POLAR DECOMPOSITION AND FUNCTIONAL CALCULUS FOR GENERALIZED TOMITA'S OBSERVABLES

HIROSHI INOUE AND CAMILLO TRAPANI

Abstract. Continuing previous studies by one of us (HI), a polar decomposition and a functional calculus for an unbounded Tomita's observable are studied. For both problems we distinguish two different cases dictated by commutation properties.

1. Introduction

The notion of Tomita's observable algebra has recently been recovered by Atsushi Inoue in [1]. Two types of observable have been considered: the first one consists of a *quartet* (A, x, y^*, μ) where A is a linear operator in Hilbert space $\mathcal{H}$, $x \in \mathcal{H}$, y^* is an element of the conjugate dual space $\mathcal{H}^*$ of $\mathcal{H}$ and μ is an *expectation*; the second type is described as a *trio* observable and it consists essentially of the triplet (A, x, y^*) which appears in a quartet observable. The structure of families of quartet observables $Q^*(\mathcal{H})$ or trio observables $T^*(\mathcal{H})$ has been studied in detail in the cited monograph, under the assumption that A is a bounded operator in $\mathcal{H}$. A trio observable (A, x, y^*) has also a physical meaning since it describes, in a certain sense, a measurement process: A represents a physical quantity, x and y are vector representing states of the system and y^* is nothing but the linear functional determined by y, the *ket* $|y\rangle$ of the Dirac formalism. However the assumption that the physical quantity A is represented by a bounded operator, is not convenient since operators representing physical quantities are often unbounded. For this reason the case when the operator A in a member of some *-algebra of unbounded operators acting on a dense domain $\mathcal{D}$ of Hilbert space has been considered in [2, 3, 4], where several properties of unbounded observable algebras have been obtained.

We do not go further in discussing the physical interpretation of Tomita's observables and we address our interest in the mathematics which this notion entails in the same line as in [2, 3, 4]. This paper is, in particular, devoted to the polar decomposition of an unbounded Tomita's observable and to the functional calculus constructed on an observable of this type. In the case of bounded Tomita's observables some results in this direction have been discussed in [1].

After some preliminaries and basic definitions, we envisage the problem of the polar decomposition of an unbounded trio observable, by which we mean of course the possibility of expressing an unbounded trio observable A as the product US of a partial isometry acting on the dense domain $\mathcal{D}$ and of an unbounded observable S which is reasonable to consider as the module $|A|$ of A. We consider two different situations (Sections 3.1 and Section 3.2) and we obtain the result for hermitian elements $A = A^\dagger$. Section 4 is devoted to the construction of a functional calculus for an unbounded trio observable. Also in this case, we distinguish two different situations (Sections 3.1 and Section 3.2) determined by the commutation properties of the involved operators and under some convenient condition we construct an unbounded trio observable $f(A)$ associated to some continuous functions on the half-line $(0, \infty)$ in such a way that $f \mapsto f(A)$ is a *-homomorphism (or, even, a *-isomorphism) of *-algebras.

2. Preliminaris: $T^\dagger$-algebras and all that

In this section we summarize the main basic definitions and facts concerning *unbounded observables* and $T^\dagger$-algebras.

Definition 2.1. Let $\mathcal{D}$ be a dense domain in Hilbert space $\mathcal{H}$ and $\mathcal{L}^\dagger(\mathcal{D})$ the maximal O^*-algebra on $\mathcal{D}$. An *unbounded trio observable* (for short, *observable*) is a triplet (A_0, ξ, η^*) where $A_0 \in \mathcal{L}^\dagger(\mathcal{D})$, $\xi \in \mathcal{D}$ and $\eta^* \in \mathcal{D}^* = \{\zeta^* \in \mathcal{H}^* : \zeta \in \mathcal{D}\}$. The set of unbounded trio observables on $\mathcal{D}$ is denoted by $T^\dagger(\mathcal{D})$.

The set $T^\dagger(\mathcal{D})$ is a complex *-algebra without identity with respect to the following operations:

$$\begin{aligned} A + B &= (A_0 + B_0, \xi + \zeta, \eta^* + \chi^*) \\ \alpha A &= (\alpha A_0, \alpha\xi, \alpha\eta) \\ AB &= (A_0 B_0, A_0\zeta, (B_0^\dagger \eta)^*) \\ A^\dagger &= (A_0^\dagger, \eta, \xi^*), \end{aligned}$$

where $A = (A_0, \xi, \eta^*)$, $B = (B_0, \zeta, \chi^*)$ and $\alpha \in \mathbb{C}$.

Definition 2.2. A *-subalgebra of the *-algebra $T^\dagger(\mathcal{D})$ is called a trio observable algebra on $\mathcal{D}$, for simplicity, a $T^\dagger$-*algebra* on $\mathcal{D}$. Let $\mathfrak{A}$ be a $T^\dagger$-algebra on $\mathcal{D}$ in $\mathcal{H}$. If the O*-algebra $\pi(\mathfrak{A})$ on $\mathcal{D}$ is closed (resp. self-adjoint, essentially self-adjoint, integrable), then $\mathfrak{A}$ is called π-closed (resp. π -self-adjoint, π-essentially self-adjoint, π-integrable).

When $\mathcal{D} = \mathcal{H}$, a trio (A_0, x, y^*) with $A_0 \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ is called a Tomita's trio observable on $\mathcal{H}$. The set $T^*(\mathcal{H})$ of all Tomita's trio observables on $\mathcal{H}$ is a Banach *-algebra without identity equipped with the above algebraic operations and the norm $\|A\| := \max\{\|A_0\|, \|x\|, \|y\|\}$. We denote the involution on $T^*(\mathcal{H})$ by $A \mapsto A^\sharp$, with $A^\sharp = (A_0^*, \eta, \xi^*)$, to distinguish $T^*(\mathcal{H})$ from $T^\dagger(\mathcal{D})$.

If $A = (A_0, \xi, \eta^*) \in T^\dagger(\mathcal{D})$, we write

$$\pi(A) = A_0, \quad \lambda(A) = \xi, \quad , \lambda^*(A) = \eta^*.$$

Then π is a (in general, unbounded) *-representation of $T^\dagger(\mathcal{D})$ onto $\mathcal{L}^\dagger(\mathcal{D})$ and λ is a linear map of $T^\dagger(\mathcal{D})$ onto $\mathcal{D}$, with

$$\lambda(AB) = \pi(A)\lambda(B), \quad \forall A, B \in T^\dagger(\mathcal{D}).$$

Similarly, λ^* is a linear map of $T^\dagger(\mathcal{D})$ onto $\mathcal{D}^*$, satisfying

$$\lambda^*(A) = (\lambda(A^\dagger))^*, \quad \lambda^*(AB) = (\pi(B)^\dagger \lambda(A^\dagger))^*, \quad \forall A, B \in T^\dagger(\mathcal{D}).$$

Let $\mathfrak{A}$ be a π-self-adjoint $T^\dagger$-algebra on $\mathcal{D}$ in $\mathcal{H}$. The commutant $\mathfrak{A}^c$ of $\mathfrak{A}$ [2, Section 5] can be defined as

$$\begin{aligned}\mathfrak{A}^c = \{&K \in T^*(\mathcal{H}); \pi(K) \in \pi(\mathfrak{A})'_w, \lambda(K), \lambda(K^\sharp) \in \mathcal{D},\\ &\pi(K)\lambda(A) = \pi(A)\lambda(K) \text{ and } \pi(K)^*\lambda(A) = \pi(A)\lambda(K^\sharp),\ \forall A \in \mathfrak{A}\}.\end{aligned}$$

3. Polar decomposition of elements of $T^\dagger(\mathcal{D})$

To study the polar decomposition of a trio observable, we define the following notions of absolute values of it as follows:

Definition 3.1. Let $\mathfrak{A}$ be a $T^\dagger$-algebra on $\mathcal{D}$ in $\mathcal{H}$. An element A of $\mathfrak{A}$ is said to be <u>positive</u> if $A = B^\dagger B$ for some B of $\mathfrak{A}$, and A is said to be <u>π-positive</u> if it is hermitian and $\pi(A)$ is positive, that is, $(\pi(A)\xi|\xi) \geqq 0$ for all $\xi \in \mathcal{D}$.

Clearly, if A is positive, it is π-positive.

Definition 3.2. If $A = B^2$ for some positive (resp. π-positive) element B of $\mathfrak{A}$, then B is said to be the <u>absolute value</u> or <u>root</u> (resp. <u>π-absolute value</u> or the <u>π-root</u>) of A and denoted by $|A|$ or $A^{\frac{1}{2}}$ (resp. $|A|_\pi$ or $A_\pi^{\frac{1}{2}}$).

In this section, let $\mathfrak{A}$ be a π-selfadjoint $T^\dagger$-algebra on $\mathcal{D}$ in $\mathcal{H}$.

3.1. **The case $K\,\eta\,\mathfrak{A}^c$.** By the symbol $K\,\eta\,\mathfrak{A}^c$ we mean the following

Definition 3.3. We say that $K \in T^\dagger(\mathcal{D})$ is affiliated with $\mathfrak{A}^c$ if

(i) $\overline{\pi(K)}\,\eta\,\pi(\mathfrak{A})'_w$;
(ii) $\pi(K)\lambda(A) = \pi(A)\lambda(K)$ and $\pi(K)^\dagger\lambda(A) = \pi(A)\lambda(K^\dagger)$, $\forall A \in \mathfrak{A}^c$.

In this case we write $K\,\eta\,\mathfrak{A}^c$. The set of all $K\,\eta\,\mathfrak{A}^c$ will be denoted by $\mathfrak{A}^c_\eta$.

Let $K\,\eta\,\mathfrak{A}^c$. As is known, if $\overline{\pi(K)} = U_K|\overline{\pi(K)}|$ is the polar decomposition of $\overline{\pi(K)}$, then

$$\pi(K)^* = |\overline{\pi(K)}|U_K^* = U_K^*U_K|\overline{\pi(K)}|U_K^* = U_K^*|\pi(K)^*|$$

is a polar decomposition of $\pi(K)^*$.

Let us put $K_0 := |\overline{\pi(K)}| = (\pi(K)^*\overline{\pi(K)})^{1/2}$ and let $K_0 = \int_0^\infty t dE_K(t)$ be its spectral decomposition. Then, from $K_0\,\eta\,\pi(\mathfrak{A})'_w$ it follows that U_K, $U_K^* \in \pi(\mathfrak{A})'_w$ and $E_K(t) \in \pi(\mathfrak{A})'_w$ for all $t \in (0,\infty)$.

