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Curves defined by a class of discrete operators: Approximation result and applications

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In approximation theory, classical discrete operators, like generalized sampling, Szász-Mirak'jan, Baskakov, and Bernstein operators, have been extensively studied for scalar functions. In this paper, we look at the approximation of curves by a class of discrete operators, and we exhibit graphical examples concerning several cases. The topic has useful implications about the computer graphics and the image processing: We discuss applications on the approximation and the reconstruction of curves in images.

KEYWORDS

approximation, curves, discrete operators, images, sampling operators

MSC CLASSIFICATION

94A20, 41A35, 65D17, 94A08

1 | INTRODUCTION

Nowadays, several types of operators have been investigated to approximate bounded and continuous real valued functions of one variable. Here, for approximation, we mean that a family of operators $\{S_n\}_{n\in\mathbb{N}}$, acting on some function space, has the property that

$$\lim_{n\to+\infty} (S_n f)(t) = f(t), \tag{1}$$

when $f : I \to \mathbb{R}$ is a bounded continuous function defined on a interval $I \subset \mathbb{R}$ (not necessarily bounded) and $t \in I$. A typical framework consists of discrete operators

$$(S_n f)(t) = \sum_{k\in J} f(v_{n,k}) K_{n,k}(t), \qquad f : I \to \mathbb{R}, \ t \in I \tag{2}$$

defined in terms of the samples $f(v_{n,k})$ of f in the point $v_{n,k}$ and the elementary functions $K_{n,k}$, where $n \in \mathbb{N}^+$ and k varies in a finite or countable set J. Classical examples include the generalized sampling operators [1–3] in the case $I = \mathbb{R}$, the Szász-Mirak'jan [4, 5] and the Baskakov operators [6] in the case $I = [0, +\infty[$, and the Bernstein operators [7] for $I = [0, 1]$. A lot of works in literature studied these operators or some variations; for instance, [8–16] (for a more complete list see Section 3).

Since a vector-valued function $\gamma : I \to \mathbb{R}^d, d > 1$, of one variable is made up of d scalar functions, that is, the components, the operators above can be used also to approximate vector-valued functions, acting individually in each component. Using a common terminology, with *curve*, we refer to a continuous vector-valued function on an interval.

We mention that some of the previous listed operators have been considered for vector-valued functions in [17–20], and moreover, Bernstein operators are at the base of the definition of the Bézier curves [21].

Taking into account the idea of approximation by components, in this paper, we look at a class of discrete operators, their properties of curves approximation, and various supporting examples. Furthermore, we discuss some consequent applications in the context of computer graphics [21] and image processing [22]. Nowadays, these fields play a crucial role in modern technology: Computer graphics studies methods to generate and visualize images; image processing investigates operations for enhancing images or for extracting useful information from them. Even though a curve in an image is represented by a finite number of points, it can be thought as discretization of a curve in the real plane; hence, we can exploit the operators for its approximation. Anyway, other processes can be done for curves in images starting by the approximations, for instance, the reconstruction with an increased resolution or affine transformations. We stress that performing these operations on the approximated curve (which possesses a mathematical expression defined by a continuous, and not discrete, variable) gives better results than those obtained applying the operations directly on the images. Other kinds of operators [23] found applications in image processing, but not for curves.

Coming back to the theoric aspects, we state the convergence result for curves, considering a class of discrete operators $\{S_n\}_{n\in\mathbb{N}}$ in a general framework in which the generalized sampling, Szász-Mirak'jan, Baskakov, and Bernstein operators fit. In details, our setting about $\{S_n\}_{n\in\mathbb{N}}$ follows, with some modifications, the assumptions adopted in [24]. Let $I \subset \mathbb{R}$ be an interval (bounded or not), $n \in \mathbb{N}^+$, $J \subset \mathbb{Z}$ an at most countable index set and $\Gamma_n = (\nu_{n,k})_{k\in J} \subset I$ a sequence of points such that

$$\lambda_n < \nu_{n,k+1} - \nu_{n,k} \leq \Lambda_n \text{ for every } k \in J, \tag{3}$$

where $\lambda_n, \Lambda_n > 0$ and $\lim_{n\to+\infty}\lambda_n = \lim_{n\to+\infty}\Lambda_n = 0$. We consider a family of continuous functions $\{K_{n,k}\}_{n\in\mathbb{N}^+,k\in J}$, $K_{n,k} : I \to \mathbb{R}$, satisfying the following conditions

$$\sum_{k\in J} K_{n,k}(t) = 1 \text{ for every } n \in \mathbb{N}^+ \text{ and } t \in I, \tag{4}$$

$$\sum_{k\in J} \left|K_{n,k}(t)\right| \text{ converges uniformly on compact sets of } I \text{ for every } n \in \mathbb{N}^+, \tag{5}$$

$$\text{there exists } M_0 \text{ such that } m_0(n) := \sup_{t\in I} \sum_{k\in J} \left|K_{n,k}(t)\right| < M_0 \text{ for every } n \in \mathbb{N}^+, \tag{6}$$

and

$$\lim_{n\to+\infty} \sum_{k\in J, |\nu_{n,k}-t|\geq\delta} \left|K_{n,k}(t)\right| = 0 \text{ for every } \delta > 0 \text{ and } t \in I. \tag{7}$$

Under this setup, for every $n \in \mathbb{N}^+$, we define the operator S_n as in (2). By the assumptions made, the convergence (1) for bounded and continuous real valued functions holds (Theorem 1), and it implies an analogous result (Corollary 1), that is, $\lim_{n\to+\infty}(S_n\gamma)(t) = \gamma(t)$, for bounded curves $\gamma : I \to \mathbb{R}^d$, $\gamma(t) = (x_1(t), \ldots, x_d(t))$, by defining

$$(S_n\gamma)(t) := ((S_n x_1)(t), \ldots, (S_n x_d)(t)), \ t \in I.$$

The paper is organized as follows. Section 2 is devoted to the approximation results related to $\{S_n\}_{n\in\mathbb{N}}$ for real valued functions and for curves. In Section 3, we show in more details that the generalized sampling, Szász-Mirak'jan, Baskakov, and Bernstein operators are special cases of our setting. Some explicit examples of approximations of curves in various cases (2D or 3D space, open or closed curves) are treated in Section 4. Finally, we discuss the applications about the approximation and the reconstruction of curves in images in Section 5 providing methods and examples.

2 | APPROXIMATION RESULTS

In this section, we state the main approximation results for the operators S_n, firstly for scalar functions and then for curves. For a bounded real function on a interval I, we write $\|f\|_\infty := \sup_{t\in I}|f(t)|$, while for a bounded function $\gamma : I \to \mathbb{R}^d$, with components $x_1, \ldots, x_d$, we write $\|\gamma\|_\infty := \max_{i=1,\ldots,d}(\|x_i\|_\infty)$. As preliminary note, we remark the following properties.

Proposition 1. *Let $f : I \to \mathbb{R}$ be a bounded function and $n \in \mathbb{N}^+$. The function $S_n f : I \to \mathbb{R}$ is well defined, continuous, and bounded.*

Proof. For every $t \in I$, by the boundedness of f and by (6), we have the following inequality:

$$|(S_n f)(t)| \leq \sum_{k \in J} |f(\nu_{n,k})||K_{n,k}(t)| \leq \|f\|_\infty m_0(n) < \infty.$$

This proves that $S_n f$ is well defined and bounded.

