On the classification of certain geproci sets

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Abstract

In this short note we develop new methods toward the ultimate goal of classifying geproci sets in $\mathbb{P}^3$. We apply these methods to show that among sets of 16 points distributed evenly on 4 skew lines, up to projective equivalence there are only two distinct geproci sets. We give different geometric distinctions between these sets. The methods we develop here can be applied in a more general set-up; this is the context of follow-up work [2].

1 Introduction

The study of geproci sets was initialized in one of the previous workshops on Lefschetz properties held in Levico Terme (Italy) in 2018. In the present note we work exclusively over the field $\mathbb{C}$ of complex numbers.

Definition 1.1 (A geproci set of points)**.** We say that a set of points $Z \subset \mathbb{P}^N_{\mathbb{C}}$ with $N \geq 3$ is *geproci* (for GEneral PROjection is a Complete Intersection), if its general projection to a hyperplane is a complete intersection.

We say that a geproci set is *trivial*, if it is already contained in a hyperplane. From now on we consider only nontrivial geproci sets.

So far nontrivial geproci sets have been discovered only in $\mathbb{P}^3$. They project to a plane, where their images are the intersection points of two curves of degrees a and b (equivalently: the ideal of the projection has exactly two generators: one of degree a and another of degree b). We assume that $a \leq b$ and we refer to such sets of points as (a, b)-geproci.

Example 1.2 (A grid)**.** Assume that we have two positive integers $a \leq b$. Let $Z \subset \mathbb{P}^3$ be a grid, i.e., the set of all intersection points among lines in two sets $\mathcal{L} = \{L_1, \ldots, L_a\}$ and $\mathcal{M} = \{M_1, \ldots, M_b\}$ such that lines from the same set, either $\mathcal{L}$ or $\mathcal{M}$, are pairwise skew but any two lines from distinct sets intersect in a point. It is elementary to see that a grid is an (a, b)-geproci set for all values of a, b and the set is nontrivial for $a, b \geq 2$ and it is contained in a unique quadric for $a, b \geq 3$.

Grids exist for any values of a and b. They were studied extensively in [3]. Here we are interested in geproci sets which are not grids but half grids.

Definition 1.3 (A half grid)**.** We say that a nontrivial (a, b)-geproci set $Z \subset \mathbb{P}^3$ is a *half grid*, if it is not a grid but one of the curves determining its general projection as a complete intersection can be taken as a union of lines.

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Remark 1.4. General projections of points can be collinear only if the points are collinear before. So an (a, b) half grid in $\mathbb{P}^3$ consists either of a-tuples of b points each distributed on a lines or of b-tuples of a points distributed on b lines. A priori it could be that the lines containing points of Z intersect. However [1, Proposition 4.14] provides an easy argument that this is not the case. Indeed, removing all lines but three from the half grid, we obtain a $(3, a)$ (or a $(3, b)$) geproci set. This set must be a grid, so that in particular the lines must be disjoint.

In particular half grids are geproci by definition. Most of the known geproci sets which are not grids are half grids. As a matter of fact, we know at the time of this writing only three exceptions – see the introduction to [1] for details. This justifies that our interest here focuses on half grids.

Our main result in this note is the following.

Theorem 1.5 (Classification of $(4, 4)$ half grids)**.** *Let $Z \subset \mathbb{P}^3$ be a $(4, 4)$ half grid. Then, up to projective change of coordinates, Z is either*

A) the anharmonic case (see Section 4.1)

$$\begin{array}{llll} (1:0:0:0), & (0:1:0:0), & (1:1:0:0), & (1:0:1:1), \\ (0:0:1:0), & (0:0:0:1), & (0:0:1:1), & (0:1:-1:0), \\ (1:0:1:0), & (0:1:0:1), & (1:1:1:1), & (1:1:0:1), \\ (1:0:\varepsilon:0), & (0:1:0:\varepsilon), & (1:1:\varepsilon:\varepsilon), & (1:1-\varepsilon:\varepsilon:1), \end{array}$$

where ε is a primitive root of unity of order six, or

B) the harmonic case (see Section 4.2)

$$\begin{array}{llll} (1:0:0:0), & (0:1:0:0), & (1:1:0:0), & (1:0:0:-1), \\ (0:0:1:0), & (0:0:0:1), & (0:0:1:1), & (0:1:1:0), \\ (1:0:1:0), & (0:1:0:1), & (1:1:1:1), & (1:1:1:-1), \\ (1:0:-1:0), & (0:1:0:-1), & (1:1:-1:-1), & (-1:1:1:1). \end{array}$$

The points in the statement of Theorem 1.5 are organized so that the half grid property is immediately visible: points in the columns are collinear. The main difference between both cases is that in the harmonic case there are exactly 4 lines containing 4 of configuration points, whereas in the anharmonic case additional collinearity can be observed for the 4 points in the bottom row.