Moreover, if $\xi \in \mathcal{D} \subset D(K_0)$, putting $E_K(n) = E_K(-n,n)$, we obtain $K_0E_K(n)\xi \to K_0\xi$ and $\pi(A)K_0E_K(n)\xi = K_0E_K(n)\pi(A)\xi \to K_0\pi(A)\xi$, for every $A \in \mathfrak{A}$. Hence, we have $K_0\xi \in \bigcap_{A\in\mathfrak{A}} D(\overline{\pi(A)}) = \mathcal{D}$. Hence

$$K_0\mathcal{D} \subset \mathcal{D}. \tag{3.1}$$

Similarly, if we put $H_0 = |\pi(K)^*|$, we get $H_0\mathcal{D} \subset \mathcal{D}$.

Let us define

$$K_0^\sim := (K_0\lceil_{\mathcal{D}}, U_K^*\lambda(K), (U_K^*\lambda(K))^*)$$
$$H_0^\sim := (H_0\lceil_{\mathcal{D}}, U_K\lambda(H), (U_K\lambda(H))^*).$$

In what follows, we will write K_0 instead of $K_0\lceil_{\mathcal{D}}$ and H_0 instead of $H_0\lceil_{\mathcal{D}}$. Since $K_0, H_0 \in \mathcal{L}^\dagger(\mathcal{D})$, $K_0^\sim$ and $H_0^\sim$ belong to $T^\dagger(\mathcal{D})$. Moreover,

Lemma 3.4.

$$K_0^\sim \in \mathfrak{A}^c_\eta, \quad (K_0^\sim)^\dagger = K_0^\sim \quad \textit{and} \quad (K_0^\sim)^2 = K^\dagger K$$
$$H_0^\sim \in \mathfrak{A}^c_\eta, \quad (H_0^\sim)^\dagger = H_0^\sim \quad \textit{and} \quad (H_0^\sim)^2 = KK^\dagger$$

Proof. Clearly, $\overline{\pi(\widetilde{K_0})} = K_0 \in \pi(\mathfrak{A})'_w$. Moreover, for every $A \in \mathfrak{A}$, we have

$$\begin{aligned}
\pi(\widetilde{K_0})\lambda(A) &= K_0\lambda(A) \\
&= U_K^* U_K K_0 \lambda(A) \\
&= U_K^* \pi(K)\lambda(A) \quad \text{since } K_0 = \overline{\pi(K)} \\
&= U_K^* \pi(A)\lambda(K) \\
&= \pi(A) U_K^* \lambda(K) \\
&= \pi(A)\lambda(\widetilde{K_0}) \qquad \text{by definition of } \widetilde{K_0}\,;
\end{aligned}$$

$$\begin{aligned}
\pi(\widetilde{K_0})^\dagger \lambda(A) &= \pi(\widetilde{K_0})\lambda(A) \\
&= \pi(A)\lambda(\widetilde{K_0}) \\
&= \pi(A)\lambda(\widetilde{K_0}^\dagger).
\end{aligned}$$

From these equalities we deduce that $\widetilde{K_0} \in \mathfrak{A}^c_\eta$. Moreover $(\widetilde{K_0})^\dagger = \widetilde{K_0}$ clearly and

$$\begin{aligned}
(\widetilde{K_0})^2 &= (K_0^2, K_0 U_K^* \lambda(K), (K_0 U_K^* \lambda(K))^*) \\
&= (\pi(K)^\dagger \pi(K), \pi(K)^\dagger \lambda(K), (\pi(K)^\dagger \lambda(K))^*) = K^\dagger K.
\end{aligned}$$

The proof for $\widetilde{H_0}$ is similar. $\square$

Lemma 3.4 shows that $\widetilde{K_0}$ and $\widetilde{H_0}$ are π-positive and they are the π-absolute values of K and $K^\dagger$, respectively, so we can write

$$\begin{aligned}
|K|_\pi &= (|\overline{\pi(K)}|, U_K^* \lambda(K), (U_K^* \lambda(K))^*) \\
|K^\dagger|_\pi &= (|\pi(K)^*|, U_K^* \lambda(K^\dagger), (U_K^* \lambda(K^\dagger))^*).
\end{aligned} \tag{3.2}$$

It is natural to ask if $|K|_\pi$ is positive, i.e., there exists a hermitian element S of $\mathfrak{A}^c_\eta$ with $\pi(S) = |\overline{\pi(K)}|^{\frac{1}{2}} \lceil_{\mathcal{D}}$. In Theorem 3.8 below we shall show this is true if $\overline{\pi(K)}$ is invertible, that is, it has a bounded inverse. In general, this does not necessarily hold.

The next step consists in considering the polar decomposition of a hermitian K. For this purpose, we first define the product of an operator $A_0 \in \mathcal{L}^\dagger(\mathcal{D})$ and $X = (\pi(X), \lambda(X), \lambda(X^\dagger)^*) \in T^\dagger(\mathcal{D})$. We put

$$A_0 X = (A_0 \pi(X), A_0 \lambda(X), (A_0 \lambda(X^\dagger))^*).$$

Then $A_0 X \in T^\dagger(\mathcal{D})$ and $T^\dagger(\mathcal{D})$ is a left-module over $\mathcal{L}^\dagger(\mathcal{D})$.

We also put

$$P_K = U_K U_K^* \quad \text{and} \quad P_{K^\dagger} = U_K^* U_K.$$

Then P_K is the projection onto the closure of the range $R(\overline{\pi(K)})$ of $\pi(K)$. Similarly, $P_{K^\dagger}$ is the projection onto the closure of the range $R(\pi(K)^*)$ of $\pi(K)^*$.

Taking into account the above mentioned facts and notations, we have

$$U_K|K|_\pi = (\pi(K), P_K\lambda(K), (P_K\lambda(K))^*).$$

We now prove the following

Lemma 3.5. *Let $K = K^\dagger$. Then $U_K|K|_\pi \, \eta \, \mathfrak{A}^c$*

Proof. Since $\pi(U_K|K|_\pi) = \pi(K)$ we clearly have $\overline{\pi(U_K|K|_\pi)} \, \eta \, \pi(\mathfrak{A})'_w$. Moreover

$$\begin{aligned} \pi(U_K|K|_\pi)\lambda(A) &= \pi(K)\lambda(A) \\ &= P_K\pi(K)\lambda(A) \\ &= P_K\pi(A)\lambda(K) \\ &= \pi(A)P_K\lambda(K) \\ &= \pi(A)\lambda(U_K|K|_\pi). \end{aligned}$$

If $K = K^\dagger$, then $(U_K|K|_\pi)^\dagger = U_K|K|_\pi$; in this case the preceding series of equalities also implies that $\pi(U_K|K|_\pi)^\dagger = \pi(A)\lambda((U_K|K|_\pi)^\dagger)$. Hence, $U_K|K|_\pi \in \mathfrak{A}^c_\eta$. □

Remark 3.6. If $K \neq K^\dagger$, we have

$$\begin{aligned} \pi(U_K|K|_\pi)^\dagger\lambda(A) &= \pi(K)^\dagger\lambda(A) \\ &= P_{K^\dagger}\pi(K)^\dagger\lambda(A) \\ &= \pi(A)P_{K^\dagger}\lambda(K^\dagger) \\ &= \pi(A)\lambda(U_K^*|K^\dagger|_\pi) \\ &\neq \pi(A)\lambda((U_K|K|_\pi)^\dagger), \qquad \text{in general.} \end{aligned}$$

Lemma 3.7. *Let $K \in T^\dagger(\mathcal{D})$ with $K \, \eta \, \mathfrak{A}^c$. Suppose that $\mathfrak{A}$ is non-degenerate, that is, the linear span $[\pi(\mathfrak{A})\mathcal{D}]$ of $\pi(\mathfrak{A})\mathcal{D}$ is dense in $\mathcal{H}$. Then*

$$P_K\lambda(K) = \lambda(K) \quad \textit{and} \quad P_{K^\dagger}\lambda(K^\dagger) = \lambda(K^\dagger).$$

Proof. For every $A \in \mathfrak{A}$,

$$\pi(A)P_K\lambda(K) = P_K\pi(K)\lambda(A) = \pi(K)\lambda(A) = \pi(A)\lambda(K).$$

The density of $[\pi(\mathfrak{A})\mathcal{D}]$ then implies that $P_K\lambda(K) = \lambda(K)$.

The proof of the equality $P_{K^\dagger}\lambda(K^\dagger) = \lambda(K^\dagger)$ is similar. □

The next theorem establishes the polar decomposition of an hermitian element K.

Theorem 3.8. *Let K be a hermitian element of $\mathfrak{A}^c_\eta$ which is affialiated with $\mathfrak{A}^c$. Then we have the following*

(i) Suppose that $\mathfrak{A}$ is nondegenerate. Then

$$K = U_K |K|_\pi. \tag{3.3}$$

(ii) Suppose that $\overline{\pi(K)}$ is invertible. Then $|K|_\pi$ is the absolute value of K, that is $|K|_\pi = |K|$ and

$$K = U_K |K|. \tag{3.4}$$

The equalities (3.3) *and* (3.4) *are called the* π-polar decomposition *and the* polar decomposition *of K, respectively.*

Proof. (i) This follows easily from Lemma 3.5 and Lemma 3.7.
(ii) In the same way as (3.1) we can show

$$|\overline{\pi(K)}^{\frac{1}{2}}|\mathcal{D} \subset \mathcal{D}. \tag{3.5}$$

Since $\overline{\pi(K)}$ is invertible, $|\overline{\pi(K)}|$, $|\overline{\pi(K)}|^{\frac{1}{2}}$, $\pi(K)^*$, $|\pi(K)^*|$ and $|\pi(K)^*|^{\frac{1}{2}}$ are invertible, and $U_K^* U_K = U_K U_K^* = I$, which implies by (3.3) that $|\overline{\pi(K)}|^{\frac{1}{2}} \, \eta \, \pi(\mathfrak{A})'_{\mathrm{w}}$ and

$$\begin{aligned} \pi(A)|\overline{\pi(K)}|^{-\frac{1}{2}} U_K^* \lambda(K) &= |\overline{\pi(K)}|^{-\frac{1}{2}} U_K^* \pi(K) \lambda(A) \\ &= |\overline{\pi(K)}|^{-\frac{1}{2}} |\overline{\pi(K)}| \lambda(A) \\ &= |\overline{\pi(K)}|^{-\frac{1}{2}} \lambda(A) \end{aligned}$$

for all $A \in \mathfrak{A}$, so that, putting

$$S := (|\overline{\pi(K)}|^{\frac{1}{2}} \lceil_{\mathcal{D}}, |\overline{\pi(K)}|^{-\frac{1}{2}} U_K^* \lambda(K), (|\overline{\pi(K)}|^{-\frac{1}{2}} U_K^* \lambda(K))^*)$$

we get

$$\begin{aligned} S &\in \mathfrak{A}^c_\eta \\ \text{and} \quad S^2 &= (|\overline{\pi(K)}|, U_K^* \lambda(K), (U_K^* \lambda(K))^*) \\ &= |K|_\pi. \end{aligned}$$

Hence $|K|_\pi$ is positive, so $|K|_\pi = |K|$ and $K = U_K|K|$. This completes the proof. □

Remark 3.9. If K is not hermitian,

$$U_K|K|_\pi = (\pi(K), P_K\lambda(K)), (P_K\lambda(K))^*) \notin \mathfrak{A}^c_\eta,$$

in general, and even if $\mathfrak{A}$ is nondegenerate,

$$U_K|K|_\pi = (\pi(K), \lambda(K)), \lambda(K)^*) \neq (\pi(K), \lambda(K)), \lambda(K^\dagger)^*) = K.$$

Theorem 3.10. *Let $K \in T^\dagger(\mathcal{D})$ be affialiated with $\mathfrak{A}^c$. Let us put*

$$K_s := (\pi(K), P_K\lambda(K), (P_{K^\dagger}\lambda(K^\dagger))^\dagger),$$
$$K_n := (0, (I - P_K)\lambda(K), ((I - P_{K^\dagger})\lambda(K^\dagger))^\dagger).$$

Then $K_s, K_n \in \mathfrak{A}^c_\eta$ and $K = K_s + K_n$.

