{u} \leq u \quad \text{for all } u \in S_\lambda^+.
\end{aligned}$$

Similarly we show that $v \leq \widetilde{v}$ for all $v \in S_\lambda^-$. □

Having these bounds, we can now produce the extremal constant sign solutions for problem (P_λ) with $\lambda \in (0, \lambda^*)$. As we already indicated these extremal constant sign solutions will lead to the existence of a nodal solution.

Proposition 5. *If hypotheses $H(f)_1$, $H(g)_1$ hold and $\lambda \in (0,\lambda^*)$, then problem (P_λ) admits a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ and a biggest negative solution $v_\lambda^* \in -\operatorname{int} C_+$.*

Proof. Recall that S_λ^+ is downward directed. Threfore we can find $\{u_n\}_{n\geq 1} \subseteq S_\lambda^+$ decreasing such that

$$\inf_{n\geq 1} u_n = \inf S_\lambda^+ \quad \text{(see [17])}.$$

We have

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle = \int_\Omega [\lambda f(z,u_n) + g(z,u_n)]hdz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \tag{43}$$

$$\widetilde{u} \le u_n \le u_1 \quad \text{for all } n \in \mathbb{N} \text{ (see Proposition 4)}. \tag{44}$$

In (43) we choose $h = u_n \in W_0^{1,p}(\Omega)$ and use (44) and hypotheses $H(f)_1\,(i)$, $H(g)_1\,(i)$. It follows that $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u_\lambda^* \text{ in } L^r(\Omega). \tag{45}$$

Choosing $h = u_n - u_* \in W_0^{1,p}(\Omega)$ in (43), passing to the limit as $n \to +\infty$ and using (45) and Lemma 4, we infer that

$$u_n \to u_\lambda^* \text{ in } W_0^{1,p}(\Omega). \tag{46}$$

So, if in (43) we pass to the limit as $n \to +\infty$ and use (46), then

$$\langle A_p(u_\lambda^*), h\rangle + \langle A(u_\lambda^*), h\rangle = \int_\Omega [\lambda f(z,u_\lambda^*) + g(z,u_\lambda^*)]hdz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$
$$\widetilde{u} \le u_\lambda^* \quad \text{(see (44))}.$$

It follows that

$$u_\lambda^* \in S_\lambda^+,\ u_\lambda^* = \inf S_\lambda^+.$$

For the negative extremal solution, recall that S_λ^- is upward directed and so we can find $\{v_n\}_{n\ge1} \subseteq S_\lambda^-$ increasing such that

$$\sup_{n\ge1} v_n = \sup S_\lambda^-.$$

Reasoning as above and since $v_1 \le v_n \le \widetilde{v}$ for all $n \in \mathbb{N}$, we obtain $v_\lambda^* \in -\mathrm{int}\, C_+$ such that

$$v_\lambda^* \in S_\lambda^-,\ v_\lambda^* = \sup S_\lambda^-.$$

□

Now that we have the extremal constant sign solutions, we can produce a nodal one. Following the plan outlined earlier, we introduce the following truncation of the reaction of problem (P_λ) $(\lambda \in (0, \lambda*))$

$$e_\lambda(z,x) = \begin{cases} \lambda f(z, v_\lambda^*(z)) + g(z, v_\lambda^*(z)) & \text{if } x < v_\lambda^*(z), \\ \lambda f(z,x) + g(z,x) & \text{if } v_\lambda^*(z) \le x \le u_\lambda^*(z), \\ \lambda f(z, u_\lambda^*(z)) + g(z, u_\lambda^*(z)) & \text{if } u_\lambda^*(z) < x. \end{cases} \tag{47}$$

Evidently this is a Carathéodory function. We also consider the positive and negative truncations of $e_\lambda(z,\cdot)$, namely the Carathéodory functions

$$e_\lambda^\pm(z,x) = e_\lambda(z, \pm x). \tag{48}$$

We set $E_\lambda(z,x) = \int_0^x e_\lambda(z,s)ds$ and $E_\lambda^\pm(z,x) = \int_0^x e_\lambda^\pm(z,s)ds$ and consider the C^1-functionals $\psi_\lambda, \psi_\lambda^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$

$$\psi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega E_\lambda(z,u)dz,$$
$$\psi_\lambda^\pm(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega E_\lambda^\pm(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also, we need to strengthen a little the condition on $g(z,\cdot)$ near zero. More precisely, the new conditions on the perturbation term are the following:

$H(g)_2$: $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $g(z,0)=0$ for a.a. $z\in\Omega$, hypotheses $H(g)_2\,(i),(ii)$ are the same as hypotheses $H(g)_1\,(i),(ii)$ and

(iii) there exist $\eta>\widehat{\lambda}_2(2)$ and $\beta_1>0$ such that
$$\eta\leq\liminf_{x\to 0}\frac{g(z,x)}{x}\leq\limsup_{x\to 0}\frac{g(z,x)}{x}\leq\widehat{\beta}_1\quad\text{uniformly for a.a. } z\in\Omega.$$

Proposition 6. *If hypotheses $H(f)_1$, $H(g)_2$ hold and $\lambda\in(0,\lambda^*)$, then problem (P_λ) admits a nodal solution $y_0\in[v_\lambda^*,u_\lambda^*]\cap C_0^1(\overline{\Omega})$.*

Proof. Using (47) and (48), the nonlinear regularity theory (see the proof of Proposition 2) and the extremality of the solutions u_λ^* and v_λ^*, we show easily that
$$K_{\psi_\lambda}\subseteq[v_\lambda^*,u_\lambda^*]\cap C_0^1(\overline{\Omega}),\ K_{\psi_\lambda^+}=\{0,u_\lambda^*\},\ K_{\psi_\lambda^-}=\{0,v_\lambda^*\}. \tag{49}$$

Claim: $u_\lambda^*\in\operatorname{int}C_+$ and $v_\lambda^*\in-\operatorname{int}C_+$ are local minimizers of the functional ψ_λ.