Now, let $t \in I$ and $\epsilon > 0$. Given $\rho > 0$, by (5), there exists $\bar{k} > 0$ such that $\sum_{|k|>\bar{k}} |K_{n,k}(\tilde{t})| < \epsilon$ for every $\tilde{t} \in I, |t-\tilde{t}| \leq \rho$. Moreover, by the continuity of $K_{n,k}$, for each $|k| \leq \bar{k}$, there exists $\delta_k > 0$ such that $|K_{n,k}(t) - K_{n,k}(s)| < \frac{\epsilon}{2\bar{k}+1}$ for every $|t - s| < \delta_k$. Therefore, putting $\delta = \min(\{\rho\} \cup \{\delta_k : |k| \leq \bar{k}\})$, we have

$$
\begin{aligned}
|(S_n f)(t) - (S_n f)(s)| &\leq \left| \sum_{k \in J} f(\nu_{n,k})(K_{n,k}(t) - K_{n,k}(s)) \right| \\
&\leq \|f\|_\infty \sum_{k \in J} |K_{n,k}(t) - K_{n,k}(s)| \\
&\leq \|f\|_\infty \left(\sum_{|k| \leq \bar{k}} |K_{n,k}(t) - K_{n,k}(s)| + \sum_{|k| > \bar{k}} |K_{n,k}(t) - K_{n,k}(s)| \right) \\
&\leq \|f\|_\infty (\epsilon + 2\epsilon) = 3\|f\|_\infty \epsilon
\end{aligned}
$$

for every $s \in I, |t - s| < \delta$. Thus, $S_n f$ is continuous. $\square$

The theorem below states that S_n are approximation operators for scalar functions when n goes to infinity (the proof follows standard steps; see, for instance, [25, Theorem 1]). We will next use it for formulating the result for curves.

Theorem 1. *Let* $f : I \to \mathbb{R}$ *be a bounded function. Then*

$$\lim_{n \to +\infty} (S_n f)(t) = f(t),$$

for every point t of continuity of f. Moreover, if f is also uniformly continuous and (7) is satisfied uniformly with respect to $t \in I$, then

$$\lim_{n \to +\infty} \|S_n f - f\|_\infty = 0.$$

Proof. The steps to prove the two statements are very similar, so we confine to the proof of the second one. Let $\epsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(\nu_{n,k}) - f(t)| < \epsilon$ for every $n \in \mathbb{N}^+, k \in J$ and $t \in I$ satisfying $|\nu_{n,k} - t| < \delta$. We denote by $J_1 := \{k \in J : |\nu_{n,k} - t| < \delta\}$, by $J_2 := \{k \in J : |\nu_{n,k} - t| \geq \delta\}$ and remark that J_1 is finite because of (3). Moreover, by (4),

$$(S_n f)(t) - f(t) = \sum_{k \in J} (f(\nu_{n,k}) - f(t)) K_{n,k}(t),$$

so we can write $|(S_n f)(t) - f(t)| \leq s_1(t) + s_2(t)$, where

$$s_1(t) = \sum_{k \in J_1} |f(\nu_{n,k}) - f(t)||K_{n,k}(t)| \quad \text{and} \quad s_2(t) = \sum_{k \in J_2} |f(\nu_{n,k}) - f(t)||K_{n,k}(t)|.$$

For hypothesis, (7) is satisfied uniformly with respect to $t \in I$. Hence, there exists $\bar{n}$ such that $\sum_{k \in J_2} |K_{n,k}(t)| < \epsilon$ for $n \geq \bar{n}$ and for any $t \in I$. Thus, by (6), we have that for $n \geq \bar{n}$ and for any $t \in I$,

$$s_1(t) < \epsilon \sum_{k \in J_1} |K_{n,k}(t)| \leq \epsilon m_0(n) \leq \epsilon M_0,$$

and that

$$s_2(t) < 2\|f\|_\infty \sum_{k\in J_2} |K_{n,k}(t)| < 2\|f\|_\infty \epsilon.$$

Summarizing, $\|S_n f - f\|_\infty < (M_0 + 2\|f\|_\infty)\epsilon$ for $n \geq \bar{n}$, and so, the proof is concluded. $\qquad\square$

In this paper, we are interested to curves in $\mathbb{R}^d$ and Proposition 1 and Theorem 1 give, as an immediate consequence, the following approximation result. For a bounded function $\gamma : I \to \mathbb{R}^d$, $\gamma(t) = (x_1(t), \dots, x_d(t))$, $t \in I$, we define $S_n\gamma : I \to \mathbb{R}^d$ by

$$(S_n\gamma)(t) := ((S_n x_1)(t), \dots, (S_n x_d)(t)), \qquad t \in I,$$

or, in a more compact form, by

$$(S_n\gamma)(t) = \sum_{k\in J} \gamma(v_{n,k})K_{n,k}(t), \qquad t \in I.$$

Corollary 1. *Let $\gamma : I \to \mathbb{R}^d$ be a bounded function. The following statements hold.*

1. *$S_n\gamma$ is a bounded and continuous function (i.e., a bounded curve);*
2. *$\lim_{n\to+\infty}(S_n\gamma)(t) = \gamma(t)$ for every point t of continuity of γ;*
3. *if γ is also uniformly continuous and (7) is satisfied uniformly with respect to $t \in I$, then $\lim_{n\to+\infty}\|S_n\gamma - \gamma\|_\infty = 0$.*

3 | SPECIAL CASES OF OPERATORS

As mentioned in the introduction, special cases of our operators setting are the generalized sampling operators, the Szász-Mirak'jan operator, the Baskakov operator, and the Bernstein operator. We give the details for each of them below.

3.1 | Generalized sampling operator

We start recalling that, following [2], a continuous function $\chi : \mathbb{R} \to \mathbb{R}$ is called a *kernel* if the following conditions are satisfied:

(i) for every $t \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \chi(t - k) = 1;$$

(ii)

$$\sum_{k\in\mathbb{Z}} |\chi(t - k)| \text{ converges uniformly for } t \in [0, 1].$$

With Equation (2) and the choice $I = \mathbb{R}$, $J = \mathbb{Z}$, $v_{n,k} = \frac{k}{n}$ and $K_{n,k}(t) = \chi(nt - k)$, where χ is a kernel, we recover the *generalized sampling operator*

$$\left(S_n^\chi f\right)(t) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{n}\right) \chi(nt - k), \quad f : \mathbb{R} \to \mathbb{R}, \ t \in \mathbb{R}.$$

This operator has been studied in [1–3] and under some variations in [11, 12, 14, 15, 26–33]. Assumption (4) follows by property (i), assumptions (5) and (6) are given by Lemma 1(b) of [2], finally (7) holds (also uniformly with respect to $t \in \mathbb{R}$) by Lemma 1(c) of [2].

Classical examples of kernels are the Fejér kernel and the B-splines [2, 12]. The *Fejér kernel*, shown in Figure 1A, is

$$F(t) = \frac{1}{2}\text{sinc}^2\left(\frac{t}{2}\right),$$
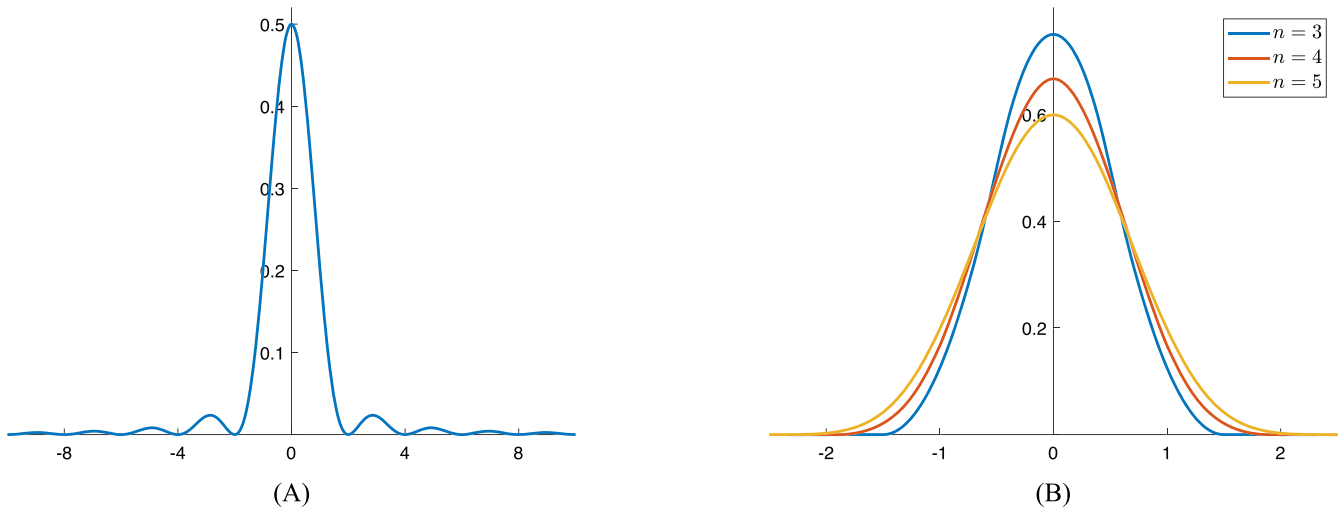
FIGURE 1 The graphs of some kernels. (A) Fejér kernel. (B) B-splines of orders 3, 4, and 5 (in blue, red, and orange, respectively). [Colour figure can be viewed at wileyonlinelibrary.com]

where sinc is the function defined by $\text{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t} & t \neq 0 \\ 1 & t = 0. \end{cases}$