Proof. We have $\overline{\pi(K_s)} = \overline{\pi(K)} \in \pi(\mathfrak{A})'_w$ and, for every $A \in \mathfrak{A}$,

$$\begin{aligned}\pi(K_s)\lambda(A) &= \pi(K)\lambda(A)\\ &= U_K U_K^* \pi(K)\lambda(A)\\ &= \pi(A)P_K\lambda(K)\\ &= \pi(A)\lambda(K_s).\end{aligned}$$

and

$$\begin{aligned}\pi(K_s)^\dagger\lambda(A) &= \pi(K)^\dagger\lambda(A)\\ &= U_K^* U_K \pi(K)^\dagger\lambda(A)\\ &= \pi(A)P_{K^\dagger}\lambda(K^\dagger)\\ &= \pi(A)\lambda K_s^\dagger).\end{aligned}$$

This implies that K_s is affiliated with $\mathfrak{A}^c$; i.e., $K_s \in \mathfrak{A}^c_\eta$.

On the other hand, $\overline{\pi(K_n)} = 0 \in \pi(\mathfrak{A})'_w$, obviously, and, for every $A \in \mathfrak{A}$

$$\begin{aligned}\pi(A)\lambda(K_n) &= \pi(A)(I - P_K)\lambda(K)\\ &= (I - P_K)\pi(K)\lambda(A)\\ &= 0\\ &= \pi(K_n)\lambda(A).\end{aligned}$$

and

$$\begin{aligned}\pi(A)\lambda(K_n^\dagger) &= \pi(A)(I - P_{K^\dagger})\lambda(K^\dagger)\\ &= (I - P_{K^\dagger})\pi(K^\dagger)\lambda(A)\\ &= 0\\ &= \pi(K_n^\dagger)\lambda(A).\end{aligned}$$

This implies that $K_n \in \mathfrak{A}^c_\eta$. It is evident that $K = K_s + K_n$. This completes the proof. $\square$

We will call K_s the *semisimple part* of K and K_n the *nilpotent part* of K. Moreover if $K = K_s$ then K is called *semisimple* and in $K = K_n$, then K is called *nilpotent*.

By Theorem 3.10 and Lemma 3.7 we get the following

Proposition 3.11. *Let $K \in \mathfrak{A}^c_\eta$. If $\mathfrak{A}$ is nondegenerate, then K is semisimple.*

3.2. **The case of $A \in \mathfrak{A}^{cs}$.** Let $\mathfrak{A}$ be a π-selfadjoint $T^\dagger$-algebra on $\mathcal{D}$. We denote by $\mathfrak{A}^{cs}$ the following *bicommutant*

$$\mathfrak{A}^{cs} = \{A \in T^\dagger(\mathcal{D}) : \; AK = KA, \; \forall K \in \mathfrak{A}^c\}.$$

Since $\mathfrak{A} \subset \mathfrak{A}^{cs}$, $\mathfrak{A}^{cs}$ is a π-selfadjoint T†-algebra on $\mathcal{D}$ in $\mathcal{H}$. Furthermore, since $\pi(\mathfrak{A}^c) \subset \pi(\mathfrak{A})'_w$ we also have $(\pi(\mathfrak{A})'_w)' \subset \pi(\mathfrak{A}^c)'$, where denotes the usual commutant of the *-algebra of bounded operators. In this subsection we will consider the polar decomposition, semisimplicity and nilpotentness for an element $A \in \mathfrak{A}^{cs}$.

Let $A \in \mathfrak{A}^{cs}$ and suppose that $\overline{\pi(A)}$ is affiliated with $\pi(\mathfrak{A}^c)'$ and let $\overline{\pi(A)} = U_A|\overline{\pi(A)}|$ be its polar decomposition. Then

$$U_A \in \pi(\mathfrak{A}^c)' \;\text{ and }\; |\overline{\pi(A)}| \,\eta\, \pi(\mathfrak{A}^c)'. \tag{3.6}$$

We notice that the inclusions

$$U_A\mathcal{D} \subset \mathcal{D}, \quad |\overline{\pi(A)}|\mathcal{D} \subset \mathcal{D}$$

are not necessarily true. For this reason, throughout this subsection we will consider elements $A \in \mathfrak{A}^{cs}$ with the property $U_A \in \mathcal{L}^\dagger(\mathcal{D})$.

Lemma 3.12. *Let us put*

$$\widetilde{|\overline{\pi(A)}|} := (|\overline{\pi(A)}|\lceil_{\mathcal{D}}, U_A^*\lambda(A), (U_A^*\lambda(A))^*).$$

Then,

$$\widetilde{|\overline{\pi(A)}|} \in \mathfrak{A}^{cs} \quad \textit{and} \quad (\widetilde{|\overline{\pi(A)}|})^2 = \pi(A)^\dagger\pi(A).$$

Proof. For simplicity we write $|\overline{\pi(A)}|$ instead of $|\overline{\pi(A)}|\lceil_{\mathcal{D}}$.

Since $U_A \in \mathcal{L}^\dagger(\mathcal{D})$, we have $|\overline{\pi(A)}|\xi = U_A^*\pi(A)\xi \in \mathcal{D}$ for every $\xi \in \mathcal{D}$; i.e., $|\overline{\pi(A)}|\mathcal{D} \subset \mathcal{D}$. Thus $\widetilde{|\overline{\pi(A)}|} \in T^\dagger(\mathcal{D})$. We shall show that $\widetilde{|\overline{\pi(A)}|} \in$

$\mathfrak{A}^{cs}$. This follows from

$$\begin{aligned}
K|\overline{\pi(A)}|^{\sim} &= (\pi(K)|\overline{\pi(A)}|, \pi(K)U_A^*\lambda(A), (|\overline{\pi(A)}|\lambda(K^\dagger))^*) \\
&= (|\overline{\pi(A)}|\pi(K), U_A^*\pi(A)\lambda(K), (U_A^*\pi(A)\lambda(K^\dagger))^*) \\
&= (|\overline{\pi(A)}|\pi(K), U_A^*U_A|\overline{\pi(A)}|\lambda(K), (U_A^*\pi(K)^\dagger\lambda(A))^*) \\
&= (|\overline{\pi(A)}|\pi(K), |\overline{\pi(A)}|\lambda(K), (\pi(K)^\dagger U_A^*\lambda(A))^*) \\
&= |\overline{\pi(A)}|^{\sim}K, \quad \forall K \in \mathfrak{A}^c.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(|\overline{\pi(A)}|^{\sim})^2 &= (|\overline{\pi(A)}|^2, |\overline{\pi(A)}|U_A^*\lambda(A), (|\overline{\pi(A)}|U_A^*\lambda(A))^*) \\
&= (\pi(A)^\dagger\pi(A), \pi(A)^\dagger\lambda(A), \pi(A)^\dagger\lambda(A))^*) = A^\dagger A
\end{aligned}$$

This completes the proof. □

By Lemma 3.12, it appears natural to define $|\overline{\pi(A)}|^{\sim}$ as $|A|_\pi$.

Let us now put $P_A = U_AU_A^*$ and $P_{A^\dagger} = U_A^*U_A$. Then P_A is the projection onto the closure of the range $R(\overline{\pi(A)})$ of $\pi(A)$. Similarly, $P_{A^\dagger}$ is the projection onto the closure of the range $R(\pi(A)^*)$ of $\pi(A)^*$. Then we have

$$\begin{aligned}
U_A|A|_\pi &= (U_A|\overline{\pi(A)}|, U_AU_A^*\lambda(A), (U_A^*U_A\lambda(A^\dagger))^*) \\
&= (\pi(A)^\dagger, P_A\lambda(A^\dagger), (P_A\lambda(A^\dagger))^*)
\end{aligned}$$

and

$$\begin{aligned}
U_A^*|A|_\pi &= (\pi(A)^\dagger, U_A^*U_A\lambda(A^\dagger), (U_A^*U_A\lambda(A^\dagger))^*) \\
&= (\pi(A)^\dagger, P_{A^\dagger}\lambda(A^\dagger), (P_{A^\dagger}\lambda(A^\dagger))^*)
\end{aligned}$$

but they are not necessarily contained in $\mathfrak{A}^{cs}$. Indeed, for any $K \in \mathfrak{A}^c$, we have

$$\begin{aligned}
U_A|A|_\pi K &= (\pi(A)\pi(K), \pi(A)\lambda(K), (\pi(K)^\dagger P_A\lambda(A))^*) \\
&= (\pi(K)\pi(A), \pi(A)\lambda(K), (\pi(A)\lambda(K^\dagger))^*);
\end{aligned} \tag{3.7}$$

but, on the other hand,

$$\begin{aligned}
KU_A|A|_\pi &= (\pi(K)\pi(A), \pi(K)P_A\lambda(A), (\pi(A)^\dagger\lambda(K^\dagger))^*) \\
&= (\pi(K)\pi(A), \pi(A)\lambda(K), (\pi(A)^\dagger\lambda(K^\dagger))^*),
\end{aligned} \tag{3.8}$$

but $\pi(A)\lambda(K^\dagger) \neq (\pi(A)^\dagger\lambda(K^\dagger)$ unless A is hermitian.