In order to put this result in some perspective, let us recall that in [1, Theorem 4.10] we proved that the only non-grid $(3, 4)$-geproci set is the half grid determined by points in the D_4 root system. Up to projective change of coordinates, its 12 points can be listed explicitly as

$$\begin{array}{llllll} (1:1:0:0), & (1:0:1:0), & (1:0:0:1), & (0:1:1:0), & (0:1:0:1), & (0:0:1:1), \\ (1:-1:0:0), & (1:0:-1:0), & (1:0:0:-1), & (0:1:-1:0), & (0:1:0:-1), & (0:0:1:-1). \end{array}$$

See Theorem 3.1 for a more precise statement.

2 Inputs from projective geometry

To fix notation let $(x : y : z : w)$ be projective coordinates on $\mathbb{P}^3$. We denote by $j(P_1, P_2; P_3, P_4)$ the cross-ratio of an ordered set of four collinear points (see [1] for the definition and first results). For integers $\{i, j, k, l\} = \{1, 2, 3, 4\}$, we write (i, j, k, l) to indicate the permutation, which sends 1 to i, 2 to j, 3 to k and 4 to l. Note that this is not cycle notation!

It is well-known that the cross-ratio is invariant under the Klein group, namely

$$j(P_1, P_2; P_3, P_4) = j(P_{\sigma(1)}, P_{\sigma(2)}; P_{\sigma(3)}, P_{\sigma(4)}),$$

where σ is one of the following permutations:

$$(1, 2, 3, 4),\ (2, 1, 4, 3),\ (3, 4, 1, 2),\ (4, 3, 2, 1). \tag{1}$$

Note that all elements in the Klein group are involutions.

There are two exceptional cases under which there are additional permutations leaving the cross-ratio invariant:

The harmonic case. If $j(P_1, P_2; P_3, P_4) \in \{-1, 1/2, 2\}$, then the following permutations leave the cross-ratio invariant:

$$\begin{array}{llll} (1,2,3,4), & (2,1,4,3), & (3,4,1,2), & (4,3,2,1), \\ (1,2,4,3), & (2,1,3,4), & (3,4,2,1), & (4,3,1,2). \end{array} \tag{2}$$

The anharmonic case. If $j(P_1, P_2; P_3, P_4) \in \left\{\frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i\right\}$, i.e., it is a primitive root of unity of order 6, then the following permutations leave the cross-ratio invariant:

$$\begin{array}{llll} (1,2,3,4), & (2,1,4,3), & (3,4,1,2), & (4,3,2,1), \\ (1,3,4,2), & (2,4,3,1), & (3,1,2,4), & (4,2,1,3), \\ (1,4,2,3), & (2,3,1,4), & (3,2,4,1), & (4,1,3,2). \end{array} \tag{3}$$

It is a classical fact in projective geometry (see, e.g., [4, Paragraph 3.4.1]) that given four lines in $\mathbb{P}^3$ not on a quadric surface, there are two (counted with multiplicities) transversals to these lines.

We saw in Example 1.2 that any (a, b) grid with $a, b \geq 3$ is contained in a quadric. Our first result here is a criterion when lines determining a $(2, 4)$ grid are contained in a quadric.

Lemma 2.1 (Quadrics and $(2,4)$ grids.)**.** *Let $R, R' \subset \mathbb{P}^3$ be a pair of skew lines. Let $P_1, \dots, P_4$ be a set of mutually distinct points on R and let $P'_1, \dots, P'_4$ be a set of mutually distinct points on R'. Let r_i be the line determined by $P_iP'_i$ for $i = 1, \dots, 4$. The lines $r_1, \dots, r_4$ are contained in a quadric if and only if*

$$j(P_1, P_2; P_3, P_4) = j(P'_1, P'_2; P'_3, P'_4).$$

Proof. The lines r_1, r_2, r_3 are pairwise skew, so they determine a unique quadric Q. This quadric contains R and R' because it has at least 3 points common with both lines. If the line r_4 is contained

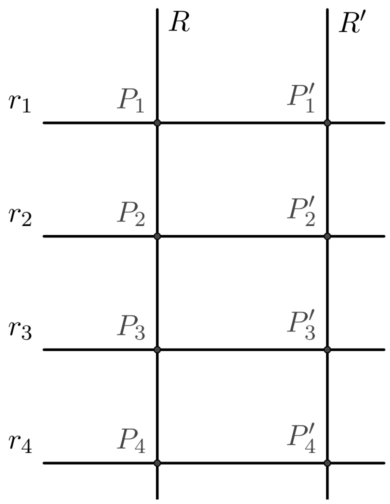

Figure 1: A $(2, 4)$ grid

in Q, then the two cross ratios are equal.

For the other direction, observe that there exists a unique line r in the ruling of Q determined by r_1 which passes through P_4. This line meets R' in the unique point P such that the cross ratios $j(P_1, P_2; P_3, P_4)$ and $j(P'_1, P'_2; P'_3, P)$ are equal. But this implies $P = P'_4$ and we are done. □

We need the following simple fact about projectivities of $\mathbb{P}^1$. We include it here with a proof, as it was difficult to track it down in the literature.

