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ORIGINAL PAPER

Mean sampling Kantorovich operators: approximation results and applications

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Abstract
As well-known, generalized sampling operators and sampling Kantorovich operators are able to approximate continuous signals and even L^p-signals in the latter case. Anyway, in the situation of a signal affected by noise, these operators are not very efficient to approximate the original signal (i.e., filtered by the noise) when the parameter goes to infinity. In order to solve this problem, we introduce a new type of operators, which we call the mean sampling Kantorovich operators. We study its approximation properties and made a comparison with the classical sampling Kantorovich operator in dealing with noisy signals.

Keywords Approximation theory · Sampling Kantorovich operators · Order of approximation · Mean · Noise

Mathematics Subject Classification 94A20 · 41A35 · 94A08

1 Introduction

1.1 Sampling-type operators and noisy signals

Generalized sampling operators were introduced and extensively studied in [2, 7–9, 11, 13–16, 21, 24–29, 31, 34] to extend the classical sampling theory based on the celebrated Whittaker–Kotelnikov–Shannon theorem [30], and one of their relevant properties is the approximation of continuous functions. Their form is the following:

$$(G_w^\chi f)(x) = \sum_{k\in\mathbb{Z}} \chi(wx-k) f\left(\frac{k}{w}\right), \qquad x \in \mathbb{R}, \tag{1.1}$$

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where $w > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a bounded function, $\chi : \mathbb{R} \to \mathbb{R}$ is a continuous kernel, as precised in Sect. 2. These operators are very sensitive to the behavior of the signal on the nodes $\frac{k}{w}$, i.e., they depend on single function values $f(\frac{k}{w})$. Moreover, real world signals are often affected by noise/interferences caused, for example, by the so-called time-jitter error, which occurs since in general it is not possible to pick up the signal exactly at the node $\frac{k}{w}$; therefore the generalized sampling operators are not very suitable to approximate f, because it would mean reconstructing f using noisy samples and this becomes even more evident if the signal is very noisy. Moreover, if the function/signal is not continuous, but e.g., integrable, these operators are again not suitable since they do not map $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$, i.e., they are not continuous, as operators, on this setting (see [6]).

Some decades after the introduction of the generalized sampling operators, new operators were introduced in [6] and a wide research line has been made since then. They are called *sampling Kantorovich operators* and their definition is given by

$$(S_w^\chi f)(x) = \sum_{k\in\mathbb{Z}} \chi(wx - t_k) \left[\frac{w}{\Delta_k} \int_{\frac{t_k}{w}}^{\frac{t_{k+1}}{w}} f(u)\,\mathrm{d}u \right], \tag{1.2}$$

where $w > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a locally integrable function (signal), $\chi : \mathbb{R} \to \mathbb{R}$ is a kernel function satisfying suitable properties, $\Pi := \{t_k\}_{k\in\mathbb{Z}}$ is a sequence of nodes, as specified in Sect. 2, and $\Delta_k := t_{k+1} - t_k$. For references on these operators, the reader can see e.g., [1, 3, 5, 17, 20, 22, 23], while for their various applications, see e.g., [18, 19, 32, 33].

In comparison with (1.1), the coefficients in (1.2) are the means $\displaystyle\frac{w}{\Delta_k}\int_{\frac{t_k}{w}}^{\frac{t_{k+1}}{w}} f(u)\,\mathrm{d}u$ of f on the intervals $\left[\frac{t_k}{w}, \frac{t_{k+1}}{w}\right]$, and this allows to reduce the time-jitter error since the average acts as a smoothing filter. Also the operators S_w^χ are approximation operators for w tending to $+\infty$, that is

$$\lim_{w\to+\infty} (S_w^\chi f)(x) = f(x), \tag{1.3}$$

at the points x of continuity for f, and the above convergence becomes uniform in case of a bounded and uniform uniformly continuous signal f. Furthermore, if $f \in L^p(\mathbb{R})$, $p \geq 1$, then $S_w^\chi f$ converges to f also with respect to the L^p norm (see e.g., [6]).

Now, in order to exploit the above properties, we need to take w sufficiently large and this implies that the measure of the intervals $\left[\frac{t_k}{w}, \frac{t_{k+1}}{w}\right]$ is small and so the noise reduction on the coefficients is not effective; furthermore in applications f is discretized, so the mean of f on an interval of small measure $\left[\frac{t_k}{w}, \frac{t_{k+1}}{w}\right]$ is calculated by the mean of few values of f, not sufficient to attenuate the noise.

In addition to this issue on the coefficients, the convergence in (1.3), for w sufficiently large, is towards $f(x)$, which means that if a signal f is affected by the noise, then $S_w^\chi f$ approximates f and not the signal unaffected by noise.

1.2 The new definition

To deal with these problems and inspired by (1.2), we introduce in this paper, for a kernel function χ and $w_1, w_2 > 0$, the operators S_{w_1,w_2}^χ defined by

$$(S_{w_1,w_2}^\chi f)(x) = \sum_{k\in\mathbb{Z}} \chi(w_1 x - t_k) \left[w_2 \int_{\frac{t_k}{w_1} - \frac{1}{2w_2}}^{\frac{t_k}{w_1} + \frac{1}{2w_2}} f(u)\,\mathrm{d}u \right], \tag{1.4}$$

for a locally integrable function $f : \mathbb{R} \to \mathbb{R}$ such that the series (1.4) is convergent for each $x \in \mathbb{R}$.

The above operators are here called *Mean sampling Kantorovich operators*.

The peculiarity in (1.4) is the presence of two independent parameters $w_1, w_2 > 0$. The role of w_1 is related to the convergence, while w_2 regulates the width of the intervals: indeed, the length of $[\frac{t_k}{w_1} - \frac{1}{2w_2}, \frac{t_k}{w_1} + \frac{1}{2w_2}]$ is w_2^{-1}. Hence, the coefficients in (1.4), which are multiplied by the kernel, are the means of f in these intervals; therefore if w_2 is sufficiently small, this means that the average is calculated in a range of non-negligible amplitude. Now, averaging means applying a smoothing filter, which implies that if the range where the average is applied has "significant" amplitude, the averaged signal and therefore the coefficients are cleaned from the noise, which is exactly what we want to have. Since the width does not depend on w_1, we do not incur into the same problem encountered for the sampling Kantorovich operators, when $w_1 \to +\infty$. Moreover, we highlight that the Mean sampling Kantorovich operators contains, as a particular case, the sampling Kantorovich operators with the mean centered in the node $\frac{t_k}{w}$ (see Remark 2.3).

1.3 Main contributions

Having explained the motivation underlying the introduction of the new operators, we now talk about the study. The main results in the paper concern the convergence of $S^{\chi}_{w_1,w_2} f$ towards $f * \rho_{w_2}$, as $w_1 \to +\infty$, both with respect to the supremum norm and to the L^p norms, $p \geq 1$. As Theorems 3.2 and 3.8 state, the convergence is not to f, but to the convolution $f * \rho_{w_2}$ between f and the function ρ_{w_2}, defined by

$$\rho_{w_2}(x) = \begin{cases} w_2, & -\frac{1}{2w_2} \leq x \leq \frac{1}{2w_2} \\ 0, & \text{otherwise.} \end{cases}$$

Explicitly, $(f * \rho_{w_2})(x) = w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u$, so it is an average of f on an interval of width w_2^{-1}. The fact that the convergence is to this function and not to f, represents a strong benefit for the problem of approximating functions/signals when they are affected by noise. And in fact, the interest is certainly not to reconstruct the signal affected by noise but rather to reconstruct the signal cleaned from the noise. As better detailed in Sect. 4, when a noise, following a distribution with zero mean, linearly overlaps a continuous signal, then the convolution of the noisy function with ρ_{w_2} gives a reasonable approximation of the original signal (unaffected by noise).

The other main contributions in the paper are some quantitative and qualitative estimates of the approximation error both in uniform and L^p norm. As well-known, these types of estimates are particularly useful to evaluate the approximation and the rate of convergence.

1.4 Article structure

In Sect. 2 we introduce some preliminaries and notations, talking about the assumption on the kernel functions and giving some classical examples. Besides the proofs of Theorems 3.2 and 3.8, Sect. 3 contains other results as continuity, estimate of the operator norm, quantitative estimates, order of approximation, related to the introduced operators (Mean sampling Kantorovich operators) and examples. In Sect. 4, we talk about the applications of the operators $S^{\chi}_{w_1,w_2}$, discussing their ability in reconstructing the signal not affected by noise, through numerical estimates of the approximation error with respect to different norms and showing how these operators are more performing compared to the sampling Kantorovich operators. All this, is also shown through graphical examples and examining different types of noise.