Theorem 3.13. *If A is hermitian and $U_A \in \mathcal{L}^\dagger(\mathcal{D})$, then*
$$U_A|A|_\pi = (\pi(A), P_A\lambda(A), (P_A\lambda(A))^*) \in \mathfrak{A}^{cs}.$$
In addition, if $\mathfrak{A}^c$ is nondegenerate (i.e., $[\pi(\mathfrak{A}^c)\mathcal{H}]$ is dense in $\mathcal{H}$), then
$$A = U_A|A|_\pi \tag{3.9}$$

Proof. Let A be hermitian. Then, by (3.9) and (3.8), $(U_A|A|_\pi)K = K(U_A|A|_\pi)$ for every $K \in \mathfrak{A}^c$: Hence, $U_A|A|_\pi \in \mathfrak{A}^{cs}$. Assume now that $\mathfrak{A}^c$ is nondegenerate. Then as shown in Lemma 3.7, we have
$$\pi(K)P_A\lambda(A) = \pi(K)\lambda(A), \quad \forall K \in \mathfrak{A}^c.$$
Since $[\pi(\mathfrak{A}^c)\mathcal{H}]$ is dense in $\mathcal{H}$, it follows that $P_A\lambda(A) = \lambda(A)$. This completes the proof. □

The equality in (3.9), $A = U_A|A|_\pi$, is called the π-*polar decomposition* of A.

Remark 3.14. If A is nonhermitian, even if $\mathfrak{A}^c$ is nondegenerate, we may have
$$U_A|A|_\pi = (\pi(A), \lambda(A), \lambda(A)^*) \neq A.$$
For example, let $\mathcal{M}$ be a self-adjoint O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$ with a cyclic and separating vector ξ_0. $\xi_0 \in \mathcal{D}$ is called cyclic (resp. separating) for $\mathcal{M}$ if $\mathcal{M}\xi_0$ (resp. $\mathcal{M}'_{\mathrm{w}}\xi_0$) is dense in $\mathcal{H}$. Then we can define a π-self-adjoint $T^\dagger$-algebra $\mathfrak{A}$ on $\mathcal{D}$ in $\mathcal{H}$ by
$$\mathfrak{A} = \{A := (A_0, A_0\xi_0, (A_0\xi_0)^*);\ A_0 \in \mathcal{M}\}.$$
Since ξ_0 is cyclic for $\mathcal{M}$, we have
$$\mathfrak{A}^c = \{(C, C\xi_0, (C^*\xi_0)^*);\ C \in \mathcal{M}'_{\mathrm{w}}\},$$
which implies by separateness of ξ_0 that for $A, B \in \mathfrak{A}$
$$\begin{aligned} A = B \quad &\text{if and only if} \quad \pi(A) = \pi(B) \\ &\text{if and only if} \quad \lambda(A) = \lambda(B). \end{aligned}$$
Hence if A is not a hermitian element of $\mathfrak{A}$, then $A \neq U_A|A|_\pi$.

As in Theorem 3.10 and Proposition 3.11, we can show that every $A \in \mathfrak{A}^{cs}$ can be decomposed into a semisimple part A_s and a nilpotent part A_n.

Theorem 3.15. *Let $A \in \mathfrak{A}^{cs}$. Suppose that $U_A \in \mathcal{L}^\dagger(\mathcal{D})$ and put*
$$\begin{aligned} A_s &:= (\pi(A), P_A\lambda(A), (P_{A^\dagger}\lambda(A^\dagger))^\dagger), \\ A_n &:= (0, (I - P_A)\lambda(A), ((I - P_{A^\dagger})\lambda(A^\dagger))^\dagger). \end{aligned}$$
Then $A_s, A_n \in \mathfrak{A}^{cs}$ and $A = A_s + A_n$.

A_s is called the *semisimple part* of A and A_n is called the *nilpotent part* of A. Furthermore, If $A = A_s$, then A is called *semisimple* and if $A = A_n$ then A is called *nilpotent*.

Even if A is hermitian and $A = U_A|A|_\pi$, $|A|_\pi$ need not be positive; in fact, the inclusion $|\overline{\pi(A)}|^{\frac{1}{2}}\mathcal{D} \subset \mathcal{D}$ also does not hold, in general. So, we shall consider when the polar decomposition $A = U_A|A|$ of A is possible. Let now $A = A^\dagger \in T^\dagger(\mathcal{D})$. Let $\overline{\pi(A)} = U_A|\overline{\pi(A)}|$ be the polar decomposition of $\overline{\pi(A)}$ and let $|\overline{\pi(A)}| = \int_0^\infty t dE_A(t)$ be the spectral resolution of $|\overline{\pi(A)}|$. Let us denote by $\mathcal{P}(A)$ the $T^\dagger$-algebra on $\mathcal{D}$ generated by A. We suppose that $\mathcal{P}(A)$ is π-closed (otherwise, replace $\mathcal{D}$ with $\widetilde{\mathcal{D}} := \bigcap_{A\in\mathcal{P}(A)} D(\overline{\pi(A)})$).

The following equalities are clear

$$\mathcal{P}(\pi(A))'_w = \{\pi(A)\}'_w; \qquad \mathcal{P}(A)^c = \{A\}^c.$$

But $\mathcal{P}(A)$ is not necessarily π-self-adjoint. By [6, Lemma 3.2] we know that $\{\pi(\mathcal{P}(A))\}'_w$ is a von Neumann algebra on $\mathcal{H}$ if, and only if, $\pi(A)$ is essentially self-adjoint. Moreover, from [5, Theorem 2.1], we know that the following statements are equivalent

(i) $\mathcal{P}(\pi(A))$ is self-adjoint;
(ii) $\pi(A^n)$ is essentially self-adjoint, for every $n \in \mathbb{N}$;
(iii) $\{\pi(A)\}'_w$ is a von Neumann algebra and $\{\pi(A)\}'_w\mathcal{D} \subset \mathcal{D}$.

Then

$$\{U_A, \{E_A(t); t \in (0,\infty)\}\}'' = (\{\pi(A)\}'_w)' \subset \{\pi(A)\}'_w.$$

Hence,

$$\{U_A, \{E_A(t);\ t \in (0,\infty)\}''\mathcal{D} \subset \{\pi(A)\}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}. \tag{3.10}$$

Now we have the following

Theorem 3.16. *Let $A = A^\dagger \in T^\dagger(\mathcal{D})$. Suppose that $\overline{\pi(A)}$ is invertible and $\pi(A^n)$ is essentially self-adjoint for every $n \in \mathbb{N}$. Then $|A|_\pi$ is positive and the polar decomposition $A = U_A|A|$ of A holds.*

Proof. By (3.10) $UA \in \mathcal{L}^\dagger(\mathcal{D})$ and $P_A = I$ because $\overline{\pi(A)}$ is invertible, which implies by Theorem 3.13 that $A = U_A|A|_\pi$. Moreover, $|\overline{\pi(A)}|^{\frac{1}{2}}\mathcal{D} \subset \mathcal{D}$ and $|\overline{\pi(A)}|^{-\frac{1}{2}}\mathcal{D} \subset \mathcal{D}$ by (3.10). Putting

$$X := (|\overline{\pi(A)}|^{\frac{1}{2}}\lceil_{\mathcal{D}}, |\overline{\pi(A)}|^{-\frac{1}{2}}U_A\lambda(A), (|\overline{\pi(A)}|^{-\frac{1}{2}}U_A\lambda(A))^*),$$

we can show that $X = X^\dagger \in \mathcal{P}(A)^{cs}$ and $X^2 = X$. Hence $|A|_\pi$ is positive and $A = U_A|A|$. This completes the proof. $\square$

Finally we show that every $A \in T^\dagger(\mathcal{D})$ can be decomposed into a semisimple part A_s and a nilpotent part A_n in $\mathcal{P}(A)^{cs}$.

Corollary 3.17. *Let $A^\dagger = A \in T^\dagger(\mathcal{D})$ and suppose that $\pi(A^n)$ is essentially self-adjoint, for every $n \in \mathbb{N}$. Put*

$$\begin{aligned} A_s &:= (\pi(A), (I - E_A(0))\lambda(A), ((I - E_A(0))\lambda(A))^*), \\ A_n &:= (0, E_A(0)\lambda(A), (E_A(0)\lambda(A))^*). \end{aligned}$$

Then $A_s, A_n \in \mathcal{P}(A)^{cs}$ and $A = A_s + A_n$. In particular, if $\mathrm{Ker}\overline{(\pi(A))} = \{0\}$, then A is semisimple and has the polar decomposition $A = U_A|A|$.

Proof. $\mathcal{P}(A)$ is π-self-adjoint and by (3.10) $U_A \in \mathcal{L}^\dagger(\mathcal{D})$ and $P_A = U_A U_A^* = I - E_A(0)$, being the projection onto Ker $(\overline{\pi(A)})$, which implies that A_s, $A_n \in \mathcal{P}(A)^{cs}$. Thus, the statement follows from Theorem 3.13 and 3.15. $\square$

4. Functional calculus

In this section we will study the functional calculus for an unbounded trio observable and consider a density theorem for $T^\dagger$-algebras.

4.1. **Case $K \,\eta\, \mathfrak{A}^c$.** Let $\overline{\pi(K)} = U_K|\pi(K)| = |\pi(K)^*|U_K$ be the polar decomposition of $\overline{\pi(K)}$ and $|\pi(K)| = \int_0^\infty t dE_K(t)$ be the spectral resolution of $|\overline{\pi(K)}|$. We put $K_0 := |\pi(K)|$ and $H_0 := |\pi(K)^*|$ for short. We recall that, by (3.2), the absolute value of K is defined as

$$|K|_\pi = (|\overline{\pi(K)}|, U_K^*\lambda(K), (U_K^*\lambda(K))^*).$$

First we will consider the functional calculus for $|K|_\pi$.

We put

$$\begin{aligned} C_\lambda^K(0,\infty) := \Bigg\{ f \in C(0,\infty) : &\int_0^\infty |f(t)|^2 d\|E_K(t)\xi\|^2 < \infty \\ &\text{and } \int_0^\infty \frac{|f(t)|^2}{t^2} d\|E_K(t)\xi\|^2 < \infty \Bigg\} \\ = \{ f \in C(0,\infty) : &D(f(K_0)) \supset \mathcal{D} \text{ and } D((f_0^{-1}f)(K_0)) \supset \mathcal{D}\} \end{aligned}$$

where $f_0(t) = t,\ t \in (0,\infty)$.

For every $f \in C_\lambda^K(0,\infty)$ we have

$$f(K_0)\mathcal{D} \subset \mathcal{D} \text{ and } (f_0^{-1}f)(K_0)U_K^* D \subset \mathcal{D}. \tag{4.1}$$

Indeed, since $f(K_0) \,\eta\, \{\pi(\mathfrak{A})\}'_w$, $E_k(n) \in \{\pi(\mathfrak{A})\}'_w$, for every $\xi \in \mathcal{D}$ we have,

$$f(K_0)E_k(n)\xi = \int_0^n f(y) dE_K(t)\xi \to f(K_0)\xi$$

and

$$\pi(A)f(K_0)E_k(n)\xi = f(K_0)E_k(n)\pi(A)\xi \to f(K_0)\pi(A)\xi$$

for all $A \in \mathfrak{A}$. Hence, $f(K_0) \in \bigcap_{A\in\mathfrak{A}} D(\overline{\pi(A)}) = \mathcal{D}$. Thus, $f(K_0)\mathcal{D} \subset \mathcal{D}$.
In very similar way, taking into account that $\mathcal{D} \subset D((f_0^{-1}f)(K_0))$ we prove that $(f_0^{-1}f)(K_0)U_K^*\mathcal{D} \subset \mathcal{D}$.