From (47) and (48) it is clear that ψ_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda^*\in W_0^{1,p}(\Omega)$ such that
$$\psi_\lambda^+(\widetilde{u}_\lambda^*)=\inf\left[\psi_\lambda^+(u)\ :\ u\in W_0^{1,p}(\Omega)\right]. \tag{50}$$

As before (see the proof of Proposition 2) for $t\in(0,1)$ small (at least that $t\widehat{u}_1\leq u_\lambda^*$, recall $u_\lambda^*\in\operatorname{int}C_+$), we have
$$\begin{aligned}&\psi_\lambda^+(t\widehat{u}_1)<0,\\ \Rightarrow\ &\psi_\lambda^+(\widetilde{u}_\lambda^*)<0=\psi_\lambda^+(0)\quad\text{(see (50))},\\ \Rightarrow\ &\widetilde{u}_\lambda^*\neq 0.\end{aligned} \tag{51}$$

From (50) we see that $\widetilde{u}_\lambda^*\in K_{\psi_\lambda^+}$. Then (51) and (49) imply that
$$\widetilde{u}_\lambda^*=u_\lambda^*\in\operatorname{int}C_+. \tag{52}$$

Clearly $\psi_\lambda^+\big|_{C_+}=\psi_\lambda\big|_{C_+}$ (see (47) and (48)). Then from (50) and (52) it follows that
$$\begin{aligned}&u_\lambda^*\text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of }\psi_\lambda,\\ \Rightarrow\ &u_\lambda^*\text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of }\psi_\lambda\end{aligned} \tag{53}$$
(see Papageorgiou-Rădulescu [19], Proposition 2.12). Similarly, using this time the functional ψ_λ^-, we show that
$$v_\lambda^*\in-\operatorname{int}C_+\text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of }\psi_\lambda. \tag{54}$$

On account of (49), we may assume that
$$K_{\psi_\lambda}\subseteq[v_\lambda^*,u_\lambda^*]\cap C_0^1(\overline{\Omega})\text{ is finite.} \tag{55}$$

Otherwise we already have an infinity of smooth nodal solutions and so we are done. Also, we may assume that $\psi_\lambda(v_\lambda^*)\leq\psi_\lambda(u_\lambda^*)$. The analysis is similar if the opposite inequality holds, using this time (54) instead of (53).

From (53), (55) and Theorem 5.7.6, p. 449, of Papageorgiou-Rădulescu-Repovš [21], we know that we can find $\rho\in(0,1)$ small such that
$$\psi_\lambda(v_\lambda^*)\leq\psi_\lambda(u_\lambda^*)<\inf[\psi_\lambda(u)\ :\ \|u-u_\lambda^*\|=\rho]=m_\lambda,\ \|v_\lambda^*-u_\lambda^*\|>\rho. \tag{56}$$

Since ψ_λ is coercive, we know that

$$\psi_\lambda(\cdot) \text{ satisfies the } C\text{-condition} \tag{57}$$

(see Papageorgiou-Rădulescu-Repovš [21], Proposition 5.1.15, p. 369). Then (56) and (57) permit the use of the mountain pass theorem. So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\psi_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}) \text{ (see (49))}, \ m_\lambda \leq \psi_\lambda(y_0). \tag{58}$$

From (47), (56), (58) it follows that $y_0 \notin \{u_\lambda^*, v_\lambda^*\}$ and it is a solution of problem (P_λ) ($\lambda \in (0, \lambda^*)$). It remains to show that y_0 is nontrivial. From the mountain pass theorem, we know that

$$\psi_\lambda(y_0) = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \psi_\lambda(\gamma(t)), \tag{59}$$

where $\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = v_\lambda^*, \gamma(1) = u_\lambda^*\}$. Then according to (59), in order to show the nontriviality of the solution y_0, it suffices to construct a path $\gamma_* \in \Gamma$ such that

$$\psi_\lambda\big|_{\gamma_*} < 0.$$

To this end let $M_c = M \cap C_0^1(\overline{\Omega})$ (see Section 2, before Lemma 3). We introduce the following set of paths

$$\widehat{\Gamma}_c = \{\widehat{\gamma} \in C([-1,1], M_c) \,:\, \widehat{\gamma}(-1) = -\widehat{u}_1, \widehat{\gamma}(1) = \widehat{u}_1\}\,.$$

From Papageorgiou-Winkert [26], we know that $\widehat{\Gamma}_c$ is dense in $\widehat{\Gamma}$. So, from Lemma 3, we see that we can find $\widehat{\gamma}_0 \in \widehat{\Gamma}_c$ such that

$$\max_{-1\leq t\leq 1} \|\nabla\widehat{\gamma}_0(t)\|_2^2 \leq \widehat{\lambda}_2(2) + \delta \quad \text{with } \delta \in (0, \eta - \widehat{\lambda}_2(2)). \tag{60}$$

From hypothesis $H(g)_2\,(iii)$, we see that given $\widehat{\eta} \in (\widehat{\lambda}_2(2) + \delta, \eta)$, we can find $\delta_0 > 0$ such that

$$G(z,x) \geq \frac{1}{2}\widehat{\eta}x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0. \tag{61}$$

Since $\widehat{\gamma}_0 \in \Gamma_c$ and $u_\lambda^* \in \operatorname{int} C_+$, $v_\lambda^* \in -\operatorname{int} C_+$, we can find $\widehat{\tau} \in (0,1)$ small such that

$$\widehat{\tau}\widehat{\gamma}_0(t) \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}), \quad \widehat{\tau}|\widehat{\gamma}_0(t)(z)| \leq \delta_0 \quad \text{for all } t \in [-1,1] \text{ and all } z \in \overline{\Omega}. \tag{62}$$