2 Notations and preliminaries

Throughout the paper we denote by $L^{\infty}(\mathbb{R})$ and $L^p(\mathbb{R})$, $p \geq 1$, the usual spaces of measurable functions with norms $\|\cdot\|_{\infty}$ and $\|\cdot\|_p$, respectively. Moreover, we denote by $C(\mathbb{R})$ the space of bounded and uniformly continuous functions on $\mathbb{R}$ and by $C_c(\mathbb{R})$ the space of bounded and compactly supported functions $f : \mathbb{R} \to \mathbb{R}$.

As concerns $\Pi := \{t_k\}_{k\in\mathbb{Z}}$, it is a sequence of real numbers such that $-\infty < t_k < t_{k+1} < +\infty$ for every $k \in \mathbb{Z}$, $\lim\limits_{k\to\pm\infty} t_k = \pm\infty$ and there exist $\underline{\Delta}, \overline{\Delta} > 0$ satisfying $\underline{\Delta} \leq \Delta_k := t_{k+1} - t_k \leq \overline{\Delta}$ for every $k \in \mathbb{Z}$, that is $\Pi := \{t_k\}_{k\in\mathbb{Z}}$ is a not (necessarily) uniformly distributed sequence.

We now give the assumptions and basic properties on the kernel of the operators we will deal with, whose definition is recalled below.

Definition 2.1 [6] A function $\chi : \mathbb{R} \to \mathbb{R}$ is called a *kernel* if $\chi \in L^1(\mathbb{R})$, χ is bounded in a neighbourhood of the origin and the following conditions are satisfied:

(i) for every $u \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \chi(u - t_k) = 1; \tag{2.1}$$

(ii) for some $\beta > 0$, the absolute moment of order β is finite, i.e.,

$$m_{\beta,\Pi}(\chi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u - t_k)||u - t_k|^{\beta} < +\infty. \tag{2.2}$$

We now introduce the Mean sampling Kantorovich operator as follows.

Definition 2.2 Let χ be a kernel and $w_1, w_2 > 0$. The operator $S^{\chi}_{w_1,w_2}$ defined by

$$(S^{\chi}_{w_1,w_2} f)(x) = \sum_{k\in\mathbb{Z}} \chi(w_1 x - t_k) \left[w_2 \int_{\frac{t_k}{w_1} - \frac{1}{2w_2}}^{\frac{t_k}{w_1} + \frac{1}{2w_2}} f(u)\, \mathrm{d}u \right], \tag{2.3}$$

for a locally integrable function $f : \mathbb{R} \to \mathbb{R}$ such that the series (2.3) is convergent for each $x \in \mathbb{R}$, is called *Mean sampling Kantorovich operator*.

Remark 2.3 Note that if $\Pi = \{t_k\}_{k\in\mathbb{Z}}$ is uniformly distributed, i.e. $\Delta = t_{k+1} - t_k$ for every $k \in \mathbb{Z}$, then taking $w_1 = w$ and $w_2 = \frac{w}{\Delta}$ in (2.3) we recover a sampling Kantorovich operator with the mean centered in the node $\frac{t_k}{w}$, i.e.,

$$(S^{\chi}_{w,\frac{w}{\Delta}} f)(x) = \sum_{k\in\mathbb{Z}} \chi(wx - t_k) \left[\frac{w}{\Delta} \int_{\frac{t_k}{w} - \frac{\Delta}{2w}}^{\frac{t_k}{w} + \frac{\Delta}{2w}} f(u)\, \mathrm{d}u \right].$$

Concerning the above assumption (2.1), we remark that in the case of equally spaced sequence $t_k = k$, $k \in \mathbb{Z}$, condition (2.1) is equivalent to

$$\widehat{\chi}(2k\pi) = \begin{cases} 1, & k = 0, \\ 0, & k \in \mathbb{Z}\backslash\{0\}, \end{cases}$$

where $\widehat{\chi}(v) = \int_{-\infty}^{+\infty} \chi(u)e^{-ivu}\,\mathrm{d}u$, $v \in \mathbb{R}$, is the Fourier transform of χ (see [12, Lemma 4.2] and [10]). We also observe that the moment condition (2.2) is clearly satisfied if $\chi(x) = \mathcal{O}(x^{-1-\beta-\epsilon})$ for $x \to \pm\infty$ and some $\epsilon > 0$. Note that, in the case of equally spaced sequence $t_k = k$, $k \in \mathbb{Z}$, we have that $\Delta = 1$, and (2.3) gives in particular the sampling Kantorovich operators $S_w^\chi f$ centered in the node $\frac{k}{w}$.

Lemma 2.4 [6, Lemma 3.1] *Let χ be a kernel. The following statements hold*

(i) $m_{0,\Pi}(\chi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u - t_k)| < +\infty$;

(ii) for every $\gamma > 0$,

$$\lim_{w\to+\infty} \sum_{|wx-t_k|>\gamma w} |\chi(wx - t_k)| = 0,$$

uniformly with respect to $x \in \mathbb{R}$;

(iii) for every $\gamma > 0$ and $\epsilon > 0$ there exists a constant $M > 0$ such that

$$\int_{|x|>M} w|\chi(wx - t_k)|\,\mathrm{d}x < \epsilon$$

for every sufficiently large $w > 0$ and t_k such that $\frac{t_k}{w} \in [-\gamma, \gamma]$.

Some classical examples of kernels satisfying the above assumptions, with $t_k = k$, are reported here. The *Fejér's kernel* (Fig. 1a) is defined by

$$F(x) = \frac{1}{2}\mathrm{sinc}^2\left(\frac{x}{2}\right),$$

where the well-known sinc function is

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

The *Jackson kernels* (Fig. 1b) are defined by

$$J_n(x) = c_n \mathrm{sinc}^{2n}\left(\frac{x}{2n\pi\alpha}\right)$$

where $n \in \mathbb{N}^+$, $\alpha \geq 1$ and $c_n = \left(\int_{-\infty}^{+\infty} \mathrm{sinc}^{2n}\left(\frac{u}{2n\pi\alpha}\right)\mathrm{d}u\right)^{-1}$ are normalization coefficients.

Finally, the *B-spline of order n* (see e.g., Fig. 1c for $n = 3$) is the kernel defined by

$$M_n(x) = \frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\max\left(\frac{n}{2} + x - j, 0\right)^{n-1}.$$

Note that the B-splines have compact supports (more precisely, the interval $[-\frac{n}{2}, \frac{n}{2}]$, where n is the order of the B-spline), while the Fejér and the Jackson kernels have unbounded supports. A kernel with compact support is useful if one want to avoid the truncation error in the series (1.1), (1.2), or (1.4), see e.g., [4, 6].

Many other examples of kernel functions satisfy the assumptions above formulated, as the reader can see in e.g., [6, 10].

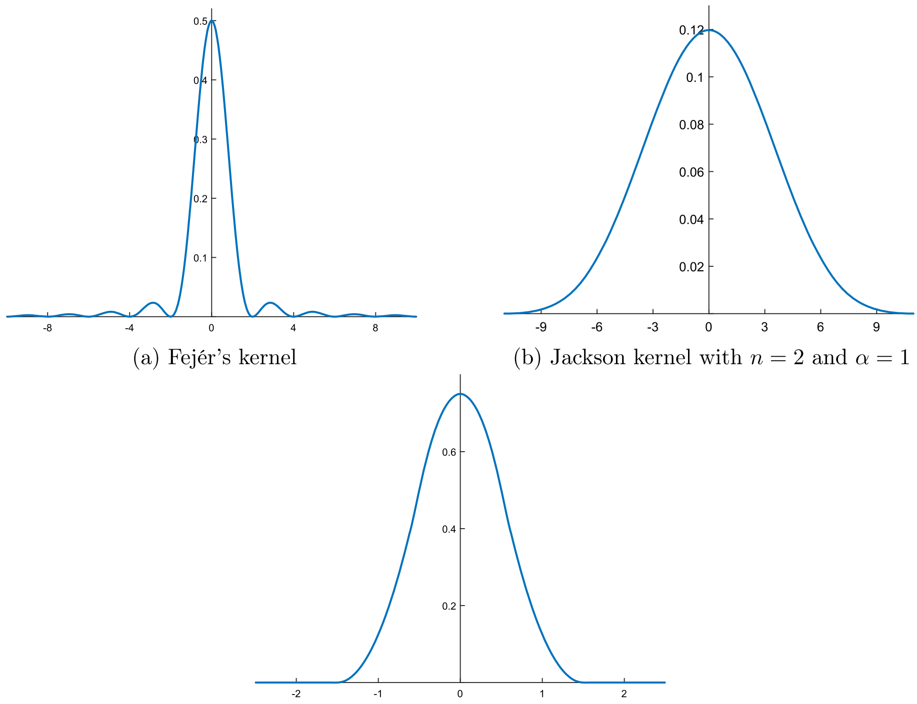

Fig. 1 Some examples of kernels

3 Convergence and quantitative results

3.1 Convergence and quantitative results with respect to the uniform norm

Before to prove regularization and convergence results about the above operator, we remark that by Lemma 2.4(i), $S^{\chi}_{w_1,w_2} f$ is well-defined if $f \in L^{\infty}(\mathbb{R})$ and, in particular,

$$|(S^{\chi}_{w_1,w_2} f)(x)| \leq m_{0,\Pi}(\chi) \| f \|_{\infty}, \qquad \forall x \in \mathbb{R}. \tag{3.1}$$

The Mean sampling Kantorovich operators possess good properties of continuity as stated by the following result.