Now we put,
(4.2)
$$f(|K|_\pi) := (f(K_0), (f_0^{-1}f)(K_0)U_K^*\lambda(K), ((f_0^{-1}f)(K_0)^*U_K^*\lambda(K))^*).$$

Then we have the following

Lemma 4.1. *$f(|K|_\pi)\,\eta\,\mathfrak{A}^c$ for every $f \in C_\lambda^K(0,\infty)$.*

Proof. By (4.1), we have $f(|K|_\pi) \in T^\dagger(\mathcal{D})$ and furthermore $\pi(f(|K|)) = f(K_0)\,\eta\,\pi(\mathfrak{A})'_w$. For every $A \in \mathfrak{A}$, we have

$$\begin{aligned}\pi(f(|K|_\pi))\lambda(A) &= f(K_0)\lambda(A)\\ &= (f_0^{-1}f)(K_0)K_0\lambda(A)\\ &= (f_0^{-1}f)(K_0)U_K^*\pi(K)\lambda(A)\\ &= (f_0^{-1}f)(K_0)U_K^*\pi(A)\lambda(K)\\ &= \pi(A)(f_0^{-1}f)(K_0)U_K^*\lambda(K)\\ &= \pi(A)\lambda(f(|K|_\pi)).\end{aligned}$$

Similarly,

$$\begin{aligned}\pi(f(|K|_\pi))^\dagger\lambda(A) &= \overline{f}(K_0)\lambda(A)\\ &= (f_0^{-1}\overline{f})(K_0)U_K^*\lambda(K)\\ &= \pi(A)\lambda(f(|K|_\pi^\dagger)).\end{aligned}$$

Hence, $f(|K|_\pi)\,\eta\,\mathfrak{A}^c$. □

Lemma 4.2. *$C_\lambda^K(0,\infty)$ is a *-subalgebra of $C(0,\infty)$.*

Proof. Let $f, g \in C_\lambda^K(0,\infty)$ and $\alpha \in \mathbb{C}$. It is clear that $f+g, \alpha f, f^* = \overline{f} \in C_\lambda^K(0,\infty)$. We show that $fg \in C_\lambda^K(0,\infty)$.
For every $\xi \in \mathcal{D}$,

$$(fg)(K_0)\xi = f(K_0)g(K_0)\xi.$$

By (4.1), $g(K_0)\xi \in \mathcal{D}$.
Thus, $g(K_0)\xi \in D(f(|K|_\pi))$ and $f(K_0)g(K_0)\xi \in \mathcal{D}$. □

By Lemma 4.1 and Lemma 4.2 we have the following theorem which establishes the functional calculus for $|K|_\pi$.

Theorem 4.3. *Let $K \in \mathfrak{A}^c_\eta$. There exists a map*

$$f \in C^K_\lambda(0,\infty) \to f(|K|_\pi) \in \mathfrak{A}^c_\eta$$

with the properties

$$\begin{aligned}(f+g)(|K|_\pi) &= f(|K|_\pi) + g(|K|_\pi)\\ (\alpha f)(|K|_\pi) &= \alpha f(|K|_\pi),\\ (fg)(|K|_\pi) &= f(|K|_\pi)g(|K|_\pi),\\ f^*(|K|_\pi) &= \bar{f}(|K|_\pi),\end{aligned}$$

for every $f, g \in C^K_\lambda(0,\infty)$, $\alpha \in \mathbb{C}$.

Remark 4.4. In other words, the map $f \in C^K_\lambda(0,\infty) \to f(|K|_\pi) \in \mathfrak{A}^c_\eta$ is a *-homomorphism.

Let us now denote by $C_c(0,\infty)$ the set of all continuous functions on $(0,\infty)$ with compact support.

For every $K \,\eta\, \mathfrak{A}^c$, $C_c(0,\infty)$ is a *-subalgebra of $C^K_\lambda(0,\infty)$. Thus, using (4.2), we define

$$C_c(|K|_\pi) = \{f(|K|_\pi); f \in C_c(0,\infty)\}.$$

The functional calculus of $|K|_\pi$ when restricted to $C_c(0,\infty)$ exhibits more regularity.

Theorem 4.5. *Suppose that $K \,\eta\, \mathfrak{A}^c$. Then*

*(i) $C_c(|K|_\pi)$ is a commutative *-subalgebra of $\mathfrak{A}^c$ and the map $f \in C_c(0,\infty) \to f(|K|_\pi) \in C_c(|K|_\pi)$ is a *-isomorphism.*

(ii) For $f \in C_c(0,\infty)$, $Kf(|K|_\pi) \in \mathfrak{A}^c$ and

$$\left|\left\langle \lambda(Kf(|K|_\pi)f(|K|_\pi)^\dagger)^\dagger \mid \lambda((K)^\dagger)\right\rangle\right| = \|\overline{f}(K_0)\lambda(K^\dagger)\|^2.$$

Proof. (i) Let $f \in C_c(0,\infty)$. Since $f(K_0) \in \pi(\mathfrak{A})'_w$, it follows from Theorem 4.3 that $f(|K|_\pi) \in \mathfrak{A}^c$ and since $C_c(0,\infty)$ is a *-subalgebra of $C^K_\lambda(0,\infty)$, it follows from Theorem 4.3 that the map $f \in C_c(0,\infty) \to f(|K|_\pi) \in C_c(|K|_\pi)$ is a *-isomorphism.

(ii) For $K \in \mathfrak{A}^c$ we have

$$\begin{aligned}Kf(|K|_\pi) &= (\pi(K)f(K_0), \pi(K)(f_0^{-1}f)(K_0)U_K^*\lambda(K), (\overline{f}(K_0)\lambda(K^\dagger))^*)\\ &= (U_K K_0 f(K_0), U_K K_0(f_0^{-1}f)(K_0)U_K^*\lambda(K), (\overline{f}(K_0)\lambda(K^\dagger))^*)\\ &= (U_K K_0 f(K_0), U_K f(K_0)U_K^*\lambda(K), (\overline{f}(K_0)\lambda(K^\dagger))^*)\end{aligned}$$

which implies that $Kf(|K|_\pi) \in T^\dagger(\mathcal{D})$.

For every $A \in \mathfrak{A}$ we have

$$
\begin{aligned}
&A(Kf(|K|_\pi)) = \\
&= (\pi(A)\pi(K)f(K_0), \pi(A)\pi(K)(f_0^{-1}f)(K_0)U_K^*\lambda(K), ((\pi(K)f(K_0))^*\lambda(A^\dagger))^*) \\
&= (\pi(K)f(K_0)\pi(A), U_K f(K_0)U_K^*\pi(K)\lambda(A), (\overline{f}(K_0)\pi(K)^*\lambda(A^\dagger))^*) \\
&= (\pi(K)f(K_0)\pi(A), U_K f(K_0)U_K^*U_K K_0\lambda(A), (\overline{f}(K_0)\pi(A)^\dagger\lambda(K^\dagger))^*) \\
&= (\pi(K)f(K_0)\pi(A), U_K f(K_0)K_0\lambda(A), (\pi(A)^\dagger\overline{f}(K_0)\lambda(K^\dagger))^*) \\
&= (\pi(K)f(K_0)\pi(A), U_K K_0 f(K_0)\lambda(A), (\pi(A)^\dagger\overline{f}(K_0)\lambda(K^\dagger))^*) \\
&= (\pi(K)f(K_0)\pi(A), \pi(K)f(K_0)\lambda(A), (\pi(A)^\dagger\overline{f}(K_0)\lambda(K^\dagger))^*) \\
&= (Kf(|K|_\pi))A.
\end{aligned}
$$

Thus we have $Kf(|K|_\pi) \in \mathfrak{A}^c$ and

$$
\lambda(Kf(|K|_\pi)) = U_k f(K_0)U_K^*\lambda(K), \qquad \lambda(Kf(|K|_\pi)^\dagger) = \overline{f}(K_0)\lambda(K^\dagger).
$$

Moreover we have

$$
\begin{aligned}
|\left\langle \lambda(Kf(|K|_\pi)f(|K|_\pi)^\dagger)^\dagger \mid \lambda((K)^\dagger)\right\rangle| &= |\left\langle \lambda(f(|K|_\pi)^\dagger) \mid \lambda(K^\dagger\right\rangle| \\
&= |\left\langle \pi(f(|K|_\pi))\lambda(Kf(|K|_\pi)^\dagger) \mid \lambda(K^\dagger\right\rangle| \\
&= |\left\langle f(K_0)\overline{f}(K_0)\lambda(K^\dagger) \mid \lambda(K^\dagger\right\rangle| \\
&= \|\overline{f}(K_0)\lambda(K^\dagger)\|^2.
\end{aligned}
$$

□

Theorem 4.6. *Let $\mathfrak{A}$ be a nondegenerate π-self-adjoint $T^\dagger$-algebra on $\mathcal{D}$ in $\mathcal{H}$. Then, for every $K \in \mathfrak{A}^c$ there exists a sequence $\{K_n\} \subset C_c(|K|_\pi)$ such that $KK_n^2 \in \mathfrak{A}^c \xrightarrow{\tau_{s^*}} K$, in the following sense:*

$$
\begin{aligned}
\pi(KK_n^2) &\to \pi(K) \quad \textit{strongly}, \\
\pi((KK_n^2)^\dagger) &\to \pi(K^\dagger) \quad \textit{strongly}, \\
\lambda(KK_n^2) &\to \lambda(K), \\
\lambda((KK_n^2)^\dagger) &\to \lambda(K^\dagger).
\end{aligned}
$$

Proof. We will show that, if $f \in C_c(0,\infty)$,

$$
U_K f(K_0) = f(H_0)U_K. \tag{4.3}
$$

Indeed, by Weierstrass approximation theorem, there exists a sequence $\{p_n\}$ of polynomials such that $p_n \to f$, uniformly. Since $U_K K_0 = H_0 U_K$ it follows that $U_K p_n(K_0) = p_n(H_0)U_K$ and $U_K p_n(K_0) \to U_K f(K_0)$, $p_n(H_0)U_K \to f(H_0)U_K$ uniformly (i.e., in the norm of bounded operators).