The *B-spline of order* $m \in \mathbb{N}^+$ is the function M_m defined by

$$M_1(t) = \begin{cases} 1 & t \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0 & \text{otherwise}, \end{cases}$$

and

$$M_m(t) = \frac{1}{(m-1)!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \max\left(\frac{m}{2} + t - j, 0\right)^{m-1}, \qquad m \geq 2,$$

or, equivalently, by the recursive formula

$$M_m = M_{m-1} * M_1, \qquad m \geq 2,$$

where $*$ is the convolution product. In contrast to the Fejér kernel, the B-spline of order m has compact support (namely, $\left[-\frac{m}{2}, \frac{m}{2}\right]$). The graphs of some B-splines are displayed in Figure 1B.

3.2 | Szász-Mirak'jan operator

From (2) and with $I = [0, +\infty[$, $J = \mathbb{N}$, $v_{n,k} = \frac{k}{n}$ and $K_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}$, we get the *Szász-Mirak'jan operator*:

$$(\mathsf{S}_n f)(t) = \sum_{k=0}^{+\infty} f\left(\frac{k}{n}\right) e^{-nt} \frac{(nt)^k}{k!}, \quad f : [0, +\infty[\to \mathbb{R}, \ t \in [0, +\infty[.$$

Szász-Mirak'jan operator has been studied in [4, 5] and, with some modifications, also in [8-10,34,35].

The continuous functions $K_{n,k}$ satisfy the assumptions. Indeed, (4) follows by the power series of the exponential function and trivially implies (6) since $K_{n,k} > 0$ for every $n \in \mathbb{N}^+$ and $k \in \mathbb{N}$. Moreover, let $n \in \mathbb{N}^+$, U a compact of $[0, +\infty[$ and $\epsilon > 0$. There exists $0 \leq a < b$ such that $U \subset [a, b]$ and, since $K_{n,k}$ is strictly increasing on $\left[0, \frac{k}{n}\right]$, there exists $\tilde{k}_1$ such that

$$\sup_{t \in U} K_{n,k}(t) \leq \sup_{t \in [a,b]} K_{n,k}(t) = K_{n,k}(b) = e^{-nb} \frac{(nb)^k}{k!}, \ \text{ for all } k > \tilde{k}_1.$$

Furthermore, there exists $\tilde{k}_2$ such that $\sum_{k > \tilde{k}_2} e^{-nb} \frac{(nb)^k}{k!} < \epsilon$. Therefore, if $\tilde{k} = \max(\tilde{k}_1, \tilde{k}_2)$,

$$\sum_{k > \tilde{k}} K_{n,k}(t) \leq \sum_{k > \tilde{k}} e^{-nb} \frac{(nb)^k}{k!} < \epsilon, \ \text{for all } t \in U.$$

In conclusion, also (5) holds true. Finally, (7) is satisfied because the proof of [24, Corollary 2] shows that for $\delta > 0$

$$\sum_{k \in \mathbb{N}, |\nu_{n,k} - t| \geq \delta} K_{n,k}(t) = o(n^{-1}) \ \text{for} \ n \to +\infty.$$

3.3 | Baskakov operator

The choice $I = [0, +\infty[$, $J = \mathbb{N}$, $\nu_{n,k} = \frac{k}{n}$ and $K_{n,k}(t) = \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}$ leads to the *Baskakov operator* (introduced in [6], see also [10, 13, 36–38])

$$(\mathsf{B}_n f)(t) = \sum_{k=0}^{+\infty} f\left(\frac{k}{n}\right) \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}, \ f : [0, +\infty[\to \mathbb{R}, \ t \in [0, +\infty[.$$

Concerning the assumptions about the continuous functions $K_{n,k}$, (4) holds as consequence of the identity $\frac{1}{1-q} = \sum_{k=0}^{+\infty} q^k$, with $q = \frac{t}{1+t}$. Then, (6) is verified since $K_{n,k} > 0$ for every $n \in \mathbb{N}^+$ and $k \in \mathbb{N}$. To prove (5), one can follow steps similar to those made for Szász-Mirak'jan operators. Finally, in [24, Section 5.2], it is proved that for every $\delta > 0$ and $t \in I$

$$\sum_{k \in J, |\nu_{n,k} - t| \geq \delta} K_{n,k}(t)(\nu_{n,k} - t)^2 = o(n^{-2}) \ \text{as} \ n \to +\infty.$$

Thus, since $\delta^2 K_{n,k}(t) \leq K_{n,k}(t)(\nu_{n,k} - t)^2$ if $|\nu_{n,k} - t| \geq \delta$, the remaining assumption (7) holds too.

3.4 | Bernstein operator

If $I = [0, 1]$, $J = \{0, 1, \dots, n\}$, $\nu_{n,k} = \frac{k}{n}$ and $K_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$, we recover the *Bernstein operator* [7] (see also [16, 39–42] for related works)

$$(\mathcal{B}_n f)(t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} t^k (1-t)^{n-k}, \qquad f : [0, 1] \to \mathbb{R}, \ t \in [0, 1].$$
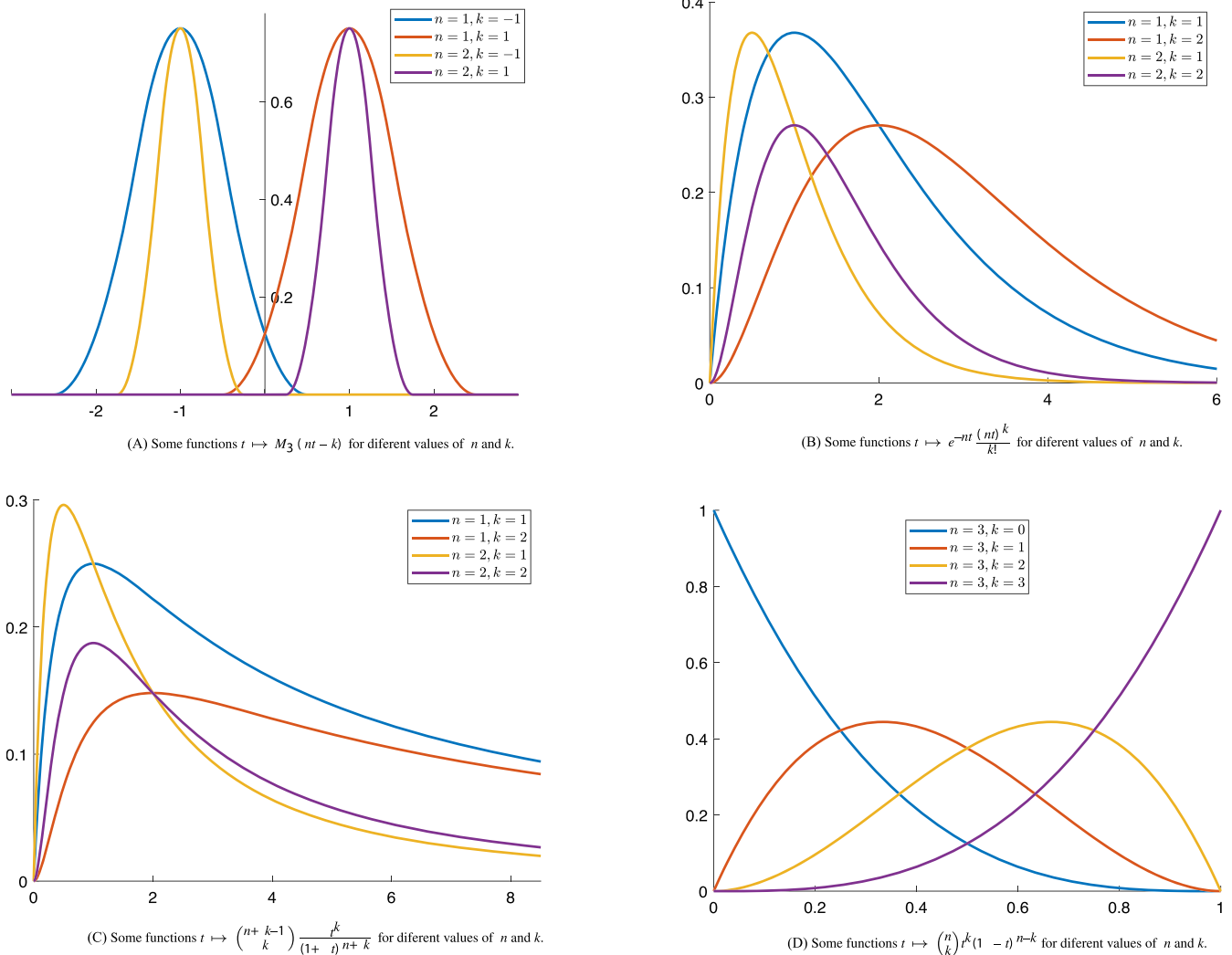
The assumption (4) can be easily checked, and (5)–(7) are trivially satisfied since J is finite and I is bounded. We also remark that if $\gamma : [0, 1] \to \mathbb{R}^d$ is a curve, then $\mathcal{B}_n \gamma$ is a so-called Bézier curve [21] with control points $\left\{ \gamma\left(\frac{k}{n}\right) \right\}_{k \in J}$.