Proposition 3.1 *Let $f \in L^{\infty}(\mathbb{R})$ and $w_1, w_2 > 0$.*

(i) If χ is a continuous, then $S^{\chi}_{w_1,w_2} f$ is continuous.
(ii) If χ is a uniformly continuous, then $S^{\chi}_{w_1,w_2} f$ is uniformly continuous.

Proof Let $\epsilon > 0$. By Lemma 2.4(i), there exists $K \in \mathbb{N}^{+}$ such that $\sum_{|k|>K} |\chi(w_1 x - t_k)| < \epsilon$ for every $x \in \mathbb{R}$. Let us define

$$s_K(x) = \sum_{|k|\leq K} \chi(w_1 x - t_k) \left[w_2 \int_{\frac{t_k}{w_1} - \frac{1}{2w_2}}^{\frac{t_k}{w_1} + \frac{1}{2w_2}} f(u)\, \mathrm{d}u \right].$$

Clearly, if χ is (uniformly) continuous, then the same holds for s_K. Moreover, we have

$$\begin{aligned}|(S^{\chi}_{w_1,w_2}f)(x) - s_K(x)| &\leq \sum_{|k|>K} |\chi(w_1 x - t_k)| \left| w_2 \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} f(u)\,\mathrm{d}u \right|, \\ &\leq \|f\|_\infty \sum_{|k|>K} |\chi(w_1 x - t_k)| < \epsilon \|f\|_\infty,\end{aligned}$$

for every $x \in \mathbb{R}$, i.e. $S^{\chi}_{w_1,w_2} f$ is the uniform limit of s_K as $K \to +\infty$. Therefore, $S^{\chi}_{w_1,w_2} f$ inherits the property of s_K, i.e. it is (uniformly) continuous if χ is (uniformly) continuous. □

We now present our first approximation result. More precisely, we prove that $S^{\chi}_{w_1,w_2} f$ converges, for $w_1 \to +\infty$, to the convolution $f * \rho_{w_2}$ with respect to the norm $\|\cdot\|_\infty$, where $(f * \rho_{w_2})(x) = w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u$ is the mean of f in the interval of length $\frac{1}{w_2}$, centered in x. Note that $\int_{\mathbb{R}} \rho_{w_2}(x)\,\mathrm{d}x = 1$ and $f * \rho_{w_2} \in C(\mathbb{R})$, since it is an absolutely continuous function.

Theorem 3.2 *Let $f \in L^\infty(\mathbb{R})$ and $w_2 > 0$. Then,*

$$\lim_{w_1 \to +\infty} \|S^{\chi}_{w_1,w_2} f - f * \rho_{w_2}\|_\infty = 0.$$

Proof Let $\epsilon > 0$. By Lemma 2.4(ii), there exists $\overline{w}_1$ such that

$$\sum_{|w_1 x - t_k| > \epsilon w_1} |\chi(w_1 x - t_k)| < \epsilon, \tag{3.2}$$

for any $w_1 > \overline{w}_1$ and any $x \in \mathbb{R}$. Let us fix $w_1 > \overline{w}_1$ and $x \in \mathbb{R}$. By (2.1), we can write

$$(f * \rho_{w_2})(x) = \sum_{k \in \mathbb{Z}} \chi(w_1 x - t_k)(f * \rho_{w_2})(x)$$

and so

$$|(S^{\chi}_{w_1,w_2} f)(x) - (f * \rho_{w_2})(x)| \leq I_1 + I_2, \tag{3.3}$$

where

$$I_1 = \sum_{|w_1 x - t_k| \leq w_1 \epsilon} |\chi(w_1 x - t_k)| w_2 \left| \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} f(u)\,\mathrm{d}u - \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u \right|$$

and

$$I_2 = \sum_{|w_1 x - t_k| > w_1 \epsilon} |\chi(w_1 x - t_k)| w_2 \left| \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} f(u)\,\mathrm{d}u - \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u \right|.$$

For I_1, we will use the following estimate

$$\begin{aligned}\left| \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} f(u)\,\mathrm{d}u - \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u \right| &= \left| \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{x-\frac{1}{2w_2}} f(u)\,\mathrm{d}u - \int_{\frac{t_k}{w_1}+\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u \right| \\ &\leq 2\|f\|_\infty \left| x - \frac{t_k}{w_1} \right|,\end{aligned}$$

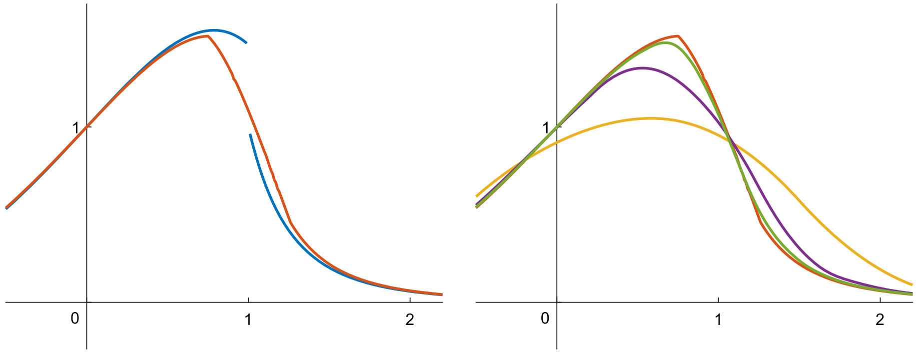

Fig. 2 On the left: a function f (in blue) and its convolution $f * \rho_1$ (in red), in case of $w_2 = 1$. On the right: Comparison between $f * \rho_1$ (in red) and the functions $S^{M_3}_{w_1,1} f$ for different values of w_1 (orange $w_1 = \frac{1}{2}$, purple $w_1 = 1$ and green $w_1 = 2$) (Color figure online)

so that

$$I_1 \leq 2w_2 \|f\|_\infty \epsilon \sum_{|w_1 x - t_k| \leq w_1 \epsilon} |\chi(w_1 x - t_k)| < 2w_2 \|f\|_\infty m_{0,\Pi}(\chi)\epsilon. \tag{3.4}$$

Moreover, by (3.2),

$$I_2 \leq 2\|f\|_\infty \epsilon. \tag{3.5}$$

In conclusion, the statement follows by the estimates (3.3), (3.4) and (3.5). □

Remark 3.3 Note that in Theorem 3.2 the convergence is with respect to the uniform norm.

In Fig. 2 we can visualize the result of Theorem 3.2, i.e. the approximation of $f * \rho_{w_2}$ by the operators $S^{M_3}_{w_1,w_2} f$ (with B-spline kernel M_3) as w_1 increases and $w_2 = 1$. Here, the function f is of the form

$$f(x) = \begin{cases} e^x \cos x & x \leq 1 \\ x^{-4} & x > 1 \end{cases}$$

and $t_k = k$ for every $k \in \mathbb{Z}$.

Note that Fig. 2, also shows that even though f is not continuous, $S^{\chi}_{w_1,w_2} f$ is continuous since $\chi = M_3$ is continuous, according to Prop. 3.1.

Remark 3.4 As shown in Theorem 3.2, when $w_1 \to +\infty$ the operators $S^{\chi}_{w_1,w_2} f$ are not reconstruction operators for the function f, but actually for the convolution $f * \rho_{w_2}$. However, the operators $S^{\chi}_{w_1,w_2}$ reconstruct f if we also let w_2 to $+\infty$ and the function f is continuous. More precisely, if $f : \mathbb{R} \to \mathbb{R}$ is bounded, then

$$\lim_{w_2 \to +\infty} \lim_{w_1 \to +\infty} (S^{\chi}_{w_1,w_2} f)(x) = f(x), \tag{3.6}$$

for any point $x \in \mathbb{R}$ of continuity of f. The result is a consequence of Theorem 3.2 and of the mean integral theorem and it can be visualized in Fig. 3, where the plots of the function $f(x) = 2e^{-x^2} + \arctan x$ and of $S^{M_3}_{w_1,w_2} f$ with large value of w_1 ($w_1 = 100$) and increasing values of w_2 (namely, $w_2 = \frac{1}{3}$, $\frac{1}{2}$ and 1) are shown.