Using (60), (61), (62) and the fact that $F \geq 0$, we have

$$\begin{aligned}\psi_\lambda(\widehat{\tau}\widehat{\gamma}_0(t)) &\leq \frac{\widehat{\tau}^p}{p}\|\nabla\widehat{\gamma}_0(t)\|_p^p + \frac{\widehat{\tau}^2}{2}\left[\widehat{\lambda}_2(2) + \delta - \widehat{\eta}\right]\\ &\leq c_{19}\widehat{\tau}^p - c_{20}\widehat{\tau}^2 \quad \text{for some } c_{19}, c_{20} > 0, \text{ all } t \in [-1,1].\end{aligned}$$

Choosing $\widehat{\tau} \in (0,1)$ even smaller if necessary, we will have

$$\psi_\lambda(\widehat{\tau}\widehat{\gamma}_0(t)) < 0 \quad \text{for all } t \in [-1,1].$$

We set $\widetilde{\gamma}_0 = \widehat{\tau}\widehat{\gamma}_0$, then $\widetilde{\gamma}_0$ is a continuous path in $C_0^1(\overline{\Omega}) \subseteq W_0^{1,p}(\Omega)$ which connects $-\widehat{\tau}\widehat{u}_1$ and $\widehat{\tau}\widehat{u}_1$ and such that

$$\psi_\lambda\big|_{\widetilde{\gamma}_0} < 0. \tag{63}$$

Next we will construct a continuous path in $W_0^{1,p}(\Omega)$ which connects $\widehat{\tau}\widehat{u}_1$ and u_λ^* and along which ψ_λ is negative.

Let $a = \psi_\lambda^+(u_\lambda^*) = \inf \psi_\lambda^+ < 0 = \psi_\lambda^+(0)$. From (49) we see that

$$K^0_{\psi_\lambda^+} = \{0\} \text{ and } (\psi_\lambda^+)^a = \{u_\lambda^*\}. \tag{64}$$

The second deformation theorem (see Papageorgiou-Rădulescu-Repovš [21], Theorem 5.3.12, p. 386) says that we can find a deformation $h_\lambda^+ : [0,1] \times ((\psi_\lambda^+)^0 \setminus K^0_{\psi_\lambda^+}) \to (\psi_\lambda^+)^0$ such that

$$h_\lambda^+(0,u) = u \quad \text{for all } u \in (\psi_\lambda^+)^0 \setminus K^0_{\psi_\lambda^+}, \tag{65}$$

$$h_\lambda^+(1,u) = u_\lambda^* \quad \text{(see (64))}, \tag{66}$$

$$\psi_\lambda^+(h_\lambda^+(t,u)) \leq \psi_\lambda^+(h_\lambda^+(s,u)) \quad \text{for all } 0 \leq s \leq t \leq 1, \text{ all } u \in (\psi_\lambda^+)^0 \setminus K^0_{\psi_\lambda^+}. \tag{67}$$

We have

$$\begin{aligned} & \psi_\lambda^+(\widehat{\tau}\widehat{u}_1) = \psi_\lambda(\widehat{\tau}\widehat{u}_1) = \psi_\lambda(\widetilde{\gamma}_0(1)) < 0 \quad \text{(see (63))}, \\ \Rightarrow \quad & \widehat{\tau}\widehat{u}_1 \in (\psi_\lambda^+)^0 \setminus \{0\} = (\psi_\lambda^+)^0 \setminus K^0_{\psi_\lambda^+} \quad \text{(see (64))}. \end{aligned}$$

Therefore, we can define

$$\widetilde{\gamma}_+(t) = h_\lambda^+(t, \widehat{\tau}\widehat{u}_1)^+ \quad \text{for all } t \in [0,1].$$

This is a continuous path in $W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} &\widetilde{\gamma}_+(0) = \widehat{\tau}\widehat{u}_1 \text{ (see (65))}, \\ &\widetilde{\gamma}_+(1) = u_\lambda^* \text{ (see (66))}, \end{aligned}$$

and

$$\psi_\lambda(\widetilde{\gamma}_+(t)) = \psi_\lambda^+(\widetilde{\gamma}_+(t)) < \psi_\lambda^+(\widehat{\tau}\widehat{u}_1) < 0 \text{ for all } t \in [0,1] \text{ (see (67), (63))}.$$

Therefore $\widetilde{\gamma}_+$ is a continuous path connecting $\widehat{\tau}\widehat{u}_1$ and u_λ^* and

$$\psi_\lambda\big|_{\widetilde{\gamma}_+} < 0. \tag{68}$$

In a similar fashion, we construct $\widetilde{\gamma}_-$ a continuous path in $W_0^{1,p}(\Omega)$ which connects $-\widehat{\tau}\widehat{u}_1$ and v_λ^* and

$$\psi_\lambda\big|_{\widetilde{\gamma}_-} < 0. \tag{69}$$

Let $\gamma_* = \widetilde{\gamma}_- \cup \widetilde{\gamma}_0 \cup \widetilde{\gamma}_+$. Then $\gamma_* \in \Gamma$ and

$$\begin{aligned} & \psi_\lambda\big|_{\gamma_*} < 0 \quad \text{(see (63), (68), (69))} \\ \Rightarrow \quad & y_0 \neq 0. \end{aligned}$$

Therefore $y_0 \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega})$ is a nodal solution of (P_λ). □

By strengthening the regularity of $g(z,\cdot)$ at zero, we can relax the strict nonresonance with respect to $\widehat{\lambda}_2$ (see hypothesis $H(g)_2\,(iii)$) and allow nonuniform nonresonance.