Figure 2 shows the graphs of some elementary functions appearing in the formulations of the generalized sampling, Szász-Mirak'jan, Baskakov and Bernstein operators.

4 | EXAMPLES

In this section, we illustrate some graphical examples of the approximation of curves, as stated by Corollary 1, via the operators of type (2). We first remark that a curve with bounded domain can be approximated by the operator S_n defined by (2) even if the domain of the curve is not the interval I. To explain better this idea with some examples, let us assume that $\gamma : [a, b] \to \mathbb{R}^d$ is a bounded curve where $[a, b]$ is a bounded interval ($a < b$). We have to analyze different cases depending on the type of interval I.

If I is a bounded closed interval, then we can consider the new curve $\tilde{\gamma} : I \to \mathbb{R}^d$, $\tilde{\gamma} = \gamma \circ \sigma$, where $\sigma : I \to [a, b]$ is a bijective function (for instance, an affine transformation between the domains). Since $\tilde{\gamma}$ has domain I, we can approximate

(A) Some functions $t \mapsto M_3(nt - k)$ for different values of n and k.

(B) Some functions $t \mapsto e^{-nt} \frac{(nt)^k}{k!}$ for different values of n and k.

(C) Some functions $t \mapsto \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}$ for different values of n and k.

(D) Some functions $t \mapsto \binom{n}{k} t^k (1-t)^{n-k}$ for different values of n and k.

FIGURE 2 Each subfigure shows some elementary functions $K_{n,k}$ appearing in the expression of (A) generalized sampling operator with B-spline kernel of order 3; (B) Szász-Mirak'jan operator; (C) Baskakov operator; (D) Bernstein operator. [Colour figure can be viewed at wileyonlinelibrary.com]
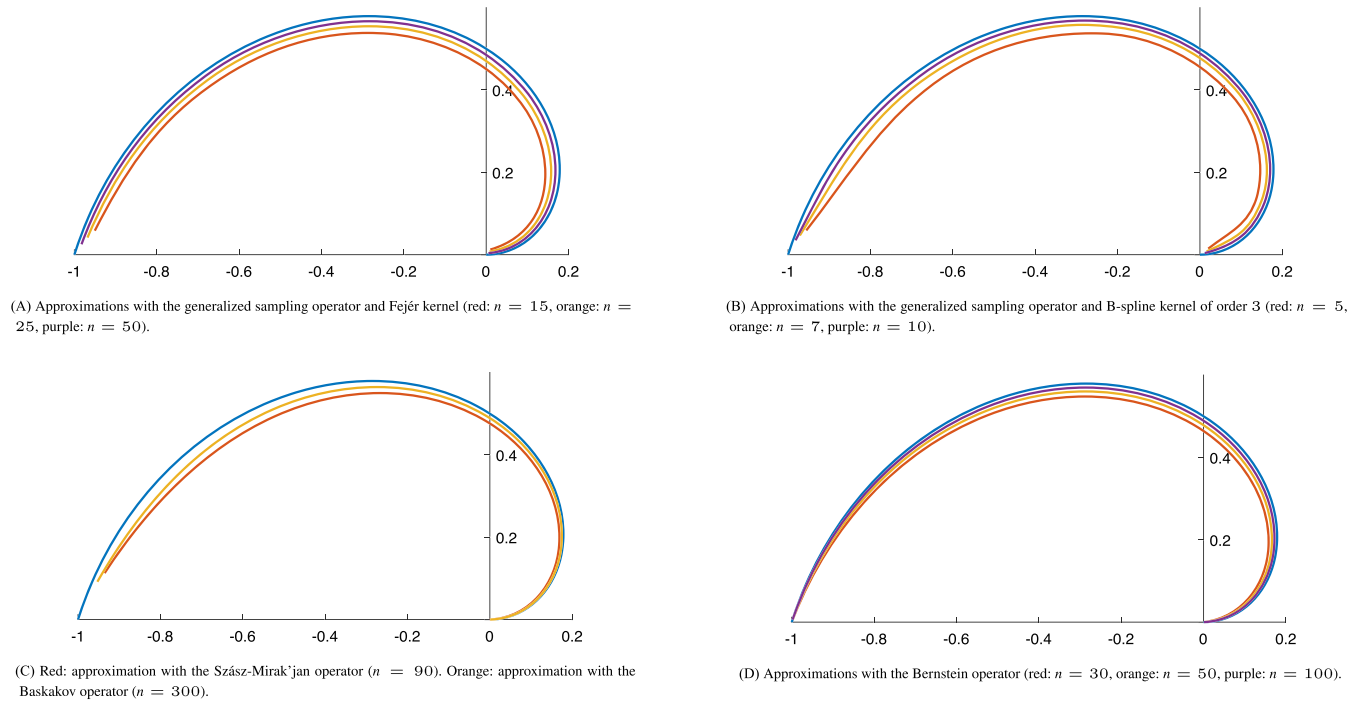
it by a curve $S_n\tilde{\gamma}$. Finally, to come back to the domain $[a, b]$, we take into account the inverse transformation σ^{-1}. In other words, we consider the curves $S_n\tilde{\gamma} \circ \sigma^{-1} = S_n(\gamma \circ \sigma) \circ \sigma^{-1}$ in order to approximate γ. The result follows by the fact that for every $t \in [a, b]$

$$\lim_{n \to +\infty} \left(S_n(\gamma \circ \sigma) \circ \sigma^{-1} \right)(t) = \lim_{n \to +\infty} (S_n(\gamma \circ \sigma))(\sigma^{-1}(t)) = (\gamma \circ \sigma)(\sigma^{-1}(t)) = \gamma(t).$$

This is exactly the case of Bernstein operators; in fact, $\mathcal{B}_n$ is defined for functions with domain $I = [0, 1]$, so to approximate a curve $\gamma : [a, b] \to \mathbb{R}^d$ we can take the curve $\mathcal{B}_n(\gamma \circ \sigma) \circ \sigma^{-1}$ where $\sigma : I \to [a, b]$ is given by $\sigma(s) = (1 - s)a + sb$.