Let us now consider an increasing sequence $\{f_n\}$ of nonnegative functions in $C_c(0,\infty)$ such that $\lim_{n\to\infty} f_n^2(t) = 1$, $0 < t < \infty$. We put

$$K_n := f_n(|K|_\pi) = (f_n(K_0), (f_0^{-1} f_n)(K_0)U_K^*\lambda(K), ((f_0^{-1} f_n)(K_0)^* U_K^*), \ n \in \mathbb{N}.$$

By Theorem 4.5,

$$K_n \in C_c(|K|_\pi) \text{ and } KK_n \in \mathfrak{A}^c. \tag{4.4}$$

By (4.3) and (4.4) we get, for all $n \in \mathbb{N}$,

$$\begin{aligned} \pi(KK_n^2) &= \pi(K)\pi(K_n^2) \\ &= H_0 U_K f_n^2(K_0) \\ &= H_0 f_n^2(K_0) U_K \\ &= f_n^2(H_0) H_0 U_K. \end{aligned}$$

We have $f_n^2(K_0) \to Q^K$ and $f_n^2(H_0) \to Q^H$, where Q^K and Q^H denote the projections onto the $\overline{R(K_0)}$ and $\overline{R(H_0)}$, respectively. From this we obtain

$$\pi(KK_n^2) \to Q^H H_0 U_K = H_0 U_K = \pi(K).$$

and

$$\begin{aligned} \pi((KK_n^2)^\dagger) = \pi(K_n^2)\pi(K)^\dagger = f_n(K_0^2) K_0 U_K^* \to \\ Q^K K_0 U_K^* = K_0 U_K^* = \pi(K)^\dagger = \pi(K^\dagger). \end{aligned}$$

Now we prove that

$$\lambda(KK_n^2) = U_K f_n(K_0)^2 U_K^* \lambda(K) \to U_K Q^K U_K^* \lambda(K) = \lambda(K).$$

Indeed, for every $A \in \mathfrak{A}$, we have

$$\begin{aligned} \pi(A) U_K Q^K U_K^* \lambda(K) &= U_K Q^K U_K^* \pi(K)\lambda(A) \\ &= U_K Q^K U_K^* U_K K_0 \lambda(A) \\ &= U_K Q^K K_0 \lambda(A) \\ &= \pi(K)\lambda(A) \\ &= \pi(A)\lambda(K). \end{aligned}$$

Since $[\pi(\mathfrak{A})\mathcal{D}]$ is dense in $\mathcal{H}$, $U_K Q^K U_K^* \lambda(K) = \lambda(K)$. Finally, we prove that

$$\lambda((KK_n^2)^\dagger) = f_n(K_0)^2 \lambda(K^\dagger) \to \lambda(K^\dagger).$$

Indeed, since $f_n(K_0)^2\lambda(K^\dagger) \to Q^K\lambda(K^\dagger)$, it suffices to show $Q^K\lambda(K^\dagger) = \lambda(K^\dagger)$. For every $A \in \mathfrak{A}$, we have

$$\begin{aligned}\pi(A)Q^K\lambda(K^\dagger) &= Q^K\pi(K^\dagger)\lambda(A)\\ &= Q^K K_0 U_K^*\lambda(A)\\ &= K_0U_K^*\lambda(A)\\ &= \pi(K^\dagger)\lambda(A)\\ &= \pi(A)\lambda(K^\dagger).\end{aligned}$$

Again, since $[\pi(\mathfrak{A})\mathcal{D}]$ is dense in $\mathcal{H}$, we get

$$Q^K\lambda(K^\dagger) = \lambda(K^\dagger).$$

In conclusion, $KK_n^2 \in \mathfrak{A}^c \xrightarrow{\tau_{s^*}} K$. □

4.2. **The case of $A \in \mathfrak{A}^{cs}$.** Let $\mathfrak{A}$ be a π-self-adjoint $T^\dagger$-algebra on $\mathcal{D}$. We consider here a functional calculus for $A \in \mathfrak{A}^{cs}$. Let $\overline{\pi(A)} = U_A|\overline{\pi(A)}|$ be the polar decomposition of $\overline{\pi(A)}$ and $A_0 := |\overline{\pi(A)}| = \int_0^\infty t dE_A(t)$ be the spectral resolution of $|\overline{\pi(A)}|$. Then $\{U_A, E_A(t); t \in (0,\infty)\} \subset (\pi(\mathfrak{A})'_w)'$ but, in general, they are not contained in $\mathcal{L}^\dagger(\mathcal{D})$. In what follows we will assume that

$$(\pi(\mathfrak{A})'_w)'\,\mathcal{D} \subset \mathcal{D}. \tag{4.5}$$

Then $\{U_A, E_A(t); t \in (0,\infty)\} \subset \mathcal{L}^\dagger(\mathcal{D})$. Furthermore, since $f(A_0) \in (\pi(\mathfrak{A})'_w)'$, for every $f \in C_c(0,\infty)$ then, by (4.5), $f(A_0)\mathcal{D} \subset \mathcal{D}$. Hence

$$f(|A|_\pi) := (f(A_0), (f_0^{-1}f)(A_0)U_A^*\lambda(A), (\overline{f}(A_0)U_A^*\lambda(A))^*) \in T^\dagger(\mathcal{D})$$

and

$$\pi(f(|A|_\pi)) = f(A_0) \in (\pi(\mathfrak{A})'_w)'. \tag{4.6}$$

Lemma 4.7. *For every $f \in C_c(0,\infty)$,*

$$f(|A|_\pi), Af(|A|_\pi) \in \mathfrak{A}^{cs}.$$

Proof.

$$\begin{aligned}Kf(|A|_\pi) &= (\pi(K)f(A_0), \pi(K)(f_0^{-1}f)(A_0)U_A^*\lambda(A), (\overline{f}(A_0)\lambda(K^\dagger))^*)\\ &= (\pi(K)f(A_0), (f_0^{-1}f)(A_0)U_A^*\pi(A)\lambda(K), (\overline{f}(A_0)\lambda(K^\dagger))^*)\\ &= (\pi(K)f(A_0), (f_0^{-1}f)(A_0)A_0\lambda(K), (\overline{f}(A_0)\lambda(K^\dagger))^*)\\ &= (\pi(K)f(A_0), f(A_0)\lambda(K), (\overline{f}(A_0)\lambda(K^\dagger))^*);\end{aligned}$$

$$\begin{aligned} f(|A|_\pi)K &= (f(A_0)\pi(K), f(A_0)\lambda(K), (\pi(K)^\dagger \overline{f}(A_0)U_A^*\lambda(K^\dagger))^*) \\ &= (f(A_0)\pi(K), f(A_0)\lambda(K), (\overline{f}(A_0)U_A^*\pi(A)\lambda(K^\dagger))^*) \\ &= (f(A_0)\pi(K), f(A_0)\lambda(K), (\overline{f}(A_0)\lambda(K^\dagger))^*). \end{aligned}$$

Therefore, $f(|A|_\pi) \in \mathfrak{A}^{cs}$. The proof for $Af(|A|_\pi)$ is similar. □

We put

$$C_c(|A|_\pi) := \{f(|A|_\pi); f \in C_c(0,\infty)\}.$$

By Lemma 4.7, $C_c(|A|_\pi)$ is a *-subalgebra of $\mathfrak{A}^{cs}$.

Moreover, we have

Theorem 4.8. *Let $\mathfrak{A}$ be a π-self-adjoint $T^\dagger$-algebra on $\mathcal{D}$ in $\mathcal{H}$. Suppose that $(\pi(\mathfrak{A})'_w)'\mathcal{D} \subset \mathcal{D}$ and that $\pi(\mathfrak{A}^c)$ is nondegenerate. Then,*

*(i) For every $A \in \mathfrak{A}^{cs}$ the map $f \in C_c(0,\infty) \to f(|A|_\pi) \in C_c(|A|_\pi)$ is a *-isomorphism.*

(ii) For every $A \in \mathfrak{A}^{cs}$ there exists a sequence $\{A_n\} \subset \mathfrak{A}^{cc}$ such that $A_n \xrightarrow{\tau_{s^*}} A$.

Proof. (i) can be proved as in Theorem 4.5. (ii) We take, similarly to what we did in Theorem 4.6, an increasing non negative sequence $\{f_n\} \subset C_c(0,\infty)$ such that $\lim_{n\to\infty} f_n^2(t) = 1$, $0 < t < \infty$ and, in the very same way as in Theorem 4.6, we prove that $Af_n(|A|_\pi) \in \mathfrak{A}^{cc}$ and $Af_n(|A|_\pi) \xrightarrow{\tau_{s^*}} A$. □

Now we consider a functional calculus of $A \in \mathfrak{A}^{cs}$ for the following subspace $C_\lambda^A(0,\infty)$ of $C(0,\infty)$ We put

$$C_\lambda^A(0,\infty) := \left\{ f \in C(0,\infty) : \int_0^\infty |f(t)|^2 d\|E_A(t)\xi\|^2 < \infty \right.$$
$$\left. \text{and } \int_0^\infty \frac{|f(t)|^2}{t^2} d\|E_A(t)\xi\|^2 < \infty \right\}.$$

By Lemma 3.12, $|A|_\pi = (A_0, U_A^*\lambda(A), (U_A^*\lambda(A))^*) \in \mathfrak{A}^{cs}$; but, differently from the case $K \in \mathfrak{A}^c$, the inclusion $f(A_0)\mathcal{D} \subset \mathcal{D}$ does not necessarily hold. We have the following

Theorem 4.9. *Let $\mathfrak{A}$ be a π-self-adjoint $T^\dagger$-algebra on $\mathcal{D}$. Suppose that $(\pi(A)'_w)'\mathcal{D} \subset \mathcal{D}$, $\pi(\mathfrak{A}^c)$ is nondegenerate and $\pi(\mathfrak{A}^c_\eta)$ is closed; i.e., $\mathcal{D} = \bigcap_{K\in\mathfrak{A}^c_\eta} D(\overline{\pi(K)})$. If $A \in \mathfrak{A}^{cs}$, then*

$$f(|A|_\pi) := (f(A_0), (f_0^{-1}f)(A_0)U_A^*\lambda(A), (\overline{f}(A_0)U_A^*\lambda(A))^*) \in \mathfrak{A}^{cs},\ f \in C_\lambda^A(0,\infty)$$

Proof. We prove that $f(A_0)\mathcal{D} \subset \mathcal{D}$ and that $f(|A|) \in T^\dagger(\mathcal{D})$, for every $f \in C_\lambda^A(0,\infty)$. If $K \in \mathfrak{A}^c_\eta$ and $\xi \in \mathcal{D}$, we have $f(A_0)E_A(n)\xi \in$

$(\pi(A)'_w)'\mathcal{D} \subset \mathcal{D}$ and $f(A_0)E_A(n)\xi \to f(A_0)\xi$; $\pi(K)f(A_0)E_A(n)\xi = f(A_0)E_A(n)\pi(K)\xi \to f(A_0)\pi(K)\xi$. This implies that

$$f(A_0)\xi \in \bigcap_{K\in\mathfrak{A}^c_\eta} D(\overline{\pi(K)}) = \mathcal{D}.$$

Hence, $f(A_0)\mathcal{D} \subset \mathcal{D}$ and $f(|A|) \in T^\dagger(\mathcal{D})$; as in Lemma 4.7 we can prove that in fact $f(|A|_\pi) \in \mathfrak{A}^{cs}$ and, in the same way as in Theorem 4.3, that $f \in C^A_\lambda(0,\infty) \to f(|A|_\pi) \in \mathfrak{A}^{cs}$ is a *-isomorphism. □

For $A = A^\dagger \in T^\dagger(\mathcal{D})$, we get the following

Corollary 4.10. *Let $A = A^\dagger \in T^\dagger(\mathcal{D})$. Let us denote by $\mathcal{P}(A)$ the π-closed $T^\dagger$- algebra generated by A. Assume that*

(i) $\pi(A)^n$ is essentially self-adjoint, for every $n \in \mathbb{N}$;
(ii) $\pi(A)$ is nonsingular; i.e., $[\pi(A)\mathcal{D}]$ is dense in $\mathcal{H}$.