The new conditions on the perturbation $g(z,x)$ are the following:

$H(g)_3$: $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(z,0) = 0$ for a.a. $z \in \Omega$, hypotheses $H(g)_3\,(i),(ii)$ are the same as the corresponding hypotheses $H(g)_1\,(i),(ii)$ and

(iv) $\widehat{\lambda}_2(2) \leq g'_x(z,0) = \lim_{x\to 0} \dfrac{g(z,x)}{x}$ uniformly for a.a. $z \in \Omega$, $g'_x(\cdot,0) \in L^\infty(\Omega)$ and $g'_x(\cdot,0) \not\equiv \widehat{\lambda}_k(2)$ for all $k \geq 2$.

Remark 3. So, now at $\widehat{\lambda}_2(2)$ we require only nonuniform nonresonance. But the same is required for every other higher eigenvalue. Moreover, we assume that $g(z,\cdot)$ is differentiable at $x=0$.

Proposition 7. *If hypotheses $H(f)_1$, $H(g)_3$ hold and $\lambda \in (0,\lambda^*)$, then problem (P_λ) admits a nodal solution $y_0 \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega})$.*

Proof. In what follows $\xi(z) = g_x'(z,0)$. By hypothesis $\xi \in L^\infty(\Omega)$. Let $\widehat{\sigma} : H_0^1(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\widehat{\sigma}(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_\Omega \xi(z)u^2dz \quad \text{for all } u \in H_0^1(\Omega).$$

On account of Lemma 1 and hypothesis $H(g)_3\,(iii)$, we see that the Morse index of $u=0$ is ≥ 2. Then Corollary 6.2.14, p. 486 of Papageorgiou-Rădulescu-Repovš [21], implies that

$$C_1(\widehat{\sigma},0) = 0. \tag{70}$$

Let $\sigma = \widehat{\sigma}\big|_{W_0^{1,p}(\Omega)}$. Since $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$ densely, Theorem 6.6.26, p. 545, of Papageorgiou-Rădulescu-Repovš [21], implies that

$$\begin{aligned} & C_k(\widehat{\sigma},0) = C_k(\sigma,0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow \quad & C_1(\sigma,0) = 0 \quad \text{(see (70))}. \end{aligned} \tag{71}$$

Let $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega [\lambda F(z,u) + G(z,u)]dz$$

for all $u \in W_0^{1,p}(\Omega)$, with $F(z,x) = \int_0^x f(z,s)ds$, $G(z,x) = \int_0^x g(z,s)ds$.

Evidently $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$. We introduce the homotopy $\widehat{h}_\lambda(t,u)$ defined by

$$\widehat{h}_\lambda(t,u) = (1-t)\varphi_\lambda(u) + t\sigma(u) \quad \text{for all } (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Suppose that we could find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t,\ u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ and } (\widehat{h}_\lambda)_u'(t_n,u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{72}$$

From the equation in (72), we have

$$\begin{aligned} (1-t_n)\langle A_p(u_n),h\rangle + \langle A(u_n),h\rangle = {} & (1-t_n)\int_\Omega [\lambda f(z,u_n) + g(z,u_n)]hdz \\ & + t_n\int_\Omega \xi(z)u_nhdz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{73}$$

Let $\|\cdot\|_{1,2}$ denote the norm of $H_0^1(\Omega)$. We set $y_n = \dfrac{u_n}{\|u_n\|_{1,2}}$, $n \in \mathbb{N}$. Then $\|y_n\|_{1,2} = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H_0^1(\Omega) \text{ as } n \to +\infty. \tag{74}$$

From (73) we have

$$\begin{aligned} (1-t_n)\|u_n\|_{1,2}^{p-2}\langle A_p(y_n),h\rangle + \langle A(y_n),h\rangle = {} & (1-t_n)\int_\Omega \left[\lambda\frac{f(z,u_n)}{\|u_n\|_{1,2}} + \frac{g(z,u_n)}{\|u_n\|_{1,2}}\right]hdz \\ & + t_n\int_\Omega \xi(z)y_nhdz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{75}$$

Note that on account of hypotheses $H(f)_1\,(iii)$ and $H(g)_3\,(iii)$, we have

$$\frac{f(\cdot,u_n(\cdot))}{\|u_n\|_{1,2}} \xrightarrow{w} 0 \text{ in } L^{r'}(\Omega) \text{ and } \frac{g(\cdot,u_n(\cdot))}{\|u_n\|_{1,2}} \xrightarrow{w} \xi(z)y \text{ in } L^{p'}(\Omega). \tag{76}$$

So, if in (75) we pass to the limit as $n\to+\infty$ and use (74), (76) and the fact that $\|u_n\|_{1,2}\to 0$ (see (72)), then

$$\begin{aligned} &\langle A(y),y\rangle = \int_\Omega \xi(z)yh dz \quad \text{for all } h\in W_0^{1,p}(\Omega),\\ \Rightarrow\quad & -\Delta y(z) = \xi(z)y(z) \quad \text{for a.a. } z\in\Omega,\ y\big|_{\partial\Omega}=0. \end{aligned} \tag{77}$$

On account of hypothesis $H(g)_3\,(iii)$, there exists integer $m\geq 2$ such that

$$\xi(z)\in[\widehat{\lambda}_m(2),\widehat{\lambda}_{m+1}(2)] \quad \text{for a.a. } z\in\Omega,\ \xi\not\equiv\widehat{\lambda}_m(2),\ \xi\not\equiv\widehat{\lambda}_{m+1}(2). \tag{78}$$

From (78) and Lemma 2, we have

$$\widetilde{\lambda}_m(2,\xi)<\widetilde{\lambda}_m(2,\widetilde{\lambda}_m(2))=1 \text{ and } 1=\widetilde{\lambda}_{m+1}(2,\widetilde{\lambda}_{m+1}(2))<\widetilde{\lambda}_{m+1}(2,\xi). \tag{79}$$