If, instead, $I = \mathbb{R}$ (in particular, in the case of generalized sampling operators) we do not have necessity of transform $[a, b]$. Indeed, since $[a, b]$ is a subset of $\mathbb{R}$, we can simply extend the curve γ to a new curve $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^d$ setting

$$\tilde{\gamma}(t) = \begin{cases} \gamma(a) & t < a \\ \gamma(t) & t \in [a, b] \\ \gamma(b) & t > b. \end{cases} \tag{8}$$

(A) Approximations with the generalized sampling operator and Fejér kernel (red: $n = 15$, orange: $n = 25$, purple: $n = 50$).

(B) Approximations with the generalized sampling operator and B-spline kernel of order 3 (red: $n = 5$, orange: $n = 7$, purple: $n = 10$).

(C) Red: approximation with the Szász-Mirak'jan operator ($n = 90$). Orange: approximation with the Baskakov operator ($n = 300$).

(D) Approximations with the Bernstein operator (red: $n = 30$, orange: $n = 50$, purple: $n = 100$).

FIGURE 3 In each subfigures, the open curve in blue is $\gamma : [0, 1] \to \mathbb{R}^2$, $\gamma(t) = (t\cos(\pi t), t\sin(\pi t))$. The curves in red, orange, and purple are some approximations given by different operators. [Colour figure can be viewed at wileyonlinelibrary.com]

Thus, to have an approximation of γ we can simply take an approximation $S_n\tilde{\gamma}$ of $\tilde{\gamma}$ and then a restriction of it in the interval $[a, b]$. Note that we have extended γ as in (8) to have a continuous function (so a curve) and then to apply Corollary 1 for every point of $\mathbb{R}$.

Finally, let us consider $I = [0, +\infty[$ as in the case of Szász-Mirak'jan or Baskakov operators. First of all, $[a, b]$ is not necessarily a subset of I, so we can appeal to the translation $\sigma : [0, b - a] \to [a, b]$, $\sigma(s) = s + a$, and secondly, we need an extension to the interval $[0, +\infty[$. Summarizing, we define the curve

$$\tilde{\gamma}(t) = \begin{cases} \gamma \circ \sigma(t) = \gamma(t + a) & t \in [0, b - a] \\ \gamma(b) & t > b - a, \end{cases} \tag{9}$$

we approximate it with $S_n\tilde{\gamma}$, we take the restriction $R(S_n\tilde{\gamma})$ of $S_n\tilde{\gamma}$ on the interval $[0, b - a]$ and, coming back to the original domain exploiting σ^{-1}, we are able to approximate γ with the curve $R(S_n\tilde{\gamma}) \circ \sigma^{-1}$.

As example, in Figure 3 we consider a curve of the real plane (in blue) and its approximations by the operators of Section 3 for different values of the parameter n. In particular, we apply the method explained above in the cases of generalized sampling, Szász-Mirak'jan and Baskakov operators. In Figure 4 we show a curve in the tridimensional space and an approximation by the generalized sampling operator (the kernel chosen is the B-spline of order 3 and n is equal to 10).

If $\gamma : [0, 1] \to \mathbb{R}^d$ is, in particular, a closed curve (i.e., $\gamma(0) = \gamma(1)$ is satisfied), we would like approximations of γ which are closed too. For this end, we extend γ not as in (8), but periodically, that is, we define

$$\tilde{\gamma}(t) = \gamma(t - m), \quad \text{for every } t \in \mathbb{R}, m \in \mathbb{Z} \text{ such that } t - m \in [0, 1]. \tag{10}$$

Hence, the approximation $S_n^\chi \tilde{\gamma}$ with the generalized sampling operator is a closed curve as a consequence of the following result.

Proposition 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and periodic function with period 1. Then, for every kernel χ and every $n \in \mathbb{N}^+$, the function $S_n^\chi f$ is periodic with period 1.*
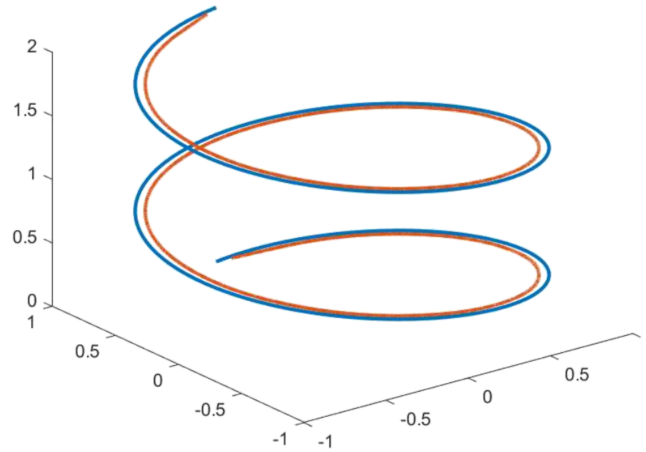
FIGURE 4   The curve in blue is $\gamma : [0,2] \to \mathbb{R}^3$, $\gamma(t) = (\cos(2\pi t), \sin(2\pi t), t)$. The curve in red is the approximation given by the generalized sampling operator with B-spline kernel of order 3 ($n = 10$). [Colour figure can be viewed at wileyonlinelibrary.com]

Proof. Let χ be a kernel and $n \in \mathbb{N}^+$. For every $t \in \mathbb{R}$ and $m \in \mathbb{Z}$, we have

$$\left(S_n^\chi f\right)(t - m) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{n}\right) \chi(n(t-m) - k)$$

$$= \sum_{k \in \mathbb{Z}} f\left(\frac{k}{n}\right) \chi(nt - nm - k)$$

$$= \sum_{k \in \mathbb{Z}} f\left(\frac{k + nm}{n}\right) \chi(nt - nm - k)$$

where in the last equality, we used the periodicity of f. Thus, with a change $k' = k + nm$, we can write

$$\left(S_n^\chi f\right)(t - m) = \sum_{k' \in \mathbb{Z}} f\left(\frac{k'}{n}\right) \chi(nt - k') = \left(S_n^\chi f\right)(t),$$

that is, $S_n^\chi f$ is periodic with period 1. $\qquad\square$

We remark that if $\gamma : [0,1] \to \mathbb{R}^d$ is a closed curve, then also the approximations $\mathcal{B}_n \gamma$ by the Bernstein operator are closed curve, because $(\mathcal{B}_n \gamma)(0) = \gamma(0) = \gamma(1) = (\mathcal{B}_n \gamma)(1)$. In Figure 5, a closed curve of the real plane and some approximations by the generalized sampling operator (making use of Proposition 2) and the Bernstein operator are presented.

5 | APPLICATIONS

The approximation of curves by discrete operators (2) lends itself to some applications about computer graphics and image processing. In these fields, several types of curves have found utility for problems of approximations, for instance, trigonometric polynomials [43], Bezier curves, splines and rational Bezier curves [21], and NURBS curves [44]. Coming back to the general operators (2), we want to discuss some direct consequences of Corollary 1.