Lemma 2.2. *Let φ be a projective transformation of $\mathbb{P}^1$ with exactly 1 fixed point P. Then φ has no other finite orbit but that of P.*

Proof. Without loss of generality we may assume that $P = (1 : 0)$. Let M be a matrix representing φ. Since $\varphi(P) = P$ and φ has no other fixed points, it has the form

$$M = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$$

with $\varepsilon \in \mathbb{C} \setminus \{0\}$, because φ is not the identity. For a point $Q \neq P$ we have $Q = (q : 1)$ for some $q \in \mathbb{C}$ and for a positive integer n we have

$$\varphi^n(Q) = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} q + n\varepsilon \\ 1 \end{pmatrix},$$

so the orbit of Q is infinite. □

The next Lemma characterizes projective involutions (i.e., a projective transformations φ with $\varphi^2 = \mathrm{Id}$ but $\varphi \neq \mathrm{Id}$) with two fixed points.

Lemma 2.3 (Involution of $\mathbb{P}^1$ with 2 fixed points)**.** *Let P, P' be two distinct points in $\mathbb{P}^1$. Then there exists a unique involution φ with fixed points at P and P'.*

Proof. We can assume that $P = (1 : 0)$ and $P' = (0 : 1)$. Then any matrix fixing (projectively) these two points has the shape

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}.$$

The condition $M^2 = \mathrm{Id}$ (and $M \neq \mathrm{Id}$) forces $\varepsilon = -1$. □

Our next observation is that projective transformations on two skew lines in $\mathbb{P}^3$ are always induced by a projective transformation of the ambient space.

Lemma 2.4 (Extending projectivities on a pair of skew lines)**.** *Let R, R' be skew lines in $\mathbb{P}^3$. Let φ be a projectivity on R and let φ' be a projectivity on R'. Then there exists a projective transformation Φ of $\mathbb{P}^3$, which restricts to φ on R and to φ' on R'. In particular the two lines are invariant under Φ.*

Proof. Up to change of coordinates we can assume that R is the $z = w = 0$ line and R' is defined by equations $x = y = 0$. Let M be a matrix defining φ in $(x : y)$ coordinates and let M' be a matrix defining φ' in $(z : w)$ coordinates. Then the matrix

$$\begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix}$$

defines a projectivity Φ in coordinates $(x : y : z : w)$, which satisfies the requirements of the Lemma. □

3 On the $(4, 4)$ half grids

In [1, Theorem 4.10] we showed the following classification result for $(3, 4)$-geproci sets which is crucial in the sequel.

Theorem 3.1 (Classification of $(3, 4)$-geproci sets)**.** *Let $Z \subset \mathbb{P}^3$ be a $(3, 4)$-geproci set. Then either*

a) Z is a grid, or

b) Z is the D_4 configuration of points.

Taking into account that D_4 does not contain any four collinear points and does not lie on a quadric surface, we derive the following immediate consequence.

Corollary 3.2 (Subsets of $(4,4)$ half grids)**.** *Let Z be a $(4,4)$ half grid and let R_a, R_b, R_c, R_d be four lines covering Z. Then for any index $x \in \{a,b,c,d\}$ the set $Z \setminus R_x$ is a $(3,4)$ grid.*

For any three mutually different symbols $x, y, z \in \{a,b,c,d\}$ we denote by Q_{xyz} the quadric generated by R_x, R_y and R_z. Since the lines are skew, these quadrics are smooth. They are also mutually distinct, because Z is not contained in a quadric (it would be a grid otherwise).

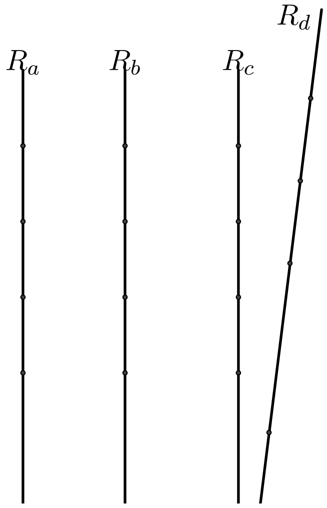

Figure 2: A (4,4) half grid

Before we impose specific coordinates on the points in Z, we want to determine additional collinearities. To this end we begin to label points in Z on the lines provided in Figure 2. We begin with the line R_c and we denote the points on this line with $c_1, \ldots, c_4$, see Figure 3. Since Z_{abc} is a grid on Q_{abc},