Then from (77) and (79) it follows that $y=0$. On the other hand if in (75) we choose $h=y_n-y$ and pass to the limit as $n\to+\infty$, we obtain

$$\begin{aligned} &\lim_{n\to+\infty}\langle A(y_n),y_n-y\rangle = 0,\\ \Rightarrow\quad & \|\nabla y_n\|_2 \to \|\nabla y\|_2,\\ \Rightarrow\quad & y_n\to y \text{ in } H_0^1(\Omega) \text{ (see (74) and use the Kadec-Klee property of Hilbert spaces)},\\ \Rightarrow\quad & \|y\|_{1,2}=1, \text{ a contradiction.} \end{aligned}$$

Therefore (72) can not happen and so from the homotopy invariance property of critical groups (see Papageorgiou-Rădulescu-Repovš [21], Theorem 6.3.8, p. 505), we have

$$\begin{aligned} &C_k(\varphi_\lambda,0)=C_k(\sigma,0) \quad \text{for all } k\in\mathbb{N}_0,\\ \Rightarrow\quad & C_1(\varphi_\lambda,0)=0 \quad \text{(see (71))}. \end{aligned} \tag{80}$$

Next using the extremal constant sign solutions $u_\lambda^*\in \operatorname{int}C_+$ and $v_\lambda^*\in -\operatorname{int}C_+$ and reasoning as in the first part of the proof of Proposition 6, via the mountain pass theorem, we produce a solution $y_0\in[v_\lambda^*,u_\lambda^*]\cap C_0^1(\overline{\Omega})$ for (P_λ) when $\lambda\in(0,\lambda^*)$. Since y_0 is a critical point of ψ_λ (see the proof of Proposition 6) which is of mountain pass type, invoking Theorem 6.5.8, p. 527, of Papageorgiou-Rădulescu-Repovš [21], we have

$$C_1(\psi_\lambda,y_0)\neq 0. \tag{81}$$

We consider the homotopy

$$\widetilde{h}_\lambda(t,u)=(1-t)\varphi_\lambda(u)+t\psi_\lambda(u) \quad \text{for all } (t,u)\in[0,1]\times W_0^{1,p}(\Omega).$$

Suppose that we can find $\{t_n\}_{n\geq1}\subseteq[0,1]$ and $\{u_n\}_{n\geq1}\subseteq W_0^{1,p}(\Omega)$ such that

$$t_n\to t,\ u_n\to 0 \text{ in } W_0^{1,p}(\Omega) \text{ and } (\widetilde{h}_\lambda)'_u(t_n,u_n)=0 \quad \text{for all } n\in\mathbb{N}. \tag{82}$$

We have

$$\begin{aligned} \langle A_p(u_n),h\rangle+\langle A(u_n),h\rangle = {} & (1-t_n)\int_\Omega[\lambda f(z,u_n)+g(z,u_n)]hdz\\ & + t_n\int_\Omega e_\lambda(z,u_n)hdz \quad \text{for all } h\in W_0^{1,p}(\Omega),\text{ all } n\in\mathbb{N}. \end{aligned}$$

From Theorem 7.1, p. 286, of Ladyzhenskaya-Ural′tseva [14], we have

$$u_n \in L^\infty(\Omega),\ \|u_n\|_\infty \le c_{21} \quad \text{for some } c_{21} > 0, \text{ all } n \in \mathbb{N}.$$

Then invoking Theorem 1 of Lieberman [15], we can find $\alpha \in (0,1)$ and $c_{22} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}),\ \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le c_{22} \quad \text{for all } n \in \mathbb{N}.$$

The compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (82) imply that

$$\begin{aligned} & u_n \to 0 \text{ in } C_0^1(\overline{\Omega}), \\ \Rightarrow\quad & u_n \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}) \quad \text{for all } n \ge n_0, \\ \Rightarrow\quad & \{u_n\}_{n \ge n_0} \subseteq K_{\psi_\lambda} \quad (\text{see } (47)), \end{aligned}$$

which contradicts our assumption that K_{ψ_λ} is finite (see (55)). Hence (82) can not happen and once again the homotopy invariance property of critical groups implies that

$$\begin{aligned} & C_k(\varphi_\lambda, 0) = C_k(\psi_\lambda, 0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow\quad & C_1(\psi_\lambda, 0) = 0 \quad (\text{see } (80)). \end{aligned} \tag{83}$$

Comparing (81) and (83), we conclude that $y_0 \neq 0$. Therefore $y_0 \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega})$ is a nodal solution of (P_λ), $\lambda \in (0, \lambda^*)$. □

If we add an additional mild hypothesis, we can improve the conclusion in Propositions 6 and 7.

The extra hypothesis is the following one:

H_0: For every $\rho > 0$ and $\lambda > 0$, there exists $\widehat{\xi}_\rho^\lambda > 0$ such that for a.a. $z \in \Omega$, the function

$$x \to \lambda f(z,x) + g(z,x) + \widehat{\xi}_\rho^\lambda |x|^{p-2}x$$

is nondecreasing on $[-\rho, \rho]$.

Proposition 8. *If hypotheses $H(f)_1$, $H(g)_2$ or $H(g)_3$ and H_0 hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) admits a nodal solution $y_0 \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*]$.*

Proof. According to Proposition 6 (if $H(g)_2$ is in effect) or Proposition 7 (if $H(g)_3$ is in effect), we have a nodal solution $y_0 \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega})$. Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be defined by

$$a(y) = |y|^{p-2}y + y \quad \text{for all } y \in \mathbb{R}^N.$$

Since $p > 2$, we have $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and

$$\nabla a(y) = |y|^{p-2}y + y \left[\mathrm{id} + (p-2) \frac{y \otimes y}{|y|^2} \right] + \mathrm{id} \quad \text{for all } y \in \mathbb{R}^N.$$

Therefore

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge |\xi|^2 \quad \text{for all } y, \xi \in \mathbb{R}^N.$$