To begin with, we recall that a gray-scale image is, under the mathematical point of view, a matrix containing in each entry (*pixel*) the corresponding level gray. Curves in images (like, for instance, object contours or sharp transitions of gray levels) can be considered as discretization[1] of curves in the real plane. In particular, if we are interested only to a curve itself, then we can represent it in a *binary* image, that is, a matrix with values 0 (for the background) and 1 (for the curve). In this paper, we consider only image curves that are discretization of curves that does not intersect themselves. Figure 6

[1]For discretization, we mean the process that round off the coordinates values towards the nearest integer.
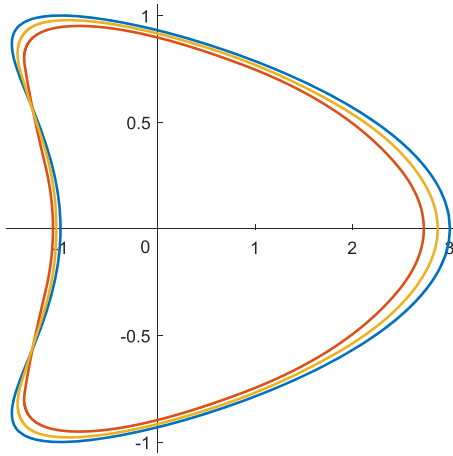
(A) Approximations with the generalized sampling operator and B-spline kernel of order 3 (red: $n = 10$, orange: $n = 15$).
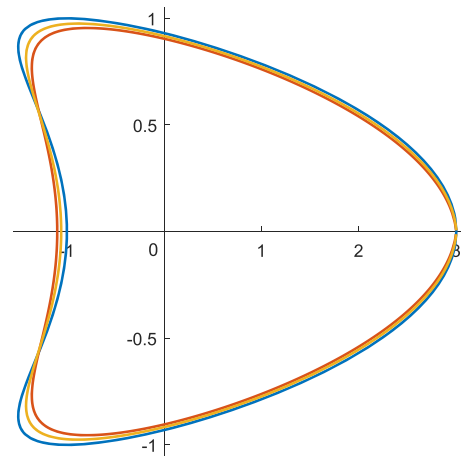
(B) Approximations with the Bernstein operator (red: $n = 80$, orange: $n = 150$).

FIGURE 5 In each subfigures the closed curve in blue is $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ defined by $\gamma(t) = (\cos(4\pi t) + 2\cos(2\pi t), \sin(2\pi t))$. The curves in red, orange, and purple are some approximations given by different operators. [Colour figure can be viewed at wileyonlinelibrary.com]
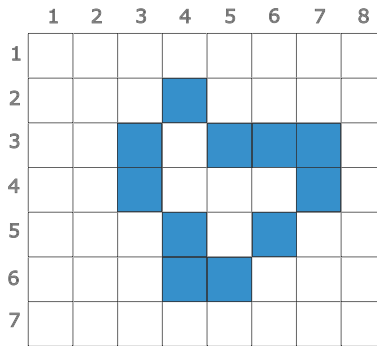
FIGURE 6 An example of a curve in an image. [Colour figure can be viewed at wileyonlinelibrary.com]

shows a binary image with value 0 in the entries represented by a white square and value 1 in the entries represented by a colored square. The pixels with value equals to 1 constitute a closed curve.

5.1 │ Approximations of curves in images

The approximation method of curves in images by means of an operator S_n of type (2) consists in the extraction of the coordinates of the curve points, from which two continuous functions $x_1 : I \rightarrow \mathbb{R}$ and $x_2 : I \rightarrow \mathbb{R}$ can be defined and then in the approximation of x_1 and x_2 in terms of S_n. By Corollary 1, the curve $S_n\gamma = (S_n x_1, S_n x_2)$ is, for n large enough, an approximation of the curve $\gamma = (x_1, x_2)$, and as consequence, it determines an approximation of the image curve via discretization.

With the help of Figure 7, we give more details, confining to the case of closed curves (the other case, open curves, can be treated with little and intuitive changes). First of all, Figure 7A shows a pixel, the eight directions to the nearest neighborhoods and, following [45] (see also [43]), the ordering from 0 to 7 of them. Figure 7B contains the same curve of Figure 6 (the square colored in red is the starting and final point of the curve which is, by our convention, the point of the contour with maximum ordinate and minimum abscissa). Based on the ordering of Figure 7A, a path that runs through the curve from and to the initial point can be created as in Figure 7B (among the possible directions, the chosen one is represented by the lowest number).

Collecting the labels of the directions as a sequence, we obtain the so-called *chain code* [43]. Anyway, for our applications, we are actually interested to the sequences of abscissas and ordinates.[2] For example, the curve represented

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[2]For a point of coordinates (i, j) of an image curve, we call j and i the abscissa and the ordinate of the point, respectively. Note also that we enumerate the ordinates from above to bottom (see Figure 7B) as it usually done for matrices, that is, for the representation of images.

(A)   (B)
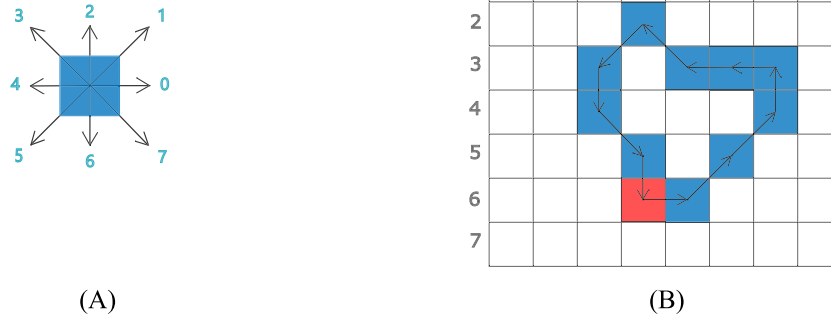
FIGURE 7 (A) The eight directions from a pixel to the nearest neighborhoods and the corresponding ordering as defined in [45]. (B) An image representing a closed curve with starting and final point in red. The path indicated by the arrows in the second figure is obtained starting from the red pixel and moving to a nearest neighborhood following the ordering of the directions. [Colour figure can be viewed at wileyonlinelibrary.com]
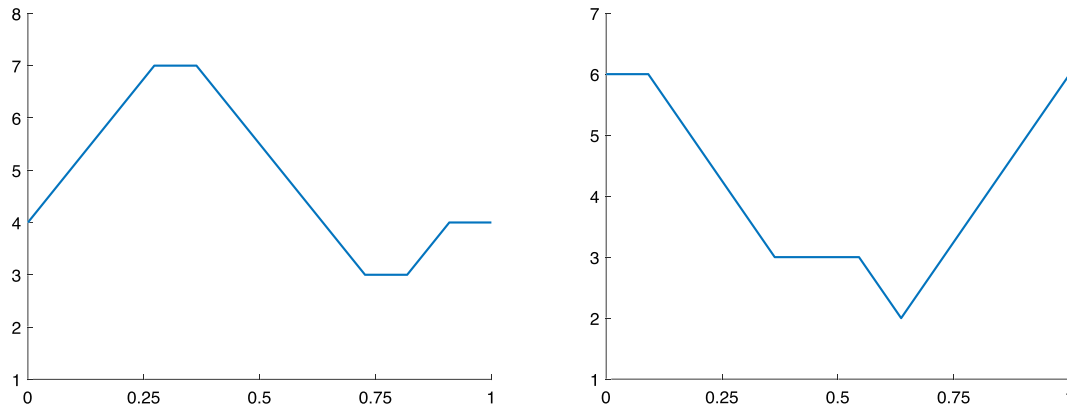
FIGURE 8 The piece-wise *linear* functions determined by the abscissas and ordinates of the curve of Figure 7B on the left and on the right, respectively. [Colour figure can be viewed at wileyonlinelibrary.com]

in Figure 7B, and with starting point in red, has chain code $c = (0, 1, 1, 2, 4, 4, 3, 5, 6, 7, 6)$, sequence of abscissas $u = (4, 5, 6, 7, 7, 6, 5, 4, 3, 3, 4)$ and sequence of ordinates $v = (6, 6, 5, 4, 3, 3, 3, 2, 3, 4, 5)$.

Once the sequences of abscissas $u = (u_j)_{j=1}^N$ and of ordinates $v = (v_j)_{j=1}^N$ are determined, we define a piece-wise linear function $x_1 : [0, 1] \rightarrow \mathbb{R}$ associated to u as

$$\begin{cases} x_1(t) = (j - Nt)(u_j - u_{j+1}) + u_{j+1} & \text{if } t \in \left[\frac{j-1}{N}, \frac{j}{N}\right[\text{ and } 1 \leq j \leq N-1, \\ x_1(t) = N(1-t)(u_N - u_1) + u_1 & \text{if } t \in \left[\frac{N-1}{N}, 1\right] \end{cases} \tag{11}$$

and, similarly, a function $x_2 : [0, 1] \rightarrow \mathbb{R}$ associated to v. In Figure 8, we show the functions x_1 and x_2 for the example of Figure 7B.