*Then, $f(|A|_\pi) \in \mathcal{P}(A)^{cs}$, for every $f \in C^A_\lambda(0,\infty)$ and the map $f \in C^A_\lambda(0,\infty) \to f(|A|_\pi) \in \mathcal{P}(A)^{cs}$ is a *-isomorphism.*

Proof. By (i) $\mathcal{P}(A)$ is a π-self-adjoint $T^\dagger$-algebra on $\mathcal{D}$ and $(\mathcal{P}(A)'_w)' \subset \mathcal{P}(A)'_w$; this implies that that $(\mathcal{P}(A)'_w)'\mathcal{D} \subset \mathcal{D}$. Furthermore from the inclusion $\mathcal{P}(A) \subset \mathcal{P}(A)^c_\eta$ it follows that $\mathcal{P}(A)^c_\eta$ is π-closed. For every $n \in \mathbb{N}$, we put

$$A_n = (\pi(A)E_A(n), \lambda(A), \lambda(A)^*).$$

Then, $A_n \in \mathcal{P}(A)^c$ and $\pi(A_n)\xi = \pi(A)E_A(n)\xi \to \pi(A)\xi$. By the assumption (ii) it follows that $\pi(\mathcal{P}(A)^c)$ is nondegenerate. The statement then follows from Theorem 4.9. □

5. Examples

Example 5.1. Let $\mathcal{M}$ be a self-adjoint O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$ with identity I and $\xi_0 \neq 0 \in \mathcal{D}$. We define a π-self-adjoint $T^\dagger$-algebra $\mathfrak{A}$ on $\mathcal{D}$ in $\mathcal{H}$ by

$$\mathfrak{A} = \{X = (X_0, X_0\xi_0, (X_0^\dagger\xi_0)^*);\ X_0 \in \mathcal{M}\}.$$

Then it is easily shown that

$$\begin{aligned}
\mathfrak{A}^c &= \{C = (C_0, C_0\xi_0, (C_0^*\xi_0)^*);\ C_0 \in \mathcal{M}'_{\mathrm{w}}\},\\
\mathfrak{A}^c_\eta &= \{K = (K_0, K_0\xi_0, (K_0^\dagger\xi_0)^*);\ \bar{K}_0\,\eta\,\mathcal{M}'_{\mathrm{w}}\},\\
\mathfrak{A}^{cs} &= \{A = (A_0, A_0\xi_0, (A_0^\dagger\xi_0)^*);\ A_0 \in (\mathcal{M}'_{\mathrm{w}})^s\}.
\end{aligned}$$

(1) Let $K = K^\dagger \in \mathfrak{A}^c_\eta$. Then

$$|K|_\pi = (|\bar{K}_0|\lceil_{\mathcal{D}}, |\bar{K}_0|\xi_0, (|\bar{K}_0|\xi_0)^*)$$

and

$$K = U_K |K|_\pi.$$

Furthermore, putting

$$S := (|\bar{K}_0|^{\frac{1}{2}}\lceil_{\mathcal{D}}, |\bar{K}_0|^{\frac{1}{2}}\xi_0, (|\bar{K}_0|^{\frac{1}{2}}\xi_0)^*),$$

$S \in \mathfrak{A}^c_\eta$ and $S^2 = |K|_\pi$. Hence $|K|_\pi$ is positive, so $K = U_K|K|$.

(2) Let $A = A^\dagger \in \mathfrak{A}^{cs}$. Suppose that A_0^n is essentially self-adjoint for all $n \in \mathbb{N}$. By (3.10), $U_A\lceil_{\mathcal{D}}$, $|\bar{A}_0|\lceil_{\mathcal{D}}$, $|\bar{A}_0|^{\frac{1}{2}}\lceil_{\mathcal{D}} \in \mathcal{L}^\dagger(\mathcal{D})$, so

$$\begin{aligned} |A|_\pi &= (|\bar{A}_0|\lceil_{\mathcal{D}}, |\bar{A}_0|\xi_0, (|\bar{A}_0|\xi_0)^*) \\ &= (|\bar{A}_0|^{\frac{1}{2}}\lceil_{\mathcal{D}}, |\bar{A}_0|^{\frac{1}{2}}\xi_0, (|\bar{A}_0|^{\frac{1}{2}}\xi_0)^*)^2 \\ &= |A| \end{aligned}$$

and $A = U_A|A|$. This shows that Theorem 3.8 for $K = K^\dagger\, \eta\, \mathfrak{A}^c_\eta$ and Theorem 3.16 for $A = A^\dagger \in \mathfrak{A}^{cs}$ hold without assumption of the invertivility of $\overline{\pi(K)}$ and $\overline{\pi(A)}$.

Example 5.2. [2, Example 2] Let $\mathcal{S} := \mathcal{S}(\mathbb{R})$ denote the Schwartz space of all C^∞ rapidly decreasing functions. It is well known that the operators q and p defined for $\phi \in \mathcal{S}$ by

$$(q\phi)(t) = t\phi(t),$$
$$(p\phi)(t) = -i\frac{d\phi}{dt}$$

leave $\mathcal{S}$ invariant, are essentially self-adjoint on $\mathcal{S}$ and satisfy the Canonical Commutation Relation (CCR) $qp\phi - pq\phi = i\phi$; this implies that the self-adjoint O*-algebra $\mathcal{M}_{\mathrm{S}}$ that they generate on $\mathcal{S}$ is constituted by elements of the form

$$a = \sum_{k=0}^{N}\sum_{h=0}^{M} \alpha_{kn} q^k p^h.$$

Here we treat with the following π-self-adjoint $T^\dagger$-algebras $\mathfrak{A}_{(1)}$, $\mathfrak{A}_{(2)}$ and $\mathfrak{A}_{(3)}$ on $\mathcal{S}$:

$$\begin{aligned} \mathfrak{A}_{(1)} &= \{(a, \varphi, \psi^*);\ a \in \mathcal{M}_{\mathcal{S}},\ \varphi, \psi \in \mathcal{S}\}, \\ \mathfrak{A}_{(2)} &= \{(a, a\varphi, (a\psi)^*);\ a \in \mathcal{M}_{\mathcal{S}}\} \quad \text{for} \quad \varphi, \psi \in \mathcal{S} \\ \mathfrak{A}_{(3)} &= \{(a, 0, 0);\ a \in \mathcal{M}_{\mathcal{S}}\}. \end{aligned}$$

Since, as is well known, $(\mathcal{M}_{\mathcal{S}})'_{\mathrm{w}} = \mathbb{C}I$, we can show

(i) $\mathfrak{A}^c_{(1)} = \{0\}$ and $\mathfrak{A}^{cs}_{(1)} = T^\dagger(\mathcal{S})$,

(ii) $\mathfrak{A}^c_{(2)} = \{(\alpha I, \alpha\varphi, (\bar{\alpha}\psi)^*);\ \ \alpha \in \mathbb{C}\}$ and $\mathfrak{A}^{cs}_{(2)} = \{(x, x\varphi, (x^\dagger\psi)^*);\ x \in \mathcal{L}^\dagger(\mathcal{S})\}$,

(iii) $\mathfrak{A}^c_{(3)} = \{(\alpha I, 0, 0);\ \alpha \in \mathbb{C}\}$ and $\mathfrak{A}^{cs}_{(3)} = \{(x, 0, 0);\ x \in \mathcal{L}^\dagger(\mathcal{S})\}$,

so the results of Section 3.2 on polar decomposition and of Section 4.2 on functional calculus can be applied to every element A of $\mathfrak{A}^{cs}_{(i)}$ $(i = 1, 2, 3)$ satisfying

$$U_A \in \mathcal{L}^\dagger(\mathcal{S}). \tag{5.1}$$

where U_A is the partial isometry defined by the polar decomposition of $\overline{\pi(A)}$. We give examples where the property (5.1) is fulfilled.
Let

$$a_- = \frac{1}{\sqrt{2}}(q + ip) \quad , \quad a_+ = \frac{1}{\sqrt{2}}(q - ip)$$
$$h = a_- a_+ \quad \text{and} \quad k = a_+ a_-.$$

Then $\mathcal{M}_\mathcal{S}$ is generated by I, a_- and a_+. Let $\{\varphi_n\}$ be the ONB in the Hilbert space $L^2 := L^2(\mathbb{R})$ contained in $\mathcal{S}$ defined by

$$\varphi_n(t) = \pi^{-\frac{1}{2}}(2^n n!)^{-\frac{1}{2}}(t - \frac{d}{dt})^n e^{-\frac{t^2}{2}}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Then $\varphi_0 \in \mathcal{S}$ is a strongly cyclic vector for $\mathcal{M}_\mathcal{S}$, that is, $\mathcal{M}_\mathcal{S}\varphi_0$ is $t_{\mathcal{M}_\mathcal{S}}$-dense in $\mathcal{S}$ and

$$\begin{aligned}
a_-\varphi_n &= \begin{cases} 0 & , \quad n = 0 \\ \sqrt{n}\varphi_{n-1} & , \quad n \in \mathbb{N} \end{cases} \\
a_+\varphi_n &= \sqrt{n+1}\varphi_{n+1}, \quad n \in \mathbb{N}_0 \\
h\varphi_n &= (n+1)\varphi_n, \quad n \in \mathbb{N}_0 \\
k\varphi_n &= n\varphi_n, \quad n \in \mathbb{N}_0
\end{aligned}$$

hold, so a_- and a_+ are called the lowering operator and the raising operator, respectively, and h and k are called the number operators, and

$$\begin{aligned}
a_- &= \sum_{n \in \mathbb{N}_0} \sqrt{n+1}\ \varphi_n \otimes \bar{\varphi}_{n+1}, \\
a_+ &= \sum_{n \in \mathbb{N}_0} \sqrt{n+1}\ \varphi_{n+1} \otimes \bar{\varphi}_n, \\
h &= \sum_{n \in \mathbb{N}_0} (n+1)\ \varphi_n \otimes \bar{\varphi}_n, \\
k &= \sum_{n \in \mathbb{N}_0} n\ \varphi_n \otimes \bar{\varphi}_n
\end{aligned}$$