Invoking the tangency principle of Pucci-Serrin [27] (Theorem 2.5.2, p. 35), we infer that

$$v_\lambda^*(z) < y_0(z) < u_\lambda^*(z) \quad \text{for all } z \in \Omega. \tag{84}$$

Let $\rho = \max\{\|u_\lambda^*\|_\infty, \|v_\lambda^*\|_\infty\}$ and let $\widehat{\xi}_\rho^\lambda > 0$ be as postulated by hypothesis H_0. Take $\widetilde{\xi}_\rho^\lambda > \widehat{\xi}_\rho^\lambda$. We have

$$-\Delta_p y_0(z) - \Delta y_0(z) + \widetilde{\xi}_\rho^\lambda |y_0(z)|^{p-2} y_0(z)$$

$$
\begin{aligned}
&= \lambda f(z, y_0(z)) + g(z, y_0(z)) + \widehat{\xi}_\rho^\lambda |y_0(z)|^{p-2} y_0(z) + (\widetilde{\xi}_\rho^\lambda - \widehat{\xi}_\rho^\lambda)|y_0(z)|^{p-2} y_0(z) \\
&\leq \lambda f(z, u_\lambda^*(z)) + g(z, u_\lambda^*(z)) + \widehat{\xi}_\rho^\lambda u_\lambda^*(z)^{p-1} + (\widetilde{\xi}_\rho^\lambda - \widehat{\xi}_\rho^\lambda) u_\lambda^*(z)^{p-1} \\
&\qquad\qquad\qquad\qquad \text{(see (84) and hypothesis } H_0) \\
&= -\Delta_p u_\lambda^*(z) - \Delta u_\lambda^*(z) + \widetilde{\xi}_\rho^\lambda u_\lambda^*(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \qquad (85)
\end{aligned}
$$

From (85) we see that

$$(\widetilde{\xi}_\rho^\lambda - \widehat{\xi}_\rho^\lambda)|y_0|^{p-2} y_0 \prec (\widetilde{\xi}_\rho^\lambda - \widehat{\xi}_\rho^\lambda)(u_\lambda^*)^{p-1}.$$

Then from (84) and Proposition 2.2 of Papageorgiou-Rădulescu [17], we have

$$u_\lambda^* - y_0 \in \operatorname{int} C_+.$$

Similarly we show that

$$y_0 - v_\lambda^* \in \operatorname{int} C_+.$$

We conclude that

$$y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [v_\lambda^*, u_\lambda^*].$$

□

Summarizing the results of this section, we can state the following multiplicity theorem.

Theorem 1. *(a) If hypotheses $H(f)_1$ and $H(g)_2$ or $H(g)_3$ hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) admits at least five nontrivial smooth solutions $u_0, \widehat{u} \in \operatorname{int} C_+$, $v_0, \widehat{v} \in -\operatorname{int} C_+$, $y_0 \in C_0^1(\overline{\Omega})$ nodal; moreover, there exist extremal constant sign solutions $u_\lambda^* \in \operatorname{int} C_+$, $v_\lambda^* \in -\operatorname{int} C_+$ and $y_0 \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega})$.*

(b)If hypotheses $H(f)_1$, $H(g)_2$ or $H(g)_3$. H_0 hold and $\lambda \in (0, \lambda^*)$, then the nodal solution y_0 satisfies $y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [v_\lambda^*, u_\lambda^*]$.

4. Seven Solutions

In this section by strengthening the regularity of the nonlinearities $f(z, \cdot)$, $g(z, \cdot)$ and by employing also flow invariance arguments, we produce two more nodal solutions, for a total of seven nontrivial smooth solutions, all with sign information.

The new conditions on $f(z, x)$ and $g(z, x)$ are the following:

$H(f)_2$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that for all $x \in \mathbb{R}$ $z \to f(z, x)$ is measurable, for a.a. $z \in \Omega$ $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f_x'(z, x)| \leq a(z)[1 + |x|^{r-2}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;
(ii) this is the same as hypothesis $H(f)_1$ *(ii)*;
(iii) $f_x'(z, 0) = \lim_{x \to 0} \dfrac{f(z, x)}{x} = 0$ uniformly for a.a. $z \in \Omega$.

Remark 4. Hypothesis $H(f)_2$ *(iii)* implies that $f(z, 0) = 0$ for a.a. $z \in \Omega$.

$H(g)_4$: $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that for all $x \in \mathbb{R}$ $z \to g(z, x)$ is measurable, for a.a. $z \in \Omega$ $g(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|g_x'(z, x)| \leq a(z)[1 + |x|^{r-2}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;
(ii) $\lim_{x \to \pm\infty} \dfrac{g(z, x)}{|x|^{p-2} x} = 0$ uniformly for a.a. $z \in \Omega$;
(iii) this is the same as hypothesis $H(g)_3$ *(iii)*.

Remark 5. On account of hypothesis $H(g)_4\,(iii) = H(g)_3\,(iii)$ we have $g(z,0)=0$ for a.a. $z \in \Omega$.

In this case the energy (Euler) functional of problem (P_λ) $\varphi_\lambda : W_0^{1,p} \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega [\lambda F(z,u) + G(z,u)]dz \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

belongs in $C^2(W_0^{1,p}(\Omega))$.

Finally we strengthen hypothesis H_0 and require that it is global in $x \in \mathbb{R}$.

H_0': For every $\lambda > 0$, there exists $\widehat{\xi}_\lambda > 0$ such that for a.a. $z \in \Omega$, the function

$$x \to \lambda f(z,x) + g(z,x) + \widehat{\xi}_\lambda |x|^{p-2}x$$

is nondecreasing on $\mathbb{R}$.