Next, we approximate x_1 and x_2 by means of an operator S_n for some $n \in \mathbb{N}^+$; that is, we consider $S_n x_1$ and $S_n x_2$ (we proceed, in particular, as explained in Section 4 if S_n is not defined for functions with domain $[0, 1]$). Finally, putting what we need together, the curve $(S_n x_1, S_n x_2)$ constitutes an approximation of (x_1, x_2) and hence of the image curve after that the discretization is made.

The entire process is illustrated in Figure 9 with an image (shown in (A)) containing a curve with an higher number of points. In particular, following the steps above, we extracted the point coordinates according to the path made by the direction ordering and we defined the abscissas and ordinates functions x_1 and x_2, which are displayed in blue in Figure 9C–D. We employed the generalized sampling operator S_n^F with Fejér kernel and $n = 100$ to approximate x_1 and
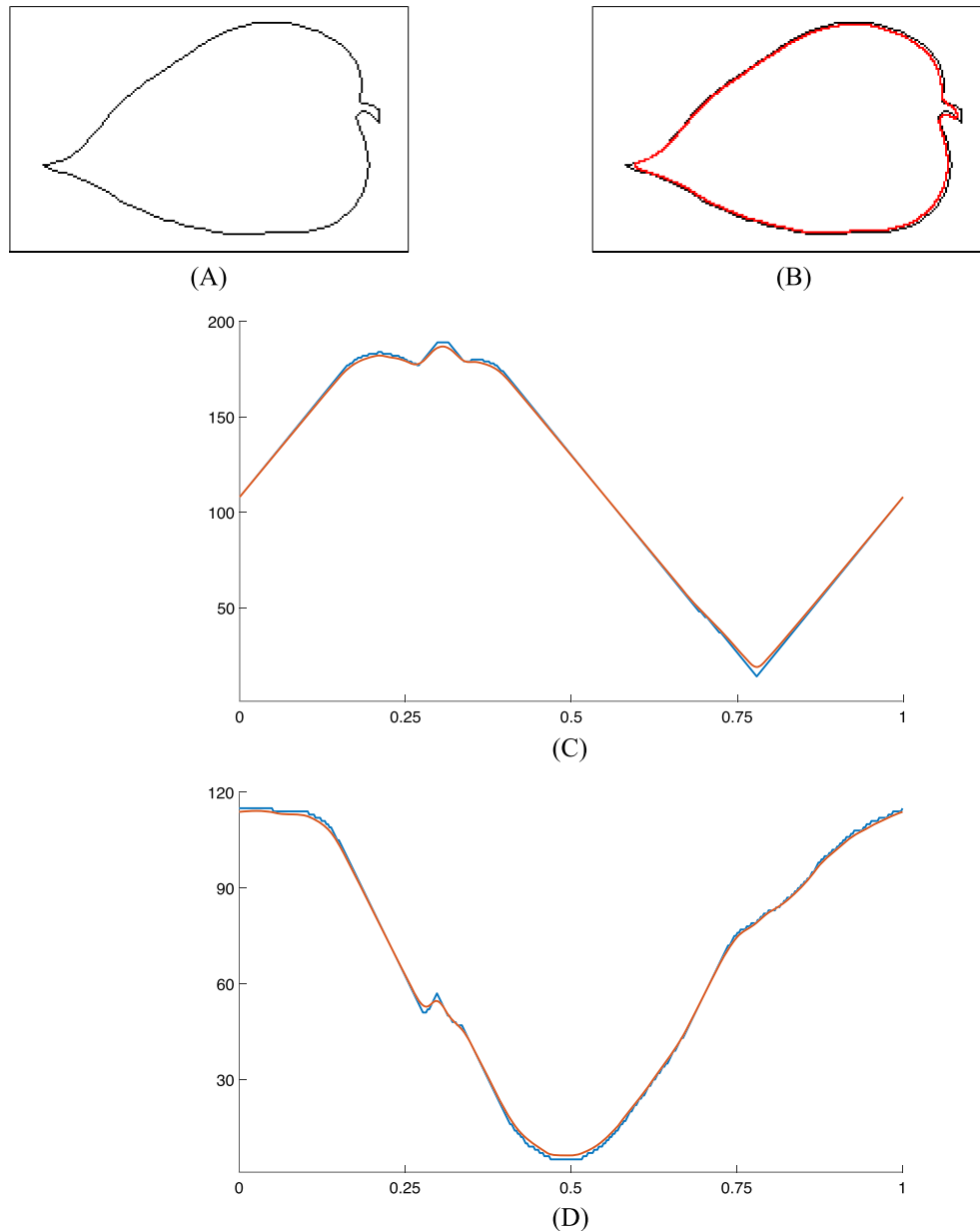
FIGURE 9 An image with a curve is shown in (A) (the resolution of the image is 200×120). The functions x_1 and x_2 giving the abscissas and ordinates of the points of the curve are plotted, in blue, in (C) and (D), respectively. The functions in red in (C) and (D) are, instead, the corresponding approximations $S_{100}^F x_1$ and $S_{100}^F x_2$ obtained by the generalized sampling operator S_{100}^F with Fejér kernel. The discretization of the curve $(S_{100}^F x_1, S_{100}^F x_2)$ is shown, in red color, in (B) and is overlaid on the original curve. [Colour figure can be viewed at wileyonlinelibrary.com]

x_2 (the functions $S_n^F x_1$ and $S_n^F x_2$ are shown in Figure 9C–D in red). Figure 9B is an image containing the original curve in black and the approximating curve $\left(S_n^F x_1, S_n^F x_2\right)$ in red after discretization.

Remark 1. A variation of the method consists in defining x_1 and x_2 as piece-wise constant functions. More precisely, if $u = (u_j)_{j=1}^N$ is the sequence of abscissas of a closed curve, then we define $x_1 : [0, 1] \to \mathbb{R}$ associated to u as

$$x_1(t) = u_j \text{ if } t \in \left[\frac{j-1}{N}, \frac{j}{N}\right[\text{ and } 1 \le j \le N-1, \qquad x_1(t) = u_N \text{ if } t \in \left[\frac{N-1}{N}, 1\right]$$

and x_2 in a similar way in terms of $v = (v_j)_{j=1}^N$ (Figure 10 shows the new cases for the same example of Figure 7B). As before, we consider $S_n x_1$ and $S_n x_2$, with n large enough, as approximations of x_1 and x_2, respectively. Anyway,
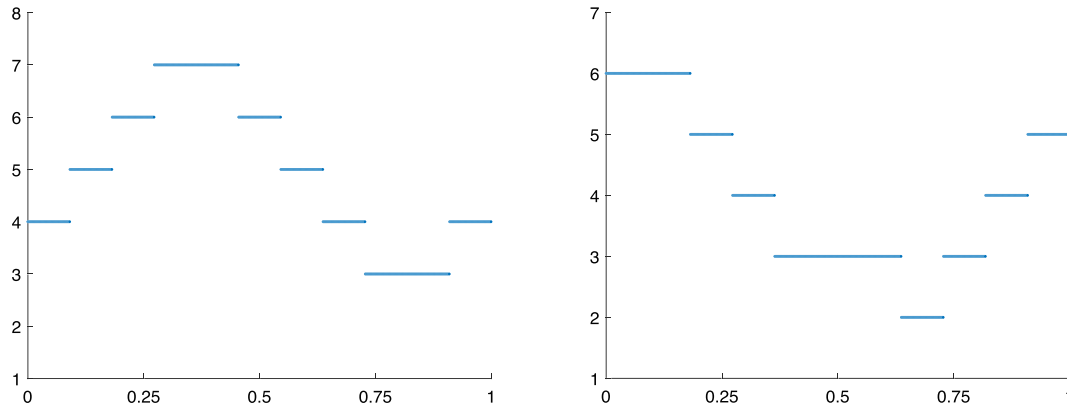
FIGURE 10 The piece-wise *constant* functions of abscissas and ordinates of the curve of Figure 7B on the left and on the right, respectively. [Colour figure can be viewed at wileyonlinelibrary.com]
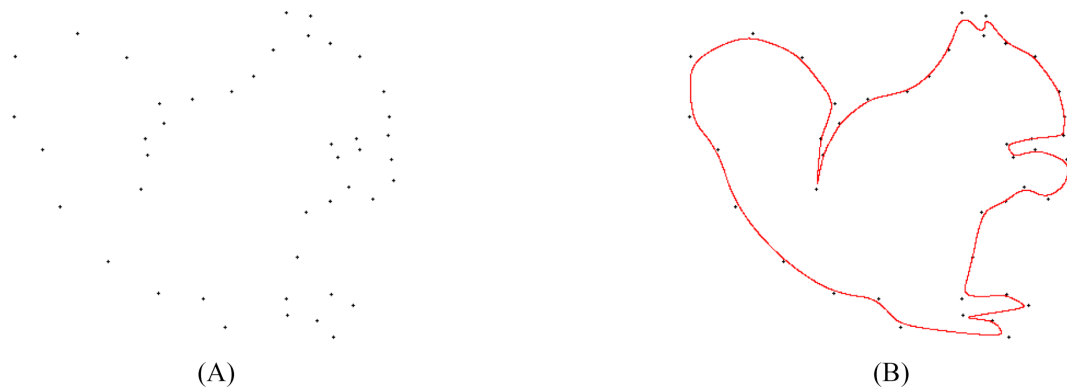
FIGURE 11 Approximation of a curve from a sequence of few points. (A) A set of points (a given ordering is also defined). (B) The same set of points and a curve created by applying a generalized sampling operator. [Colour figure can be viewed at wileyonlinelibrary.com]