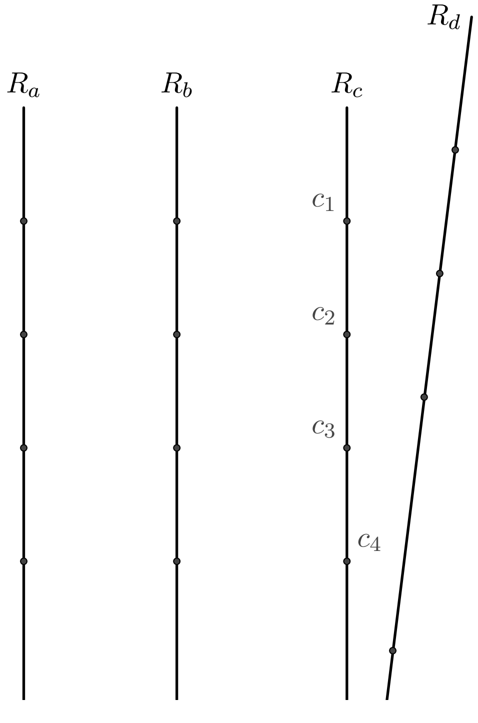

Figure 3: A (4,4) half grid with numbered points on line R_c

there are four lines r_1, r_2, r_3, r_4 in the ruling complementary to that determined by the lines R_a, R_b

and R_c, each of them passing through the point c_i with the same index. The intersections of each r_i with R_a, R_b determine a labelling for the points of Z on the lines R_a and R_b, see Figure 4.

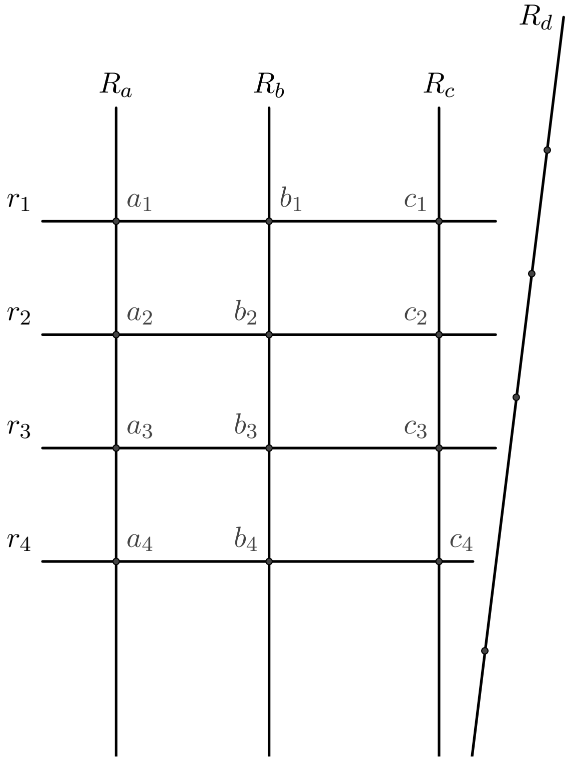

Figure 4: A $(4, 4)$ half grid with numbered points on lines R_a, R_b and R_c

Since $r_1, \ldots, r_4$ are contained in a quadric we have

$$j(a_1, a_2; a_3, a_4) = j(b_1, b_2; b_3, b_4) = j(c_1, c_2; c_3, c_4) \tag{4}$$

by Lemma 2.1. This is a considerable constraint and we want to combine it with similar conditions determined by the remaining three quadrics.

The set Z_{bcd} is a grid in Q_{bcd}. So there are four lines in the ruling of Q_{bcd} complementary to the ruling containing R_c cutting out the points of Z on the union $R_b \cup R_c \cup R_d$. We call them $L_1, \ldots, L_4$, where the numbering is determined by the numbering of points on R_c, see Figure 5. This numbering determines also numbering of points on the line R_d. Thus all points in Z are now labeled. For $i = 1, \ldots, 4$ we denote by $b_{\beta(i)}$ the point of intersection $R_b \cap L_i$ (in Figure 5 we took as an example $\beta(1) = 2$, $\beta(2) = 3$ and so on). In any case we have

$$j(b_{\beta(1)}, b_{\beta(2)}; b_{\beta(3)}, b_{\beta(4)}) = j(c_1, c_2; c_3, c_4) = j(d_1, d_2; d_3, d_4) \tag{5}$$

by Lemma 2.1. Now, the key point here is to see that the permutation β can be assumed not to be an involution, which forces one of cases: (2) or (3). We will first exclude the possibility that β is the identity.

Proposition 3.3. *The permutation β is not the identity.*

Proof. If $\beta(i) = i$, then $L_i = r_i$, so that $d_i \in Q_{abc}$. If this happens for all i, we obtain that Z is contained in a quadric, hence it is a grid. A contradiction. □

Since β preserves the cross-ratio of four points on R_b, there exists a projective transformation φ_β of R_b, which restricts to β on $Z \cap R_b$. We show in the next Lemma that this projectivity has exactly two fixed points.