Furthermore, as said in Remark 2.3, if Π is uniformly distributed and $w_1 = w$, $w_2 = \frac{w}{\Delta}$, then $S^{\chi}_{w,\frac{w}{\Delta}}$ is a sampling Kantorovich operator (with mean centered in the node) and so we

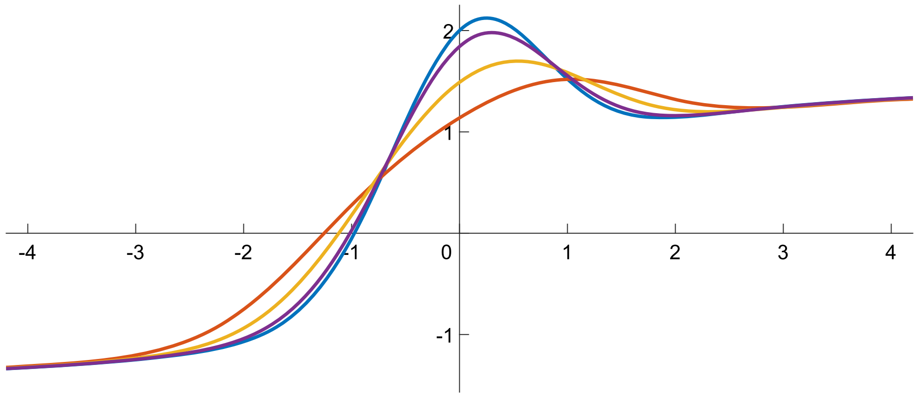

Fig. 3 Comparison between the function $f(x) = 2e^{-x^2} + \arctan x$ (in blue) and functions $S^{M_3}_{100,w_2} f$ for different values of w_2 (red $w_2 = \frac{1}{3}$, orange $w_2 = \frac{1}{2}$ and purple $w_2 = 1$) (Color figure online)

obtain the classical result ([6])

$$\lim_{w\to+\infty} (S^{\chi}_{w,\frac{w}{\Delta}} f)(x) = f(x).$$

Now, in relation to Theorem 3.2, we state a quantitative result for the rate of convergence in terms of the usual modulus of continuity, defined, for $f : \mathbb{R} \to \mathbb{R}$ bounded, by

$$\omega(f,\delta) = \sup_{|x-y|<\delta} |f(x) - f(y)|.$$

Theorem 3.5 *Let χ be a kernel such that $m_{1,\Pi}(\chi) < +\infty$ and $w_1, w_2 > 0$. Then, if $f \in L^\infty(\mathbb{R})$ and $\delta > 0$, we have*

$$\|S^{\chi}_{w_1,w_2} f - f * \rho_{w_2}\|_\infty \le \left(m_{0,\Pi}(\chi) + \frac{m_{1,\Pi}(\chi)}{\delta w_1} \right) \omega(f,\delta).$$

In particular, for $\delta = \frac{1}{w_1}$, $\|S^{\chi}_{w_1,w_2} f - f * \rho_{w_2}\|_\infty = \mathcal{O}(\omega(f, \frac{1}{w_1}))$ as $w_1 \to +\infty$.

Proof By using the well-known property

$$\omega(f,\lambda\delta) \le (1+\lambda)\omega(f,\delta), \qquad \lambda,\delta > 0,$$

for any $x \in \mathbb{R}$ we have

$$\begin{aligned}
&|(S^{\chi}_{w_1,w_2} f)(x) - (f * \rho_{w_2})(x)| \\
&\le \sum_{k\in\mathbb{Z}} |\chi(w_1 x - t_k)| w_2 \left| \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} f(u)\,\mathrm{d}u - \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u \right| \\
&= \sum_{k\in\mathbb{Z}} |\chi(w_1 x - t_k)| w_2 \left| \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} f(u)\,\mathrm{d}u - \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} f\left(u + x - \frac{t_k}{w_1}\right)\mathrm{d}u \right| \\
&\le \sum_{k\in\mathbb{Z}} |\chi(w_1 x - t_k)| w_2 \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} \left| f(u) - f\left(u + x - \frac{t_k}{w_1}\right)\right| \mathrm{d}u \\
&\le \sum_{k\in\mathbb{Z}} |\chi(w_1 x - t_k)| \omega\left(f, \left| x - \frac{t_k}{w_1}\right|\right)
\end{aligned}$$

$$\leq \sum_{k\in\mathbb{Z}} |\chi(w_1 x - t_k)| \left(1 + \frac{1}{\delta}\left|x - \frac{t_k}{w_1}\right|\right) \omega(f, \delta)$$
$$\leq \left(m_{0,\Pi}(\chi) + \frac{m_{1,\Pi}(\chi)}{\delta w_1}\right) \omega(f, \delta).$$

Finally, the second statement is obtained by taking $\delta = \frac{1}{w_1}$. □

Remark 3.6 (i) Theorem 3.5, for $\delta = \frac{1}{w_1}$, gives another proof of the convergence of $S^{\chi}_{w_1,w_2} f$ to $f * \rho_{w_2}$ as $w_1 \to +\infty$ and with respect the uniform norm, under the assumptions that $m_{1,\Pi}(\chi) < +\infty$ and that f is uniformly continuous (for which we have that $\omega(f, \frac{1}{w_1}) \to 0$ as $w_1 \to +\infty$).

(ii) Under the assumption of Theorem 3.5, we obtain also qualitative estimates for the rate of convergence when the function belongs to the Lipschitz class

$$\mathrm{Lip}_\infty(\alpha) = \{f \in C(\mathbb{R}) : \omega(f, \delta) = \mathcal{O}(\delta^\alpha), \text{ for } \delta \to 0\},$$

where $0 < \alpha \leq 1$.
Indeed, if $f \in \mathrm{Lip}_\infty(\alpha)$, then $\|S^{\chi}_{w_1,w_2} f - f * \rho_{w_2}\|_\infty = \mathcal{O}(\frac{1}{w_1^\alpha})$ for $w_1 \to +\infty$.

3.2 Convergence and quantitative results in L^p norms

In the setting of the $L^p(\mathbb{R})-$spaces, we can state the following results. We firstly remark that if $f \in L^p(\mathbb{R})$, then $f * \rho_{w_2} \in L^p(\mathbb{R})$.

Proposition 3.7 *Let $w_1, w_2 > 0$ and $1 \leq p < +\infty$. Then, $S^{\chi}_{w_1,w_2} f$ is well-defined for any $f \in L^p(\mathbb{R})$ and*

$$\|S^{\chi}_{w_1,w_2} f\|_p \leq \left(\frac{w_2}{w_1}\right)^{\frac{1}{p}} \left\lceil \frac{w_1}{\underline{\Delta} w_2} \right\rceil^{\frac{1}{p}} m_{0,\Pi}(\chi)^{\frac{p-1}{p}} \|\chi\|_1^{\frac{1}{p}} \|f\|_p,$$

where $\underline{\Delta}$ is the lower bound of the sequence $\{\Delta_k\}_{k\in\mathbb{Z}}$.
In particular, $S^{\chi}_{w_1,w_2} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is a bounded operator.

Proof Making use of Jensen inequality twice, being $|\cdot|^p$, $p \geq 1$, convex and of Fubini–Tonelli theorem, we can write

$$\int_{\mathbb{R}} |S^{\chi}_{w_1,w_2} f(x)|^p \,\mathrm{d}x \leq \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{|\chi(w_1 x - t_k)|}{m_{0,\Pi}(\chi)} \left[w_2 \int_{\frac{t_k}{w_1} - \frac{1}{2w_2}}^{\frac{t_k}{w_1} + \frac{1}{2w_2}} m_{0,\Pi}(\chi)^p |f(u)|^p \,\mathrm{d}u \right] \mathrm{d}x$$
$$\leq m_{0,\Pi}(\chi)^{p-1} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} |\chi(w_1 x - t_k)| \,\mathrm{d}x \left[w_2 \int_{\frac{t_k}{w_1} - \frac{1}{2w_2}}^{\frac{t_k}{w_1} + \frac{1}{2w_2}} |f(u)|^p \,\mathrm{d}u \right]$$
$$\leq \frac{w_2}{w_1} \left\lceil \frac{w_1}{\underline{\Delta} w_2} \right\rceil m_{0,\Pi}(\chi)^{p-1} \|\chi\|_1 \|f\|_p^p,$$

where we used the fact that $\int_{\mathbb{R}} |\chi(w_1 x - t_k)| \,\mathrm{d}x = w_1^{-1} \int_{\mathbb{R}} |\chi(u)| \,\mathrm{d}u = w_1^{-1} \|\chi\|_1$ and

$$\sum_{k\in\mathbb{Z}} \int_{\frac{t_k}{w_1} - \frac{1}{2w_2}}^{\frac{t_k}{w_1} + \frac{1}{2w_2}} |f(u)|^p \,\mathrm{d}u \leq \left\lceil \frac{w_1}{\underline{\Delta} w_2} \right\rceil \int_{\mathbb{R}} |f(u)|^p \,\mathrm{d}u, \tag{3.7}$$

being $\lceil\cdot\rceil$ the ceiling function. In the previous inequality, the number $\left\lceil \frac{w_1}{\Delta w_2} \right\rceil$ takes into account the possible overlapping of the intervals $\left[\frac{t_k}{w_1} - \frac{1}{2w_2}, \frac{t_k}{w_1} + \frac{1}{2w_2}\right]$. Hence

$$\|S^{\chi}_{w_1,w_2} f\|_p \leq \left(\frac{w_2}{w_1}\right)^{\frac{1}{p}} \left\lceil \frac{w_1}{\Delta w_2} \right\rceil^{\frac{1}{p}} m_{0,\Pi}(\chi)^{\frac{p-1}{p}} \|\chi\|_1^{\frac{1}{p}} \|f\|_p.$$

□

As stated in the following theorem, the convergence of the Mean sampling Kantorovich operator, for $w_1 \to +\infty$, holds also in the L^p setting.