Proposition 9. *If hypotheses $H(f)_2$, $H(g)_4$, H_0' hold and $\lambda \in (0,\lambda^*)$, then problem (P_λ) has at least two more nodal solutions $\widehat{y} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*]$ and $\widetilde{y} \in C_0^1(\overline{\Omega})$.*

Proof. From Theorem 1 we already have a nodal solution

$$y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*]. \tag{86}$$

As in the proof of Proposition 7, using the homotopy $\widetilde{h}_\lambda(t,u)$ and (86) we show that

$$C_k(\varphi_\lambda, y_0) = C_k(\varphi_\lambda, y_0) \quad \text{for all } k \in \mathbb{N}_0, \tag{87}$$

$$\Rightarrow \quad C_1(\varphi_\lambda, y_0) \neq 0 \quad \text{(see (81))}. \tag{88}$$

From (88) and Papageorgiou-Rădulescu [17] (see Claim 3 in the proof of Proposition 3.6), we have

$$C_k(\varphi_\lambda, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0,$$

$$\Rightarrow \quad C_k(\psi_\lambda, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (87))}. \tag{89}$$

Recall that $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ are local minimizers of ψ_λ (see the Claim in the proof of Proposition 6). So, we have

$$C_k(\psi_\lambda, u_\lambda^*) = C_k(\psi_\lambda, v_\lambda^*) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{90}$$

Recall that $\psi_\lambda(\cdot)$ is coercive. Therefore we have

$$C_k(\psi_\lambda, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \tag{91}$$

(see Papageorgiou-Rădulescu-Repovš [21], Proposition 6.2.24(a), p. 491).

As before $\xi(z) = g_x'(z,0)$, $\xi \in L^\infty(\Omega)$. Hypothesis $H(g)_4\,(iii) = H(g)_3\,(iii)$ implies that

$$\xi(z) \in [\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2)] \text{ for a.a. } z \in \Omega, \text{ some } m \geq 2,\ \xi \not\equiv \widehat{\lambda}_m(2),\ \xi \not\equiv \widehat{\lambda}_{m+1}(2).$$

If $\sigma(\cdot)$ is the functional from the proof of Proposition 7, we have

$$\begin{aligned} & C_k(\sigma,0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0, \text{ with } d_m = \dim \overline{H}_m = \dim \textstyle\bigoplus_{k=1}^m E(\widehat{\lambda}_k(2)) \geq 2, \\ \Rightarrow\ & C_k(\varphi_\lambda,0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see the proof of Proposition 7)}, \\ \Rightarrow\ & C_k(\psi_\lambda,0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (87))}. \end{aligned} \tag{92}$$

Suppose that $K_{\psi_\lambda} = \{0, u_\lambda^*, v_\lambda^*, y_0\}$. Then from (92), (90), (89), (91) and the Morse relation with $t = -1$ (see (5)), we have

$$\begin{aligned} & (-1)^{\mathrm{d_m}} + 2(-1)^0 + (-1)^1 = (-1)^0, \\ \Rightarrow \quad & (-1)^{\mathrm{d_m}} = 0, \text{ a contradiction.} \end{aligned}$$

So, there exists $\widehat{y} \in K_{\psi_\lambda}$, $\widehat{y} \notin \{0, u_\lambda^*, v_\lambda^*, y_0\}$. From (49) we have

$$\begin{aligned} & \widehat{y} \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}), \\ \Rightarrow \quad & \widehat{y} \text{ is nodal.} \end{aligned}$$

Moreover, as in the proof of Proposition 8, via the strong comparison principle (see Proposition 2.2 of [17]), we have

$$\widehat{y} \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*].$$

To produce a third nodal solution, we follow a flow invariance argument motivated by the nonlinear work of He-Lei-Zhang-Sun [13] and the earlier semilinear work of Papageorgiou-Papalini [16].

So, let $K_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle K_\lambda(u), h\rangle = \langle A_p(u), h\rangle + \langle A(u), h\rangle + \widehat{\xi}_\lambda \int_\Omega |u|^{p-2} uh dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

Clearly $K_\lambda(\cdot)$ is continuous, uniformly monotone (hence maximal monotone too) and coercive. Therefore $K_\lambda(\cdot)$ is surjective (see Papageorgiou-Rădulescu-Repovš [21], Corollary 2.8.7, p. 135) and of course it is injective (on account of the uniform monotonicity). So, we can define $K_\lambda^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$. Let

$$V_\lambda(u) = K_\lambda^{-1}(\lambda N_f(u) + N_g(u) + \widehat{\xi}_\lambda |u|^{p-2} u).$$

Then $V_\lambda(\cdot)$ is locally Lipschitz and using the Sobolev embedding theorem, we see that $V_\lambda(\cdot)$ is compact. Moreover, the nonlinear regularity theory implies that

$$V_\lambda(W_0^{1,p}(\Omega)) \subseteq C_0^1(\overline{\Omega}),\ V_\lambda(C_+) \subseteq C_+ \quad (\text{see hypothesis } H_0').$$

Clearly the fixed point set of $V_\lambda(\cdot)$ is K_{φ_λ}.

We consider the following abstract Cauchy problem

$$\frac{d\sigma(t,u)}{dt} + \sigma(t,u) = V_\lambda(\sigma(t,u)) \text{ on } \mathbb{R}_+,\ \sigma(0,u) = u. \tag{93}$$

On account of the coercivity of $K_\lambda(\cdot)$, the vector field in (93) has sublinear growth. So, the flow $\sigma(\cdot, u)$ of the Cauchy problem (93) is global (see Gasiński-Papageorgiou [11], Theorem 5.1.22, p. 618) and of course is unique.

We introduce the set

$$E_1 = \{u \in C_0^1(\overline{\Omega}) : \text{ there exists } t_0 \geq 1 \text{ such that } \sigma(t,u) \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*] \text{ for all } t \geq t_0\}.$$

The fact that the flow depends continuously on the Cauchy datum implies that $E_1 \subseteq W_0^{1,p}(\Omega)$ is open. Also $0 \in E_1$ and the semigroup property of the flow implies that E_1 is σ-invariant.