Lemma 3.4 (Transversals to $R_a, \ldots, R_d$)**.** *There are two distinct transversals S, S' to $R_a, \ldots, R_d$. Moreover, the intersection points of these transversals with R_b are the fixed points of φ_β.*

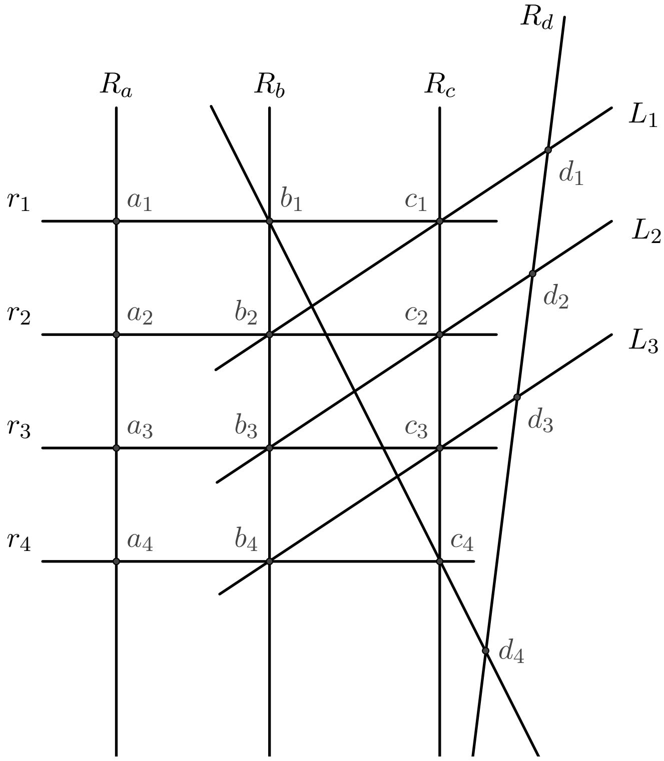

Figure 5: Grid lines on Q_{abc} and Q_{bcd}

Proof. The intersection $Q_{abc} \cap Q_{bcd}$ contains two skew lines R_b and R_c, so it must contain either one line S (counted with multiplicity 2) or two lines S, S' from the complementary rulings on both quadrics. Hence these lines must be transversals of $R_a, \ldots, R_d$.

Let us denote their intersection points:

$$a_s = S \cap R_a,\ b_s = S \cap R_b,\ c_s = S \cap R_c,\ d_s = S \cap R_d$$

and similarly

$$a_{s'} = S' \cap R_a,\ b_{s'} = S' \cap R_b,\ c_{s'} = S' \cap R_c,\ d_{s'} = S' \cap R_d,$$

see Figure 6. Now we want to exclude the possibility that $S = S'$. To this end we prove the following
Claim.
The projectivity φ_β has two fixed points at b_s and $b_{s'}$.
Proof of the Claim.
By Lemma 2.4 there exists a projective transformation Φ of $\mathbb{P}^3$, which restricts to φ_β on R_b and to the identity on R_c. This projectivity maps lines r_i joining b_i and c_i to lines joining $\Phi(b_i)$ and $\Phi(c_i)$ for $i = 1, \ldots, 4$. But $\Phi(b_i) = b_{\beta(i)}$ and $\Phi(c_i) = c_i$, so that $\Phi(r_i) = L_i$ for all $i = 1, \ldots, 4$. It follows that

$$\Phi(Q_{abc}) = Q_{bcd}.$$

Moreover, since Φ leaves R_b and R_c invariant by Lemma 2.4 it must leave also the union $S \cup S'$ invariant, because

$$Q_{abc} \cap Q_{bcd} = R_b \cup R_c \cup S \cup S'.$$

It must be in fact $\Phi(S) = S$ because Φ restricts to the identity on R_c so it cannot swap the points c_s and $c_{s'}$ (the claim $\Phi(S) = S$ remains valid also if $S = S'$).

Since φ_β is not the identity, it has at most two fixed points. It has also at least two fixed points by Lemma 2.2 because it is a transformation of finite order, equal to the order of β, so that all its orbits are finite. Suppose now that $S = S'$. Then there exists another point $b_s \neq P \in R_b$ fixed by φ_β. Let r_P be the line in the same ruling as r_1 on Q_{abc} passing through P. This line meets R_c in some point