Theorem 3.8 *Let $w_2 > 0$ and $1 \leq p < +\infty$. Then, for any $f \in L^p(\mathbb{R})$, we have*

$$\lim_{w_1 \to +\infty} \|S^{\chi}_{w_1,w_2} f - f * \rho_{w_2}\|_p = 0. \tag{3.8}$$

Proof We prove the statement in two steps. Firstly, let us assume that $f \in C_c(\mathbb{R})$. There exists an interval $[-\gamma, \gamma]$ containing the support of f; therefore, if $I = \{k \in \mathbb{Z} : [\frac{t_k}{w_1} - \frac{1}{2w_2}, \frac{t_k}{w_1} + \frac{1}{2w_2}] \cap [-\gamma, \gamma] \neq \emptyset\}$, then I is a finite set and we have

$$\int_{\frac{t_k}{w_1} - \frac{1}{2w_2}}^{\frac{t_k}{w_1} + \frac{1}{2w_2}} |f(u)|\, du = 0, \qquad k \notin I.$$

Let $\epsilon > 0$. Then, by Theorem 3.2, $\|S^{\chi}_{w_1,w_2} f - f * \rho_{w_2}\|_\infty < \epsilon$ for sufficiently large $w_1 > 0$ and, by Lemma 2.4(iii), there exists $M > 0$ such that

$$\int_{|x|>M} w_1 |\chi(w_1 x - t_k)|\, dx < \epsilon$$

for sufficiently large $w_1 > 0$ and any $t_k \in [-\gamma w_1, \gamma w_1]$. Moreover, by Jensen inequality (applied twice) and Fubini–Tonelli theorem, we have

$$\begin{aligned}
&\int_{|x|>M} |S^{\chi}_{w_1,w_2} f(x)|^p\, dx \\
&\quad \leq \frac{w_2\, m_{0,\Pi}(\chi)^p}{w_1\, m_{0,\Pi}(\chi)} \sum_{k \in I} \int_{\frac{t_k}{w_1} - \frac{1}{2w_2}}^{\frac{t_k}{w_1} + \frac{1}{2w_2}} |f(u)|^p\, du \left(\int_{|x|>M} w_1 |\chi(w_1 x - t_k)|\, dx\right) \\
&\quad \leq \epsilon \frac{w_2}{w_1} m_{0,\Pi}(\chi)^{p-1} \sum_{k \in I} \int_{\frac{t_k}{w_1} - \frac{1}{2w_2}}^{\frac{t_k}{w_1} + \frac{1}{2w_2}} |f(u)|^p\, du \\
&\quad \leq \left(\left\lfloor \frac{w_1}{\Delta}\left(2\gamma + \frac{1}{w_2}\right)\right\rfloor + 1\right) \frac{1}{w_1} m_{0,\Pi}(\chi)^{p-1} \|f\|_\infty^p \epsilon,
\end{aligned}$$

where $\left(\left\lfloor \frac{w_1}{\Delta}\left(2\gamma + \frac{1}{w_2}\right)\right\rfloor + 1\right)$ is a bound for the cardinality of the set I ($\lfloor\cdot\rfloor$ denotes the integer part). Moreover, we stress that $\left(\left\lfloor \frac{w_1}{\Delta}\left(2\gamma + \frac{1}{w_2}\right)\right\rfloor + 1\right)\frac{1}{w_1}$ can be obviously bounded by a number independent of w_1, for $w_1 \geq 1$.
Furthermore, let $B \subset \mathbb{R}$ a measurable set such that $|B| < \frac{\epsilon}{m_{0,\Pi}(\chi)\|f\|_\infty}$. Then, by (3.1),

$$\int_B |S^{\chi}_{w_1,w_2} f(x)|\, dx \leq \int_B m_{0,\Pi}(\chi)\|f\|_\infty\, dx < \epsilon.$$

Hence, $(S^{\chi}_{w_1,w_2} f)_{w_1>0}$ has equi-absolutely continuous integrals and we can apply the Vitali convergence theorem to obtain (3.8).

We have proved (3.8) for $f \in C_c(\mathbb{R})$. Now we extend the validity of (3.8) to any $f \in L^p(\mathbb{R})$ taking into account that $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ and that $S^{\chi}_{w_1,w_2} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ and the convolution product are bounded operators. More precisely, let $f \in L^p(\mathbb{R})$ and $\epsilon > 0$. Then there exists $g \in C_c(\mathbb{R})$ such that

$$\|f - g\|_p < \epsilon$$

and there exists $\overline{w_1} > 0$ such that

$$\|S^{\chi}_{w_1,w_2} g - g * \rho_{w_2}\|_p < \epsilon, \qquad \forall w_1 > \overline{w_1},$$

by Theorem 3.2. Therefore,

$$\|S^{\chi}_{w_1,w_2} f - S^{\chi}_{w_1,w_2} g\|_p \leq \|f - g\|_p \|S^{\chi}_{w_1,w_2}\|_p < \epsilon \|S^{\chi}_{w_1,w_2}\|_p$$

and

$$\|g * \rho_{w_2} - f * \rho_{w_2}\|_p \leq \|g - f\|_p < \epsilon.$$

Hence, for $w_1 > \overline{w_1}$, we have

$$\begin{aligned}\|S^{\chi}_{w_1,w_2} f - f * \rho_{w_2}\|_p &\leq \|S^{\chi}_{w_1,w_2} f - S^{\chi}_{w_1,w_2} g\|_p + \|S^{\chi}_{w_1,w_2} g - g * \rho_{w_2}\|_p \\ &\quad + \|g * \rho_{w_2} - f * \rho_{w_2}\|_p < \epsilon(\|S^{\chi}_{w_1,w_2}\|_p + 2).\end{aligned}$$

This completes the proof. □

Finally, we state a corresponding theorem for the rate of convergence in L^p norms. We will formulate it in terms of the modulus of smoothness of $f \in L^p(\mathbb{R})$, defined by,

$$\omega(f, \delta)_p = \sup_{0<h<\delta} \left(\int |f(x+h) - f(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad \delta > 0.$$

Theorem 3.9 *Let χ be a kernel such that*

$$M_p(\chi) = \left(\int_{\mathbb{R}} |u|^p |\chi(u)| \, du \right)^{\frac{1}{p}} < +\infty, \tag{3.9}$$

$w_1, w_2 > 0$ and $1 \leq p < +\infty$. Then, if $f \in L^p(\mathbb{R})$ and $\delta > 0$, we have

$$\|S^{\chi}_{w_1,w_2} f - f * \rho_{w_2}\|_p \leq \left(\frac{w_2}{w_1}\right)^{\frac{1}{p}} \left\lceil \frac{w_1}{\underline{\Delta} w_2} \right\rceil^{\frac{1}{p}} m_{0,\Pi}(\chi)^{\frac{p-1}{p}} \left(\|\chi\|_1^{\frac{1}{p}} + \frac{M_p(\chi)}{w_1 \delta} \right) \omega(f, \delta)_p,$$

where $\underline{\Delta}$ is the lower bound of the sequence $\{\Delta_k\}_{k\in\mathbb{Z}}$.

In particular, for $\delta = \frac{1}{w_1}$, $\|S^{\chi}_{w_1,w_2} f - f * \rho_{w_2}\|_p = \mathcal{O}(\omega(f, \frac{1}{w_1})_p)$ as $w_1 \to +\infty$.