Let ∂E_1 be the boundary of E_1. We show that it is σ-invariant. If this is not true, then we can find $\overline{u} \in \partial E_1$ and $\overline{t} > 0$ such that

$$\sigma(\overline{t}, \overline{u}) \notin \partial E_1.$$

If $\sigma(\bar{t},\overline{u}) \in E_1$, then setting $\overline{v} = \sigma(\bar{t},\overline{u})$, we have

$$\begin{aligned}
&\sigma(t,\overline{v}) \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*] \quad \text{for all } t \geq \widehat{t}_0,\\
\Rightarrow\quad &\sigma(t+\bar{t},\overline{u}) \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*] \quad \text{for all } t \geq \widehat{t}_0,\\
&\qquad\text{(by the semigroup property of the flow)},\\
\Rightarrow\quad &\overline{u} \in E_1, \text{ a contradiction (recall } E_1 \text{ is open)}.
\end{aligned}$$

If $\sigma(\bar{t},\overline{u}) \notin \overline{E}_1 = E_1 \cup \partial E_1$, then we consider a sequence $\{u_n\}_{n\geq 1} \subseteq E_1$ such that $u_n \to \overline{u}$ in $W_0^{1,p}(\Omega)$ (recall $\overline{u} \in \partial E_1$). Then the σ-invariance of E_1 implies that

$$\begin{aligned}
&\sigma(\bar{t},u_n) \in E_1,\\
\Rightarrow\quad &\sigma(\bar{t},\overline{u}) \in \overline{E}_1 \quad \text{(continuous dependence of the flow on the Cauchy datum)},
\end{aligned}$$

again a contradiction.

Therefore we have proved that ∂E_1 is σ-invariant. Now let

$$E_2 = \{u \in C_0^1(\overline{\Omega}) : \text{ there exists } \widehat{t}_0 \geq 1 \text{ such that } \sigma(t,u) \in \mathrm{int}\, C_+ \text{ for all } t \geq \widehat{t}_0\}.$$

Again this is an open set in $W_0^{1,p}(\Omega)$. Moreover, as before (see the proof of Proposition 8), using the tangency principle (see [27], p. 35) and the strong comparison principle (see [17], Proposition 2.2), we have

$$u \leq v,\ u \neq v \ \Rightarrow\ V_\lambda(v) - V_\lambda(u) \in \mathrm{int}\, C_+.$$

Therefore, we have

$$C_+ \setminus \{0\} \subseteq E_2.$$

Also, we have $0 \in \partial E_2$ and as we did for ∂E_1, we can check that ∂E_2 is σ-invariant. We have

$$\begin{aligned}
&C_+ \subseteq \overline{E}_2 \quad \text{and} \quad \partial E_1 \cap C_+ \neq \emptyset,\\
\Rightarrow\quad &\partial E_1 \cap \overline{E}_2 \neq \emptyset.
\end{aligned}$$

Since $\partial E_1 \cap (-C_+) \neq \emptyset$, we infer that $\partial E_1 \cap \partial E_2 \neq \emptyset$ and so we can define

$$\widehat{c} = \inf[\varphi_\lambda(u) : u \in \partial E_1 \cap \partial E_2] > -\infty. \tag{94}$$

We show that $\widehat{c}$ is a critical value of φ_λ. Again we argue by contradiction. So, suppose that $\widehat{c}$ is not a critical value of φ_λ. On account of Proposition 1, K_{φ_λ} is closed. So, we can find $\varepsilon > 0$ small such that

$$(\widehat{c} - \varepsilon, \widehat{c} + \varepsilon) \cap \varphi_\lambda(K_{\varphi_\lambda}) = \emptyset. \tag{95}$$

Let $\widehat{u} \in \partial E_1 \cap \partial E_2$ be such that

$$\varphi_\lambda(\widehat{u}) \leq \widehat{c} + \varepsilon \quad \text{(see (94))}.$$

As in the proof of the deformation theorem (see, for example, Papageorgiou-Rădulescu-Repovš [21], Theorem 5.3.7(e), p. 382), because of (95) we have

$$\varphi_\lambda(\sigma(1,\widehat{u})) \leq \widehat{c} - \varepsilon. \tag{96}$$

But recall that the sets ∂E_1 and ∂E_2 are σ-invariant. Therefore $\varphi_\lambda(\sigma(1,\widehat{u})) \geq \widehat{c}$, a contradiction to (96). This proves that $\widehat{c}$ is a critical value of φ_λ.

Let $\widetilde{y} \in \partial E_1 \cap \partial E_2$ be such that

$$\varphi_\lambda'(\widetilde{y}) = 0,\ \varphi_\lambda(\widetilde{y}) = \widehat{c},\ \widetilde{y} \neq 0.$$

We have $\widetilde{y} \notin \operatorname{int} C_+ \cup (-\operatorname{int} C_+)$ and so $\widetilde{y}$ is nodal. Moreover, from the definition of the sets E_1 and E_2, we have that

$$\widetilde{y} \neq y_0,\ \widetilde{y} \neq \widehat{y}.$$

Finally the nonlinear regularity theory implies that $\widetilde{y} \in C_0^1(\overline{\Omega})$. $\square$

So, we can state the following multiplicity theorem for problem (P_λ).

Theorem 2. *If hypotheses $H(f)_2$, $H(g)_4$, H_0' hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least seven nontrivial smooth solutions $u_0, \widehat{u} \in \operatorname{int} C_+$, $v_0, \widehat{v} \in -\operatorname{int} C_+$, $y_0, \widehat{y}, \widetilde{y} \in C_0^1(\overline{\Omega})$ nodal; moreover there exist extremal constant sign solutions $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ and $y_0, \widehat{y} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*]$, $\widetilde{y} \in \partial[v_\lambda^*, u_\lambda^*]$.*

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