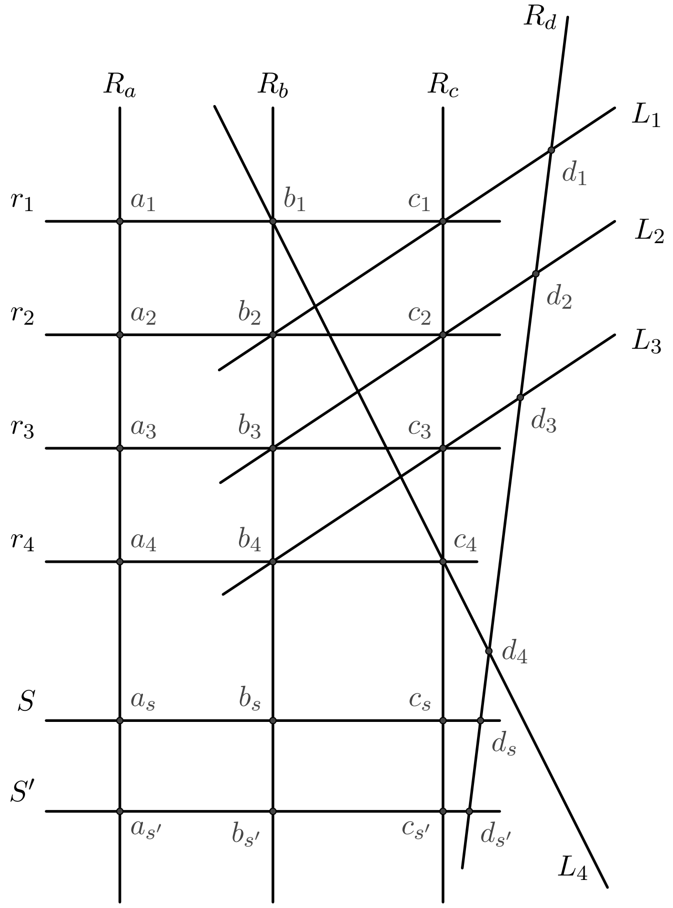

Figure 6: All lines and transversals

P_c. Since $\Phi(P_c) = P_c$ (as Φ restricted to R_c is the identity) the line r_P is invariant under Φ. But then it belongs to $Q_{abc} \cap Q_{bcd}$, so it must be the line S' distinct from S.

The two fixed points of φ_β are then b_s and $b_{s'}$. This ends the proof of the Claim and also of the Lemma. $\square$

Considering the grid Z_{abd} on the quadric Q_{abd} we obtain another permutation β' acting on points $b_1, \ldots, b_4$ determined by numbering of points on the line R_d in a way that the following triples of points are collinear:

$$a_{\alpha(1)}, b_{\beta'(1)}, d_1, \quad a_{\alpha(2)}, b_{\beta'(2)}, d_2, \quad a_{\alpha(3)}, b_{\beta'(3)}, d_3, \quad a_{\alpha(4)}, b_{\beta'(4)}, d_4, \tag{6}$$

where α is some permutation of points on R_a.

Lemma 3.5. *The permutations β and β' do not coincide and at least one of them is not an involution.*

Proof. By definition of β the following triples of points are collinear:

$$b_{\beta(1)}, c_1, d_1, \quad b_{\beta(2)}, c_2, d_2, \quad b_{\beta(3)}, c_3, d_3, \quad b_{\beta(4)}, c_4, d_4. \tag{7}$$

Combining (7) with (6) we conclude that if $\beta = \beta'$, then the lines L_i meet the line R_a at points of Z. But this implies that Z is a grid, a contradiction.

Now, replacing R_c by R_d in the proof of Lemma 3.4 we conclude that b_s and $b_{s'}$ are fixed points of β'. So β and β' induce projectivities φ and φ' on R_b which have the same pair points as their fixed points. By Lemma 2.3 at most one of these projectivities can be an involution. $\square$

Since all permutations in (1) are involutions, we conclude that the points in Z_b are either harmonic or anharmonic. Then (4) and (5) imply that points in all Z_x are simultaneously either harmonic or anharmonic for $x \in \{a, b, c, d\}$. We study both cases in more detail in the next section.

4 Proof of Theorem 1.5

To fix notation we assume in this section that the permutation β introduced before Proposition 3.3 is not an involution. This can be done since relabelling the lines R_a, R_b, R_c and R_d according to the following rule: $(a, b, c, d) \to (d, b, a, c)$ exchanges the role of β and β' from Lemma 3.5.

4.1 The anharmonic case

In this part we prove Theorem 1.5 A). We see in (3) that all non-involutions appearing there have a fixed point. Renumbering the points, if necessary, we may assume that b_4 is the fixed point of β. This implies that r_4 is one of the two lines transversal to $R_a, \ldots, R_d$. Let this line be S, so that we have $r_4 = S$.

Moreover, renumbering the remaining points, if necessary, we may assume that β is the 3-cycle $(2, 3, 1, 4)$. Now we can begin with fixing coordinates. Suppose to begin with that

$$R_b : \ x = z = 0 \ \text{ and } \ R_c : \ x - y = z - w = 0.$$

Let also

$$b_1 = (0 : 1 : 0 : 0), \ b_2 = (0 : 0 : 0 : 1), \ b_3 = (0 : 1 : 0 : 1)$$

and

$$c_1 = (1 : 1 : 0 : 0), \ c_2 = (0 : 0 : 1 : 1), \ c_3 = (1 : 1 : 1 : 1).$$

Then necessarily

$$b_4 = (0 : 1 : 0 : \varepsilon) \ \text{ and } \ c_4 = (1 : 1 : \varepsilon : \varepsilon),$$

where ε is a primitive root of unity of order 6, so that it satisfies the equation $\varepsilon^2 - \varepsilon + 1 = 0$. Then it must be

$$r_1 : \ z = w = 0, \ r_2 : \ x = y = 0, \ r_3 : \ x - z = y - w = 0, \ \text{ and } \ r_4 = S : \ \varepsilon x - z = \varepsilon y - w = 0.$$

With the given fixed β we can reproduce also equations of the L_i lines:

$$L_1 : \ x - y = z = 0, \ L_2 : \ x = y + z - w = 0, \ L_3 : \ x - z = z - w = 0 \ \text{ and } \ L_4 = r_4 = S.$$

It remains to construct the lines R_a and R_d.