Proof First of all, as in the proof of Theorem 3.5, for any $x \in \mathbb{R}$

$$\begin{aligned}&|(S^{\chi}_{w_1,w_2} f)(x) - (f * \rho_{w_2})(x)| \\ &\leq \sum_{k\in\mathbb{Z}} |\chi(w_1 x - t_k)| w_2 \int_{\frac{t_k}{w_1} - \frac{1}{2w_2}}^{\frac{t_k}{w_1} + \frac{1}{2w_2}} \left| f(u) - f\left(u + x - \frac{t_k}{w_1}\right) \right| du.\end{aligned} \tag{3.10}$$

Moreover, by Jensen inequality, Fubini–Tonelli theorem, the change of variable $y = x - \frac{t_k}{w_1}$ and inequality (3.7), we get

$$\begin{aligned}
&\int_{\mathbb{R}} |(S^{\chi}_{w_1,w_2} f)(x) - (f * \rho_{w_2})(x)|^p \, \mathrm{d}x \\
&\le m_{0,\Pi}(\chi)^{p-1} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(w_1 x - t_k)| w_2 \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} \left| f(u) - f\left(u + x - \frac{t_k}{w_1}\right)\right|^p \mathrm{d}u \, \mathrm{d}x \\
&\le m_{0,\Pi}(\chi)^{p-1} \int_{\mathbb{R}} |\chi(w_1 y)| w_2 \sum_{k\in\mathbb{Z}} \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} |f(u) - f(u+y)|^p \, \mathrm{d}u \, \mathrm{d}y \\
&\le w_2 m_{0,\Pi}(\chi)^{p-1} \int_{\mathbb{R}} |\chi(w_1 y)| \left\lceil \frac{w_1}{\Delta w_2} \right\rceil \int_{\mathbb{R}} |f(u) - f(u+y)|^p \, \mathrm{d}u \, \mathrm{d}y \\
&\le w_2 \left\lceil \frac{w_1}{\Delta w_2} \right\rceil m_{0,\Pi}(\chi)^{p-1} \int_{\mathbb{R}} |\chi(w_1 y)| \omega(f, |y|)_p^p \, \mathrm{d}y.
\end{aligned}$$

Now, we apply the well-know inequality

$$\omega(f, \lambda\delta)_p \le (1+\lambda)\omega(f,\delta)_p, \qquad \lambda, \delta > 0,$$

to obtain

$$\begin{aligned}
&\int_{\mathbb{R}} |(S^{\chi}_{w_1,w_2} f)(x) - (f * \rho_{w_2})(x)|^p \, \mathrm{d}x \\
&\le w_2 \left\lceil \frac{w_1}{\Delta w_2} \right\rceil m_{0,\Pi}(\chi)^{p-1} \omega(f,\delta)_p^p \int_{\mathbb{R}} |\chi(w_1 y)| \left(1 + \frac{|y|}{\delta}\right)^p \mathrm{d}y.
\end{aligned}$$

Hence, by Minkowski inequality,

$$\begin{aligned}
&\| S^{\chi}_{w_1,w_2} f - f * \rho_{w_2} \|_p \\
&\le w_2^{\frac{1}{p}} \left\lceil \frac{w_1}{\Delta w_2} \right\rceil^{\frac{1}{p}} m_{0,\Pi}(\chi)^{\frac{p-1}{p}} \omega(f,\delta)_p \left(\left(\int_{\mathbb{R}} |\chi(w_1 y)| \, \mathrm{d}y \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} |\chi(w_1 y)| \frac{|y|^p}{\delta^p} \, \mathrm{d}y \right)^{\frac{1}{p}} \right) \\
&\le w_2^{\frac{1}{p}} \left\lceil \frac{w_1}{\Delta w_2} \right\rceil^{\frac{1}{p}} m_{0,\Pi}(\chi)^{\frac{p-1}{p}} \left(w_1^{-\frac{1}{p}} \|\chi\|_1^{\frac{1}{p}} + w_1^{-1-\frac{1}{p}} \frac{M_p(\chi)}{\delta} \right) \omega(f,\delta)_p \\
&\le \left(\frac{w_2}{w_1}\right)^{\frac{1}{p}} \left\lceil \frac{w_1}{\Delta w_2} \right\rceil^{\frac{1}{p}} m_{0,\Pi}(\chi)^{\frac{p-1}{p}} \left(\|\chi\|_1^{\frac{1}{p}} + \frac{M_p(\chi)}{w_1 \delta} \right) \omega(f,\delta)_p.
\end{aligned}$$

This proves the main estimate and, to conclude, the second statement is obtained by choosing $\delta = \frac{1}{w_1}$. □

We remark that condition (3.9) is satisfied if $\chi(x) = \mathcal{O}(x^{-1-p-\epsilon})$ for $x \to \pm\infty$ and some $\epsilon > 0$ (in particular, by the B-spline kernels). Moreover, for $\delta = \frac{1}{w_1}$, as a consequence of Theorem 3.9 and the fact that $\lim_{w_1\to+\infty} \omega\left(f, \frac{1}{w_1}\right)_p = 0$, we obtain another proof of the convergence in L^p norms of $S^{\chi}_{w_1,w_2} f$ to $f * \rho_{w_2}$ as $w_1 \to +\infty$, under assumption (3.9). Finally, an observation similar to Remark 3.6(ii) can be made if $\omega(f,\delta)_p$ has a particular behavior for $\delta \to 0$.

4 Applications

The Mean sampling Kantorovich operator $S^{\chi}_{w_1,w_2} f$ and the sampling Kantorovich operator $S^{\chi}_{w_1} f$ share a very similar formulation: for each node $\frac{t_k}{w_1}$, the mean of the function f in a neighborhood of the node is calculated and multiplied by $\chi(w_1 x - t_k)$. The difference between $S^{\chi}_{w_1,w_2} f$ and $S^{\chi}_{w_1} f$ relies on the dependence of the interval length on the parameters. For the operator $S^{\chi}_{w_1} f$ the interval length depends on w_1, while for $S^{\chi}_{w_1,w_2} f$ the interval length depends only on w_2, which is independent of w_1. This implies, in particular, that with $S^{\chi}_{w_1,w_2} f$ we have more freedom on the choice of the interval where the mean is calculated and that the length does not change if w_1 varies.

An application of this flexibility is the approximation of a signal corrupted by a noise: what we have is only the signal $\tilde{f}$ modified by an unknown noise and we want to recover the original signal f, unaffected by the noise. This problem is frequent in the real applications because signals during transmission are often disturbed and then received with some variation (noise).

Let us imagine the following specific situation: the noise acts on a continuous signal f in an additive way, i.e. η is the noise function and $\tilde{f} = f + \eta$ is the noisy signal (by assumption, f and η are locally integrable functions). Moreover, we assume that for every x the value $\eta(x)$ is a realization of a random variable $Z(x)$ with mean μ and variance σ^2 and the distribution is independent of x. In other words, we consider η as a realization of a stochastic process Z with an independent and identically distributed random variable. This is a common mathematical formulation of noises [35].

Reconstructing f by a generalized sampling operator $G^{\chi}_{w_1} \tilde{f}$ is not a good choice since the samples $\tilde{f}(\frac{k}{w_1})$ are very sensitive to the noise, i.e. even a little perturbation of the point leads to a value $\tilde{f}(\frac{k}{w_1} + \epsilon)$ much different. This can be seen in the example of Fig. 4 where the plot of f, of the form

$$f(x) = \begin{cases} x\cos(\frac{\pi}{2}x) & 0 \le x \le 5 \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

is shown in Fig. 4a and its perturbation $\tilde{f}$ by a noise, is shown in Fig. 4b.

Also the sampling Kantorovich operator is not suitable for our problem; indeed, for $w_1 \to +\infty$, $S^{\chi}_{w_1} \tilde{f}$ converges to $\tilde{f}$ (see e.g., [6]), which is not the function we want to recover. In the following we are going to show that the Mean sampling Kantorovich operator $S^{\chi}_{w_1,w_2} \tilde{f}$, is instead more appropriate to the problem.