where $(\varphi \otimes \bar{\psi})\xi := (\xi|\psi)\varphi$. Then h and k are positive self-adjoint operators in L^2 and $a_- = u_{a_-}k^{\frac{1}{2}}$ and $a_+ = u_{a_+}h^{\frac{1}{2}}$ are the polar decompositions of a_- and a_+, respectively, satisfying

$$u_{a_-}\varphi_n = \begin{cases} 0 & , \quad n = 0 \\ \varphi_{n-1} & , \quad n \in \mathbb{N} \end{cases} \tag{5.2}$$

and

$$u_{a_-}^*\varphi_n = u_{a_+}\varphi_n = \varphi_{n+1}, \quad n \in \mathbb{N}_0, \tag{5.3}$$

which implies that

$$u_{a_-}, u_{a_+} \in \mathcal{L}^\dagger(\mathcal{S}), \tag{5.4}$$

$$u_{a_+}^* u_{a_+} = u_{a_-}u_{a_-}^* = I, u_{a_+}u_{a_+}^* = u_{a_-}^* u_{a_-} = \text{ Proj } \{\varphi_0\}^\perp \tag{5.5}$$

and

$$h \text{ is invertible.} \tag{5.6}$$

Let $A_\pm$ be any element of $T^\dagger(\mathcal{S})$ having $\pi(A_\pm) = a_\pm$ and write

$$A_\pm = (a_\pm, \varphi_\pm, (\psi_\pm)^*)$$

for some $\varphi_\pm, \psi_\pm \in \mathcal{S}$, respectively. By (5.4)

$$\begin{aligned} |A_+|_\pi &= (|a_+|, u_{a_+}^*\varphi_+, (u_{a_+}^*\varphi_+)^*) \\ &= (h^{\frac{1}{2}}, u_{a_-}\varphi_+, (u_{a_-}\varphi_+)^*), \\ |A_-|_\pi &= (|a_-|, u_{a_-}^*\varphi_-, (u_{a_-}^*\varphi_-)^*) \\ &= (k^{\frac{1}{2}}, u_{a_+}\varphi_-, (u_{a_+}\varphi_-)^*) \end{aligned}$$

are well defined. Moreover, we can show that

$$|A_\pm|_\pi \text{ is positive.} \tag{5.7}$$

Indeed, by (5.3) $u_{a_-}^*\varphi_- = u_{a_+}\varphi_- \in \{\varphi_0\}^\perp$, so $\left(\sum_{n\in\mathbb{N}} \frac{1}{\sqrt{n}}\varphi_n \otimes \bar{\varphi}_n\right) u_{a_+}\varphi_-$ is in $\mathcal{S}$ and

$$\begin{aligned} X_- &:= \left(k^{\frac{1}{2}}, \left(\sum_{n\in\mathbb{N}} \frac{1}{\sqrt{n}}\varphi_n \otimes \bar{\varphi}_n\right) u_{a_-}^*\varphi_-, \left(\left(\sum_{n\in\mathbb{N}} \frac{1}{\sqrt{n}}\varphi_n \otimes \bar{\varphi}_n\right) u_{a_-}^*\varphi_-\right)^*\right) \\ &\in T^\dagger(\mathcal{S}), \end{aligned}$$

$X_-^\dagger = X_-$ and $(X_-)^2 = |A_-|_\pi$. Hence $|A|_\pi$ is positive. By (5.6) $|A_+|_\pi$ is positive.

First we consider the polar decomposition of an element of $\mathfrak{A}_{(1)}^{cs}$.

For any $\varphi \in \mathcal{S}$ we put

$$A_{\pm,\varphi}^{(1)} = (a_\pm, \varphi, (\varphi)^*), \quad \varphi \in \mathcal{S}.$$

Then

$$\begin{aligned} |A^{(1)}_{\pm,\varphi}| &= (|a_\pm|, u^*_{a_\pm}\varphi, (u^*_{a_\pm}\varphi)^*), \\ u_{a_\pm}|A^{(1)}_{\pm,\varphi}| &= (a_\pm, u_{a_\pm}u^*_{a_\pm}\varphi, (u_{a_\pm}u^*_{a_\pm}\varphi)^*). \end{aligned}$$

By (5.5) we have

$$u_{a_-}|A^{(1)}_{-,\varphi}| = A^{(1)}_{-,\varphi}.$$

For $A^{(1)}_{+,\varphi}$ it follows from (5.5) that if $(\varphi|\varphi_0) = 0$, then

$$u_{a_+}|A^{(1)}_{+,\varphi}| = A^{(1)}_{+,\varphi},$$

and if otherwise, then

$$u_{a_+}|A^{(1)}_{+,\varphi}| \neq A^{(1)}_{+,\varphi}.$$

Second, take the following elements of $\mathfrak{A}^{cs}_{(2)}$:

$$A^{(2)}_\pm = (a_\pm, a_\pm\varphi_0, (a^*_\pm\varphi_0)^*).$$

Then since

$$\begin{aligned} A^{(2)}_- &= (a_-, 0, \varphi_1^*), \\ |A^{(2)}_-| &= (|a_-|, 0, 0) = |A|, \end{aligned}$$

it follows that

$$u_{a_-}|A^{(2)}_-| = (a_-, 0, 0) \neq A^{(2)}_-.$$

Moreover

$$\begin{aligned} A^{(2)}_+ &= (a_+, a_+\varphi_0, (a^*_+\varphi_0)^*) = (a_+, \varphi_1, 0), \\ |A^{(2)}_+| &= (|a_+|, u^*_{a_+}a_+\varphi_0, (u^*_{a_+}a_+\varphi_0)^*) \\ &= (|a_+|, \varphi_0, \varphi_0^*), \\ u_{a_+}|A^{(2)}_+| &= (a_+, u_{a_+}\varphi_0, (u_{a_+}\varphi_0)^*) \\ &= (a_+, \varphi_1, \varphi_1^*) \\ &\neq A^{(2)}_+. \end{aligned}$$

Third, let

$$A^{(3)}_\pm = (a_\pm, 0, 0).$$

Then

$$|A^{(3)}_+| = (|a_+|, 0, 0), \quad \text{and} \quad u_{a_+}|A^{(3)}_+| = (a_+, 0, 0) = A^{(3)}_+,$$

and similarly

$$|A^{(3)}_-| = u_{a_-}|A^{(3)}_-.$$

As seen above, various cases arise for unbounded trio observables A_- (resp. A_+) defined by the lowering operator a_- (resp. the raising operator a_+).

Example 5.3. Let $\mathcal{P}(h)$ be a polynomial algebra on $\mathcal{S}$ generated by I and h. Then $\mathcal{P}(h)$ is a self-adjoint O^*-algebra on $\mathcal{S}$ and

$$\mathcal{P}(h)'_{\mathrm{w}} = \{h\}'_{\mathrm{w}} = \{(\varphi_n \otimes \bar{\varphi}_n);\ n \in \mathbb{N}_0\}' = (\mathcal{P}(h)'_{\mathrm{w}})'.$$

We define the following π-self-adjoint $T^\dagger$-algebras on $\mathcal{S}$ by

$$\begin{aligned}
\mathcal{B}_{(1)} &:= \{(p(h), \varphi, \psi^*);\ p(h) \in \mathcal{P}(h) \text{ and } \varphi, \psi \in \mathcal{S}\},\\
\mathcal{B}_{(2)} &:= \{(p(h), p(h)\varphi_0, (p(h)^\dagger \varphi_0)^*);\ p(h) \in \mathcal{P}(h)\},\\
\mathcal{B}_{(3)} &:= \{(p(h), 0, 0);\ p(h) \in \mathcal{P}(h)\}.
\end{aligned}$$

Then we have the following

(1)

$$\begin{aligned}
\mathcal{B}^{c}_{(1)} &= \{(\pi(K), 0, 0);\ \pi(K) \in \mathcal{P}(h)'_{\mathrm{w}}\},\\
\mathcal{B}^{cs}_{(1)} &= \{(\pi(A), 0, 0);\ \overline{\pi(A)}\, \eta\, (\mathcal{P}(h)'_{\mathrm{w}})'\}.
\end{aligned}$$

(2)

$$\begin{aligned}
\mathcal{B}^{c}_{(2)} &= \{(\pi(K), \pi(K)\varphi_0, (\pi(K)^*\varphi_0)^*);\ \pi(K) \in \mathcal{P}(h)'_{\mathrm{w}}\},\\
\mathcal{B}^{cs}_{(2)} &= \{(\pi(A), \pi(A)\varphi_0, (\pi(A)^\dagger\varphi_0)^*);\ \overline{\pi(A)}\, \eta\, (\mathcal{P}(h)'_{\mathrm{w}})'\}.
\end{aligned}$$

(3)

$$\begin{aligned}
\mathcal{B}^{c}_{(3)} &= \{(\pi(K), 0, 0);\ \pi(K) \in \mathcal{P}(h)'_{\mathrm{w}}\},\\
\mathcal{B}^{cs}_{(3)} &= \{(\pi(A), 0, 0);\ \overline{\pi(A)}\, \eta\, (\mathcal{P}(h)'_{\mathrm{w}})'\}.
\end{aligned}$$

Hence every element A of $\mathcal{B}^{cs}_{(i)}$ $(i = 1, 2, 3)$ satisfies (5.1), so its polar decomposition and functional calculus are possible.

Acknowledgements: This work has been partially done within the activities of Gruppo UMI Teoria dell'Approssimazione e Applicazioni and of GNAMPA of the INdAM. One of us (H.I.) is supported by JSPS KAKENHI Grant Number 20K14335 and also acknoledges the financial support of CORI-Università di Palermo and the Kyushu Sangyo University.

References

[1] A.Inoue, *Tomita's Lectures on Observable Algebras in Hilbert Space*, Lecture Notes in Mathematics 2285, Springer 2021.

[2] A.Inoue and H.Inoue, An unbounded generalization of Tomita's observable algebras, Rep. Math. Phys. **89** 2022, 153-184.

[3] H.Inoue, An unbounded generalization of Tomita's observable algebras II, Rep. Math. Phys. **91** 2023, 251-276.

[4] H.Inoue, An unbounded generalization of Tomita's observable algebras III, to appear

[5] A.Inoue and K.Takesue, *Selfadjoint representations of polynomial algebras*, Trans. Amer, Math. Soc. **280** (1983) 393-400.

[6] R.T. Powers, *Self-adjoint algebras of unbounded operators*, Comm. Math. Phys. *21*(1971), 85–124.

Department of Economics, Kyushu Sangyo University, Fukuoka, Japan
Email address: `h-inoue@ip.kyusan-u.ac.jp`

Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, Via Archirafi n. 34, I-90123 Palermo, Italy
Email address: `camillo.trapani@unipa.it`