To this end note that for $i \in \{1, 2, 3\}$ the line M_i joining c_i to one of points a_1, a_2, a_3 on R_a must also meet R_d in one of the points d_1, d_2, d_3. The following Lemma restricts considerably possible collineations.

Lemma 4.1. *It is neither $a_{\beta(i)} \in M_i$ nor $a_i \in M_i$ for all $i = 1, \ldots, 4$.*

Proof. Assume to the contrary that $a_{\beta(i)} \in M_i$ for some i, see Figure 7. Since the lines M_i and L_i intersect in c_i (and are not equal, as otherwise L_i would be one of the secants S, S' contradicting the assumption that β has no fixed point), they span a plane, call it π_i. Since L_i and M_i intersect R_d in two distinct points, this line is also contained in π_i. Similarly, since $b_{\beta(i)} \in L_i$ by definition and $a_{\beta(i)} \in M_i$ by assumption, the line $r_{\beta(i)}$ is contained in π_i. But then the lines $r_{\beta(i)}$ and R_d intersect, so that $r_{\beta(i)}$ is either S or S', a contradiction.

It cannot be that $a_i \in M_i$, because then it would be either $M_i = r_i$ and we would have $M_i = S'$ contradicting β not being an involution. $\square$

Corollary 4.2. *It is $a_{\beta^2(i)} \in M_i$ for all $i = 1, \ldots, 4$.*

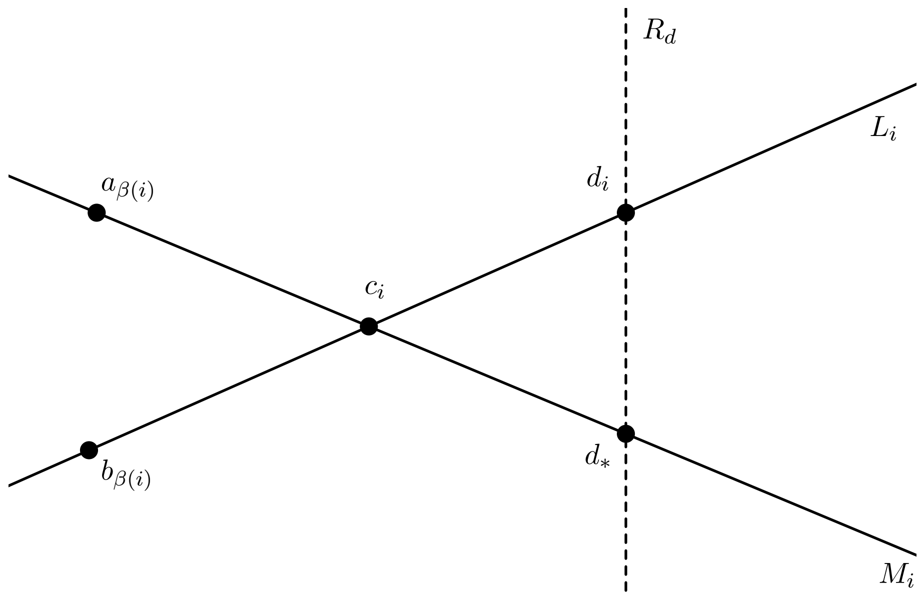

Figure 7: Ilustration for Lemma 4.1

Moreover, replacing M_i by N_i in Lemma 4.1 we conclude that neither $a_{\beta^{-1}(i)} \in N_i$ nor $a_i \in N_i$.

Corollary 4.3. *It is* $a_{\beta(i)} \in N_i$ *for all* $i = 1, \ldots, 4$.

Since we are in the position to pick the point a_1 on the r_1 line arbitrarily (but distinct from b_1 and c_1), we pick $a_1 = (1 : 0 : 0 : 0)$, this forces $R_a :\ y = w = 0$ and

$$a_2 = (0 : 0 : 1 : 0),\ a_3 = (1 : 0 : 1 : 0) \text{ and } a_4 = (1 : 0 : \varepsilon : 0).$$

Now, as we have specific coordinates for all points on lines R_a, R_b, R_c we can determine equations of lines M_i and N_j and check their intersections. We obtain

$$d_1 = (1 : 0 : 1 : 1),\ d_2 = (0 : 1 : -1 : 0), d_3 = (1 : 1 : 0 : 1).$$

Then $R_d :\ y + z - w = x - w = 0$ and $d_4 = (\varepsilon : \varepsilon - 1 : 1 : \varepsilon)$. This concludes proof of Theorem 1.5 A).

4.2 The harmonic case

In this part we prove Theorem 1.5 B). Right away we fix equations of lines R_a, R_b, R_c and the coordinates of points a_1, a_2, a_3, b_1, b_2, b_3 and c_1, c_2, c_3 as in Section 4.1. Then necessarily

$$a_4 = (1 : 0 : -1 : 0),\ b_4 = (0 : 1 : 0 : -1) \ \text{ and } \ c_4 = (1 : 1 : -1 : -1),$$

since the quadruples of points on lines R_a, R_b and R_c must be harmonic.