4.1 Case $\mu = 0$

We begin with by treating the case of $\mu = 0$ (in this case η is a so-called *white* noise). Theorem 3.2 ensures that $S^{\chi}_{w_1,w_2} \tilde{f}$ converges to the convolution $\tilde{f} * \rho_{w_2}$, which is equals in a point x to the mean of $\tilde{f}$ on the interval of the length w_2^{-1} and centered on x. To explain why $\tilde{f} * \rho_{w_2}$ gives a good approximation of f, first of all we start with the following equality

$$(\tilde{f} * \rho_{w_2})(x) = w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} \tilde{f}(u)\,\mathrm{d}u = w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u + w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} \eta(u)\,\mathrm{d}u. \tag{4.2}$$

Now, if w_2 is sufficiently large, then the interval $\left[x - \frac{1}{2w_2}, x + \frac{1}{2w_2}\right]$ is small. Hence, the mean integral theorem and the continuity of f imply that we can approximate the mean of

f on the interval with its value on x, i.e.

$$w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u \approx f(x).$$

Moving to the term concerning η in (4.2), we can argue as follows. If w_2 is sufficiently small, then the interval $\left[x - \frac{1}{2w_2}, x + \frac{1}{2w_2}\right]$ is large. Hence, there are a lot of sampling points for η, say n. Moreover, the average $w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} \eta(u)\,\mathrm{d}u$ can be discretized as the mean of n samples of η in the above interval,

$$w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} \eta(u)\,\mathrm{d}u \approx \frac{1}{n}\sum_{i=1}^{n} \eta(u_i). \tag{4.3}$$

Since the random variables $Z(u_i)$, for $u_i = 1, \ldots, n$, are independent and identically distributed with zero mean and variance σ^2, the variable $\dfrac{1}{n}\sum_{i=1}^{n} Z(u_i)$ has mean equals to zero and variance

$$\frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{\sigma^2}{n}. \tag{4.4}$$

Since n is large, the variance in (4.4) is small, so the mean (4.3) of the samples of η varies little from zero. In conclusion, in (4.2) we can neglect the term with η, i.e.

$$w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} \eta(u)\,\mathrm{d}u \approx 0.$$

Now we want to put it all together. To do this, we choose adequately w_2, in the sense that it must be neither too small nor too large. Then we can apply the considerations for f and η above, and equation (4.2) becomes

$$(\tilde{f} * \rho_{w_2})(x) = w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u + w_2 \int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} \eta(u)\,\mathrm{d}u \approx f(x). \tag{4.5}$$

Note that, the above consideration in (4.5) will be supported below by numerical estimations. Therefore we will prove that $S^{\chi}_{w_1,w_2}\tilde{f}$, for w_1 large and w_2 suitable in the above sense, gives a reasonable approximation of f.

By making a very similar argument, it is possible to prove that the coefficients $w_2 \int_{\frac{t_k}{w_1}-\frac{1}{2w_2}}^{\frac{t_k}{w_1}+\frac{1}{2w_2}} \tilde{f}(u)\,\mathrm{d}u$ in the expression of $S^{\chi}_{w_1,w_2}\tilde{f}$ are few sensitive to the noise for large w_1 and suitable w_2. In contrast, like in the case of the generalized sampling operator, the coefficients in the expression of $S^{\chi}_{w_1}\tilde{f}$ are sensitive to the noise for large w_1, (i.e., when applying the convergence theorem) because the interval $[\frac{t_k}{w_1}, \frac{t_{k+1}}{w_1}]$, on which the mean is calculated, has a small length $\frac{\Delta_k}{w_1}$ and hence the mean of η on this interval cannot be neglected according to the arguments above mentioned.

In the following, we illustrate some graphical and numerical examples to support our discussion of the approximation of a function f by the Mean sampling Kantorovich operator $S^{\chi}_{w_1,w_2}\tilde{f}$. We consider once again Fig. 4, the kernel $\chi = M_3$ and the parameters $w_1 = 100$ and $w_2 = 10$. As already said, the original function f with expression (4.1) is displayed in

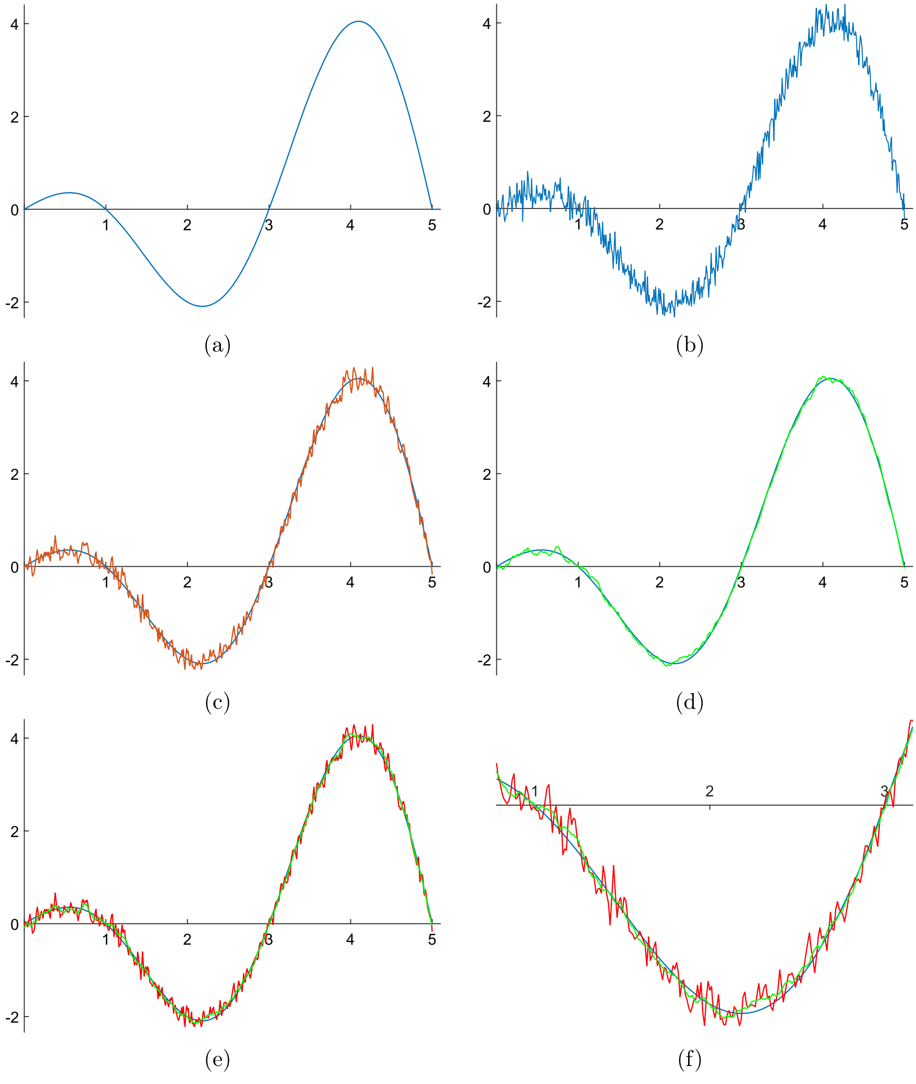

Fig. 4 Comparisons between the function f given by (4.1), the function $\tilde{f} = f + \eta$ (f corrupted by a noise η following a normal distribution with zero mean), and the functions $S_{100}^{M_3}\tilde{f}$ and $S_{100,10}^{M_3}\tilde{f}$. Here the plots are considered in the interval [0, 5], where the signal has a contribution. **a** Plot of f. **b** Plot of $\tilde{f}$. **c** Plots of f in blue and $S_{100}^{M_3}\tilde{f}$ in red. **d** Plots of f in blue and $S_{100,10}^{M_3}\tilde{f}$ in green. **e** Plots of f, $S_{100}^{M_3}\tilde{f}$ and $S_{100,10}^{M_3}\tilde{f}$ in blue, red and green, respectively. **f** Zoom of the plots of the same functions in (**e**) (colour figure online)

Fig. 4a, while its perturbation $\tilde{f}$, by the noise based on a normal distribution with zero mean, is shown in Fig. 4b. Moreover, we plotted the operators $S_{w_1}^{M_3}\tilde{f}$ and $S_{w_1,w_2}^{M_3}\tilde{f}$ overlapped to f in Fig. 4c, d, respectively, and also all together in Fig. 4e, f. As one can see from the plots, especially from Fig. 4f, the operator $S_{w_1,w_2}^{M_3}\tilde{f}$ gives a better approximation of f (i.e., the signal not affected by the noise) than $S_{w_1}^{M_3}\tilde{f}$, in coherence with the discussion above. Beyond

Table 1 L^∞ errors between f and $S^{M_3}_{w_1,10}\tilde{f}$ and between f and $S^{M_3}_{w_1}\tilde{f}$ for increasing values of w_1. The functions f and $\tilde{f}$ refer to Fig. 4

w_1	$\|f - S^{M_3}_{w_1,10}\tilde{f}\|_\infty$	$\|f - S^{M_3}_{w_1}\tilde{f}\|_\infty$
100	0.138	0.396
200	0.139	0.485
300	0.143	0.558

Table 2 L^2 errors between f and $S^{M_3}_{w_1,10}\tilde{f}$ and between f and $S^{M_3}_{w_1}\tilde{f}$ for increasing values of w_1. The functions f and $\tilde{f}$ refer to Fig. 4

w_1	$\|f - S^{M_3}_{w_1,10}\tilde{f}\|_2$	$\|f - S^{M_3}_{w_1}\tilde{f}\|_2$
100	0.126	0.289
200	0.127	0.380
300	0.127	0.421

the graphical visualization, the better performance of the operator $S^{M_3}_{w_1,w_2}\tilde{f}$ can be seen also in the L^∞ and L^2 errors:

$$\|f - S^{M_3}_{100,10}\tilde{f}\|_\infty = 0.138 \text{ against } \|f - S^{M_3}_{100}\tilde{f}\|_\infty = 0.396$$

and

$$\|f - S^{M_3}_{100,10}\tilde{f}\|_2 = 0.126 \text{ against } \|f - S^{M_3}_{100}\tilde{f}\|_2 = 0.289.$$

In Tables 1 and 2 we report the L^∞ and L^2 errors of approximation between f and the two operators for some increasing values of w_1. As one can see, not only the Mean sampling Kantorovich operator gives better approximations, but also the corresponding errors reach the stability earlier. On the contrary, for increasing values of w_1, the errors between f and $S^{M_3}_{w_1}\tilde{f}$ gets worse because $S^{M_3}_{w_1}\tilde{f}$ converges to the noisy signal $\tilde{f}$. For instance, taking $w_1 = 300$, the errors $\|f - S^{M_3}_{w_1}\tilde{f}\|_\infty$ and $\|f - S^{M_3}_{w_1}\tilde{f}\|_2$ are around the errors between f and $\tilde{f}$, which are $\|f - \tilde{f}\|_\infty = 0.557$ and $\|f - \tilde{f}\|_2 = 0.426$.