since in this case x_1 and x_2 are piece-wise constant, according to Corollary 1 the approximation holds point-wise except in a finite number of points, of the form $\frac{j}{N}$. Consequently, this happens also for the convergence of $S_n\gamma(t) = ((S_n x_1)(t), (S_n x_2)(t))$ to $(x_1(t), x_2(t))$ (note that, even though x_1 and x_2 are piece-wise constant, $S_n\gamma : [0,1] \rightarrow \mathbb{R}^2$ is still a curve by Corollary 1). In particular, the values $\lim_{n\to+\infty}(S_n x_1)\left(\frac{j}{N}\right)$ and $\lim_{n\to+\infty}(S_n x_2)\left(\frac{j}{N}\right)$ may differ from the values of x_1 and x_2 in a neighborhood of $\frac{j}{N}$. By the way, this does not necessarily constitute an issue for the application. Indeed, there are some discrete operators S_n with the following properties: if t_0 is a jump discontinuity for a function f, then $S_n f(t_0)$ converges, as $n \rightarrow +\infty$, to an intermediate value between $\lim_{t\to t_0^+} f(t)$ and $\lim_{t\to t_0^-} f(t)$ (an example is the generalized sampling operator with some hypothesis about the kernel, see [46, Theorem 2]). Thus, considering these operators and since the jumps of x_1 are equal to 1, the value of $\lim_{n\to+\infty}(S_n x_1)\left(\frac{j}{N}\right)$ after rounding off becomes u_j or u_{j+1}. A similar statement holds for $S_n x_2$. In conclusion, defining x_1 and x_2 as piece-wise constant (and not piece-wise linear) functions, the error between the image curve and the approximation curve for n large enough is, at most, of one pixel of distance.

Following the same idea of approximating the coordinates functions, another application can be made: the approximation of a curve from few given points. Let us consider the problem, as in Figure 11A, of having only few points of a curve and, in addition, an ordering of them (which follows the path of the original contour). Even though the points are separate, a piece-wise linear curve can be defined connecting the points in the given ordering. The abscissa and ordinate functions for such a possible curve can be still defined as in (11). The piece-wise linear curve may give a good representation of the original contour, but it may be at the same time undesired, since it is not smooth. Thus, we can consider an approximation operator S_n, so that $(S_n x_1, S_n x_2)$, for n large, determines a curve approximating the piece-wise linear contour and more regular (like in Figure 11B).

(A)

(B) (C)
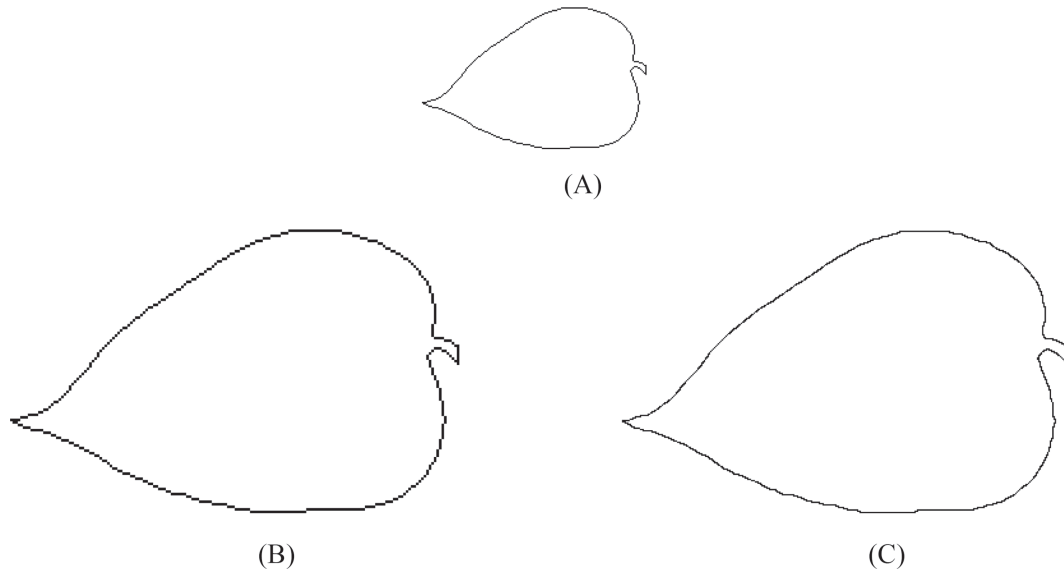
FIGURE 12 In (A): the curve in a image. In (B): the image in (A) with doubled scale. In (C): the curve in (A) reconstructed by a generalized sampling operator and dilated (the image has the same scale of (A) but doubled resolution). [Colour figure can be viewed at wileyonlinelibrary.com]

5.2 | Reconstruction of curves in images

The technique described in the previous section leads to move from a discrete object (the curve in an image) to a continuous object (the curve $S_n\gamma$). This allows to make some operations on the continuous object that give better results in comparison to working with the discrete initial object. For example, once we approximate γ with $S_n\gamma$, we can dilate $S_n\gamma$ and then convert it to a curve in an image by discretization. The result is a larger image which is not a simple scaling of the original image. Speaking in the imaging language, the operator S_n can be also used to reconstruct the curves in images with an higher resolution.

In Figure 12, we show the results of this procedure. The curve γ of the same image case of Figure 9 is shown in Figure 12A. The resolution of the image is 200×120. The image scaled by a factor equals to 2 is displayed in Figure 12B (the effect is that of a zoom, the resolution is not changed). The generalized sampling operator $S_n^{M_3}$ with B-spline kernel of order 3 and $n = 200$ was applied to approximate γ as explained in the previous section. The obtained curve $S_n^{M_3}\gamma$ is multiplied by 2 and then discretized (i.e., the values of the coordinates are rounded off towards the nearest integer). The final result is in Figure 12C, and by effect of the multiplication, the curve is contained in an image with increased resolution (400×240). Note the difference of quality of the images from a comparison between Figures 12B and 12C.

Other operations that can be performed after the approximating curve $S_n\gamma$ is obtained are, for examples, translations, rotations or dilations along specific directions, projections. Working on an auxiliary curve of the real plane as $S_n\gamma$, defined by a continuous and not discrete variable, allows to make operations with better results.

AUTHOR CONTRIBUTIONS

Rosario Corso: Conceptualization; methodology; software; investigation; formal analysis; supervision; visualization; writing—original draft; writing—review and editing. **Gabriele Gucciardi:** Formal analysis; investigation; software.

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CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

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