The only permutations in (2) of order greater than 2 are the 4-cycles $(3, 4, 2, 1)$ and $(4, 3, 1, 2)$. Renumbering the points if necessary, we may assume that $\beta = (3, 4, 2, 1)$. Then the lines L_i are determined as follows:

$$L_1 :\ z = x - y + w = 0,\ L_2 :\ x = y - z + w = 0,\ L_3 :\ x - z = y - z = 0 \ \text{ and } L_4 :\ x + w = z - w = 0$$

and

$$c_i, d_i, b_{\beta(i)} \in L_i \ \text{ for } i = 1, \ldots, 4.$$

Let, as before, M_i be the line through c_i containing a configuration point on R_a and another one on R_d. And let N_i be the line through b_i containing a configuration point on R_a and on R_d. Since Lemma 4.1 remains valid also in the situation considered here only the following collinearities are possible:

$$M_1 :\ c_1 \text{ and } \{a_2 \text{ or } a_4\}\,,\ M_2 :\ c_2 \text{ and } \{a_1 \text{ or } a_3\}\,,$$

$$M_3: \ c_3 \text{ and } \{a_1 \text{ or } a_4\}, \ M_4: \ c_4 \text{ and } \{a_2 \text{ or } a_3\}.$$

The choice of points on M_1 and the possibilities listed above determine the points on the remaining lines, so that there are only two possibilities:

$$M_1: \ c_1 - a_2 \ \text{ forces } M_4: \ c_4 - a_3 \text{ and } M_2: \ c_2 - a_1 \text{ and } M_3: \ c_3 - a_4,$$

whereas

$$M_1: \ c_1 - c_4 \ \text{ forces } M_3: \ c_3 - a_1 \text{ and } M_2: \ c_2 - a_3 \text{ and } M_4: \ c_4 - a_2.$$

By the same token for the lines N_i we obtain the following possibilities

$$N_1: \ b_1 \text{ and } \{a_2 \text{ or } a_3\}, \ N_2: \ b_2 \text{ and } \{a_1 \text{ or } a_4\},$$

$$N_3: \ b_3 \text{ and } \{a_2 \text{ or } a_4\}, \ N_4: \ b_4 \text{ and } \{a_1 \text{ or } a_3\}.$$

which provides two cases for the configuration:

$$N_1: \ b_1 - a_2 \ \text{ forces } N_3: \ b_3 - a_4 \text{ and } N_2: \ b_2 - a_1 \text{ and } N_4: \ b_4 - a_3,$$

whereas

$$N_1: \ b_1 - a_3 \ \text{ forces } N_4: \ b_4 - a_1 \text{ and } N_2: \ b_2 - a_4 \text{ and } N_3: \ b_3 - a_2.$$

We need now to determine which lines M_i and N_j intersect in points being potential configuration points on the R_d line. So these points cannot be the a_k points. Incidences between all possible lines M_i and N_j lines are summarized in Table 1. An analysis of Table 1 together with matching new points

	b_1a_2	b_2a_1	b_3a_4	b_4a_3	b_1a_3	b_2a_3	b_3a_2	b_4a_1
c_1a_2	a_2	$\emptyset$	$\emptyset$	$\emptyset$	$1:1:1:0$	$\emptyset$	a_2	$\emptyset$
c_2a_1	$\emptyset$	a_1	$\emptyset$	$\emptyset$	$\emptyset$	$-1:0:1:1$	$\emptyset$	a_1
c_3a_4	$\emptyset$	$\emptyset$	a_4	$\emptyset$	$\emptyset$	a_4	0:1:2:1	$\emptyset$
c_4a_3	$\emptyset$	$\emptyset$	$\emptyset$	a_3	a_3	$\emptyset$	$\emptyset$	2:1:0:-1
c_1a_4	$0:1:1:0$	$\emptyset$	a_4	$\emptyset$	$1:2:1:0$	a_4	$\emptyset$	$\emptyset$
c_2a_3	$\emptyset$	$1:0:0:-1$	$\emptyset$	a_3	a_3	$-1:0:1:2$	$\emptyset$	$\emptyset$
c_3a_1	$\emptyset$	a_1	-1:1:1:1	$\emptyset$	$\emptyset$	$\emptyset$	0:1:1:1	a_1
c_4a_2	a_2	$\emptyset$	$\emptyset$	1:1:1:-1	$\emptyset$	$\emptyset$	a_2	1:1:0:-1

Table 1: Incidences of potential lines M_i and N_j

to equations of lines L_i shows that it must be

$$R_d : x - z + 2w = y - z + w = 0$$

and the configuration points on this line are

$$d_1 = (2:1:0:-1), d_2 = (0:1:2:1), d_3 = (1:1:1:0), d_4 = (-1:0:1:1)$$

or

$$R_d : x - z + 2w = y - z + w = 0$$

and the configuration points are

$$d_1 = (1:0:0:-1), d_2 = (0:1:1:0), d_3 = (1:1:1:-1), d_4 = (-1:1:1:1).$$

We leave it as an exercise to a motivated reader to check that both configurations obtained this way are projectively equivalent. This ends the proof of Theorem 1.5.

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