Table 3, containing some errors $\|f - S^{M_3}_{w_1,w_2}\tilde{f}\|_\infty$ varying both w_1 and w_2, shows the importance to take for w_2 a suitable value. This can be visualize also in Fig. 5. As explained above, a small value of w_2 determines a large interval $\left[x - \frac{1}{2w_2}, x + \frac{1}{2w_2}\right]$, so the mean of η is based on a lot of numbers and then accurately near to zero (since η has a distribution with zero mean). Consequently, $S^{\chi}_{w_1,w_2}\tilde{f}$ is very smooth containing almost nothing noise, as one can see in the orange plot corresponding to $w_1 = 100$ and $w_2 = 1$. Anyway, since the interval is large, the mean of f on it is not very near to $f(x)$, so the approximation in (4.5) is not accurate. This can be also seen from the distance between $S^{\chi}_{100,1}\tilde{f}$ and f (in blue) in Fig. 5.

On the other hand, if w_2 is too large, then the interval $\left[x - \frac{1}{2w_2}, x + \frac{1}{2w_2}\right]$ is small, the mean of η is made on a few values and thus not very near to zero. Consequently, $S^{\chi}_{w_1,w_2}\tilde{f}$ contains a significant noise and so it is not a good approximation of f (for this case we took as example $w_2 = 50$, red plot in Fig. 5). Finally, as an intermediate value we considered $w_2 = 10$ (green plot) which gives a better approximation in comparison to the others although containing some, but reduced, noise. All these arguments are confirmed by the calculations of the errors $\|f - S^{M_3}_{w_1,w_2}\tilde{f}\|_\infty$ in Table 3 below.

Table 3 L^∞ errors between f and $S^{M_3}_{w_1,w_2}\tilde{f}$ for various values of w_1 and w_2. The functions f and $\tilde{f}$ refer to Fig. 4

		w_2		
		1	10	50
w_1	100	0.755	0.138	0.309
	200	0.758	0.139	0.323
	300	0.790	0.143	0.342

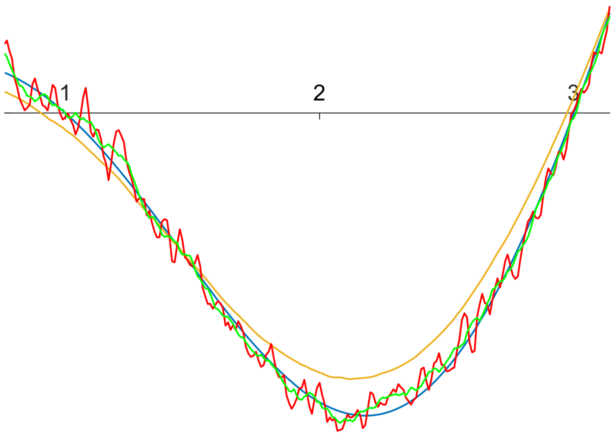

Fig. 5 Comparison between the function f defined by (4.1), in blue, and the functions $S^{M_3}_{100,w_2}\tilde{f}$ for different values of w_2: orange $w_2 = 1$ (small), green $w_2 = 10$ (midway) and red $w_2 = 50$ (large) (Color figure online)

4.2 Case $\mu \neq 0$

To conclude our discussion on the applications, we consider a different case where the noise η follows an identically and independent distribution with mean $\mu \neq 0$. Again, we are able to approximate a continuous signal f by starting from the noisy signal $\tilde{f} = f + \eta$ and by applying the Mean sampling Kantorovich operator $S^{\chi}_{w_1,w_2}\tilde{f}$.

We explain the new idea in details. By the theory, $S^{\chi}_{w_1,w_2}\tilde{f}$ converges to $\tilde{f} * \rho_{w_2}$. If w_2 is suitably chosen in the above sense, then $w_2 \displaystyle\int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} \eta(u)\,\mathrm{d}u$ is a value close to μ, so $(\tilde{f} * \rho_{w_2})(x) \approx w_2 \displaystyle\int_{x-\frac{1}{2w_2}}^{x+\frac{1}{2w_2}} f(u)\,\mathrm{d}u + \mu$, and hence $(\tilde{f} * \rho_{w_2})(x) \approx f(x) + \mu$ since f is continuous. Therefore, to reconstruct f, given $\tilde{f}$, we choose w_1 large (for the convergence), w_2 a midway value (for the reasons above), we determine $S^{\chi}_{w_1,w_2}\tilde{f}$ and we subtract μ to the result.

With the help of Fig. 6, we illustrate this process. In (a) there is the plot of the function f with expression (4.1), while in (b) the plot of function $\tilde{f} = f + \eta$ is shown, where the noise η was chosen to have a normal distribution with mean $\mu = 1$. We apply the Mean sampling Kantorovich operator $S^{M_3}_{100,10}\tilde{f}$ which is displayed in (c) overlapped to $\tilde{f}$. Finally, we showed the functions f and $S^{M_3}_{100,10}\tilde{f} - 1$ in (d), where it is possible to see the good approximation obtained.

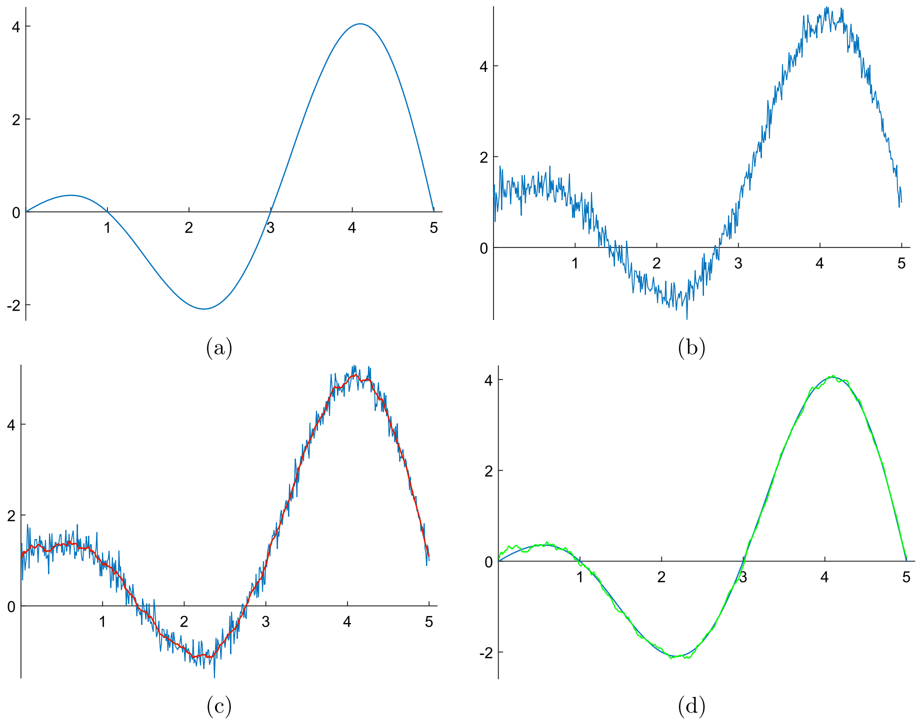

Fig. 6 Case of a noise with mean μ different from zero. **a** Plot of f. **b** Plot of $\tilde{f} = f + \eta$ where η is a noise following a normal distribution with $\mu = 1$. **c** Plots of $\tilde{f}$ in blue and $S^{M_3}_{100,10}\tilde{f}$ in red. **d** Plots of $\tilde{f}$ in blue and $S^{M_3}_{100,10}\tilde{f} - 1$ in green. As before, the plots are considered in the interval [0, 5], where the signal has a contribution (Color figure online)

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Data Availability Not applicable.

Declarations

Conflict of interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

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